Notes for

Waves II

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These notes cover the Waves II course as taught by me, (Colm-cille Caulfield) in Lent 2025. They have grown out of the course as taught previously by Professor John Lister and Dr Stephen Cowley to whom heartfelt thanks are offered for access to their versions. However, the notes have been modified non-trivially in both content and ordering. Naturally, I am responsible for any errors, and would be very glad to be told about any such errors by email to cpc12@cam.ac.uk. I hope these notes prove a useful adjunct to staring at the zoomed lectures. They are **not** intended as a substitute for a good textbook... but are hopefully easier to read than my handwriting...

There is no way to move without making waves
Malcolm Forbes

Part I

(Linear) Sound Waves

Chapter 1

Motion of an Inviscid Compressible Fluid

Too much pressure, it's getting to my head
Neol Davies

Part Overview

In this part, we learn the mathematics underlying how sound travels by considering the motion of an inviscid compressible fluid, deriving equations for mass conservation and momentum conservation. As the schedules note that "discussion of thermodynamics" is not examinable, we skate over some of the subtleties associated with the "equation of state" and the energetics, but we still construct a set of self-consistent equations, demonstrating the importance of the Mach number. We find that a little bit of compressibility in air is essential for sound to propagate. For appropriately small and irrotational perturbations so that we can introduce the concept of the acoustic velocity potential, we obtain the acoustic wave equation, and thus consider linear sound. We consider both plane and spherical waves (as solutions of the acoustic wave equation) and define an array of key concepts: amplitude; frequency; wave number; and phase speed of a wave. These definitions lead naturally to the idea of a dispersion relation, and hence to definitions of isotropy and dispersion (or not!) of waves. We also consider acoustic energy, the specific meanings of intensity, power, energy density and flux in this context, and appreciate how important it is to be careful in time-averaging. We define the decibel, and consider reflection and transmission, thus explaining why you hear more of your neighbour's music if they like reggae than if they like jazz flute, which I think we can all agree is a good thing.

Fluid Assumptions

So, enough of the musical judgements. We will consider a fluid under the continuum hypothesis. Effectively, in the first (but definitely not the last) modelling approximation we make in an attempt to describe the natural world mathematically, we assume that the "mean free path" for collisions of molecules is much, much smaller than characteristic scales of the motion of interest (such as the wavelength of the sound waves). This is reasonable in the Earth's atmosphere and the sea (so dolphins, if they could be bothered, would also be correct to make the continuum hypothesis). The fluid we will consider is assumed to be inviscid (and so $\nu = 0$) yet (crucially) is allowed to be compressible. Under these assumptions, we realise that we need five variables (in general functions of space and time) to describe the system fully:

$$\left.\begin{array}{lcl} \rho(\boldsymbol{x}, t) & : & \text{The density of the fluid at } \boldsymbol{x}, t \\ p(\boldsymbol{x}, t) & : & \text{The pressure in the fluid at } \boldsymbol{x}, t \\ \boldsymbol{u}(\boldsymbol{x}, t) & : & \text{The velocity of the fluid at } \boldsymbol{x}, t \end{array}\right\} \quad \text{5 variables in 3D and } t$$

since in general $\boldsymbol{u} = (u, v, w)$ has three non-zero components. So, we need five equations...

1.1 Mass Conservation

Consider a fixed volume $\mathcal{V}$, with surface $\mathcal{S}$ and outward unit normal $\boldsymbol{n}$. In the simplest case when there are no sources or sinks in $\mathcal{V}$ (and so mass is neither created or destroyed) the mass in this "blob" $\mathcal{V}$ can only change due to a transport of mass either in or out, and hence crossing the surface $\mathcal{S}$: i.e.

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, dV = -\int_{\mathcal{S}} \rho\, \boldsymbol{u}.\boldsymbol{n} dS,$$

where the minus sign comes from the convention that the normal is outward. As the volume is fixed, we can bring the derivative inside the integral on the LHS, while on the RHS we apply the divergence theorem to obtain:

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} dV = -\int_{\mathcal{V}} \boldsymbol{\nabla}.(\rho \boldsymbol{u})\, dV.$$

The volume is (clearly) arbitrary, and so we can obtain the two equivalent forms of the **conservation of mass** equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{u}) = 0, \tag{1.1}$$

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla}\rho = -\rho \boldsymbol{\nabla}.\boldsymbol{u}, \tag{1.2}$$

defining the convective derivative D/Dt.

Note that if ρ is constant in space and time, this equation **implies** that $\nabla.\boldsymbol{u} = 0$. So, if a fluid is incompressible, i.e. if pressure changes do not cause density changes, **then** the velocity field is solenoidal (i.e. divergence-free). Indeed, in that very special case, the "pressure" is actually a (spatio-temporally varying) Lagrange multiplier imposing the constraint on the velocity field that it is solenoidal everywhere at all times, and cannot compress anything. The more you think about that, the weirder it gets, as:

- it implies infinitely fast information propagation;
- in practice it makes numerical approximation of solutions, i.e. "simulation" quite tricky;
- "pressure" does not have a thermodynamic meaning.

Perhaps most significant, at least for this course, is that an incompressible fluid is a silent fluid, (a gift, or at least a blessing in Part II Fluids) and so we need (at least small) compressibility for sound waves to propagate, and so $\nabla.\boldsymbol{u}$ is not necessarily zero.

As we shall also find repeatedly, application of boundary conditions properly is exceptionally important, and as the fluid is inviscid, no tangential stresses can be supported, so at boundaries, we can never (in this course) impose conditions of the form $\boldsymbol{u} \times \boldsymbol{n}$. However, we can impose **kinematic** boundary conditions, i.e. that the normal component of the fluid velocity will match the normal component of a rigid boundary velocity, (loosely "the fluid moves with the boundary") and so $\boldsymbol{u}.\boldsymbol{n} = \mathbf{U}_b.\boldsymbol{n}$, where of course the boundary velocity $\mathbf{U}_b$ can be zero.

1.2 Momentum Conservation

As perhaps you will be completely unsurprised to read, our old Trinmo Homie Newton rides to the rescue again to obtain an equation for the velocity fields. Considering the forces which may cause the **momentum** of our "blob" to change, we have to remember that, as the fluid is inviscid, we can neither force it at the boundary by the imposition of tangential stresses, nor can we expect it to experience internal (viscous stresses). Therefore there are three ways momentum can change:

1. momentum can be transported in (or out) of the volume across the surface $\mathcal{S}$;

2. **normal** stresses (i.e. *pressure*) can be imposed on the surface (remembering stresses are forces per unit area);

3. body forces can apply in the volume (e.g. gravity, thanks again Isaac).

Therefore, once again assuming there are no sources or sinks in $\mathcal{V}$:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \boldsymbol{u}\, dV = -\int_{\mathcal{S}} (\rho \boldsymbol{u})\boldsymbol{u}.\boldsymbol{n} dS - \int_{\mathcal{S}} p\, \mathbf{n} dS + \int_{\mathcal{V}} \rho \mathbf{f}\, dV \ ,$$

where $\mathbf{f}$ is for notational convenience actually an acceleration, i.e. a force per unit mass, like the gravitational acceleration $\mathbf{g}$.

Due to the two velocities in the transport term (or flux term, remembering that flux is transport per unit area), it is (perhaps) clearer to apply the divergence theorem in indicial form using the Einstein summation convention. Therefore, the ith component of the momentum equation is, after applying the divergence theorem,

$$\int_{\mathcal{V}} \frac{\partial}{\partial t}(\rho u_i)\, dV = \int_{\mathcal{V}} \left(-\frac{\partial}{\partial x_j}(\rho u_i u_j) - \frac{\partial p}{\partial x_i} + \rho f_i \right) dV \ ,$$

and once again, since the volume is arbitrary:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \rho f_i \ .$$

Applying the product rule on the LHS to $u_i(\rho u_j)$ and using (1.1), it is possible to obtain the inviscid version of the Navier-Stokes equation, more commonly of course called **Euler's equation**:

$$\rho \frac{D\boldsymbol{u}}{Dt} \equiv \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{u} \quad = \quad -\boldsymbol{\nabla} p + \rho \mathbf{f}. \tag{1.3}$$

Another natural class of boundary conditions (arising from this "Newton's second law" type equation) are **dynamic** boundary conditions. For example, at a *free boundary*, like the atmosphere, $p =$ const., or wlog, $p = 0$. Alternatively, at an interface between different fluids, assuming that there is no surface tension, pressure must be continuous and so $[p]_-^+ = 0$. If there is surface tension, then $[p]_-^+ = T\kappa$, where κ is (here) the curvature and T is the (coefficient) of surface tension. The choice of notation here is very difficult, as T is also used for temperature and (as we shall see) γ (another common notation for surface tension) is also used for the ratio of specific heats. Remember dimensions: pressure is a stress (so force per unit area) while curvature is the inverse of the radius of curvature, and so has dimension L^{-1}. Therefore, surface tension has units of a force per unit length, or equivalently energy per unit area (which helps me at least to understand why surface tension tries to minimise the surface energy of bubbles...) Whatever, between (1.2) and (1.3) (a vector equation remember) we have four equations. We need to plead the fifth...

1.3 Equation of State

Several lectures, indeed several courses, could be given to construct appropriately the required equations using **thermodynamics**. The fundamental issue is that for compressible fluids, the pressure and the density are actually *parameters of state*, required to describe the *state* of matter we are considering, and a third parameter is typically needed: the classic example of course is temperature. Generically, an equation of state is of the form $f(p, V, T) = 0$ where $V = 1/\rho$ is here the (specific) volume and T is the (absolute) temperature. The classic example is of course the *ideal gas law*:

$$p = \rho R T,$$

where R is the specific gas constant. We now have a fifth equation, however unfortunately we have a sixth variable, as in general T can vary. There are (at least) two ways forward.

Approximation

The first is to make an approximation based on separation of time scales. We assume that the time scale over which the sound wave (which we are interested in) propagates is much smaller than the time scale over which (thermal) diffusion may occur. (This is distinct from assuming the fluid is inviscid.) A typical sound wave has period $O(10^{-3})s$ (i.e. with frequencies in the kilohertz) and wavelengths around $\lambda \sim 30cm$. Therefore, the thermal diffusion time scale

$$t_d \sim \frac{\lambda^2}{\kappa} \sim O(10^4)s,$$

(where $\kappa = k/\rho c_P$ is [here!] the thermal diffusivity, k is the thermal conductivity and c_P is the specific heat capacity at constant pressure) and so we can argue that temperature variations are unlikely to matter in the acoustic problems in which we are interested. Phew.

Thermodynamics: stated as dogma

A more rigorously appealing approach is to use thermodynamics. Effectively, we consider a different set of parameters of state, rather considering an equation of state $f_s(p, \rho, S) = 0$ where S is the entropy, loosely the state of disorder, and slightly more formally (in a classical context) the ratio of the amount of heat to the instantaneous temperature (and thus with SI units of Joules per Kelvin) in a body. We then assume *reversible adiabatic motion*. Adiabatic motions do not involve change in the heat (or mass), while reversible motions can be reversed. Loosely, "duh", but that's a bit like the reason for the name of Red Stripe Beer. Why's it called Red Stripe? Because there's a Red Stripe on the label. But why's there a Red Stripe on the label? Because it's called Red Stripe! Hooray Beer. But I digress...More precisely throughout the motion, formally taking an infinitely long time, the system remains in thermodynamic equilibrium with its surroundings, and both the system and its surroundings must be completely unchanged at the beginning and the end of a reversible motion. By the laws of thermodynamics, such a reversible adiabatic motion corresponds to *isentropic motion*, i.e.

$$\frac{DS}{Dt} = 0, \tag{1.4}$$

and indeed it is the second law of thermodynamics that tells us that no real finite time motion can actually be isentropic: things can only get worse after all.

More cheerfully, since the equation of state implies that p can be expressed as some function of ρ and S and so

$$p = p(\rho, S), \tag{1.5}$$

we now have six equations (adding (1.4) and (1.5)) to solve our system for the six variables p, ρ, u, v, w and S, even though the evolution of S is a bit dull.

Indeed, in this part of the course we go even further, and assume that S is constant, taking the same value everywhere, i.e. the motion is *homentropic*. (In the last Part V of the course, when we consider shocks, we need to allow the entropy to take different values either side of the shock). For homentropic flows of an ideal (also sometimes called perfect) gas, the equation of state takes a particularly simple form

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{1.6}$$

where p_0 and ρ_0 are initial/reference values, $\gamma = c_P/c_V$ is the (constant) ratio of specific heats, c_P being the specific heat capacity at constant pressure and

c_V being the specific heat capacity at constant volume. The ratio $\gamma = 5/3$ for a monatomic gas and $\gamma = 7/5$ for a diatomic gas, which (to a very good approximation) is dry air (predominantly N_2).

Laws of Thermodynamics in a bit more detail

This part closely follows the notes of Dr Stephen Cowley. It is non-examinable until further notice.

0th Law: Temperature is a thing

After many collisions over a certain time scale, the various velocities of the molecules become randomised, and the gas is in *thermal equilibrium* which can be characterized by a quantity called the *temperature* T. The equilibration time for air is 10^{-9}s and for water 10^{-12}s. A *reversible change* to the system has to be slow compared with the above times, so that the system always remains near to thermal equilibrium.

Parameters of State

Also sometimes called state variables or functions of state, these are quantities which (if the motion or process remains in glorious thermal equilibrium) vary by path-independent amounts as the process evolves from its initial to final state. Pressure, density and absolute temperature are examples.

Equations of State

An *equation of state* is a functional relation between the parameters of state for a system in thermodynamic equilibrium, e.g. $f(p, \rho, T) = 0$.

1st Law: Internal Energy is a thing, and you can change it by adding heat or doing work

The first law is a thermodynamic version of conservation of energy. A unit mass of gas with specific volume $V = 1/\rho$ has an (internal) energy $e(\rho, T)$ (due to the kinetic energy of thermal motion of molecules and possibly from the potential energy of excited vibrational modes). This can be modified by adding heat or doing (mechanical) work:

$$de = \delta Q + \delta W \ . \tag{1.7}$$

Now, e is path-independent (and so a parameter of state) while Q and W are *not*, as the changes **are** path-dependent (as shown by the *inexact* differential δ).

Adiabatic Processes, Isolated Systems and Reversible Changes

An *adiabatic process* is a process in which there is no change of heat (or mass, but here we are not considering such sources) and so $\delta Q = 0$. An *isolated system* does not interact with its environment, and so there is no transfer of heat, mass and no work can be done on it. A *reversible change* from one equilibrium state to another is an 'infinitely' slow quasi-static process during which the thermodynamic system remains infinitesimally close to thermodynamic equilibrium. At all times, the parameters of state have a well-defined meaning, and so we can write

$$de = dQ - pdV = dQ + \frac{p}{\rho^2} d\rho \ , \tag{1.8}$$

since the mechanical work done on a simple gas against pressure is $-p\,dV$, as positive work is done under compression (as anyone pumping up a bike's tyre knows all too well) i.e. $dV < 0$, and $V = 1/\rho$ because of the normalization.

Specific Heat Capacities

The specific heat of a material is defined as dQ/dT in a reversible change. In essence, it is the amount of heat input required to increase the temperature of a unit mass by 1K in a reversible change. The specific heat (capacities) at constant volume and constant pressure are therefore:

$$\begin{aligned} c_V &= \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial e}{\partial T}\right)_{V \text{ or } \rho} , \\ c_p &= \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial e}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p . \end{aligned}$$

2nd Law: Entropy is a thing, and it always increases

The *Second Law of Thermodynamics* states that all thermodynamic systems have another (extremely important) parameter of state called the *entropy*, S, such that in a *reversible* change

$$TdS = dQ \ .$$

Crucially, the entropy of an isolated system can only *increase*, i.e. for an *irreversible* adiabatic change $dS > 0$. Loosely, S is a measure of the disorder in the system, and is proportional to the number of arrangements of the molecules at a given density ρ and temperature T. The fact that heat flows from hot to cold is a manifestation of the second law, spreading chaos like a horde of orcs (or indeed football supporters).

Therefore, for a *reversible* process

$$TdS = de + pdV \ ,$$

while more generally $dS \geq \delta Q/T$ (cf. (1.7)). Therefore for an *isentropic* (i.e. reversible and adiabatic) process, where $dS = 0$, (1.8) becomes

$$de = -pdV = \frac{p}{\rho^2}d\rho \ ,$$

and after this brief interlude we are able to return to our usual programming of examinable material...

1.4 Energy Equation for Isentropic Motions

With our examinable hat back on, the internal energy (per unit mass) of a gas during an isentropic motion satisfies

$$de = -pdV = \frac{p}{\rho^2}d\rho \ , \tag{1.9}$$

where $V = 1/\rho$ is the specific volume (for a unit mass) and the integral of this can be understood to be the work done on that unit mass. Therefore for reversible adiabatic motions

$$\left(\frac{\partial e}{\partial \rho}\right)_{adia} = \frac{p}{\rho^2}.$$

Therefore, for an ideal gas

$$\left(\frac{\partial e}{\partial \rho}\right)_{adia} = \frac{p_0}{\rho^2}\left(\frac{\rho}{\rho_0}\right)^{\gamma} \rightarrow e = \frac{1}{\gamma - 1}\frac{p}{\rho} + C, \tag{1.10}$$

where the constant C is (usually) normalized/assumed to be zero. The *enthalpy* H (loosely the amount of heat exchanged with the surroundings) is defined as

$$H \equiv e + \frac{p}{\rho} = e + pV, \tag{1.11}$$

remembering that $V = 1/\rho$ is the specific volume, and so the second term can be (very loosely!) interpreted as the pressure work done to "make the space" for this volume of gas. However it is interpreted, for an ideal gas

$$H = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}. \tag{1.12}$$

This quantity is useful when considering heating and cooling systems (like refrigerators), and perhaps more relevantly for this course, when considering shock waves in Part V.

The internal (or compressive) energy **per unit volume** ρe is often labelled as W...

Chapter 2

(Linear) Sound

Drum's quite good, the bass is too loud, and I can't hear the words
Nicky Tesco

2.1 Linear Wave Equation

We consider a base state with $\boldsymbol{u} = \mathbf{0}$, $p = p_0$, $\rho = \rho_0$, $\mathbf{f} = \mathbf{0}$ in an ideal gas where we assume homentropic motion, and p_0 and ρ_0 are constants, independent of both space and time. Therefore, this base state is a solution of the closed (because the motion is homentropic) system of equations (1.2), (1.3) and (1.6) repeated here for convenience:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla}.\boldsymbol{u}, \tag{2.1}$$

$$\rho\frac{D\boldsymbol{u}}{Dt} \equiv \rho\frac{\partial \mathbf{u}}{\partial t} + \rho\boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}p, \tag{2.2}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{2.3}$$

Now consider a "small" (and "smooth") perturbation to this base flow, so that:

$$\boldsymbol{u} = \mathbf{0} + \boldsymbol{u}, \tag{2.4}$$

$$\rho = \rho_0 + \tilde{\rho}, \qquad |\tilde{\rho}| \ll \rho_0, \tag{2.5}$$

$$p = p_0 + \tilde{p}, \qquad |\tilde{p}| \ll p_0, \tag{2.6}$$

where the smallness and smoothness allows us to ignore products of small *perturbation* quantities compared to quantities which are *linear* in the perturbation quantities.

For (2.5) and (2.6) to be consistent with (2.3), we construct the Taylor expansion of p, remembering that $p(\rho, S) = p(\rho, S_0)$ since we assume the motion is homentropic:

$$\begin{aligned} p_0 + \tilde{p} &= p(\rho_0 + \tilde{\rho}, S_0) \\ &= p(\rho_0, S_0) + \tilde{\rho}\frac{\partial p}{\partial \rho}(\rho_0, S_0) + ... \\ &= p_0 + \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0, S_0} \tilde{\rho} + ... \end{aligned}$$

Remembering that $[p] = ML^{-1}T^{-2}$ since it is a force per unit area, $[p/\rho] = L^2T^{-2}$ and so the partial derivative (which could also be written as a total derivative if the entropy is understood to be kept constant) can be recognised to be a square of a speed: the *sound speed* c_0 (at density ρ_0), and so

$$\tilde{p} = c_0^2 \tilde{\rho}\,, \tag{2.7}$$

showing that here pressure and density perturbation can be linearly related.

Substituting (2.4)-(2.6) into (2.1) and only keeping linear terms, we obtain

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \boldsymbol{\nabla}.\boldsymbol{u} = 0. \tag{2.8}$$

Implicitly here we have quantified what we mean by $\boldsymbol{u}$ being "small" and "smooth", as we have ignored terms involving $\boldsymbol{u}$ and a single spatial derivative compared to the time derivative. So, assuming that $|\boldsymbol{u}| \sim \mathcal{U}$, and that the motion varies on a typical length scale $\mathcal{L}$ and time scale $\mathcal{T}$, we are assuming that $\mathcal{U} \ll \mathcal{L}/\mathcal{T}$. But what are these scales?

Now, making the same substitution into (2.2), once again only keeping first order quantities, we find that

$$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \tilde{p}, \tag{2.9}$$

where the exact same implicit scaling has been made for the velocity to ignore the advective term i.e. that $\mathcal{U} \ll \mathcal{L}/\mathcal{T}$.

Pressure can be eliminated from (2.9) using (2.7), and then taking the divergence of the modifed (2.9) eliminates $\mathbf{u}$ to obtain the linear, three-dimensional wave equation for $\tilde{\rho}$:

$$\frac{\partial^2}{\partial t^2}\tilde{\rho} = c_0^2 \nabla^2 \tilde{\rho}. \tag{2.10}$$

From (2.7) it is obvious that $\tilde{p}$ satisfies the wave equation.

From this equation we can now revisit the scaling assumption that the velocity is sufficiently small, as we can now consistently estimate the characteristic ratio $\mathcal{L}/\mathcal{T}$. "Clearly" (refuge of the professorial scoundrel I know) for the right hand side and left hand side to balance $c_0 \sim \mathcal{L}/\mathcal{T}$. Therefore, for our approximations to be internally consistent

$$\mathcal{U} \ll c_0 \to M \equiv \frac{\mathcal{U}}{c_0} \ll 1, \tag{2.11}$$

i.e. the *Mach number* must be small. Interestingly (at least to me) M has to be small but strictly positive for sound waves, as $\mathcal{U} = 0$ is pretty boring, and $c_0 \to \infty$ implies that ρ doesn't depend on pressure ...

For an ideal gas, since $p = p_0(\rho/\rho_0)^\gamma$,

$$c_0^2 = \frac{\gamma p_0}{\rho_0}. \tag{2.12}$$

Ballpark figures, at atmospheric pressure $c_0 \simeq 300ms^{-1}$ in air and $c_0 \simeq 1500ms^{-1}$ in water.

Exercise

It is a little more subtle to establish that $\boldsymbol{u}$ also satisfies the wave equation. First, show that $\partial\boldsymbol{u}/\partial t$ and $\boldsymbol{\nabla}.\boldsymbol{u}$ satisfy the wave equation. Integrate the wave equation involving $\partial\boldsymbol{u}/\partial t$ with respect to t, show that the integrating factor $\boldsymbol{\alpha}$ is divergence free, and hence, under the assumption that the motion starts from rest, show that $\boldsymbol{u}$ does indeed satisfy the wave equation!

2.2 The Acoustic Velocity Potential

From taking the curl of the linearised Euler equation (2.9), the vorticity $\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \mathbf{u}$ remains constant in time. Therefore, in particular if $\boldsymbol{\omega} = \mathbf{0}$ at $t = 0$ it remains zero $\forall t$. Therefore we can assume that $\boldsymbol{u}$ is *irrotational* and so can be expressed as a gradient of a potential, ϕ, the *acoustic velocity potential*:

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi. \tag{2.13}$$

Substituting this expression into (2.9) and integrating with respect to space we obtain (wlog):

$$-\rho_0 \frac{\partial \phi}{\partial t} = \tilde{p} = c_0^2 \tilde{\rho}, \tag{2.14}$$

using (2.7). (We can set the emergent integrating factor $\beta(t)$ to zero as we can add an arbitrary funcion of t, e.g. $\int \beta dt$, to ϕ without changing its gradient, i.e. $\boldsymbol{u}$, the only thing we care about.) Differentiating (2.14) with respect to time and using (2.8) and (2.13), we show that ϕ also satisfies the wave equation:

$$\frac{\partial^2}{\partial t^2}\phi = c_0^2\nabla^2\phi. \tag{2.15}$$

Of course, since this equation is linear, we can superpose solutions, i.e. if ϕ_1 and ϕ_2 are solutions, then so too is $\lambda\phi_1 + \mu\phi_2$ where λ and ϕ are scalars.

2.3 Plane Waves

Remember the joys of our youth (as in years past I have "taught" both Methods and DE). If ϕ is assumed to be a function only of x and t (and so $\partial\phi/\partial y = \partial\phi/\partial z = 0$, we recover the 1D wave equation of Methods and Differential Equations:

$$\frac{\partial^2}{\partial t^2}\phi = c_0^2\frac{\partial^2}{\partial x^2}\phi, \tag{2.16}$$

with the general *D'Alembert solution*

$$\phi = f(x - c_0t) + g(x + c_0t), \tag{2.17}$$

i.e. the sum of two arbitrary disturbances that propagate to the right and left with speed c_0 and no change of shape.

This idea can be generalised to a wave that propagates in the $\hat{\mathbf{k}}$ direction, where $\mathbf{k} = (k, l, m)$ is an arbitrary vector (the *wavevector*) and $\hat{\mathbf{k}}$ has been normalised by the *wavenumber* κ:

$$\hat{\mathbf{k}} = \frac{\mathbf{k}}{\sqrt{k^2 + l^2 + m^2}} = \frac{\mathbf{k}}{\kappa}, \tag{2.18}$$

where, I know, I know, κ now means something else again.

Any function $\phi = f(\hat{\mathbf{k}}\cdot\mathbf{x} - c_0t)$ satisfies the wave equation, and the wave propagates in the $\hat{\mathbf{k}}$ direction at speed c_0 independently of the particular form of the function f (think about rotating the coordinate system so that the $\hat{\mathbf{k}}$ direction is one of the coordinate directions). They are called *plane waves* since ϕ takes a constant value on the plane $kx + ly + mz = C$.

Remembering that $\mathbf{u} = \boldsymbol{\nabla}\phi = \hat{\mathbf{k}}f'$ (where the prime of course denotes differentiation with respect to argument) and so $\boldsymbol{u}$ is parallel to $\hat{\mathbf{k}}$. Therefore, these waves are *longitudinal*, unlike (transverse) waves on a string for example. Furthermore,

$$\tilde{p} = -\rho_0\frac{\partial\phi}{\partial t} = \rho_0c_0f' = \rho_0c_0\boldsymbol{u}.\hat{\mathbf{k}}, \tag{2.19}$$

(for such waves propagating exclusively in the positive $\hat{\mathbf{k}}$ direction) and so $\tilde{p}$ and $\boldsymbol{u}$ are in phase. Indeed the ratio $\tilde{p}/|\mathbf{u}|$ is called the *acoustic impedance*, (loosely, the response of the system to the pressure perturbation) and here is equal to $\rho_0 c_0$.

2.4 Harmonic Plane Waves

The special case $f = A\exp[i(\mathbf{k}.\mathbf{x} - \omega t)]$ is called a *harmonic plane wave*. It is critically important to remember that the real part is always understood in such expressions, and so $\phi = Re(A\exp[i(\mathbf{k}.\mathbf{x} - \omega t)])$. Such waves are important, since (in the linear regime) any disturbance can be written as the sum of harmonic plane waves using a classical Fourier decomposition. Key points of notation/jargon:

- A is the (complex) *amplitude*;
- ω is the *frequency*;
- $\mathbf{k}$ is the *wavevector*;
- $|\mathbf{k}| = \kappa$ is the *wavenumber*;
- $\lambda = 2\pi/\kappa$ is the *wavelength* (3cm $\leftrightarrow$ 30m for human-audible sound);
- $\theta(\mathbf{x}, t) = \mathbf{k}.\mathbf{x} - \omega t$ is the *phase*;
- Wave crests (and troughs) are levels of constant phase.
- The wavevector is perpendicular to crests;
- If the plane wave satisfies the wave equation, then ω and $\mathbf{k}$ must satisfy the *dispersion relation*:
$$\omega^2 = c_0^2 \kappa^2. \tag{2.20}$$
- In general, the *phase speed* $c_p = \omega/\kappa$.
- The *phase velocity* $\mathbf{c}_p = \omega\mathbf{k}/\kappa^2 = c_p\hat{\mathbf{k}}$.
- If the phase speed c_p is independent of frequency (or equivalently independent of wavenumber, or equivalently independent of wavelength, as frequency and wavenumber are related by a dispersion relation remember) the waves are said to be *non-dispersive* (as in this case, as $|c_p| = c_0$).
- If the dispersion relation depends only on κ (and not the vector direction $\hat{\mathbf{k}}$) it is said to be *isotropic*.

Chapter 3

Energetics

Anger is an energy
John Lydon

3.1 Total Energy Density

Remembering (1.8), the total *energy density* (i.e. the energy per unit volume) is given by

$$E = \rho \left[\frac{1}{2} |\boldsymbol{u}|^2 + \int^{\rho} \frac{p(\hat{\rho}, S)}{\hat{\rho}^2} d\hat{\rho} \right] \,, \tag{3.1}$$

where the second term on the RHS is recognisable as the integral of $(-pdV)$ at constant entropy (where V is the volume per unit mass and $\rho = V^{-1}$).

Liberally applying the product rule and Leibniz' rule, we can show that

$$\begin{aligned} \frac{\partial E}{\partial t} + \boldsymbol{\nabla}.(\boldsymbol{u}E + p\boldsymbol{u}) &= \rho \boldsymbol{u}.\frac{\partial \boldsymbol{u}}{\partial t} + \frac{p}{\rho}\frac{\partial \rho}{\partial t} + \frac{E}{\rho}\frac{\partial \rho}{\partial t} + E\boldsymbol{\nabla}.\boldsymbol{u} + \frac{E}{\rho}(\boldsymbol{u}.\boldsymbol{\nabla})\rho \\ &\quad + \rho(\boldsymbol{u}.\,\boldsymbol{\nabla})\left(\frac{1}{2}|\boldsymbol{u}|^2\right) + \frac{p}{\rho}(\boldsymbol{u}.\boldsymbol{\nabla})\rho + (\boldsymbol{u}.\boldsymbol{\nabla})p + p\nabla.\boldsymbol{u}, \\ &= 0\,, \end{aligned} \tag{3.2}$$

if we carefully also apply conservation of mass (1.2) and Euler's equation (1.3) with no body force.

In integral form for a fixed volume $\mathcal{V}$, we have the exact result

$$\frac{d}{dt}\int_{\mathcal{V}} E\,dV = -\int_{\mathcal{S}} E\boldsymbol{u}.\boldsymbol{n}\,dS - \int_{\mathcal{S}} p\boldsymbol{u}\cdot\boldsymbol{n}\,dS, \tag{3.3}$$

i.e. the (time rate of) change of energy in $\mathcal{V}$ is equal to the rate of the advection of energy into $\mathcal{V}$ and the rate of working by the pressure p at the surface $\mathcal{S}$.

3.2 Acoustic Energy Equation for Linear Waves

The energy equation for linear sound waves can be calculated directly (as we are perturbing about a state of rest) though we actually rely on some subtle (and convenient cancellations). If we add $\boldsymbol{u}$.(2.9) and $(\tilde{p}/\rho_0)\times$ (2.8), remembering (2.7), i.e. $c_0^2\tilde{\rho} = \tilde{p}$, we obtain the *acoustic energy equation* for linear sound waves:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho_0 |\boldsymbol{u}|^2 + \frac{1}{2}\frac{c_0^2\tilde{\rho}^2}{\rho_0}\right] + \boldsymbol{\nabla}.(\tilde{p}\boldsymbol{u}) \quad = \quad 0, \tag{3.4}$$

$$\frac{\partial}{\partial t}[K + W] + \boldsymbol{\nabla}.\mathbf{I} = 0, \tag{3.5}$$

defining the *kinetic energy density* K, the *wave-energy flux* or *acoustic intensity* $\mathbf{I}$ and the *potential energy density due to compression* W, which using (2.7) can also be written as

$$W \equiv \frac{1}{2}\frac{\tilde{p}^2}{c_0^2\rho_0}. \tag{3.6}$$

Integrating over a fixed volume $\mathcal{V}$ analogously to (3.3), we obtain

$$\frac{d}{dt}\int_{\mathcal{V}}(K + W)\,dV = -\int_{\mathcal{V}} \boldsymbol{\nabla}.\mathbf{I}dV = -\int_{\mathcal{S}} \tilde{p}\boldsymbol{u}.\boldsymbol{n}\,dS \tag{3.7}$$

i.e. the (time rate of) change of the *acoustic energy* in $\mathcal{V}$ is equal to the rate of working by the pressure p at the surface $\mathcal{S}$, as the advection of energy into the volume is at higher (third) order as the base state is at rest.

3.3 Comments

There are (at least) five comments I wish to make.

3.3.1 Loudness

Remember $\mathbf{I} \equiv \tilde{p}\boldsymbol{u}$ and so it has units of watts per square metre. The *intensity* or *loudness* in decibels is defined as

$$\text{loudness} \equiv 120 + 10\log_{10}|\mathbf{I}|\,, \quad \text{with}$$

$$I = \begin{cases} 10^{-12}\,\mathrm{Wm}^{-2} & \to 0\,dB \quad : \text{ which can just be heard at 3000Hz;} \\ 1\,\mathrm{Wm}^{-2} & \to 120\,dB \quad : \text{ the threshold of pain.} \end{cases}$$

Normal levels of conversation have total power (integrated over a surface) of $\approx 10^{-5}$W. As shown on the first example sheet, the intensity depends on distance like $1/r^2$, explaining why it gets so much quieter towards the back of a festival (remember them?) crowd (remember them?)

3.3.2 Acoustic Potential

Remembering the various expressions involving the acoustic potential ϕ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, it is possible to establish (another refuge of the professorial scoundrel) that

$$K = \frac{1}{2}\rho_0|\boldsymbol{\nabla}\phi|^2; \; W = \frac{1}{2}\frac{\rho_0}{c_0^2}\left|\frac{\partial\phi}{\partial t}\right|^2; \; \mathbf{I} = -\rho_0\frac{\partial\phi}{\partial t}\boldsymbol{\nabla}\phi. \tag{3.8}$$

3.3.3 (Unidirectional) Plane Waves

For any unidirectional plane wave $\phi = f(\hat{k}.\mathbf{x} - c_0 t)$ for some unit vector $\hat{\mathbf{k}}$, and so, using (3.8), there is *instantaneous equipartition* between kinetic energy and potential energy, as

$$K = \frac{\rho_0[f']^2}{2} = W, \tag{3.9}$$

at all times (and indeed all points in space). This is a very special result (and property) of plane acoustic waves. Furthermore,

$$\mathbf{I} = \tilde{p}\boldsymbol{u} = \rho_0[f']^2 c_0\hat{\mathbf{k}} = (K + W)c_0\hat{\mathbf{k}}. \tag{3.10}$$

Therefore the acoustic energy is transported in the direction of the wave at the speed c_0. The velocity of energy propagation of wave fields is called the *group velocity* $\mathbf{c_g}$ (for reasons which will hopefully become clearer later in the course) and **in this special case** is equal to the phase velocity as $\mathbf{c}_g = c_0\hat{\mathbf{k}} = \mathbf{c}_p$.

3.3.4 Harmonic Plane Waves and Averaging

Remember for harmonic plane waves $f = A\exp[i(\mathbf{k}.\mathbf{x} - \omega t)]$ with the real part understood. It is **critical** to take the real part **before** calculating the energy. There is a useful property of the system when one averages over one period $T = 2\pi/\omega$, as denoted by

$$\langle\cdot\rangle \equiv \frac{\omega}{2\pi}\int^{2\pi/\omega} \cdot dt. \tag{3.11}$$

It is straightforward to establish/remember from Methods L1 that

$$\langle\cos^2\omega t\rangle = \frac{1}{2} = \langle\sin^2\omega t\rangle,$$

and of course $\langle \cos \omega t \sin \omega t \rangle = 0$. Therefore, for functions $Ae^{i\omega t}$, $Be^{i\omega t}$, real part understood of course, where $A = A_r + iA_i$, and $B = B_r + iB_i$,

$$\langle Re[A \exp(i\omega t)] Re[B \exp(i\omega t)] \rangle = \frac{A_r B_r + A_i B_i}{2} = \frac{1}{2} Re[AB^*],$$

where $*$ denotes complex conjugate.

Exercise

For harmonic plane waves with f as above, show that

$$\langle K \rangle = \langle W \rangle = \frac{\rho_0 k^2}{4} |A|^2.$$

3.3.5 Consistent Energy Derivation

To get from (3.2) to (3.4) is a bit fiddly as one needs to be careful about all the terms up to quadratic. If we write $\rho = \rho_0 + \tilde{\rho}$, $p = p_0 + \tilde{p}$ as before etc., and expand (3.1) *including* quadratic terms:

$$\begin{array}{rcccccccccc} E & = & \frac{1}{2} & + & \rho \int^{\rho_0} \frac{p}{\rho^2} d\rho & + & \rho \frac{p_0}{\rho_0^2} \tilde{\rho} & + & \frac{1}{2} \rho \frac{\partial}{\partial \rho} \left(\frac{p}{\rho^2} \right)_0 \tilde{\rho}^2 & + & \dots \\ & = & \frac{1}{2} \rho_0 \boldsymbol{u}^2 & + & \rho \int^{\rho_0} \frac{p}{\rho^2} d\rho & + & \frac{p_0}{\rho_0} \tilde{\rho} & + & \frac{1}{2} \frac{c_0^2}{\rho_0} \tilde{\rho}^2 & + & \dots \\ & = & K & + & E_0 & + & E_r & + & W & + & \dots\ , \end{array}$$

where the quadratic terms K and W are as in (3.4), and

$$E_0 = \rho_0 \int^{\rho_0} \frac{p}{\rho^2} d\rho \qquad \text{and} \qquad E_r = \left(\int^{\rho_0} \frac{p}{\rho^2} d\rho + \frac{p_0}{\rho_0} \right) \tilde{\rho}\,. \qquad (3.12)$$

Internal energy per unit volume when fluid at rest — Change in energy per unit volume due to change in mass

Note in particular the careful and consistent handling of the density premultiplying the integral, and in the limits of integration. From the conservation of mass equation (1.1) it follows, after some mathematical keepy-uppy, that $(E_0 + E_r)$ **exactly** satisfies

$$\frac{\partial}{\partial t}(E_0 + E_r) + \nabla . \left(\boldsymbol{u}(E_0 + E_r) + p_0 \boldsymbol{u} \right) = 0\,, \qquad (3.13)$$

Hence from (3.2) and (3.13), it follows that consistent to second order we can indeed use the acoustic energy equation for linear waves,

$$\frac{\partial}{\partial t}(K + W) + \nabla .(\tilde{p} \boldsymbol{u}) \approx 0\ .$$

Chapter 4

Transmission and Spherical Symmetry

No language, just sound, that's all we need know
Ian Curtis

In this chapter, we apply what we have learnt to solve important problems concerning transmission and spherically-symmetric geometries. These important phenomena are also considered at length on the example sheet.

4.1 Reflection and Transmission

We will consider this very important wave phenomenon here by a simple example. (Part II of the course has many more complicated examples.) Consider the problem of sound bouncing off a "thin" wall (in the sense that the thickness of the wall is thin compared to characteristic wavelengths λ of the sound). Without loss of generality, consider plane waves with a normal angle of incidence with frequency $\omega > 0$ and sound speed c_0 in a fluid of density ρ_0. The wall has uniform mass m per unit area, and its equilibrium position is $x = 0$.

Now, without loss of generality, the incoming or *incident* wave (in $x < 0$) has associated pressure perturbation

$$\tilde{p} = A \exp\left[i\omega\left(t - \frac{x}{c_0}\right)\right],$$

where A is a known amplitude (which we can assume is real, just by shifting the origin of time). Note that the choice of signs in the argument implies that this incident wave is travelling from left to right.

When this pressure perturbation (remember a force per unit area) interacts with the wall three things happen in general:

1. some of the energy is *reflected*, so that there is a reflected wave with pressure perturbation
$$\tilde{p} = R \exp\left[i\omega \left(t + \frac{x}{c_0} \right) \right],$$
travelling from right to left in $x < 0$, where R is in general complex;

2. the pressure perturbations may become imbalanced between $x < 0$ and $x > 0$ to make the wall move;

3. some of the energy may be *transmitted*, setting up a wave field in $x > 0$ travelling from left to right in $x > 0$:
$$\tilde{p} = T \exp\left[i\omega \left(t - \frac{x}{c_0} \right) \right].$$

4. The transmitted wave is being *radiated* from the oscillating boundary of the thin wall.

5. This requirement that the waves in $x > 0$ are travelling exclusively to the right is called a *radiation condition.*

6. It is effectively an example of a *causality condition*: there is nothing to cause waves to be coming leftwards from large values of positive x.

Since $\tilde{p} = -\rho_0 \partial\phi/\partial t$ and $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ where ϕ is the acoustic potential, the x-component of the velocity is:

$$\begin{aligned} u &= \frac{1}{\rho_0 c_0} \left[A e^{i\omega(t - x/c_0)} - R e^{i\omega(t + x/c_0)} \right] \text{ for } x < 0; \\ &= \frac{1}{\rho_0 c_0} T e^{i\omega(t - x/c_0)} \text{ for } x > 0. \end{aligned}$$

The unknown coefficients (R and T) are determined by applying boundary conditions (a recurring theme of this course...) We are in a linear world, so we assume that the deflections of the wall are sufficiently small that we can apply the boundary conditions not at the wall's present position $x = X(t)$ but rather at its equilibrium position $x = 0$. The two boundary conditions are our old friends Kinematic and Dynamic:

$$\begin{aligned} \dot{X}(t) &= u(0_+, t) = u(0_-, t) : \text{kinematic;} \\ m\ddot{X} &= \tilde{p}(0_-, t) - \tilde{p}(0_+, t) = -[\tilde{p}]_-^+ : \text{dynamic;} \end{aligned}$$

remembering that if the pressure is higher to the left that will tend to accelerate the wall in the positive $x-$direction. Since the terms on the right hand sides of these equations are proportional to $e^{i\omega t}$, it seems reasonable to assume that $X = X_0 e^{i\omega t}$, and so

$$\begin{aligned} i\omega\rho_0 c_0 X_0 &= A - R = T; \\ -m\omega^2 X_0 &= A + R - T, \end{aligned}$$

and so

$$R = \frac{A}{1-2i\alpha}; \; T = \frac{-2i\alpha A}{1-2i\alpha}; \; \alpha = \frac{\rho_0 c_0}{\omega m}.$$

Note that α is a nondimensional quantity, and as is typical in such problems, identifying such nondimensional groupings is very useful for the interpretation of the system.

Here, remembering that $\omega = c_0 k$ from the dispersion relation,

$$\begin{aligned} \alpha &= \frac{\left[\frac{\rho_0}{k}\right]}{m}, \\ &\sim \frac{\text{mass of gas in a wavelength}}{\text{mass of wall}}. \end{aligned}$$

The limits clearly match:

- $m = 0 : \alpha \to \infty$, $R \to 0$, $T \to A$. So if there is no wall, there is no reflection. (There is no spoon.)

- $m \to \infty : \alpha \to 0$, $R \to A$, $T \to 0$. So if the wall is so massive that it doesn't move, there is no transmission.

But we know more. High frequencies have small α, and so are more likely to be reflected, while lower frequencies have larger α, and are more likely to be transmitted. Therefore, if your neighbour has the (selfish) good sense to play Peter Tosh's masterful cover of "Johnny B. Goode", you are more likely to hear Sly and Robbie than you are to hear the guitar solo, or indeed the words...

Exercise

Show that the wave-energy flux is independent of x by showing that

$$\begin{aligned} \langle \tilde{p}u \rangle &= \frac{1}{2\rho_0 c_0}\left(|A|^2 - |R|^2\right) \text{ for } x < 0; \\ &= \frac{1}{2\rho_0 c_0}|T|^2 \text{ for } x > 0. \end{aligned}$$

4.2 Spherically symmetric waves

Let us consider a spherically symmetric pressure perturbation, $\tilde{p}(r,t)$. Remembering the $r-$dependent component of the Laplacian operator (which of course is engraved on your heart):

$$\nabla^2\tilde{p} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\tilde{p}).$$

This is particularly convenient, since upon multiplying across by r, the 3D wave equation neatly reduces to a 1D wave equation for $r\tilde{p}$

$$\frac{\partial^2}{\partial t^2}(r\tilde{p}) - c_0^2\frac{\partial^2}{\partial r^2}(r\tilde{p}) = 0\,.$$

Therefore, remembering our good friend D'Alembert, the general solution for $\tilde{p}$ is

$$\tilde{p}(r,t) = \underbrace{\frac{f(r-c_0t)}{r}}_{\text{outgoing wave}} + \underbrace{\frac{g(r+c_0t)}{r}}_{\text{incoming wave}}\,. \tag{4.1}$$

There is **often** a radiation condition of no incoming waves from ∞, and so in infinite domains $g\equiv 0$. Conversely, in a finite domain (e.g. inside a spherical shell) there can be a *regularity* condition at $r=0$ (if it's in the domain of course) implying $g(x) = -f(-x)$.

Conventionally, the outgoing solution is written as:

$$\tilde{p}(r,t) = \frac{\dot{q}(t-r/c_0)}{4\pi r}\,. \tag{4.2}$$

The density perturbation is then given by

$$\tilde{\rho} = \frac{1}{c_0^2}\tilde{p} = \frac{\dot{q}(t-r/c_0)}{4\pi c_0^2 r}\,, \tag{4.3}$$

and since $\tilde{p} = -\rho_0\partial\phi/\partial t$, the acoustic potential is given by

$$\phi = -\frac{q(t-r/c_0)}{4\pi\rho_0 r}\,. \tag{4.4}$$

The velocity is the gradient of this potential, so be sure and remember the product rule to get **two** terms:

$$u = \boldsymbol{\nabla}\phi \;=\; \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} = \frac{1}{4\pi\rho_0}\left[\frac{\dot{q}(t-r/c_0)}{c_0 r} + \frac{q(t-r/c_0)}{r^2}\right]\hat{\mathbf{r}}\,. \tag{4.5}$$

where the first term is expected to dominate in the *far field* for large r, while the second term is expected to dominate in the *near field* for small r. "Large" and "small" should be defined relative to a characteristic wave length, which would be expected to scale with c_0/ω.

Note that the mass flux out of a sphere of radius r is thus

$$\begin{aligned} 4\pi r^2 \rho_0 \, u(r,t) &= \frac{r}{c_0}\,\dot{q}(t - r/c_0) + q(t - r/c_0), \\ &\to q(t), \end{aligned}$$

as $r \to 0$, and so $q(t)$ is the mass flux from a point source at the origin, which is the underlying reason behind the "conventional" choice. Many more interesting aspects are worked through on the example sheet.

4.3 Solution for a pulsating sphere

Here, at the end of this part of the course, we will consider a simple example. A sphere of equilibrium radius a performs small (in the sense that $\epsilon \ll a$) radial pulsations at frequency ω:

$$r = a + \epsilon e^{i\omega t},$$

where (as usual) we take the real part of the equation. On the surface of the sphere, $r = a + \varepsilon e^{i\omega t}$, the velocity of the sphere is

$$u = \frac{\partial \phi}{\partial r} = i\omega\epsilon e^{i\omega t}\,.$$

Naturally, we would like to apply the boundary condition at the equilibrium radius $r = a$. We can justify this by taking a Taylor series expansion:

$$u(a + \epsilon e^{i\omega t}, t) \approx u(a,t) + \underbrace{\epsilon e^{i\omega t}\frac{\partial}{\partial r}u(a,t)}_{\text{doubly small}} + \dots\,.$$

thus ignoring higher order terms.

From the particular form of the forcing, we expect a solution with $q(t) \propto e^{i\omega t}$, and so we assume from (4.4)

$$\phi = A\frac{e^{i\omega(t - \frac{r-a}{c_0})}}{4\pi\rho_0 r}\,,$$

where the particular structure for the exponent has been chosen to make applying the boundary condition neater. (Such a cunning plan is to be encouraged in this course.) Then from (4.5)

$$u = -\frac{A}{4\pi\rho_0}\left(\frac{1}{r^2} + \frac{i\omega}{c_0 r}\right) e^{i\omega(t-\frac{r-a}{c_0})} .$$

Matching the velocity in the fluid to the velocity of the sphere at $r = a$, we obtain

$$-\frac{A}{4\pi\rho_0 a^2}\left(1 + \frac{i\omega a}{c_0}\right) = i\omega\epsilon ,$$

and so

$$A = -\frac{4\pi i\rho_0 a^2\omega\varepsilon}{(1 + i\omega a/c_0)} .$$

Therefore, remembering the relationship between perturbation pressure and acoustic potential:

$$\tilde{p} = -i\omega\rho_0\phi = -\frac{\varepsilon a^2\omega^2\rho_0}{(1 + i\omega a/c_0)}\frac{e^{i\omega(t-(r-a)/c_0)}}{r} .$$

Compact equals inefficient

It is important to compare the scale of the sphere to the characteristic wavelength $\lambda = 2\pi c_0/\omega$.

- A sphere is *large* if $a \gg \lambda = 2\pi c_0/\omega$, i.e. if $\omega a/c_0 = ka \gg 1$.
- Conversely, a sphere is *small* if $a \ll \lambda = 2\pi c_0/\omega$, i.e. if $\omega a/c_0 = ka \ll 1$.
- Such small spheres are called *compact sources*, and they are inefficient sound sources because $\tilde{p}$ and u are out of phase "near" such a source, and so $\mathbf{I}$ is small. Nevertheless, in either case the oscillating sphere is doing work on the fluid, driving the sound waves.

This is essentially worked through on the example sheet.

Part II

Elastic Waves

Chapter 5

Equations of Linear Elasticity

Are you all alone? Are you made of stone?
Ian Brown

5.1 Deformation of a Solid

After 4 lectures of madness, we are paid in full, having finished our journey into sound. We now start thinking about substances that are solid, solid as a rock. We again make the *continuum approximation*, and so we assume that we can average over volumes both large enough to contain many molecules but also much smaller than the scales in interest. This approximation enables us to define fields for important quantities, such as the density $\rho(\boldsymbol{x}, t)$.

As we are now interested in the mechanics of solids, we think about deformations. As a body is deformed, a particle moves from its original 'reference' position $\mathbf{X}$ to a new position $\boldsymbol{x}(t)$. Now, since the position must be invertible, $\mathbf{X} \equiv \mathbf{X}(\boldsymbol{x}, t)$, as we can always follow the time evolution of the particle at the reference position for all times. A key, indeed distinguishing feature of an elastic solid is that the solid **remembers** the original reference configuration $\mathbf{X}$. There is a conventional notation for solid mechanics to denote the

$$
\begin{aligned}
\text{particle displacement by} \qquad & \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X} \,, \\
\text{particle velocity by} \qquad & \boldsymbol{v}(\boldsymbol{x}, t) = \frac{d\boldsymbol{u}}{dt} \,, \\
\text{particle acceleration by} \qquad & \boldsymbol{a}(\boldsymbol{x}, t) = \frac{d\boldsymbol{v}}{dt} \,,
\end{aligned}
$$

where

$$
\frac{d}{dt} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}}
$$

is the rate of change moving with the particle, written more usually in fluid dynamics as

$$\frac{D}{Dt} \equiv \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \,.$$

Warning: Although there is potential for confusion, in solid mechanics $\boldsymbol{u}$ denotes *displacement* **not** *velocity*. One just has to deal with it, like the existence of *Mamma Mia*. It is convenient to write all fields as functions of the current (Eulerian) position $\boldsymbol{x}$, rather than the original (Lagrangian) position $\mathbf{X}$, but we must always remember that elastic solids are like elephants and folk singers: they never forget where they've come from.

5.2 The Stress Tensor

Just as in fluid dynamics, the forces acting the material can be divided into

1. *body forces* acting on and proportional to the volume, like gravity, where we denote the body force by $\mathbf{F}$, or $\rho\mathbf{f}$;

2. *surface forces*, or tractions/stresses, acting on and proportional to the surface area, like pressure;

Consider a small area element $\boldsymbol{n}\,dS$ at position $\boldsymbol{x}$ within a solid body. The force exerted **by the outside on the solid inside** of the element (with 'outside' defined by the **outward normal** $\boldsymbol{n}$) is assumed to be a surface force $\boldsymbol{\tau}(\boldsymbol{x}, t; \boldsymbol{n})\,dS$; by Newton three, $\boldsymbol{\tau}(\boldsymbol{x}, t; -\boldsymbol{n}) = -\boldsymbol{\tau}(\boldsymbol{x}, t; \boldsymbol{n})$. The *traction* or *stress*, $\boldsymbol{\tau}$, acting on the area element $\boldsymbol{n}\,dS$ depends on the orientation $\boldsymbol{n}$. (The units of traction are force/area, and forces are obtained by integrating tractions over an area.) In an inviscid fluid $\boldsymbol{\tau} = -p\boldsymbol{n}$.

In general, $\boldsymbol{\tau}(\boldsymbol{x}, t; \boldsymbol{n})$ is linearly related to $\boldsymbol{n}$ by a second-rank tensor $\boldsymbol{\sigma}$. Consider a small material tetrahedron with three faces aligned parallel to the Cartesian coordinate planes and a fourth "top" sloping face with area ϵ^2 and normal $\boldsymbol{n}$, as shown in figure 5.1. By geometry, the areas of the other faces are $\epsilon^2 n_i$ and the normals are $-\mathbf{e}_i$, where $\mathbf{e}_i$ is the i^{th} basis vector.

The surface forces on the tetrahedron are $\propto \epsilon^2$, whereas both the body forces and the inertia of the tetrahedron are $\propto \epsilon^3$. Therefore, the surface forces must balance each other as $\epsilon \to 0$, i.e.

$$\epsilon^2 \big\{ \boldsymbol{\tau}(\boldsymbol{x}, t; \boldsymbol{n}) - \boldsymbol{\tau}(\boldsymbol{x}, t; \mathrm{e}_1) n_1 - \boldsymbol{\tau}(\boldsymbol{x}, t; \mathrm{e}_2) n_2 - \boldsymbol{\tau}(\boldsymbol{x}, t; \mathrm{e}_3) n_3 \big\} = O(\epsilon^3) \,.$$

Therefore, the traction $\boldsymbol{\tau}$ exerted by the outside on the inside of a surface with outward normal $\boldsymbol{n}$ is given by

$$\boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \,, \tag{5.1}$$

where $\boldsymbol{\sigma}$ is the second-rank tensor, the (Cauchy) *stress tensor*, given by

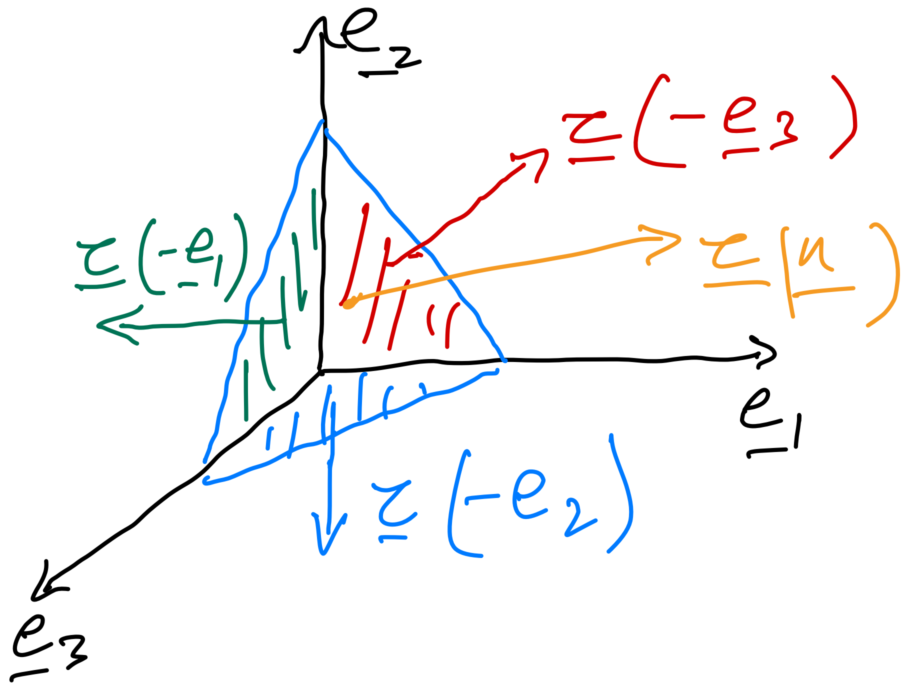

Figure 5.1: Schematic of tetrahedron showing various surface forces.

$$\sigma_{ij}(\boldsymbol{x},t) = \tau_i(\boldsymbol{x},t;\mathbf{e}_j)\,. \tag{5.2}$$

Warning: 'stress' is also often used to refer to the vector $\boldsymbol{\tau}$ as well as the tensor $\boldsymbol{\sigma}$, and also note that the vector $\boldsymbol{\tau}$ in general is defined in terms of the related normal to the surface under question.

5.3 Momentum, angular momentum & energy

5.3.1 Momentum

Consider an arbitrary *material* control volume $\mathcal{V}(t)$ (i.e. a particular labelled "blob", whose volume may change with time) with surface $\mathcal{S}(t)$ and outward normal $\boldsymbol{n}$. Integrate '$\boldsymbol{F} = m\boldsymbol{a}$' over all the particles in $\mathcal{V}$ to obtain

$$\int_{\mathcal{V}} \rho(\boldsymbol{x},t)\boldsymbol{a}(\boldsymbol{x},t)\,dV = \int_{\mathcal{V}} \boldsymbol{F}(\boldsymbol{x},t)\,dV + \int_{\mathcal{S}} \boldsymbol{\sigma}(\boldsymbol{x},t)\cdot\boldsymbol{n}\,dS\,. \tag{5.3}$$

Using the divergence theorem and the arbitrariness of $\mathcal{V}$, we obtain the *momentum equation*

$$\rho a_i = F_i + \frac{\partial \sigma_{ij}}{\partial x_j}\,, \tag{5.4}$$

as (5.3) is a vector equation, and so the $i-$th component of the last term on the RHS is $\sigma_{ij}n_j$. Taking the divergence is then equivalent to taking $\partial/\partial x_j$ as in the second term on the RHS of (5.4).

Exercise

Check that (5.4) is consistent with momentum budget for a **fixed** control volume.

5.3.2 Angular Momentum

Provided that there are no long-range forces to exert body couples on the material, the stress tensor can be shown to symmetric through considering the angular momentum balance. (A counter-example is a solid containing magnetic particles in an external magnetic field.) Taking moments about $\mathbf{0}$,

$$\int_{\mathcal{V}} \rho \boldsymbol{x} \times \boldsymbol{a} \, dV = \int_{\mathcal{V}} \boldsymbol{x} \times \mathbf{F} \, dV + \int_{\mathcal{S}} \boldsymbol{x} \times \boldsymbol{\sigma} . \boldsymbol{n} \, dS \, .$$

Considering this in component form, remembering our alternating tensors, and product rules:

$$\epsilon_{pqi} x_q (\rho a_i - F_i) = \frac{\partial}{\partial x_j}(\epsilon_{pqi} x_q \sigma_{ij}) = \epsilon_{pqi} x_q \frac{\partial \sigma_{ij}}{\partial x_j} + \epsilon_{pji}\sigma_{ij} \, .$$

From (5.4), we recognise that the LHS is equal to the first term on the RHS. Therefore,

$$\epsilon_{pji}\sigma_{ij} = 0 \, ,$$

and so from the properties of the alternating tensor $\boldsymbol{\sigma}$ is symmetric:

$$\sigma_{ij} = \sigma_{ji}. \tag{5.5}$$

5.3.3 Energy

Startlingly unsurprisingly, the energy equation is obtained by multiplying the momentum equation (5.4) by the velocity v_i, and then remembering to apply the Einstein summation convention on repeated indices. Remember

$$v_i a_i = \frac{d}{dt}(\frac{1}{2}v^2) \, ,$$

$$\frac{d}{dt}(\rho \, dV) = 0 \qquad \text{for a material volume element} \, ,$$

$$v_i \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial (v_i \sigma_{ij})}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\sigma_{ij} = \frac{\partial (v_i \sigma_{ij})}{\partial x_j} - \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\sigma_{ij} \, ,$$

using the symmetry of $\boldsymbol{\sigma}$. Integrating over a *material* volume (where we can exploit conservation of mass of that particular volume), we obtain

$$\underbrace{\frac{d}{dt}\int_{\mathcal{V}(t)} \frac{1}{2}\rho v^2\, dV}_{\text{Change in KE}} = \underbrace{\int_{\mathcal{V}} \boldsymbol{v}.\mathbf{F}\, dV}_{\substack{\text{Rate of work}\\ \text{by body forces}}} + \underbrace{\int_{\mathcal{S}} \boldsymbol{v}.\boldsymbol{\sigma}.\boldsymbol{n}\, dS}_{\substack{\text{Rate of work}\\ \text{by surface forces}}} - \underbrace{\int_{\mathcal{V}} \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\sigma_{ij}\, dV}_{\substack{\text{Rate of release}\\ \text{of internal energy}}} . \tag{5.6}$$

The final term is a bit subtle, and we shall discuss it further later: just wait until your (insert most scary adult carer) gets home! For the moment, note that in the simple case $\sigma_{ij} = -p\delta_{ij}$ the final term reduces to $+\int p\, \boldsymbol{\nabla}\cdot\boldsymbol{v}\, dV$, the result for the reversible adiabatic compression (or decompression: think about the signs) of a gas.

Chapter 6

Stress and Strain

Do everything at least twice
Dave Wakeling

In this chapter, we show how to relate stress to strain, and consider some simple situations.

6.1 Infinitesimal strain assumption

Elastic stresses result from the *change* in the separation of two neighbouring material elements:

$$\begin{aligned}\delta x_i - \delta X_i &= u_i(\boldsymbol{x} + \boldsymbol{\delta x}, t) - u_i(\boldsymbol{x}, t), \\ &= \frac{\partial u_i}{\partial x_j}\,\delta x_j + \text{h.o.t.}\,.\end{aligned}$$

For now, we assume that the deformation is small (hence *infinitesimal* strain)

$$|\boldsymbol{\delta x} - \boldsymbol{\delta X}| \ll |\boldsymbol{\delta x}|$$

and so

$$\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1\,.$$

This is appropriate for (well) hard solids like metal and rock, where a small deformation produces a large restoring force (but *not* true for rubber [soul]...)

We can decompose

$$\begin{aligned}\frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \\ &= e_{ij} + \omega_{ij}\,,\end{aligned}$$

where e_{ij} and ω_{ij} are the symmetric and antisymmetric parts of the tensor $\frac{\partial u_i}{\partial x_j}$. Indeed

$$\omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\,\omega_k, \quad \text{where} \quad \boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}\ ,$$

then

$$\omega_{ij}\,\delta x_j \;=\; \frac{1}{2}\left(\boldsymbol{\omega} \times \boldsymbol{\delta x}\right)_i \ .$$

With small deformations this term gives only a rigid rotation, and hence induces no elastic stresses. $\boldsymbol{\nabla} \times \mathbf{u}$ is the *rotation.* The tensor $\mathbf{e}$ is the (Cauchy or infinitesimal) *strain* tensor. "Obviously" $\mathbf{e}$ is symmetric:

$$\mathbf{e} = \frac{1}{2}\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right].$$

Note that this leads to the potential for confusion between solids mechanics and fluid mechanics. In fluids of course, $\boldsymbol{u}$, $\boldsymbol{\omega}$ and $\mathbf{e}$ are the velocity, vorticity and the strain rate...sometimes marked with an over-dot).

If $\mathbf{e}$ is zero then to first order there is no deformation. $\mathbf{e}$ is called the (Cauchy, or infinitesimal) *strain* tensor.

Exercise

Show that with small deformations, $\mathbf{e}$ is related to changes in length by

$$|\boldsymbol{\delta x}|^2 - |\boldsymbol{\delta X}|^2 = 2\,\boldsymbol{\delta X}.\mathrm{e}.\boldsymbol{\delta X}\ .$$

Consequences of infinitesimal strain

Since $|\boldsymbol{\nabla u}| \ll 1$:

1. It is consistent to neglect the convective part of (material) time derivatives:

   $$\frac{d}{dt} \;=\; \frac{\partial}{\partial t} + \frac{du_i}{dt}\frac{\partial}{\partial x_i} \simeq \frac{\partial}{\partial t},$$

   and so:

   $$v_i \simeq \frac{\partial u_i}{\partial t};\ a_i \simeq \frac{\partial^2 u_i}{\partial t^2}.$$

2. Volumes can be assumed to be constant to leading order as

   $$\frac{\Delta V}{V} \;\simeq\; \frac{\int_{\mathcal{S}} \boldsymbol{u}.\boldsymbol{n} dS}{V} \;=\; \frac{\int_{\mathcal{V}} \boldsymbol{\nabla}.\boldsymbol{u} dV}{V} \simeq |\boldsymbol{\nabla}.\boldsymbol{u}| \ll 1.$$

3. Density changes can be neglected in the momentum equation as

$$\tilde{\rho} = -\rho \boldsymbol{\nabla} . \boldsymbol{u} \ll \rho .$$

6.2 Constitutive equation for a linear elastic solid

The constitutive equation is the relationship between stress and strain. We wish to consider a linear elastic solid subject to "appropriately small" strains about the reference state $\mathbf{e} = 0$.

1. We assume that the relationship is *local* and *instantaneous* so that $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}(\mathbf{e})$, i.e. there is no dependence on

$$\frac{\partial e_{ij}}{\partial t} \quad \text{or} \quad \frac{\partial e_{ij}}{\partial x_k} \quad \text{or} \quad \ldots \; ;$$

2. We also assume that the relationship is *linear* (since $|\mathbf{e}| \ll 1$) and so

$$\sigma_{ij}(\boldsymbol{x}, t) \;=\; c_{ijk\ell}\, e_{k\ell}(\boldsymbol{x}, t) \,,$$

   where $\mathbf{c}$ is a fourth-order tensor that is a property of the material (cf. Hooke's law). Since $\sigma_{ij} = \sigma_{ji}$, and $e_{k\ell} \;=\; e_{\ell k}$,

$$c_{ijk\ell} \;=\; c_{jik\ell} \;=\; c_{ij\ell k} .$$

   Nevertheless, we are still in a world of hurt (unless materials float our boat) as there are up to 36 parameters in a general anisotropic material!

3. To make progress (at least in Tripos questions) we also assume that the material is *isotropic* and *uniform*. Therefore, the general form for $\mathbf{c}$ is

$$c_{ijk\ell} \;=\; \lambda\, \delta_{ij}\, \delta_{k\ell} + \mu\, \delta_{ik}\, \delta_{j\ell} + \mu'\, \delta_{i\ell}\, \delta_{jk},$$

   where λ, μ and μ' are constants. From the symmetry of $\boldsymbol{\sigma}$, $\mu' = \mu$ and so

$$\sigma_{ij} \;=\; \lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij} = \; \lambda\, \delta_{ij}\, \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) , \tag{6.1}$$

   where λ and μ are the *Lamé constants/moduli.* In vector notation

$$\boldsymbol{\sigma} \;=\; \lambda(\boldsymbol{\nabla} . \boldsymbol{u})\boldsymbol{I} + 2\mu\mathbf{e}, \tag{6.2}$$

$$\mathbf{e} \;=\; \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T \right] . \tag{6.3}$$

This expression determines stress from strain. It can be inverted, since (using Einstein's summation convention as usual)

$$\sigma_{kk} = (3\lambda + 2\mu)e_{kk},$$

and so (6.1) is equivalent to

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\delta_{ij}\sigma_{kk}\right), \tag{6.4}$$

thus expressing strain in terms of stress.

6.3 Simple deformations

a) Dilatation & Pressure

Consider a small change of material volume

$$\Delta V = \int_{\mathcal{S}} \boldsymbol{u}.\mathbf{n}\, dS = \int_{\mathcal{V}} \nabla.\boldsymbol{u}\, dV\,.$$

We define the [local] *dilatation* to be

$$\theta \equiv \boldsymbol{\nabla}.\boldsymbol{u} = \frac{\partial u_k}{\partial x_k} = e_{kk}\;.$$

Note how dilatation appears in the vector description of the stress (6.2).

We define the *pressure* in general, even in a solid, as the trace of the stress tensor, i.e.

$$p \;=\; -\frac{1}{3}\sigma_{kk}\,,$$

and so

$$\sigma_{ij} \;=\; -p\,\delta_{ij} + \overline{\sigma}_{ij}\,,$$

where $\overline{\sigma}_{ij}$ is the *deviatoric stress*. In the case of *hydrostatic stress*, i.e. when the stress is "like in an (inviscid or stationary) fluid",

$$\sigma_{ij} \;=\; -p\,\delta_{ij} \quad \text{and} \quad \boldsymbol{\tau}(\boldsymbol{x}, t; \mathbf{n}) \;=\; -p\mathbf{n}\,,$$

and so

$$p \;=\; -\left(\lambda + \frac{2\mu}{3}\right) e_{kk} = -\kappa\theta\,,$$

where κ is the *bulk modulus* or *modulus of incompressibility*. On physical grounds we expect that $\kappa > 0$, with positive pressure thus corresponding to *compression*. Hydrostatic stresses change volume, deviatoric stresses change shape.

b) Simple Shear

Consider the *simple shear* $\boldsymbol{u} = (\gamma y, 0, 0)$. Then

$$\mathbf{e} = \gamma \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma} = \gamma\mu \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

μ is referred to as the *shear modulus* or *modulus of rigidity*. In a fluid $\mu = 0$ as an inviscid elastic solid cannot "support" any tangential stresses. More generally we expect on physical grounds that $\mu \geqslant 0$.

c) Uniaxial extension

Consider a uniaxial extension induced by a stress

$$\boldsymbol{\sigma} = \frac{F}{A} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then from (6.4),

$$\mathbf{e} = \frac{F}{2A(3\lambda + 2\mu)\mu} \begin{pmatrix} 2(\lambda+\mu) & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = \frac{F}{EA} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{pmatrix}.$$

Here

$$E = \frac{(3\lambda + 2\mu)\mu}{\lambda + \mu} = \frac{3\kappa\mu}{\lambda + \mu} \geqslant 0$$

is *Young's modulus*, a measure of tensile stiffness, while

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

is *Poisson's ratio*. ν is *not* the kinematic viscosity, and can actually be negative in *auxetic* materials, which are typically crystalline. It is a measure of the effect of deformation of materials in the direction **perpendicular** to loading: push down on a cube of blu-tack (other wall-damaging materials are available) and it gets "fatter", while stretch an elastic band and it gets "thinner", though weirder things can happen for crystals.

Chapter 7

Wave Equations for Elastic Solids

Hold me, (be)cause I'm free
Mick Jagger

We now have all the tools we need to construct linearised wave equations for linear elastic solids.

7.1 Linearised Equations

Linearising (5.4) for the case $\mathbf{F} = \mathbf{0}$ (or considering sufficiently small perturbations relative to a static equilibrium), the left hand side simplifies to

$$\rho a_i \simeq \rho_0 \frac{\partial^2}{\partial t^2} u_i = \frac{\partial \sigma_{ij}}{\partial x_j}\,.$$

Substituting in (6.1) remembering that we have assumed the solid is uniform, so that λ and μ are constants we obtain

$$\rho_0 \frac{\partial^2 u_j}{\partial t^2} = +\lambda \frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial x_k}\right) + \mu \frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Using the vector identity (which we all know how to prove...)

$$\nabla^2 \boldsymbol{q} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{q}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{q}),$$

we obtain the two (equivalent) forms for the linearised Cauchy momentum equation for linear isotropic and uniform elastic materials:

$$\begin{aligned}
\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} &= (\lambda + \mu)\, \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u}, && (7.1)\\
&= (\lambda + 2\mu)\, \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{u}) - \mu \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u})\,. && (7.2)
\end{aligned}$$

7.2 Boundary Conditions

As we shall see, the vital issue to address to solve these equations is (usually) to apply the appropriate boundary conditions. Four common sets are as follows.

1. At a *rigid* or *clamped* boundary, there can be no displacements in **any** direction, and so the three components of **u** are zero:

$$\boldsymbol{u} = \mathbf{0}\,.$$

2. At a *free* boundary, the three components of traction/stress $\boldsymbol{\tau}$ are zero (as the boundary is unstressed)

$$\boldsymbol{\tau} = \mathbf{0}.$$

3. At a *solid/solid* interface, no relative motion and Newton 3 imply continuity of **all components of** displacement $\boldsymbol{u}$ and traction/stress $\boldsymbol{\tau}$ and so *six* conditions

$$[\boldsymbol{u}]_-^+ = [\boldsymbol{\sigma}\cdot\boldsymbol{n}]_-^+ = 0\,.$$

4. At a *solid/inviscid fluid* interface there is no relative *normal* motion, but it is critical to remember that in an inviscid fluid, the shear modulus μ is zero, and the fluid cannot support tangential or shear stresses, but only has a normal (pressure stress). Therefore there are only *four* conditions (3 traction/stress, two of which are zero, and one on normal displacement):

$$[\boldsymbol{u}.\boldsymbol{n}]_-^+ = 0\,, \quad \boldsymbol{\sigma}.\boldsymbol{n} = -p_0\boldsymbol{n}\,, \quad \boldsymbol{n}\times\boldsymbol{\sigma}.\boldsymbol{n} = \mathbf{0}.$$

7.3 Energy equation

We can now revisit the energy equation (5.6) in the absence of body forces applying the linearisation relating $\boldsymbol{v} = \dot{\boldsymbol{u}}$, and the notation we now use:

$$\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\rho_0|\dot{\boldsymbol{u}}|^2 dV \;=\; \int_{\mathcal{S}}\dot{\boldsymbol{u}}.\boldsymbol{\sigma}.\boldsymbol{n}dS - \int_{\mathcal{V}}\sigma_{ij}\dot{e}_{ij}.$$

The last term can now be interpreted using the stress tensor expression (6.1), since

$$\begin{aligned}
\sigma_{ij}\dot{e}_{ij} &= \lambda e_{kk}\dot{e}_{ll} + 2\mu e_{ij}\dot{e}_{ij}, \\
&= \dot{\sigma}_{ij}e_{ij}, \\
&= \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\lambda e_{kk}^2 + 2\mu e_{ij}e_{ij}\right)\right], \\
&= \frac{\partial}{\partial t}\left[\frac{1}{2}\sigma_{ij}e_{ij}\right].
\end{aligned}$$

Therefore we can recognise the term in the square bracket (in alternative formulations) as the *elastic potential energy* and we have the energy equation

$$\frac{d}{dt}\int_V \frac{1}{2}\left(\rho_0 \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} + \sigma_{ij}e_{ij}\right) dV - \int_S \dot{\boldsymbol{u}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dS = \int_V \dot{\boldsymbol{u}} \cdot \boldsymbol{F}\, dV\,.$$

As usual, since the volume is arbitrary,

$$\frac{\partial}{\partial t}(K + W) \quad + \quad \boldsymbol{\nabla} \cdot \boldsymbol{I} = 0, \tag{7.3}$$

where (assuming no strain energy when the material is undeformed):

$$\begin{aligned}
\text{Kinetic Energy} = K &= \frac{1}{2}\rho_0 \mid \dot{\mathbf{u}}|^2\,, && (7.4)\\
\text{Elastic Potential Energy} = W &= \frac{1}{2}\sigma_{ij}e_{ij}\,, && (7.5)\\
\text{Energy Flux Vector} = \boldsymbol{I} &= -\dot{\boldsymbol{u}} \cdot \boldsymbol{\sigma}\,. && (7.6)
\end{aligned}$$

Exercise

Show

$$\begin{aligned}
W &= \frac{1}{2}\left(\lambda e_{kk}e_{jj} + 2\mu e_{ij}e_{ij}\right) \\
&= \frac{1}{2}\left(\lambda + \frac{2}{3}\mu\right)\left(e_{kk}e_{jj}\right) + \mu\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right)\left(e_{ij} - \frac{1}{3}e_{mm}\delta_{ij}\right).
\end{aligned}$$

Therefore W is positive definite iff

$$\kappa = \lambda + \frac{2}{3}\mu > 0\,, \quad \mu > 0\,. \tag{7.7}$$

7.4 Elastic Waves

We can now obtain two different classes of waves.

Primary/compressional/pressure waves

Take the divergence of (7.1) (or (7.2) actually) to obtain

$$\frac{\partial^2\theta}{\partial t^2} - c_P^2 \boldsymbol{\nabla}^2\theta = 0\,, \tag{7.8}$$

where $\theta = \boldsymbol{\nabla}\cdot\boldsymbol{u}$ (the dilatation) and

$$c_P^2 = \frac{\lambda + 2\mu}{\rho_0} = \left(\frac{\kappa + \frac{4}{3}\mu}{\rho_0}\right) > 0\,. \tag{7.9}$$

c_P is the dilatational (or compressional, or primary) wave speed. Note this is a scalar equation.

Secondary/shear waves

Alternatively, take the curl of (7.2) ((or (7.1) actually) to obtain

$$\frac{\partial^2\boldsymbol{\omega}}{\partial t^2} - c_S^2 \boldsymbol{\nabla}^2\boldsymbol{\omega} = 0, \tag{7.10}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ (the rotation) and

$$0 < \frac{\mu}{\rho} = c_S^2 < c_P^2\,. \tag{7.11}$$

c_S is the shear wave speed. Note that this is a vector equation

Hence a general disturbance, e.g. as generated by an earthquake, propagates [at least] at two speeds. The first to arrive, the primary P wave, is the dilatational part. The second to arrive, the secondary S wave, is the shear wave.

Elastic fluids

If $\mu = 0$ (i.e. an elastic fluid) then $c_S = 0$ and only compressional, i.e. acoustic/sound, waves are supported with

$$c_P^2 = \frac{\kappa}{\rho} = \frac{dp}{d\rho}\,,$$

since for an elastic fluid

$$p = -\frac{1}{3}\sigma_{kk} = -\lambda e_{kk} = -\kappa\theta\,.$$

Example

For the Earth's mantle $c_p \simeq 13\text{km s}^{-1}$, $c_s \simeq 7\text{km s}^{-1}$.

7.5 Plane waves

Look for solutions to (7.2) of the form

$$\boldsymbol{u} = \mathbf{f}\left(\widehat{\boldsymbol{k}} \cdot \boldsymbol{x} - ct\right) ,$$

where $\widehat{\boldsymbol{k}}$ is a unit vector in the direction of the wavevector. Then we require

$$c^2 \mathbf{f}'' = c_P^2\, \widehat{\boldsymbol{k}}(\widehat{\boldsymbol{k}} \cdot \mathbf{f}'') - c_S^2\, \widehat{\boldsymbol{k}} \times \left(\widehat{\boldsymbol{k}} \times \mathbf{f}''\right) .$$

Taking dot products and cross-products with respect to $\widehat{\boldsymbol{k}}$ we obtain

$$\begin{aligned} (c^2 - c_P^2)\, \widehat{\boldsymbol{k}} \cdot \mathbf{f}'' &= 0\,, \\ (c^2 - c_S^2)\, \widehat{\boldsymbol{k}} \times \mathbf{f}'' &= 0\,. \end{aligned}$$

Since $c_P \neq c_S$ there are thus two different possibilities.

Longitudinal plane waves

If $c = c_P$, then $\widehat{\boldsymbol{k}} \times \mathbf{f}'' = 0$, and we actually let $\boldsymbol{u}$ be defined as $\widehat{\boldsymbol{k}}$ times a *scalar* function

$$\begin{aligned} \boldsymbol{u} &= \widehat{\boldsymbol{k}}\, f\left(\widehat{\boldsymbol{k}} \cdot \boldsymbol{x} - c_P t\right) , \qquad (7.12) \\ \theta &= \boldsymbol{\nabla} \cdot \boldsymbol{u} = f'\left(\widehat{\boldsymbol{k}} \cdot \boldsymbol{x} - c_P t\right), \\ \boldsymbol{\omega} &= \boldsymbol{\nabla} \times \mathbf{u} = 0\,. \end{aligned}$$

This is a plane wave of arbitrary shape with $\boldsymbol{u}$ parallel to $\widehat{\boldsymbol{k}}$, i.e. the displacement is parallel to the direction of travel of the wave. This is a *longitudinal* wave.

Transverse plane waves

If $c = c_S$, then $\widehat{\boldsymbol{k}} \cdot \mathbf{f}'' = 0$. We can then let $\boldsymbol{u}$ be defined as a *vector* function $\mathbf{g}$:

$$\begin{aligned} \boldsymbol{u} &= \widehat{\boldsymbol{k}} \times \mathbf{g}\left(\widehat{\boldsymbol{k}} \cdot \boldsymbol{x} - c_S t\right) , \qquad (7.13) \\ \theta &= \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \\ \boldsymbol{\omega} &= \boldsymbol{\nabla} \times \mathbf{u} = -\mathbf{g}'\left(\widehat{\boldsymbol{k}} \cdot \boldsymbol{x} - c_S t\right) . \end{aligned}$$

Without loss of generality, we can assume that $\widehat{\boldsymbol{k}}\cdot\mathbf{g} = 0$. This is a plane wave of arbitrary shape with $\boldsymbol{u}\cdot\widehat{\boldsymbol{k}} = 0$, i.e. the displacement is perpendicular to the direction of travel of the wave. This is a *transverse* wave. If the direction of $\widehat{\boldsymbol{k}} \times \mathbf{g}$ is fixed, then the wave is said to be *polarized*.

7.6 Energies of Plane Waves

For such plane waves, the stress tensor $\boldsymbol{\sigma}$ is

$$\sigma_{ij} = \begin{cases} \left(\lambda\delta_{ij} + 2\mu\widehat{k}_i\widehat{k}_j\right) f' & : \text{P wave,} \\ \mu\left(\epsilon_{i\ell m}\widehat{k}_j + \epsilon_{j\ell m}\widehat{k}_i\right)\widehat{k}_\ell g'_m & : \text{S wave.} \end{cases}$$

Therefore, the kinetic energy (density, as it's per unit volume) K and the potential energy (density, ditto) W are

$$K = \frac{1}{2}\rho_0\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}} = \begin{cases} \frac{1}{2}\rho_0 c_P^2 f'^2 & : \text{P wave,} \\ \frac{1}{2}\rho_0 c_S^2(\widehat{\boldsymbol{k}}\times\mathbf{g}')\cdot(\widehat{\boldsymbol{k}}\times\mathbf{g}') & : \text{S wave,} \end{cases}$$

$$W = \frac{1}{2}\lambda e_{kk}e_{jj} + \mu e_{ij}e_{ij} = \begin{cases} \frac{1}{2}\left(\lambda + 2\mu\right) f'^2 & : \text{P wave,} \\ \frac{1}{2}\mu(\widehat{\boldsymbol{k}}\times\mathbf{g}')\cdot(\widehat{\boldsymbol{k}}\times\mathbf{g}') & : \text{S wave,} \end{cases}$$

But remember, by definition $\rho_0 c_P^2 = \lambda + 2\mu$ and $\rho_0 c_S^2 = \mu$ for both P and S waves there is (again) instantaneous equi-partition of energy:

$$K = W.$$

Furthermore, the energy flux

$$\boldsymbol{I} = -\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}} = \begin{cases} (\lambda + 2\mu)c_P\, f'^2\,\widehat{\boldsymbol{k}} & : \text{P wave,} \\ \mu c_S\,(\widehat{\boldsymbol{k}}\times\mathbf{g}')\cdot(\widehat{\boldsymbol{k}}\times\mathbf{g}')\,\widehat{\boldsymbol{k}} & : \text{S wave.} \end{cases}$$

Therefore for either type of plane wave *separately*,

$$\boldsymbol{I} = (K + W)\,c\,\widehat{\boldsymbol{k}}\,,$$

where c is the appropriate phase speed (i.e. c_P or c_S). Hence energy is propagated at the wavespeed in the direction of travel of the waves, which is completely unsurprising since the waves are non-dispersive. However, it is very important to remember that while linear displacements can be added, quadratic energies cannot in general (a point investigated in the example sheet).

Chapter 8

Harmonic Plane Waves and Polarization

And where is the harmony, sweet harmony
Nick Lowe

Harmonic plane waves are a special case of plane waves where the functions f and $\mathbf{g}$ of the previous chapter are restricted to complex exponential form.

1. A harmonic P-wave takes the form

$$\boldsymbol{u} = \mathbf{A}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,,$$

   where $\kappa = |\boldsymbol{k}|$, $\omega = \kappa c_P$ and $\mathbf{A}$ is parallel to $\boldsymbol{k}$.

2. A harmonic S-wave takes the form

$$\boldsymbol{u} = \mathbf{B}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,,$$

   where $\omega = \kappa c_S$ and $\mathbf{B}$ is a now perpendicular to $\boldsymbol{k}$. "Clearly", there are two directions perpendicular to $\boldsymbol{k}$, and so there are two different ways in which S-waves can be *polarized*.

8.1 Polarization of S-waves

To fix ideas, let us consider a plane boundary at $z = 0$, thus defining the $z-$direction to be the "vertical". Without loss of generality, we then consider plane waves with wave vector $\boldsymbol{k}$ in the $x - z$ plane, and so

$$\boldsymbol{k} = \kappa(\sin\theta, 0, \cos\theta), \tag{8.1}$$

where θ is the *angle of incidence*, i.e. the angle the wave vector makes with the vertical $z-$direction (or more precisely the normal to the plane boundary). Therefore the P-wave has displacement

$$\boldsymbol{u} = A(\sin\theta, 0, \cos\theta) \exp i \left[\kappa(x\sin\theta + z\cos\theta) - \omega t\right], \tag{8.2}$$

where A is a (complex in general) constant.

8.1.1 SH waves

There are two natural ways to consider displacements which are perpendicular to the wave vector $\boldsymbol{k}$ defined in (8.1). An SH wave has *purely horizontal* motions (i.e. in the $y-$direction within this coordinate system), and so:

$$\boldsymbol{u} = B_H(0, 1, 0) \exp i \left[\kappa(x\sin\theta + z\cos\theta) - \omega t\right], \tag{8.3}$$

where B_H is again a (complex in general) constant. SH waves are said to be *horizontally polarized.*

8.1.2 SV waves

The other natural perpendicular direction is in the same plane as the wave vector, but with a nontrivial vertical (i.e. z) component. An SV wave has displacements

$$\boldsymbol{u} = B_v(\cos\theta, 0, -\sin\theta) \exp i \left[\kappa(x\sin\theta + z\cos\theta) - \omega t\right], \tag{8.4}$$

where B_V is yet again a (complex in general) constant. SV waves are said to be *vertically polarized*, although there displacements are not purely in the vertical direction.

In general, a disturbance contains P, SV and SH waves, though there is a natural decoupling of SH waves from P and SV waves.

8.2 Reflecton and Refraction of Harmonic Plane Waves

Consider a wave with wave vector $\boldsymbol{k} = \kappa(\sin\theta, 0, \cos\theta)$ incident on an interface between two elastic half-planes, with (in general) different physical properties:

- ρ, λ, μ and hence c_p and c_s for $z < 0$;

- ρ', λ', μ' and hence c_p' and c_s' for $z > 0$.

We assume that displacements are sufficiently small and smooth that we can linearize and impose boundary conditions at the undisturbed equilibrium interface (at $z = 0$ with unit normal $\boldsymbol{n} = (0, 0, 1)$). There are then several different situation.

1. In general we have six conditions:

   - continuity of displacement at $z = 0$: $[\boldsymbol{u}]_-^+ = \boldsymbol{0}$;
   - continuity of traction at $z = 0$: $[\boldsymbol{\tau}]_-^+ = [\boldsymbol{\sigma} \cdot \boldsymbol{n}]_-^+ = \boldsymbol{0}$.

   These six conditions can determine the three transmitted (P, SV and SH) and three reflected (P, SV and SH) amplitudes. Potentially a lot of algebra...

2. If the upper layer is instead an inviscid elastic fluid, we only have four conditions:

   - continuity of *normal* displacement at $z = 0$: $[\boldsymbol{u} \cdot \boldsymbol{n}]_-^+ = 0$;
   - continuity of traction at $z = 0$: $[\boldsymbol{\tau}]_-^+ = [\boldsymbol{\sigma} \cdot \boldsymbol{n}]_-^+ = \boldsymbol{0}$,

   with the two tangential components in the plane of the boundary being zero. Four conditions is all we need, as there is only a P-wave in the upper layer: phew!

3. Rigid boundary problems with the three conditions $\boldsymbol{u} = \boldsymbol{0}$ or (stress-) free boundary problems where $\boldsymbol{\tau} = \boldsymbol{0}$ are also well-posed to determine the three reflected amplitudes.

Decoupling of SH waves

Note that SH waves are the only waves with displacements u_y in the horizontal ($y-$) direction. Therefore boundary conditions on both u_y and σ_{yz} (which involves a term $\partial u_y/\partial z$) only apply to SH waves. Therefore, SH waves decouple from P waves and SV waves, in the sense that SH waves only excite SH waves, while P and SV waves can excite each other. Sounds like fun...and it makes it somewhat more straightforward to solve reflection and refraction problems.

Chapter 9

Reflection and Refraction Examples

With the record selection and the mirror's reflection,
I'm a-dancing with myself
Billy Idol

To end this part of the course, we are going to consider three important examples of reflection, transmission (and in general refraction) and evanescence of non-dispersive waves in elastic solids when waves propagate in the vicinity of plane boundaries, where the properties of the medium change discontinuously.

9.1 Reflection/Refraction of SH waves

Due to the decoupling mentioned in the previous chapter, perhaps the simplest case is the situation shown in figure 9.1. There are three waves:

- An incoming SH wave in $z < 0$, with complex amplitude B, angle of incidence θ, frequency ω and wavenumber (magnitude) κ;

- A reflected SH wave also in $z < 0$, with complex amplitude R, (outgoing) angle of incidence $\bar{\theta}$, frequency $\bar{\omega}$ and wavenumber (magnitude) $\bar{\kappa}$;

- A transmitted SH wave in $z > 0$, with complex amplitude T, angle of incidence θ', frequency ω' and wavenumber (magnitude) κ'.

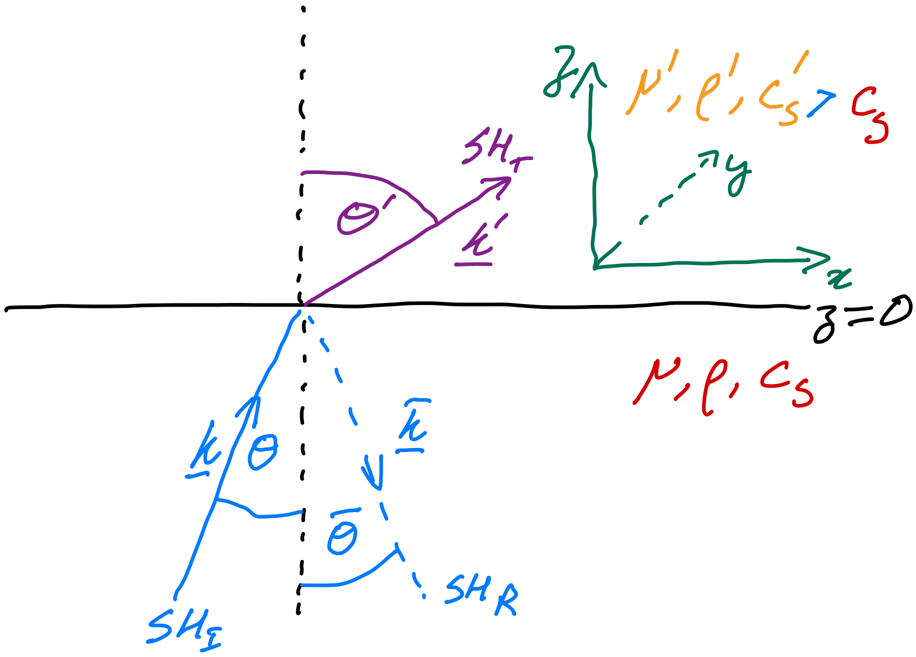

Figure 9.1: Schematic showing an incident SH wave at a boundary between two elastic half planes, which excites a reflected SH wave and a transmitted SH wave. The upper half plane has the larger shear wave phase speed $c_s' > c_s$.

Therefore, the disturbance is *only* in the y direction, and takes the form:

$$\begin{aligned} u_y &= B\exp i\left[\kappa(x\sin\theta + z\cos\theta) - \omega t\right] && (9.1)\\ &\quad +R\exp i\left[\bar{\kappa}(x\sin\bar{\theta} - z\cos\bar{\theta}) - \bar{\omega}t\right] \text{ for } z<0; \\ u_y &= T\exp i\left[\kappa'(x\sin\theta' + z\cos\theta') - \omega' t\right] \text{ for } z>0. && (9.2)\end{aligned}$$

Note in particular the sign of the term involving z in the reflected wave, showing that its wave vector is pointing downwards (towards negative z). Furthermore, from the dispersion relation:

$$\frac{\omega}{\kappa} = \frac{\bar{\omega}}{\bar{\kappa}} = c_s, \tag{9.3}$$

$$\tag{9.4}$$

while

$$\frac{\omega'}{\kappa'} = c_s'. \tag{9.5}$$

The objective is to determine appropriately all the various parameters (e.g the angles, wavenumber magnitudes, frequencies) and the disturbance amplitudes.

9.1.1 Angles and Snell's Law

The key boundary conditions are that

$$[u_y]_-^+ = [\sigma_{xy}]_-^+ = 0. \tag{9.6}$$

"Clearly", these must be satisfied for all values of x and t. Therefore each of the three waves must depend on x and t like the incoming wave, and so must depend on $\exp i[\kappa x\sin\theta - \omega t]$. Therefore:

1. All the waves must have the same *frequency*: $\omega = \bar{\omega} = \omega'$.
2. Now from (9.3) $\kappa = \bar{\kappa}$.
3. Now, from the required $x-$dependence for $z = 0^-$, $\bar{\theta} = \theta$, and so the reflected wave vector has the same magnitude and makes the same angle with the vertical as the incoming wave vector.
4. For the *refraction* of the transmitted wave, the required $x-$dependence combined with (9.5) yields *Snell's Law*:

$$\frac{\sin\theta}{c_s} = \frac{\sin\theta'}{c_s'}. \tag{9.7}$$

By combining (9.3) and (9.5), the $z-$component for the transmitted wave vector can then be rewritten as

$$
\begin{aligned}
\kappa' \cos\theta' &= \frac{\kappa c_s}{c_s'}\left(1-\sin^2\theta'\right)^{1/2}, \\
&= \frac{\kappa c_s}{c_s'}\left(1-\left[\frac{c_s'\sin\theta}{c_s}\right]^2\right)^{1/2}, \\
&= \kappa\left(\left[\frac{c_s}{c_s'}\right]^2-\sin^2\theta\right)^{1/2} = m'. \qquad (9.8)
\end{aligned}
$$

[This uses the common convention for the components of a wave vector: $\boldsymbol{k} = (k, l, m)$ and so $\kappa = \sqrt{k^2 + l^2 + m^2}$.]

Evanescence

From Snell's Law, it is apparent that (as in the figure) if $c_s' > c_s$, then $\theta' > \theta$. Indeed, from (9.8) if $\sin\theta > c_s/c_s'$, there is no real solution for θ', and m' is purely imaginary $m' = i\hat{m}'$ where $\hat{m}' > 0$ since the transmitted disturbance takes the form

$$
u_y = T \exp i\left[\kappa x \sin\theta - \omega' t\right] e^{-\hat{m}' z},
$$

and the wave is *evanescent* in $z < 0$. The amplitude decays exponentially (i.e. there is *attenuation* in the amplitude), with an e-folding height of $O(1/\hat{m}')$. There is *total internal reflection* for $\theta > \theta_c$ where

$$
\sin\theta_c = \frac{c_s}{c_s'},
$$

which clearly only makes sense for $c_s/c_s' < 1$.

9.1.2 Amplitudes

The various amplitudes can now be determined from applying (9.6) to (9.1) and (9.2). Continuity of displacement implies

$$
B + R = T.
$$

Continuity of $[\sigma_{yz}]$ requires

$$
\begin{aligned}
\left[\mu \frac{\partial}{\partial z} u_y\right]_-^+ &= 0, \\
\rightarrow i\mu\kappa(B - R)\cos\theta &= i\mu'\kappa' T \cos\theta',
\end{aligned}
$$

and so

$$R = \frac{(1-Z)}{1+Z}B, \tag{9.9}$$

$$T = \frac{2}{1+Z}B, \tag{9.10}$$

where Z is the ratio of appropriate impedances (loosely the ratio of the stress applied to the vertical velocity response, analogously to the concept we encountered for acoustic waves in Part I):

$$\begin{aligned} Z &= \frac{\mu'\kappa'\cos\theta'}{\mu\kappa\cos\theta}, \\ &= \frac{\rho' c_s'\cos\theta'}{\rho c_s\cos\theta}. \end{aligned} \tag{9.11}$$

So, if the impedances are matched (and so $Z = 1$) there is complete transmission with no reflection. Conversely, if $\cos\theta'$ is purely imaginary $|R| = |B|$, and there is total internal reflection.

9.1.3 Energy Flux

Indeed in general, we expect the $z-$component of the (wave) energy flux to be constant with z, at least when averaged over a period as defined in (3.11). We know for such waves that in general

$$\mathbf{I} = -\boldsymbol{\sigma}.\frac{\partial \mathbf{u}}{\partial t},$$

and so, in this particular case, the averaged $z-$component of from the incident wave, the reflected wave and the transmitted wave are (remembering the cunning averaging trick):

$$\begin{aligned} \langle \mathbf{I}_z\rangle_I &= -\frac{1}{2}\Re\left[\sigma_{yz}\left(\dot{\boldsymbol{u}}\right)_y^*\right]_I = \frac{\omega}{2}\mu\kappa\cos\theta|B|^2, \\ \langle \mathbf{I}_z\rangle_R &= -\frac{\omega}{2}\mu\kappa\cos\theta|R|^2, \\ \langle \mathbf{I}_z\rangle_T &= \frac{\omega}{2}\mu'\kappa'\cos\theta'|T|^2 \text{ if } \cos\theta' \text{ is real}, \\ &= 0 \text{ if } \cos\theta' \text{ is imaginary}. \end{aligned}$$

Therefore (as expected of course) if $\cos\theta'$ is real,

$$|B|^2 - |R|^2 = Z|T|^2,$$

while if $\cos\theta'$ is imaginary,

$$|B|^2 = |R|^2.$$

9.2 Reflection and Refraction of P/SV waves

In general, problems considering incoming P (or SV) waves to an interface (wlog at $z = 0$ with the incoming wave vector being restricted to the $x - z$ plane) between different media involve one incoming wave (either P or SV) exciting 4 waves: transmitted P **and** SV waves, as well as reflected P and SV waves.

9.2.1 Angles and Snell's Law

As the boundary conditions (at $z = 0$) once again need to be satisfied for all x and t, we once again have that the frequency of oscillation ω must be the same for each of the waves. Then, the combination of the dispersion relation(s) and the requirement for the $x-$dependence of the disturbances to match leads to the appropriate version of Snell's law. As an example, let us suppose there is an incoming P wave with angle of incidence θ. Let us also suppose (similarly to above) that the phase speeds are c_p and c_s in the lower half plane, and c'_p and c'_s in the upper half plane. Therefore

$$\begin{aligned} \frac{\sin\theta}{c_p} &= \frac{\sin\bar{\theta}_P}{c_p} = \frac{\sin\bar{\theta}_s}{c_s}, \\ &= \frac{\sin\theta'_P}{c'_p} = \frac{\sin\theta'_s}{c'_s}. \end{aligned}$$

In these expressions, the reflected P wave has (outgoing) angle of incidence $\bar{\theta}_p$, and the reflected SV wave has (outgoing) angle of incidence $\bar{\theta}_s$. In the upper half plane, the transmitted P wave has angle of incidence θ'_p and the transmitted SV wave has angle of incidence θ'_s. Since $c_p > c_s$ (and $c'_p > c'_s$) it is clear that $\bar{\theta}_p = \theta > \bar{\theta}_s$ and $\theta'_p > \theta'_s$.

Amplitudes

Once again, the (now 4) boundary condtions can be used to determine the 4 amplitudes. It can be a fun calculation however, as in general the boundary conditions will lead to a 4×4 matrix. This subject is not for those who are put off my a bit of mathematical manipulation. However, life can be made a little easier in situations where:

- the boundary is either free or rigid (and so there are no waves in $z > 0$);
- one of the shear moduli μ, μ' is zero, thus meaning in such an (inviscid) elastic fluid, there are no SV waves;

- both $\mu = \mu' = 0$ and we recover the acoustic problems of Part I.

Fun fact: the (observed) absence of S waves was the way by which it was first inferred that the earth had a liquid outer core.

9.3 Evanescent Waves

Evanescent waves are (sometimes) said to be *trapped* at the interface, as the amplitude decays away from the boundary. Let us consider such waves, fixing the requirement that the frequency ω is real, yet the wave vector $\boldsymbol{k}$ is allowed to be complex, so that $\boldsymbol{k} = \boldsymbol{k}_r + i\boldsymbol{k}_i$. Therefore, the displacement

$$\boldsymbol{u} \;=\; \mathrm{k}\exp(i\boldsymbol{k}_r.\boldsymbol{x} - \boldsymbol{k}_i.\boldsymbol{x} - i\omega t)\,,$$

and so there is *propagation* in the $\boldsymbol{k}_r$ direction and *attenuation* in the $\boldsymbol{k}_i$ direction.

From the appropriate dispersion relation (for either P waves or SV waves)

$$\begin{aligned} \omega^2 &= c_{p/s}^2(\boldsymbol{k}_r + i\boldsymbol{k}_i)\cdot(\boldsymbol{k}_r + i\boldsymbol{k}_i) \\ &= c_{p/s}^2(\kappa_r^2 - \kappa_i^2) + 2ic_{p/s}^2\,\boldsymbol{k}_r\cdot\boldsymbol{k}_i\,. \end{aligned}$$

Since ω is real (by construction):

- $k_r^2 > k_i^2$;
- $\boldsymbol{k}_r\cdot\boldsymbol{k}_i = 0$;
- the phase speed of the disturbance (wlog assumed positive)

$$c \equiv \frac{\omega}{\kappa_r} = c_{p/s}\left(1 - \tfrac{\kappa_i^2}{\kappa_r^2}\right)^{1/2} < c_{p/s}.$$

Therefore, the (trapped) propagation along (or more precisely in the vicinity of) the boundary is slower than through the interior. This (general) phenomenon is important in seismology.

9.3.1 Rayleigh waves

As a final example of such a “trapped” wave (as well as some tortuous mathematical keepy-uppy) we consider *Rayleigh waves*. As shown (effectively) in the first section of this chapter, it is not possible to have a self-sustained SH wave at a boundary, as an incoming wave has to force the system continually. Curiously, it is actually possible to have a self-contained combination of P and SV waves, called a Rayleigh wave (and it is these waves that typically cause the most damage in earthquakes, as well as to your paper supplies if you set the problem up untidily).

The problem of interest is for an semi-infinite (for $z < 0$) region of an elastic solid of density ρ, with Lamé moduli λ, μ and associated phase speeds

$$c_s^2 = \frac{\mu}{\rho}; \; c_p^2 = \frac{\lambda + 2\mu}{\rho}.$$

The boundary at $z = 0$ is stress-free, (for example with a vacuum in $z > 0$) and we seek a solution as a particular combination of a P-wave and and SV-wave, trapped at the boundary and so evanescent as $z \to -\infty$. With perhaps not the best choice of notation, we then define the two wave vectors as

$$\begin{aligned} \boldsymbol{k}_p &= k(1, 0, -ia), \\ \boldsymbol{k}_s &= k(1, 0, -ib), \end{aligned}$$

where k, a and b are all real and positive.

Then, the displacement can be written as

$$\begin{aligned} \boldsymbol{u} &= A(i, 0, a)e^{ik(x-ct)+kaz} \text{ (the P wave)} \\ &\quad + B(b, 0, -i)e^{ik(x-ct)+kbz} \text{ (the SV wave)}, \end{aligned}$$

which has the required properties of decay as $z \to -\infty$ and parallel/perpendicular orientation relative to the wave vectors.

From the previous section, we also can see that the dispersion relation(s) imply that

$$c^2 \equiv \frac{\omega^2}{k^2} = c_p^2(1 - a^2), \tag{9.12}$$

$$= c_s^2(1 - b^2). \tag{9.13}$$

There are two boundary conditions to impose at $z = 0$. The first is quite straightforward.

$$\begin{aligned} 0 = \sigma_{xz}|_{z=0} &= \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)_{z=0} \\ &= \mu k[2iaA + (b^2 + 1)B]. \end{aligned} \tag{9.14}$$

The second is a bit more fiddly, and a few cunning substitutions make things much neater. In particular, we relate phase speeds to each other (via the dispersion relations) and to the Lamé moduli, and also remember that S waves are effectively waves of the rotation, so their divergence is zero by construction. Therefore:

$$
\begin{aligned}
0 = \sigma_{zz}\big|_{z=0} &= \left[\lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_z}{\partial z}\right]_{z=0}, \\
&= \left[(\lambda + 2\mu)\boldsymbol{\nabla}\cdot\boldsymbol{u} - 2\mu\frac{\partial u_x}{\partial x}\right]_{z=0}, \\
&= \rho k\left[c_p^2(a^2-1)A - 2c_s^2(-A + ibB)\right], \\
&= \rho k c_s^2\left[(b^2+1)A - 2ibB\right]. \qquad (9.15)
\end{aligned}
$$

The two equations (9.14) and (9.15) of course define a simple matrix problem:

$$
\begin{pmatrix} 2ia & b^2+1 \\ b^2+1 & -2ib \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

This has a solution when the determinant of the matrix is zero of course, and so when

$$
4ab = (b^2+1)^2.
$$

Using (9.12) and (9.13), we can thus derive the *dispersion relation* for c:

$$
4\left(1 - \frac{c^2}{c_p^2}\right)^{1/2}\left(1 - \frac{c^2}{c_s^2}\right)^{1/2} = \left(2 - \frac{c^2}{c_s^2}\right)^2. \qquad (9.16)
$$

As c_p can be related to c_s via, for example *Poisson's ratio* ν, this equation can be written as a cubic for c^2/c_s^2, with one real root for $0 < c < c_s$.

Although it is appropriate to call (9.16) a dispersion relation, the waves do not disperse, as the phase speed is the same whatever the (horizontal) wavenumber k (and hence frequency). Indeed, we really have solved an eigenvalue problem for c. These waves are really important in seismology, but we need to move on to Part III of the course, where we consider *dispersive waves*. How exciting!

Part III

Dispersive Waves

Chapter 10

Dispersive Wave Examples

I believe what the old man said
Philip Oakey

Up to now we have considered *non-dispersive* waves, i.e. waves that all travel at the same speed irrespective of wavelength. But that is of course is a special case, and we now turn our attention to systems where the waves do indeed *disperse*, i.e. waves with different wavelengths travel at different speeds, and indeed we need to think about different speeds, distinguishing (perhaps) between the speed of phase propagation and the speed of energy propagation. Before getting to some theory, let's first consider a couple of useful illustrative examples: one involving acoustic waves as considered in Part I; and then involving waves in solids. In both cases the geometry gives one or more length scales which affect the possible wave numbers, and so lead to dispersive dispersion relations.

10.1 The acoustic waveguide

Consider the model problem of acoustic waves in a rectangular duct, $0 \le y \le a$, $0 \le z \le b$ so that boundary conditions of zero normal velocity apply at the walls. We then solve the (acoustic) wave equation with these boundary conditions, expecting propagatioon in the $x-$direction. (The waves are "guided" down the duct: many musical instruments work on this principle.) Therefore, we solve the following problem for the velocity potential:

$$\frac{\partial^2}{\partial t^2}\phi = c_0^2 \nabla^2 \phi \ ,$$

$$\left.\begin{array}{lll} \phi_y = 0 & \text{on} & y = 0, a \\ \phi_z = 0 & \text{on} & z = 0, b \end{array}\right\} \forall\ x \ .$$

As we assume that the waves propagate in the $x-$direction, we seek a separable harmonic wave solution of the form

$$\phi = e^{ikx-i\omega t} f(y,z) \ .$$

This satisfies the wave equation and the boundary conditions if

$$\begin{aligned} f_{yy} + f_{zz} + \left(\frac{\omega^2}{c_0^2} - k^2 \right) f &= 0 \ , \\ f_y &= 0 \qquad \text{on} \quad y = 0, h \ , \\ f_z &= 0 \qquad \text{on} \quad z = 0, b \ . \end{aligned}$$

These equations actually define an *eigenvalue* problem (cf. Schrödinger bound states), i.e. for given k seek eigenvalues ω_*, and associated eigenfunctions (commonly referred to as modes) f_*.

Blah blah blah well-posed blah blah blah unique blah blah blah we make the ansatz (German for educated guess) that the solution is separable and takes a form which naturally satisfies the boundary conditions:

$$f_{mn} = A_{mn} \cos \frac{m\pi y}{h} \cos \frac{n\pi z}{b} \ , \quad m, n \in \mathbb{Z} \, ,$$

where, indeed wlog, $m, n \in \mathbb{N}$. Then the wave equation is satisfied when ω is given by the *dispersion relation*

$$\omega^2 = \omega_{mn}^2(k) \ \equiv \ c_0^2 \left(k^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right) \ . \qquad (10.1)$$

Phase speed

For a given mode (m, n), the phase speed is given by

$$c = \frac{\omega}{k} = \pm \, c_0 \left(1 + \left(\frac{m\pi}{ka}\right)^2 + \left(\frac{n\pi}{kb}\right)^2 \right)^{\frac{1}{2}} \ .$$

(The *phase velocity* is clearly in the $x-$direction.) Hence, unless the waves are plane waves (i.e. $m = n = 0$), the phase speed, c, varies with wavenumber; the waves are then said to be *dispersive.*

Cut-off Frequency

A mode of a given (reasonably assumed real) frequency ω *propagates* only if k is real. Hence to excite the (m, n) mode we require

$$|\omega| > c_0 \left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^{\frac{1}{2}} \ . \qquad (10.2)$$

This minimum frequency is called the *cut-off frequency.* Attempts to excite the (m, n) mode ($m \neq 0$, $n \neq 0$) below this frequency yield exponentially decaying evanescent modes with imaginary k. (Such a phenomenon occurs when you blow over the top of a bottle: you must do it sufficiently gently to excite the sound.)

Hence for given ω, there exist only a finite number of propagating waves: the plane wave ($m = n = 0$) and a limited number of "cross-modes" (called so for reasons which will be discussed below). The other modes are "cut-off" and do not propagate.

10.1.1 Energetics

It is now approproate to average across the cross-sectional area of the duct, denoted by

$$\overline{\quad\cdot\quad} = \frac{1}{ab}\int_0^b \int_0^a (\quad\cdot\quad)dydz, \tag{10.3}$$

as well as over a period (as defined in (3.11) and denoted with left and right angles). Therefore, the (appropriately averaged) kinetic energy is

$$\begin{aligned}
\langle\overline{K}\rangle &= \frac{1}{ab}\int_0^b dz \int_0^a dy\, \frac{1}{2}Re[\frac{1}{2}\rho_0 \boldsymbol{u}\cdot\boldsymbol{u}^*] \\
&= \frac{1}{4\chi_{mn}}\rho_0|A_{mn}|^2\left(k^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right) \\
&= \frac{\rho_0|A_{mn}|^2\omega^2}{4\chi_{mn}c_0^2}.
\end{aligned}$$

Here $\chi_{mn} = (2 - \delta_{m0})(2 - \delta_{n0})$ accounts for the difference between averaging $(\cos 0)^2$ compared with $(\cos \frac{\pi y}{a})^2$ and so on, and we have used the dispersion relation as well as the cunning trick for calculating the energy from u (and hence from $\boldsymbol{\nabla}\phi$).

Similarly, remembering the definition of potential energy in this context, as well as the relationship (here) between pressure perturbation, density perturbation and velocity potential $\tilde{p} = c_0^2\tilde{\rho} = i\omega\rho_0\phi$, we obtain

$$\begin{aligned}
<\overline{W}> &= \frac{1}{ab}\int_0^b dz \int_0^a dy\, \frac{1}{2}Re\left[\frac{1}{2}c_0^2\frac{\tilde{\rho}\tilde{\rho}^*}{\rho_0}\right] \\
&= \frac{\rho_0|A_{mn}|^2\omega^2}{4\chi_{mn}c_0^2} = <\overline{K}>\ .
\end{aligned}$$

When averaged in this way, there is again equi-partition of energy.

Acoustic Energy Flux

The (appropriately averaged) acoustic energy flux in the $x-$direction is then

$$
\begin{aligned}
< I_x > &= \frac{1}{ab}\int_0^b dz \int_0^a dy\, \frac{1}{2} Re\left[\tilde{p}\varphi_x^*\right] \\
&= \frac{1}{ab}\int_0^b dz \int_0^a dy\, \frac{1}{2}\omega\rho_0 k|A_{mn}|^2 \cos^2\frac{m\pi y}{a}\cos^2\frac{n\pi z}{b} \\
&= \frac{\omega\rho_0 k|A_{mn}|^2}{2\chi_{mn}}\,.
\end{aligned}
$$

We can now define the *mean energy propagation velocity* to be

$$U(k) \equiv \frac{<\overline{I_x}>}{<\overline{K}+\overline{W}>} = \frac{kc_o^2}{\omega}\,.$$

We can also define the *group velocity* (please excuse the sloppy use of language, here "clearly" these velocities are in the $x-$direction) to be

$$c_g(k) \equiv \frac{\partial\omega}{\partial k}\,,$$

From the dispersion relation (10.1), choosing the positive root (dealing with our sloppiness about velocity and speed: the energy here should go "down" the duct to large positive x)

$$c_g(k) = \frac{c_0}{\left(1+\left(\frac{m\pi}{ka}\right)^2+\left(\frac{n\pi}{kb}\right)^2\right)^{\frac{1}{2}}} = \frac{kc_0^2}{\omega}\,.$$

Hence for this example

$$U(k) = c_g(k) = \frac{\partial\omega}{\partial k}\,.$$

Is this a coincidence? Err no, and we shall see why it holds much more generally (e.g. there exists a vector equivalent) below, and indeed why it is called the *group velocity.*

There are some other observations worthy of note.

1. Here,
$$c_g(k) \neq c(k),$$
and so the energy propagation velocity $\neq$ the wave crest velocity.

2. Other than for plane waves ($m = n = 0$):

- $c_g < c_0$, while $c > c_0$ (focussing on the positive roots for the waves propagating "down" the duct.
- For "short" waves, $c_g \to c_0$ and $c \to c_0$ as $k \to \infty$.
- For long waves, $c_g \to 0$ while $c \to \infty$ as $k \to 0$. That the phase speed can, in principle, be larger than the speed of light, should not stress you out, because as Madonna never sang, we are living in a classical world...

Superposition

Finally, it is actually possible to interpret the waveguide as a superposition of reflecting plane waves. To fix ideas (and to make life easier) let's choose $n = 0$ but $m \neq 0$. Therefore, if we define $\sin\theta = m\pi c_0/\omega a$, then, from the dispersion relation (10.1)

$$\frac{kc_0}{\omega} = \cos\theta \to \frac{\omega}{k} = \frac{c}{\cos\theta}.$$

Therefore the *apparent* phase speed in the x-direction is

$$c = \frac{c_0}{\cos\theta} > c_0\,,$$

while from the expression for the group velocity

$$c_g = \frac{c_0^2 k}{\omega} = c_0\cos\theta < c_0,$$

by definition of $\cos\theta$. Finally, the wave field can be interpreted as a superposition of plane waves tilted "upwards" at a positive angle θ relative to the horizontal, and plane waves tilted "downwards" at a negative angle $-\theta$ relative to the horizontal, as the velocity potential takes the form

$$\phi \propto \left[e^{\frac{i\omega}{c_0}(x\cos\theta + y\sin\theta - c_0 t)} + e^{\frac{i\omega}{c_0}(x\cos\theta - y\sin\theta - c_0 t)}\right].$$

10.2 Love Waves

As we encountered at the end of Part II, a Rayleigh wave is a special brew of a perfectly mixed interfacial P/SV wave. The question might be posed whether it is possible to have an interfacial SH waves. The answer is yes, but only within a *layered* half space, where we (effectively) have a *wave guide* along which the waves can propagate.

Love waves are a class of waves in elastic solids which are trapped in a finite layer (cf. bound states in a finite well in QM). We consider a layer $0 < z < h$ with shear modulus μ_1 and density ρ_1 so that the shear wave (phase) speed $c_s = c_1$ is relatively "slow". The upper interface at $z = h$ is (stress) free so that (for SH waves)

$$\mu \frac{\partial u_y}{\partial z}\bigg|_{z=h} = 0.$$

The layer overlays a semi-infinite region ($z < 0$) with shear modulus μ_2 and density ρ_2 so that the shear wave (phase) speed $c_s = c_2$ is relatively "fast", in the well-defined sense that $c_2 > c_1$. Then we can have Love waves (these are sometimes called Q waves as *quer* is *transverse* in German) with phase speed squeezed in $c_1 < c < c_2$ with reflection at the free interface, and total internal reflection at the lower interface, with the waves evanescent in the "fast" region beneath the "slow crust".

So we look for SH waves (i.e. solutions of the wave equation with the appropriate choice of c_s) with displacement

$$\mathbf{u} = (0, u, 0) = (0, f(z)e^{ik(x-ct)}, 0)$$

in both $0 < z < h$ and $z < 0$, as the time- and $x-$dependence must match. Let's consider each region in turn.

Propagating region $0 < z < h$

Here

$$\begin{aligned} -c^2k^2 f &= c_1^2(f'' - k^2 f), \\ \to f'' &= -k^2\left(\frac{c^2}{c_1^2} - 1\right) f \\ &= -m_1^2 f;\ m_1 = k\left(\frac{c^2}{c_1^2} - 1\right)^{1/2}, \end{aligned}$$

where m_1 must be real (and wlog positive) for there to be *propagating* waves in this region (and so the solutions to the ODE are sines and cosines). Since $\partial u/\partial z = 0$ at the stress-free boundary $z = h$, it is "clear" that the (propagating) SH wave solution in this region is then

$$u_1 = Ae^{ik(x-ct)}\cos m_1(h - z);\ m_1 = k\left(\frac{c^2}{c_1^2} - 1\right)^{1/2}, \tag{10.4}$$

for some complex amplitude A. Note for propagating waves we require $c > c_1$.

Evanescent region $z < 0$

Here

$$\begin{aligned} -c^2k^2 f &= c_2^2(f'' - k^2 f), \\ \to f'' &= +k^2\left(1 - \frac{c^2}{c_2^2}\right) f \\ &= +m_2^2 f;\ m_2 = k\left(1 - \frac{c^2}{c_2^2}\right)^{1/2}, \end{aligned}$$

where m_2 must be real (and wlog positive) for there to be *evanescent* waves in this region (and so the solutions to the ODE are now exponentials). Since $u \to 0$ as $z \to -\infty$, it is "clear" that the evanescent solution in this region is then

$$u_2 = Be^{ik(x-ct)}e^{m_2 z};\ m_2 = k\left(1 - \frac{c^2}{c_2^2}\right)^{1/2}, \tag{10.5}$$

for some complex amplitude A. Note for evanescent waves we require $c < c_2$, and so we require $c_1 < c < c_2$.

Dispersion Relation

The dispersion relation is now determined from applying the boundary conditions at $z = 0$:

$$\begin{aligned} [u]_-^+ = 0 \to A\cos m_1 h &= B, \\ \left[\mu\frac{\partial u}{\partial z}\right]_-^+ = 0 \to A\mu_1 m_1 \sin m_1 h &= \mu_2 m_2 B, \end{aligned}$$

and so we obtain the dispersion relation:

$$\tan\left[\left(\frac{c^2}{c_1^2} - 1\right)^{1/2} kh\right] = \frac{\mu_2}{\mu_1}\left(\frac{1 - \frac{c^2}{c_2^2}}{\frac{c^2}{c_1^2} - 1}\right)^{1/2}. \tag{10.6}$$

This equation can be solved graphically by plotting the LHS (green line, for a particular choice of k) and RHS (blue line) for $c_1 < c < c_2$, noting that over this range, $\phi = (c^2/c_1^2 - 1)^{1/2}\, kh$ increases from 0 to $(c^2/c_1^2 - 1)^{1/2}\, kh$, as shown in figure 10.1

There are several "interesting" (perhaps in the sense that Steve Davis is interesting) properties of this dispersion relation.

1. From the graph we see that there is at least one solution with $c_1 < c < c_2$ for all k.

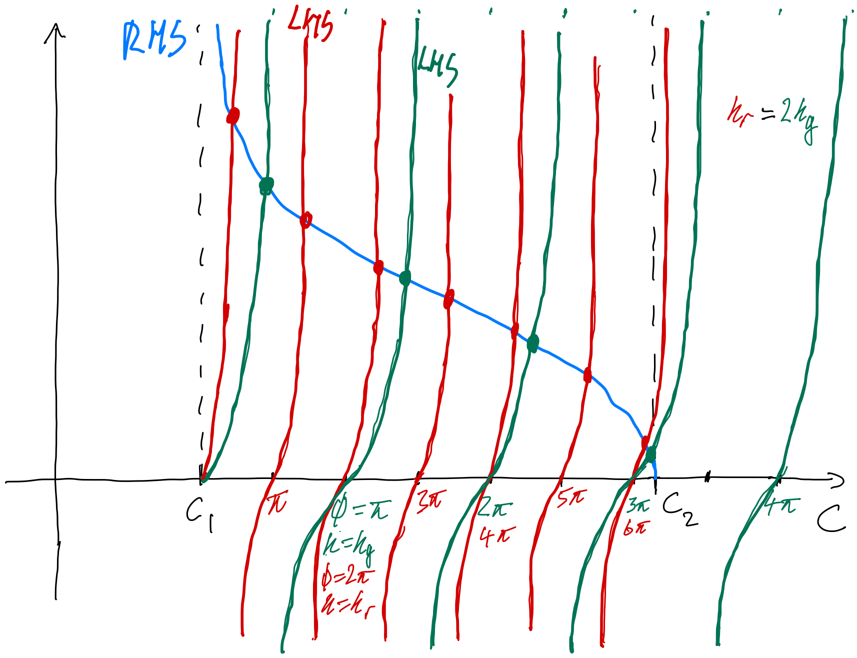

Figure 10.1: Plot of RHS (blue line) and LHS (green line for a particular choice of $k = k_g$, and red line for $k = k_r = 2k_g$). Note there is always at least one solution, and increasing k "squeezes" the tangent curves to the left, thus increasing the number of modes.

2. There are n solutions if

$$(n-1)\pi \leqslant \left(\frac{c_2^2}{c_1^2}-1\right)^{1/2} kh < n\pi \,.$$

3. Of course, since c depends on k, i.e. $c \equiv c(k)$, the waves are *dispersive.*

4. As k and ϕ increase, the tan curves are 'squeezed' to the left like a concertina, which implies that each new mode appears with $c = c_2$ and the phase speed then decreases towards c_1.

5. The n^{th} mode has a minimum "cut-off" frequency (using the fact that $\omega \equiv kc = kc_2$ at cut-off)

$$\omega_n^{(c)} = \frac{n\pi c_2}{h}\left(\frac{c_2^2}{c_1^2}-1\right)^{-1/2} = k_n^{(c)} c_2 .$$

6. Since in general $\omega = kc$, as k increases from $k_n^{(c)}$, $c_2 k < \omega < c_1 k$, and $\omega \to c_1 k$ as $k \to \infty$. Cool eh?

7. It is possible to confirm that the mean energy propagation velocity is indeed $c_g = \partial\omega/\partial k$, but the calculation is heroic, even by the standards of this course. It might be nice to have a bit of theory to establish this relationship...

Chapter 11

Dispersive Wave Packets

And the beat goes on, you'd better believe it
Leon Sylvers III

In this chapter we consider some of the key characteristics of dispersive wave "packets", and gain some further understanding (hopefully) of the group velocity.

11.1 Beats and Modulation

Consider the superposition of two waves of equal (wlog unit) amplitude and nearly equal wavenumbers and frequencies:

$$\begin{aligned}\phi &= \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t), \\ &= 2\cos\left[\frac{(k_1+k_2)}{2}x - \frac{(\omega_1+\omega_2)}{2}t\right]\cos\left[\frac{(k_1-k_2)}{2}x - \frac{(\omega_1-\omega_2)}{2}t\right],\end{aligned}$$

where $k_1 \simeq k_2 \simeq k$ and $\omega_1(k_1) \simeq \omega_2(k_2) \simeq \omega$, as there is a dispersion relation $\omega(k)$. Wlog $k_1 > k_2$.

Therefore there is a *modulated* wave. The *carrier wave* with frequency $(\omega_1+\omega_2)/2$ and wavenumber $(k_1+k_2)/2$ has phase speed $(\omega_1+\omega_2)/(k_1+k_2) \simeq \omega/k$. The amplitude of this carrier wave is *modulated* on a much longer wavelength $\pi/(k_1-k_2)$. This modulation in amplitude (and hence loudness) is often called *beating* in music. The envelope of this modulation moves at the speed $(\omega_1-\omega_2)/(k_1-k_2) \simeq d\omega/dk$. Since the energy of the modulated wave is propagating with the peaks of this modulated envelope, which may be thought of as a "group" of the carrier waves' peaks and troughs, we can now see why c_g is indeed called the group velocity, and also why it is related to energy propagation.

11.2 Fourier Transforms for IVPs

We now start to build the set of tools needed to solve initial value problems for dispersive wave systems. Fundamentally, we want to consider a linear system on $-\infty < x < \infty$ with a known dispersion relation $\omega(k)$, in the sense that the system has a governing PDE which admits wavelike solutions for particular choices of ω and k. If there is no disturbance arriving from $\pm\infty$, how does an initial disturbance evolve in time? The central idea is to exploit superposition of linear waves using Fourier transforms.

The appropriate FT/inverse FT pair in this situation is different from that you might have encountered in Methods. We want to think of the disturbance ϕ as being composed of (infinitely many) waves with the amplitude of the wave with wavenumber k being given by some function $\hat{\phi}(k)$, sometimes called (in this context) the *spectrum* of ϕ. Therefore

$$\phi(x,t) = \int_{-\infty}^{\infty} \hat{\phi}(k,t) e^{ikx} dk. \tag{11.1}$$

We recognise that this is a perfectly reasonable definition for an *inverse Fourier transform*, provided the (forward) Fourier transform determining $\hat{\phi}(k)$ is

$$\hat{\phi}(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x,t) e^{-ikx} dx. \tag{11.2}$$

Thus ϕ can be split into (many) waves of the form e^{ikx}. Each of these waves evolves with time like $\exp(-i\omega(k)t)$, and so by (re-)superposition we can then determine the time evolution of the initial disturbance, provided we are able to determine $\hat{\phi}(k,t)$. Let's consider two examples which illustrate some key ideas.

First order in time PDE

Let us assume that the (linear) PDE only involves $\partial\phi/\partial t$ (though is unconstrained in the order of the partial derivatives with respect to x). Then a well-posed problem only requires an initial condition $\phi(x,0)$, and the problem is then solved as follows.

- At $t=0$, $\hat{\phi}(k,0)$ is determined from (11.2):

$$\hat{\phi}(k,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x,0) e^{-ikx} dx.$$

- At general time t,

$$\hat{\phi}(k,t) = \hat{\phi}(k,0)e^{-i\omega(k)t}.$$

- This expression is then plugged into (11.1) to determine $\phi(x,t)$:

$$\phi(x,t) = \int_{-\infty}^{\infty} \hat{\phi}(k,0)e^{i[kx-\omega(k)t]}dk.$$

Observations

It is very important to note that k ranges from $-\infty$ to ∞ in the superposition, so we have to allow for negative wavenumbers. Furthermore, $\hat{\phi}(k,0)$ is clearly key to the solution. Since (usually) we will be interested in real disturbances $\phi(x,t)$, in particular $\phi(x,0)$ is real. Then, from the properties of FTs, $\hat{\phi}(-k,0) = \hat{\phi}^*(k,0)$, where the asterisk denotes complex conjugation. Furthermore, if the initial disturbance is real and *even* in x, i.e. $\phi(x,0) = \phi(-x,0)$, then $\hat{\phi}(k,0)$ is also real. These properties are often useful in solving problems of (ahem) interest, such as Tripos questions.

Second order in time PDE

The situation is a little more complicated in problems where the underlying PDE is second order in time, and so involves a term $\partial^2\phi/\partial t^2$.

- Two initial conditions are now required (remember D'Alembert's solution for example):

$$\phi(x,0) \text{ and } \frac{\partial\phi}{\partial t}(x,0).$$

- Also, the dispersion relation has the form

$$\omega^2 = f(k),$$

  for some function $f(k)$.

- So (just as in D'Alembert's solution) an initial wavelike disturbance e^{ikx} gives rise to waves $\exp[i(kx \mp \omega(k)t)]$ where here $\omega \equiv \sqrt{f}$, and so **in the formula** $\omega \geq 0$.

- It is necessary to be very careful with signs (and in descriptions of the properties of the waves). Just to take one example, the phase velocity is determined by the sign of $\pm\omega/k$, but remember k can be negative.

With all those various caveats, the solution for such second order problems can then be written as

$$\phi(x,t) = \int_{-\infty}^{\infty} \left[A(k)e^{i[kx-\omega(k)t]} + B(k)e^{i[kx+\omega(k)t]}\right] dk; \ \omega(k) \equiv \sqrt{f(k)},$$

where $A(k)$ and $B(k)$ can be determined from the two expressions:

$$\begin{aligned} A+B &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x,0)e^{-ikx}dx, \\ -i\omega A + i\omega B &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial\phi}{\partial t}(x,0)e^{-ikx}dx, \ \omega \equiv \sqrt{f(k)}. \end{aligned}$$

These expressions can be simplified for "nice" initial conditions:

$$\begin{aligned} \frac{\partial\phi}{\partial t}(x,0) = 0 \quad &\rightarrow \quad A = B = \frac{1}{2}\hat{\phi}(k,0), \\ \phi(x,0) = 0 \quad &\rightarrow \quad B = -A = \frac{1}{2i\omega}\frac{\partial\hat{\phi}}{\partial t}(k,0); \ \omega \equiv \sqrt{f(k)}. \end{aligned}$$

Chapter 12

Method of Stationary Phase

It's just a silly phase I'm going through
Eric Stewart

In this chapter we consider waves to approximate the dominant contributions to a certain class of integrals which regularly arise in wave problems, and find that there is a "deep" connection to the idea of group velocity. If this lecture were a book, it would have a yellow and black cover. To read a proper treatment of this beautiful technique, please consult *Perturbation Methods'* by E. J. Hinch, or go to Asymptotic Methods in Part II. You have been warned...

12.1 Stationary Phase

As one shown in the last chapter, integrals of the form

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{ikx - i\omega(k)t} \, dk \tag{12.1}$$

naturally arise in initial value problems of wave systems. For large times $t \gg 1$ we reasonably expect x to be large, since $x = O(c_g t)$ as the energy involved in the waves is propagating at that speed. Of course if $c_g(k)$ is not constant, waves with different wavelengths will travel at different speeds, and so they will spread out, i.e. they will *disperse*.

We are curious what will be observed by an observer travelling at speed V at large t, and so we write

$$x = Vt,$$

and investigate the limit as $t \to \infty$ with V remaining $O(1)$. The key integral (12.1) can then be rewritten as

$$\phi = \int_{-\infty}^{\infty} A(k) e^{i\theta} \, dk \,, \tag{12.2}$$

where $\theta(k)$ is the *phase*:

$$\theta(k) = [kV - \omega(k)]t \ . \tag{12.3}$$

Therefore

$$\frac{d\theta}{dk} = \left(V - \frac{d\omega}{dk} \right) t, \tag{12.4}$$

and remember we are considering large t.

We then have two different situations.

1: Varying Phase

If

$$\frac{d\omega}{dk} \neq V,$$

then the phase θ changes rapidly with k and the integrand oscillates wildly (like the three Smiths who are still cool) and there is so much cancellation that the integral is *exponentially small.* The characteristic scale of the oscillation (i.e. when θ varies by 2π) is $O(1/t)$, so really small for large t.

* Non-Examinable Example of *Exponentially Small* *

Consider as a particularly simple example $\theta = kt$, and the associated integral

$$J(t) = \int_{-\infty}^{\infty} A(k) e^{itk} \, dk \ ,$$

where A is C^∞, (such posh Maths lingo is a hint that this is non-examinable) and $A^{(n)}(k) \to 0$ as $|k| \to \infty$ (e.g. $A(k) = e^{-k^2}$). Then, from integration by parts,

$$\begin{aligned} J(t) &= -\frac{1}{it} \int_{-\infty}^{\infty} A'(k) e^{itk} \, dk \\ &= \left(\frac{i}{t}\right)^2 \int_{-\infty}^{\infty} A''(k) e^{itk} \, dk \ , \\ &= \left(\frac{i}{t}\right)^n \int_{-\infty}^{\infty} A^{(n)}(k) e^{itk} \, dk \ . \end{aligned}$$

If $\int_{-\infty}^{\infty} \left|A^{(n)}(k)\right| \, dk$ is well-behaved, it follows that as $t \to \infty$, $J(t) \to 0$ faster than any algebraic power of t. Colloquially (i.e. to the satisfaction of an *applied* mathmo, if that isn't an oxymoron...) $J(t)$ is said to be *exponentially small* as $t \to \infty$. It is possible to prove similar exponential decay for $\phi(x,t)$ as $t \to \infty$ if $\theta(k)$ is a *monotonic* function of k. However, if that is not the case, it is time to return to our previous (examinable) programming.

2: Stationary Phase

If, at $k = k_0$,

$$c_g(k_0) = \frac{d\omega}{dk}(k_0) = V,$$

then there is a point of *stationary phase* at $k = k_0$ and the cancellation is *much* weaker near $k = k_0$. Near $k = k_0$,

$$\theta = [k_0 V - \omega(k_0)]\, t - \frac{1}{2}(k - k_0)^2 \omega'' t + \dots$$

since the linear $(k - k_0)$ term is zero. Therefore, we can conclude that θ changes rapidly with k when

$$|k - k_0| \gg \left(|\omega'' t|\right)^{-1/2},$$

and so only contributions for wavenumbers in the range

$$|k - k_0| = O\left(\frac{1}{\sqrt{|\omega''| t}}\right)$$

are significant.

As only this range is significant, the dominant asymptotic contribution (denoted by the symbol $\sim$, colloquially known by applied mathmos at least as "twiddles", which seems oddly appropriate) to the key integral (12.2) is

$$\phi \sim A(k_0) e^{i[k_0 V - \omega(k_0)]t} \int_{-\infty}^{\infty} \exp\left[\frac{-i\omega''(k_0)(k - k_0)^2 t}{2}\right] d(k - k_0). \quad (12.5)$$

The nicest proof of why the limits on the integral can actually be sent all the way to $\pm\infty$ (which you have to admit, is a long way) as opposed to just very much larger in magnitude than $1/\sqrt{|\omega''| t}$ relies on *steepest descent*, as described beautifully in Hinch's book.

The integral in (12.5) can now be evaluated by remembering (from Complex Methods for example) that

$$\int_{-\infty}^{\infty} e^{-a\xi^2} d\xi = \sqrt{\frac{\pi}{a}} \text{ for } Re(a) \geq 0.$$

Therefore, if there is just one such point of stationary phase, the dominant contribution as $t \to \infty$ to the integral (12.1) is

$$\phi(Vt,t) \sim A(k_0)\sqrt{\frac{2\pi}{i\omega''(k_0)t}}e^{i[k_0V-\omega(k_0)]t}; \ \omega'(k_0) = V. \tag{12.6}$$

12.2 Notes and Observations

There are (at least) six observations to be made.

1. We have assumed that there is a single point of stationary phase k_0. If there are more, they each give contributions to the key integral, which must be added together. Conversely, if there are no points of stationary phase, the integral ϕ is *exponentially small.*

2. For second order (in time) equations with $\partial^2\phi/\partial t^2$ terms, we must always remember to add contributions from waves of form $\exp[i(kx + \omega t)]$. These should be understood to have **negative** frequency.

3. If at the point of stationary phase $\omega''(k_0) = 0$, one has to take a deep breath, remember one is a mathmo (as well as a natural number...) and include higher order terms in the Taylor series. If $\omega''(k_0) \neq 0$, the higher order terms give corrections of $O\left(t^{-3/2}\right)$ for large t.

4. The expression (12.6) can be made even more formulaic by remembering that
$$\frac{1}{\sqrt{\pm i}} = e^{\mp i\pi/4}.$$
Therefore, remembering also that $\omega'(k_0) = V$,
$$\phi(Vt,t) \sim A(k_0)\sqrt{\frac{2\pi}{|\omega''(k_0)|t}}\exp\left(i\left[k_0V - \omega(k_0)\right]t - \frac{\pi i}{4}\operatorname{sgn}\omega''\right). \tag{12.7}$$

5. Similar ideas apply to waves in 2D or 3D where we need to consider vector expressions. For an integral
$$\phi(\boldsymbol{x},t) = \int_{-\infty}^{\infty} A(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}\,d\boldsymbol{k},$$

then at $\boldsymbol{x} = t\mathbf{V}$ as $t \to \infty$, the integral "sees" waves (i.e. has contributions from points of stationary phase) where

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{k}} [\boldsymbol{k} \cdot \mathbf{V} - \omega(\boldsymbol{k})] &= \mathbf{0}, \\ \boldsymbol{\nabla}_{\boldsymbol{k}} \omega = \frac{\partial \omega}{\partial \boldsymbol{k}} = \mathbf{c}_g &= \mathbf{V}. \end{aligned}$$

So, the dominant contribution comes when the observer is travelling at the group velocity.

6. The radiation condition is that **energy** propagates outward from a source of waves, not that the crests do. So, if a source at $\boldsymbol{x} = \mathbf{0}$ starts emitting waves at $t = 0$, with wave vector $\boldsymbol{k}$ and group velocity $\mathbf{c}_g(\boldsymbol{k})$, the disturbance reaches $\boldsymbol{x} = \mathbf{c}_g(\boldsymbol{k})t$ at time t.

12.3 Interpretation

The key formula (12.7) can be interpreted as follows. An observer moving with speed V will eventually only see those waves in the initial spectrum $A(k)$ with wavenumber k_0 such that the group velocity

$$c_g(k_0) = \left. \frac{\partial \omega}{\partial k} \right|_{k_0} = V.$$

The local amplitude decreases like $1/\sqrt{|\omega''|t}$ due to the combined effects of dispersion (i.e. spreading out) of nearby wavenumbers and conservation of energy. Wavenumbers in the range $[k_0, k_0 + \delta k]$ have group velocities in the range

$$[\omega'(k_0), \omega'(k_0 + \delta k)] \simeq [\omega'(k_0), \omega'(k_0) + \delta k \omega''(k_0)].$$

Therefore, waves in this range will separate $\delta k \omega'' t$ over time t.

As we expect energy E to be both constant and proportional to the square of the amplitude, we see

$$C = E \propto \int_{Vt}^{(V+\delta k \omega'')t} \phi^2 dx \to \phi \propto \frac{1}{\sqrt{\omega'' t}},$$

where wlog $\omega''(k_0) > 0$.

With that wlog however, it is important to keep track of all possibilities of signs, as **independently** it is possible that we would need to consider:

- $\pm\omega$;
- multiple wavenumbers of stationary phase, and in particular $\pm k_0$;
- $\pm V$.

It is necessary to think carefully about all such possibilities (which may make the final answer simpler of course!)

12.4 Example: Elastic Beam

Waves can occur in elastic beams (such as a twanged ruler before damping or a poor hold makes the waves decay significantly). For small amplitude twangs, consider the following problem for the displacement from equilibrium $\phi(x,t)$:

$$\begin{aligned} \frac{\partial^2\phi}{\partial t^2} + \gamma^2\frac{\partial^4\phi}{\partial x^4} &= 0, \\ \frac{\partial\phi}{\partial t}(x,0) = 0,\ \phi(x,0) &= f(x), \end{aligned}$$

where the initial displacement $f(x)$ is real.

Solution

The dispersion relation is

$$\omega^2 = \gamma^2 k^4.$$

Since the PDE is second order in time, the general solution has two integrals, and we need to be **particularly** careful with our signs:

$$\begin{aligned} \phi(x,t) &= \int_{-\infty}^{\infty} \frac{1}{2}\hat{f}(k)\left[e^{i(kx-\gamma k^2 t)} + e^{i(kx+\gamma k^2 t)}\right] dk, \\ &= \phi_+ + \phi_-, \end{aligned}$$

where $\hat{f}(k)$ is the Fourier transform of $f(x)$ as defined in (11.2). (So, if you're interested in not dropping points like a good football fan, don't forget your pies.) In terms of the formula (12.7), it is also important to remember that the frequency in ϕ_- is *negative*.

Now put $x = Vt$. Then:

- $\phi_\pm$ has stationary phase when

$$V \pm 2\gamma k = 0 \to k_\pm = \pm k_0 = \pm\frac{V}{2\gamma}.$$

- The value of the stationary phase (which appears in the argument of the exponential in (12.7) remember) is

$$k_{\pm}V \mp \gamma(k_{\pm})^2 = \pm\frac{V^2}{4\gamma},$$

while the second derivative of the frequency at the wavenumber of stationary phase is:

$$\omega''(k_{\pm}) = \pm 2\gamma,$$

of course remembering that the relevant frequency for ϕ_- at $k_- = -k_0$ is negative.

- Therefore, we see that all the key terms have opposite signs at $\pm k_0 = k_{\pm}$ in $\phi_{\pm}$. Since ϕ is real, $\hat{f}^*(k) = \hat{f}(-k)$, and so ϕ_- has the complex conjugate contribution to ϕ_+.

- Therefore, plugging all these quantities into (12.7),

$$\phi \sim \frac{1}{2}\hat{f}\left(\frac{V}{2\gamma}\right)\left(\frac{2\pi}{2\gamma t}\right)^{1/2}\exp\left(\frac{iV^2 t}{4\gamma} - \frac{i\pi}{4}\right) + c.c. \tag{12.8}$$

- **If** furthermore, the initial condition is even so that $f(x) = f(-x)$, then $\hat{f}(k)$ is also real, and then (12.8) reduces further (but be careful with the factor 1/2):

$$\phi \sim \hat{f}\left(\frac{V}{2\gamma}\right)\left(\frac{\pi}{\gamma t}\right)^{1/2}\cos\left(\frac{V^2 t}{4\gamma} - \frac{\pi}{4}\right), \tag{12.9}$$

as $x, t \to \infty$ with x/t fixed. Simples!

Chapter 13

Linear Water Waves

Carry me on the waves to the lands I've never seen
Eithne Ní Bhraonáin

We now start to consider several different wave systems with interesting dispersive properties. The first we consider are (surface) capillary-gravity waves for a layer of fluid of finite depth where there is surface tension at the upper interface: intuitively think about a layer of water with air above.

13.1 Problem description

We assume inviscid, irrotational, incompressible 2D (for simplicity) flow of a layer of fluid of equilibrium extent $-h \le z \le 0$. Hence

$$\boldsymbol{u} = (u, 0, w) = \boldsymbol{\nabla}\phi\,, \quad \boldsymbol{\nabla}.\boldsymbol{u} = 0 \quad \text{and} \quad \nabla^2\phi = 0\,. \tag{13.1}$$

Bottom Boundary Condition

The (linear) boundary condition at the rigid bottom is

$$\phi_z = 0 \qquad \text{on} \qquad z = -h\ . \tag{13.2}$$

1: Dynamic Boundary Condition at Surface

We can assume that the atmosphere is at constant pressure and the free surface (at $z = \eta$) is subject to surface tension and so

$$p\big|_{z=\eta_-} - p_{atm} = -T\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}, \tag{13.3}$$

where T is the coefficient of surface tension (with units force per unit length or equivalently energy per unit area) and the fraction describes the (local) curvature of the surface.

2: Kinematic Boundary Condition at Surface

Fluid elements must travel at the velocity of the interface and so

$$\frac{D}{Dt}[z-\eta(x,z,t)] = 0 \qquad \text{on} \qquad z=\eta\ ,$$

i.e.

$$w = \eta_t + u\eta_x \qquad \text{on} \qquad z=\eta\ . \tag{13.4}$$

Bernoulli's equation for unsteady potential flow (correctly written...) with gravity yields our governing equation:

$$\rho\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}|\boldsymbol{\nabla}\phi|^2\right)+p-p_{atm}+\rho g z = f(t)\ . \tag{13.5}$$

To make further process we need to linearize (13.4) and (13.5), and hence also implicitly (13.3).

13.2 Linearized Problem

We linearize for sufficiently small and smooth (or not steep) waves so that

$$|\eta| \ll \min(h,\lambda) \quad \text{and} \quad |\boldsymbol{u}| \ll \omega\lambda\,,$$

where λ and ω are the wavelength and wave frequency, respectively. Then, (13.4) can be approximated by

$$w = \frac{\partial\eta}{\partial t} \qquad \text{on} \qquad z=0\ . \tag{13.6}$$

Similarly the combination of (13.3) and (13.5) can be approximated by the requirement that, at $z=0$ the following expression is independent of x:

$$\rho\frac{\partial\phi}{\partial t}+\rho g\eta - T\eta_{xx}. \tag{13.7}$$

We seek wave-like Fourier mode solutions:

$$\begin{aligned}\phi &= f(z)e^{ikx-i\omega t}\,,\\ \eta &= A\ e^{ikx-i\omega t}\ .\end{aligned}$$

From (13.1), $\nabla^2\phi=0$, which implies

$$f''-k^2f=0\,,$$

while the bottom boundary condition, (13.2), implies that $f'(-h) = 0$, and hence

$$f = B \cosh k(z+h) \ .$$

The (linearized) kinematic boundary condition (13.6) implies that

$$-i\omega A = kB \sinh kh \, ,$$

while the (linearized) dynamic boundary condition (13.7) implies that (to ensure independence with respect to x due to the e^{ikx} terms):

$$-i\omega B \cosh kh + \left(g + \frac{Tk^2}{\rho} \right) A = 0.$$

Combining these two expressions, we obtain the *dispersion relation* for gravity waves with surface tension, i.e. capillary-gravity waves:

$$\omega^2 = gk \left(1 + \frac{T}{g\rho} k^2 \right) \tanh kh = gk \tanh kh \left(1 + l_c^2 k^2 \right) , \tag{13.8}$$

where $l_c = \sqrt{T/g\rho}$ is the *capillary length.* For air/water interfaces at atmospheric pressure and 20°C, $l_c \simeq 2.7$mm.

Exercise

For such waves, show that the *group velocity* is given by

$$c_g = \frac{\partial \omega}{\partial k} \quad = \quad \frac{\omega}{2k} \left(\frac{1 + 3l_c^2 k^2}{1 + l_c^2 k^2} + \frac{2kh}{\sinh 2kh} \right) . \tag{13.9}$$

13.3 Limiting Cases

Considering the dispersion relation (13.8) and the expression for the group velocity (13.9), there are clearly two important and independent scalings. There are different regimes when $|kh| \ll 1$, and so the layer is shallow relative to the wavelength, and when $|kh| \gg 1$ and so the layer is deep. Analogously when $l_c^2 k^2 \gg 1$, surface tension is expected to be important, while when $l_c^2 k^2 \ll 1$, it is reasonable to expect that surface tension can be ignored.

A: Shallow Water $|kh| \ll 1$

In this asymptotic regime,

$$\tanh kh \simeq \sinh kh \simeq kh.$$

1. It is difficult, (particularly for water/air interfaces) for the surface-tension-important regime also to occur. This is because $l_c^2 k^2 \gg 1$ implies that the wavelength $\lambda = 2\pi/k$ must satisfy both

$$h \ll \lambda \ll l_c.$$

The implied requirement that $h \ll l_c$ requires layer depths significantly smaller than 1mm, and it is hard to justify such layers being unaffected by viscosity.

2. It is much more common (and plausible) to consider the other limit where $l_c^2 k^2 \ll 1$, and the effects of surface tension can be ignored. In this double limit, the dispersion relation and group velocity expression reduce to

$$c = \pm\sqrt{gh} = c_g, \tag{13.10}$$

i.e. non-dispersive long (surface) gravity waves, where the signs arise from the quadratic form of the dispersion relation. It is important to appreciate that these waves can be really, really fast: it is possible for waves of wavelength 100km in shelf seas with $h \sim$ 1km, thus implying speeds $\sim$ 100m/s...i.e. over 200 mph in old money!

B: Deep Water $|kh| \gg 1$

In this asymptotic regime,

$$\tanh kh \simeq \pm 1, \ \frac{2kh}{\sinh 2kh} \to 0.$$

Both possibilities for the capillary length are now much more likely.

1. $l_c^2 k^2 \gg 1$ is likely to occur in water for wavelengths much smaller than $\sim$ 2cm. (It is easy to imagine water layers much, much deeper than this.) In this double limit,

$$\omega \simeq \pm(g l_c^2 |k|^3)^{1/2}, \tag{13.11}$$

$$\to c_g = \frac{3}{2}\frac{\omega}{k} = \frac{3}{2}c. \tag{13.12}$$

These are *capillary* waves. They are dispersive. Since the group velocity is larger than the phase velocity, 'new' crests appear to appear at the front of a wave packet, and propagate backwards over the packet, disappearing at the rear. Remember that is due to **relative** differences in the speed, and the phase velocity is not in a different direction to the group velocity.

2. Conversely, $l_c^2 k^2 \ll 1$ is likely to occur in water for wavelengths much larger than $\sim$ 2cm. (It is still easy to imagine water layers much, much deeper than wavelengths still much longer than a metre, as the oceans are typically multiple kilometres deep.) In this double limit,

$$\omega \simeq \pm(g|k|)^{1/2}, \tag{13.13}$$

$$\rightarrow c_g = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}c. \tag{13.14}$$

   These are *deep water* waves. They are also dispersive. Since the group velocity is smaller than the phase velocity, 'new' crests appear to appear at the back of a wave packet, and propagate forwards over the packet, disappearing at the front. Remember again that this is due to **relative** differences in the speed, and the phase velocity is still in the same direction to the group velocity.

Connecting these different regimes has more than a few interesting implications. Here are just three.

- For these waves, the dispersion relation allows modes in all four quadrants of $k - \omega$ space: both k and ω can be negative or positive.

- Fixing ideas in the first quadrant, there must be a wavenumber k_1 where c_g is minimum, and so $\omega'' = 0$, and so stationary phase (in its simplest form) breaks down.

- Furthermore, there is another wavenumber $k_2 > k_1$ where the phase speed is minimized. Interestingly, $k_2 = 1/l_c$. Also, since

$$\frac{\partial c}{\partial k} = \frac{\partial}{\partial k}\left(\frac{\omega}{k}\right) = \frac{1}{k}\left(c_g - c\right),$$

  at l_c, $c = c_g$ and indeed

$$c_p \gtrless c_g \quad \text{according as} \quad kl_c \lesseqgtr 1,$$

  further justifying why l_c is a natural length to consider.

Chapter 14

Internal Gravity Waves

Say we can, say we will, not just another drop in the ocean
Ian McCulloch

Both the atmosphere and the ocean are *stratified*, i.e. the density on average decreases with height (or increases with depth depending on your worldview: do you look up to the heavens, or down into the abyss...) Therefore, a fluid parcel that is lifted up from its equilibrium position will be more dense than its surroundings, and (in the presence of a gravitational field) feel a (restoring) *buoyancy force*. Therefore, there is a tendency to fall back to and overshoot (due to inertia) its equilibrium position. So we have yet another a mechanism for waves, called *internal gravity waves*.

14.1 Linear Theory for Incompressible Fluids

Note, it is important to appreciate that the density differences we consider here are **not** due to pressure differences, as in acoustics. To a good approximation, water is incompressible, so the theory presented here works very well for oceans and lakes, where density differences are due to temperature and/or salinity differences (though weird things can still happen around 4°C). For the atmosphere, due to the fact that pressure changes lead (adiabatically) to changes in temperature and density (the principal reason why it is colder up a mountain) there are some technical issues which keep meteorologists amused. Nevertheless, these issues can still be dealt with relatively straightforwardly using concepts such as *potential temperature* which is covered in Part III (or indeed all over the internet/in books...)

Once we've relaxed about allowing density to vary for reasons other than pressure changes, there are three equations for an inviscid (as usual in this course) fluid. We have conservation of mass, and we **assume** that the flow is incompressible (or even more precisely that the velocity field is solenoidal):

$$\begin{aligned}\frac{D\rho}{Dt} = \rho_t + u\rho_x + v\rho_y + w\rho_z &= 0\,, \\ \boldsymbol{\nabla}.\boldsymbol{u} = u_x + v_y + w_z &= 0\,,\end{aligned}$$

where remember $\boldsymbol{u} = (u, v, w)$. We also have the momentum equation, where the gravitational body force is explicit:

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}p - \rho g\widehat{\mathbf{z}}\ . \tag{14.1}$$

Remember that the pressure here is not the thermodynamic variable of Part I, but rather a (non-local) Lagrange multiplier that ensures the application of the incompressibility constraint (14.1) on the velocity field. The usual "trick" that removes the gravitational body force is to redefine pressure to absorb a hydrostatic component. This "trick" still works even for a background density distribution $\rho_0(z)$ that is a function of z alone, as we can define a hydrostatic pressure p_0 as the solution to

$$\frac{dp_0}{dz} = -\rho_0 g\,. \tag{14.2}$$

where the RHS is not (necessarily) constant.

We thus wish to consider small perturbations to a state of rest $\boldsymbol{u} = \mathbf{0}$, with a background density distribution $\rho_0(z)$ and associated hydrostatic pressure p_0 defined by (14.2). Clearly any constant reference value of the pressure can be absorbed during the integration of (14.2). Therefore, our variables are u, v, w (as they are assumed small perturbations from the state of rest) while density and pressure are decomposed as:

$$p = p_0(z) + \tilde{p}(\boldsymbol{x}, t)\ , \quad \rho = \rho_0(z) + \tilde{\rho}(\boldsymbol{x}, t)\ ,$$

where the perturbation quantities are also assumed small.

Keeping only quantities that are first order in the perturbations, (and remembering that ρ_0' is allowed to be order one, i.e. much bigger than the perturbation quantities) we arrive at a system of five linear equations:

$$\begin{aligned}\tilde{\rho}_t + w\rho_0'(z) &= 0\ , &(14.3)\\ \rho_0 u_t &= -\tilde{p}_x\ , &(14.4)\\ \rho_0 v_t &= -\tilde{p}_y\ , &(14.5)\\ \rho_0 w_t &= -\tilde{p}_z - \tilde{\rho} g\ , &(14.6)\\ u_x + v_y + w_z &= 0\ . &(14.7)\end{aligned}$$

We can eliminate u and v by taking $\partial/\partial x$ (14.4) $+\partial/\partial y$ (14.5) and then applying (14.7) to obtain

$$\rho_0 w_{zt} = \tilde{p}_{xx} + \tilde{p}_{yy} \equiv \nabla_h^2 \tilde{p}, \tag{14.8}$$

defining the horizontal Laplacian ∇_h^2.

Pressure can be eliminated by comparing $\partial/\partial z$ (14.8) and ∇_h^2 (14.6):

$$[\rho_0 w_{zt}]_z = \nabla_h^2 \tilde{p}_z = -\rho_0 \nabla_h^2 w_t - g\nabla_h^2 \tilde{\rho},$$

and so

$$\rho_0 \frac{\partial}{\partial t}\left[\nabla^2 w + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial w}{\partial z}\right] = -g\nabla_h^2 \tilde{\rho}. \tag{14.9}$$

Using (14.3) to eliminate $\tilde{\rho}$ (by differentiating (14.9) with respect to time) we obtain an equation purely in terms of w:

$$\rho_0 \frac{\partial^2}{\partial t^2}\left[\nabla^2 w + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial w}{\partial z}\right] = g\frac{d\rho_0}{dz}\nabla_h^2 w. \tag{14.10}$$

Boussinesq Approximation

There is an annoying term on the LHS of (14.10). But let us consider a situation where we have motions so that w varies on a scale L_W and ρ_0 varies on a scale L_{ρ_0}. Then if $L_W \ll L_{\rho_0}$,

$$\left|\frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial w}{\partial z}\right| \ll \left|\frac{\partial^2 w}{\partial z^2}\right|,$$

and so we can ignore the second term in the bracket on the LHS. This assumption (that the vertical scale of velocity variation is much smaller than the scale of background density variation) is one version of what is known as the *Boussinesq approximation.*

Under this assumption, (14.10) reduces to

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\nabla^2 w - \frac{g}{\rho_0}\frac{d\rho_0}{dz}\nabla_h^2 w &= 0, \\ \frac{\partial^2}{\partial t^2}\nabla^2 w + N^2\nabla_h^2 w &= 0, \end{aligned} \tag{14.11}$$

defining the *buoyancy frequency* N. This is the frequency of vertical oscillation in a constant density gradient incompressible inviscid stratified fluid.

Typical periods of such vertical oscillation in the atmosphere and oceans are of the order of minutes. This frequency

$$N^2 \equiv -\frac{g}{\rho_0}\frac{d\rho_0}{dz}, \tag{14.12}$$

is real in situations where the density gradient is statically stable and so $d\rho_0/dz < 0$. It is also sometimes called the *Brunt–Väisälä frequency*, but since that is:

- difficult to type and say (if one is not Finnish at least);
- non-intuitive;
- and (partially) named after a Trinmo,

I prefer to refer to N as the *buoyancy frequency.*

Although the (linear) derivation above allowed the density ρ_0 on the LHS of (14.4), (14.5) and (14.6), to remain a function of z, that is really a special case here. The more general version of the Boussinesq approximation is that density variations away from a reference value are only relevant for buoyancy forcing, and not for variations in inertia: i.e. density variations only matter when multiplied by g. Formally, $|\rho - \rho_{\rm ref}| \to 0$, $g \to \infty$ such that $g\tilde{\rho}/\rho_{\rm ref}$ and $(g/\rho_{\rm ref})d\rho_0/dz$ stay finite for some reference value of density $\rho_{\rm ref}$. This is (to a good approximation) true in the ocean (where temperature and salinity variations only lead to relative density differences of a few percent) and for vertical distances of the order of a km or so in the atmosphere, yet buoyancy forces are crucially important dynamically in both situations.

Within this version of the Boussineq approximation, (14.4), (14.5) and (14.6) become:

$$\rho_{\rm ref} u_t = -\tilde{p}_x \ , \tag{14.13}$$
$$\rho_{\rm ref} v_t = -\tilde{p}_y \ , \tag{14.14}$$
$$\rho_{\rm ref} w_t = -\tilde{p}_z - \tilde{\rho} g \ . \tag{14.15}$$

Following an analogous procedure to the one used above, but now for the equation set (14.3), (14.7),(14.13)-(14.15), the "annoying" second term on the LHS of (14.10) does not arise, and we directly arrive at (14.11), with the buoyancy frequency now defined by

$$N^2 \equiv -\frac{g}{\rho_{\rm ref}}\frac{d\rho_0}{dz}. \tag{14.16}$$

14.2 Plane Wave Solutions

Of course, in general $N(z)$. However if N is constant, the situation becomes very simple. Formally N is constant implies an exponential dependence in density from (14.12):

$$\rho_0 = \bar{\rho}\exp(-z/L_{\rho_0}) \ ,$$

where $\bar{\rho}$ is a constant and $L_{\rho_0} = g/N^2$. The Boussinesq approximation implies that this length scale is very large, and so this exponential can be approximated (well) by a linear function (i.e. the first two terms of the Taylor series) and so (to a good approximation) the density varies linearly with depth for constant N. Of course, this is entirely consistent with the result for (constant) buoyancy frequency arising from the second, more general version of the Boussinesq approximation, since requiring N as defined in (14.16) to be constant implies that ρ_0 is (precisely) a linear function of z.

With such a constant N within either version of the Boussinesq approximation, (14.11) admits plane wave solutions

$$w = w_0 e^{i(\boldsymbol{k}.\boldsymbol{x}-\omega t)}, \ \boldsymbol{k} = (k, l, m); \kappa = |\boldsymbol{k}|,$$

and indeed all other variables $\boldsymbol{u}, \tilde{p}, \tilde{\rho}$ vary like plane waves. This can be "easily" established from the governing equations assuming the plane wave structure, deriving *polarisation relations* between the various amplitudes. For example, from incompressibility (14.7):

$$ku_0 + lv_0 + mw_0 = 0.$$

The dispersion relation for these plane waves is:

$$\omega^2 = \frac{N^2(k^2+l^2)}{k^2+l^2+m^2}, \tag{14.17}$$

or alternatively

$$\omega = \pm N\cos\theta, \tag{14.18}$$

where θ is the angle that the wave vector makes with the horizontal plane $z = 0$.

Comments

These *internal gravity waves* have several strange and beautiful properties.

1. They are *transverse* waves as

$$\boldsymbol{\nabla}.\boldsymbol{u} = 0 \to \boldsymbol{k}.\boldsymbol{u} = 0,$$

and so motion is normal to the wave vector and so along troughs or crests.

2. For real $\boldsymbol{k}$, $|\omega| \leq N$. As $\omega \to 0$, the motion approaches purely horizontal, as $\boldsymbol{k}$ becomes parallel with the vertical. Conversely, as $\omega \to N$, $m \to 0$ and so the wave vector is horizontal, and the motion (perhaps unsurprisingly) is purely vertical "bobbing". Forcing with $\omega > N$ thus will lead to evanescent waves, with $m^2 < 0$.

3. As usual, if the environment changes, vertical velocity and pressure must be continuous. From (14.8), for such wave-like solutions this is equivalent to the requirement that w and $\partial w/\partial z$ must be continuous.

4. ω is **independent** of $|\boldsymbol{k}| = \kappa$, but depends on **direction** of $\boldsymbol{k}$. The dispersion relation is most definitely **not** isotropic.

5. As an exercise, show that the group velocity $\mathbf{c}_g$:

$$\begin{aligned} \mathbf{c}_g &\equiv \frac{\partial \omega}{\partial \boldsymbol{k}}, \\ &= \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m}\right), \\ &= \frac{N^2}{\omega}\frac{m}{\kappa^4}(km, lm, -k^2 - l^2). \end{aligned} \tag{14.19}$$

   Therefore, $\boldsymbol{k}.\mathbf{c}_g = 0$ again, and energy propagates at right angles to phase. In particular (considering the vertical component of the phase velocity ω/m) it is clear that the vertical component of phase velocity is always of different sign from the vertical component of the group velocity. As the radiation condition is that energy propagates outwards from a source, this means that (for example) upward energy propagation will be associated with waves with downward phase velocity...

6. Indeed, if we oscillate a source (such as a horizontal cylinder, so that the flow is close to 2D with $l = 0$), the dispersion relation fixes $\hat{\boldsymbol{k}}$ and not $|\boldsymbol{k}| = \kappa$. Therefore four beams appear like the Scottish flag's Cross of St Andrew, as shown in the video. The directions of phase velocity and group velocity are as expected, but a bit weird looking, and you need to be careful about the various signs. In such a 2D circumstance, there are four beams, with appropriate combinations in each quadrant of

$$\begin{aligned} \boldsymbol{k} &= \kappa(\cos\theta, 0, \sin\theta); \ \cos\theta = \pm\frac{\omega}{N}, \\ \mathbf{c} &= \frac{\omega}{\kappa^2}\boldsymbol{k}, \\ \mathbf{c}_g &= \pm\frac{N}{\kappa}\sin\theta(\sin\theta, 0, -\cos\theta). \end{aligned}$$

Chapter 15

Moving Wave Sources & Media

Aus dem Lautsprecher klingt es dann
Emil Schult

Up to now we have assumed that both the source of waves and the medium through which they propagate are stationary. In this chapter we consider what happens when we relax those constraints, and allow either the source to move uniformly or (indeed equivalently) the ambient medium to move uniformly. These leads to an elegant mathematical explanation of the well-known *Doppler effect*, beloved by (among others) Schneider and Hütter.

15.1 $\omega(\boldsymbol{k})$ & $\mathbf{c}_g$ for Moving Sources

Consider (plane) waves of the form $\phi(\boldsymbol{x}, t) = \exp[i(\boldsymbol{k}.\boldsymbol{x} - \omega t)]$ in a uniform stationary medium with position vector $\boldsymbol{x}$. Consider an observer or a source moving with some constant velocity $\boldsymbol{U}$. Therefore in the observer's/source's (primed) frame of reference, the medium moves with velocity $-\boldsymbol{U}$ and the position vector is $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{U}t$. Therefore, the wave becomes

$$\begin{aligned} \phi(\boldsymbol{x}', t) &= e^{i(\boldsymbol{k}.\boldsymbol{x}' + \boldsymbol{k}.\boldsymbol{U}t - \omega t)}, \\ &= e^{i(\boldsymbol{k}.\boldsymbol{x}' - \omega' t)}. \end{aligned}$$

Therefore the frequency in the frame of reference of the observer/source is

$$\omega' = \omega - \boldsymbol{k}.\boldsymbol{U}.$$

And so, the dispersion relation for a medium with velocity $-\boldsymbol{U}$ is (with now a dropped prime)

$$\omega = \Omega(\boldsymbol{k}) \equiv \Omega_0(\boldsymbol{k}) - \boldsymbol{k}.\boldsymbol{U}, \tag{15.1}$$

where $\Omega_0(\boldsymbol{k})$ is the dispersion relation for the **stationary** medium.

- $\Omega_0(\boldsymbol{k})$ is often called the *intrinsic frequency* (in the sense that it is a characteristic of the medium);
- $\Omega(\boldsymbol{k})$ is often called the *extrinsic frequency* (as it depends on the particular relative velocity of the medium relative to the source or the observer).

The corresponding group velocity is

$$\mathbf{c}_g(\boldsymbol{k}) = \boldsymbol{\nabla}_k(\Omega_0(\boldsymbol{k}) - \boldsymbol{k}.\boldsymbol{U}) = \mathbf{c}_{g0}(\boldsymbol{k}) - \boldsymbol{U}. \tag{15.2}$$

It is important to appreciate that if $\Omega_0(\boldsymbol{k})$ is isotropic (and so only a function of the wavenumber $\kappa = |\boldsymbol{k}|$), $\Omega(\boldsymbol{k})$ is not because the relative velocity $\boldsymbol{U}$ inevitably breaks the symmetry by introducing a direction. Furthermore, group velocities behave as expected under a Galilean transformation, while the phase "velocity"

$$\mathbf{c} \equiv \frac{\omega}{\kappa}\hat{\boldsymbol{k}},$$

does not, as it is not really a true velocity.

15.2 Doppler Effect

As an example, consider a plane sound wave (with wave vector $\boldsymbol{k}$) emitted from a source moving at a velocity $\boldsymbol{U}$ in a medium with sound speed c_0. From (15.1), the frequency from the source in the moving frame is

$$\begin{aligned} \omega' &= c_0|\boldsymbol{k}| - \boldsymbol{U}.\boldsymbol{k}, \\ &= \omega(1 - M\cos\beta), \\ \to \omega &= \frac{\omega'}{1 - M\cos\beta}. \end{aligned}$$

where $M = |\boldsymbol{U}|/c_0$ is the Mach number, β is the angle the wave vector makes with $\boldsymbol{U}$, and ω is the intrinsic frequency $c_0|\boldsymbol{k}|$.

But now think about a situation (such as an ambulance, or a police car, or indeed any vehicle on the autobahn...) where the frequency of emitted sound is constant *in the moving frame*. Therefore, the well-known Doppler effect is manifest.

1. If the source is approaching a fixed observer, $\cos\beta > 0$, and so the frequency detected by the observer $\omega > \omega'$, and so sounds higher pitched.
2. Conversely, if the source is receding from a fixed observer, $\cos\beta < 0$, and so the frequency detected by the observer $\omega < \omega'$, and so sounds lower pitched....

Neeeeeowwww... which is a finite Mach number effect...

Chapter 16

Steady Waves from Moving Sources

Woahhh, do the duck
Ray Charles (for the crucial anatid improvisation...)

In general, waves are unsteady precisely because of the $e^{-i\omega t}$ term. However, they can be steady (or at least appear steady) in the frame of reference of a moving source, or equivalently when the medium is moving uniformly. Remembering (15.1), and once again dropping the prime from the extrinsic frequency

$$\begin{aligned} \omega = \Omega(\boldsymbol{k}) \equiv \Omega_0(\boldsymbol{k}) - \boldsymbol{k}.\boldsymbol{U} &= 0, \\ \to c_{p0}(\boldsymbol{k}) &= |\boldsymbol{U}|\cos\beta, \end{aligned}$$

where β is the angle between the velocity of the moving source and the wave vector. So the frequency is zero in the moving frame if the *intrinsic* phase velocity takes a particular value. This particular value is so that this phase velocity precisely matches the component of the source's velocity in the direction of the wave vector, and so the wave seems stationary in that frame. Simples!

In the rest of the chapter, to end Part III of the course, (though some of this is actually listed in Part IV of the Schedules) we consider three particularly interesting examples: the langorous finger; the supersonic plane (without gin and tonic); and the apogee of the entire Mathematical Tripos, the wave pattern behind a duck (more prosaicly known as Kelvin ship waves).

16.1 Example 1: Capillary-Gravity Waves in 1D

Consider a cm-sized object moving at speed U in "deep" (i.e. metres or more) water. The Cantabrigian example is the finger of a langorous passenger in a punt. If $|U|$ is sufficiently large, we will see a *steady* wave pattern in the frame of the object. Let us focus on the simplest case of 1D flow. Therefore, the wave vector and the **velocity** of the moving object are in the same direction. We only need to consider the speed U and the wavenumber k, and the angle $\beta = 0$ so we need to find a situation where the extrinsic frequency $\omega(k) = 0$, or equivalently where the intrinsic phase speed $c_0 = \Omega_0/k = U$, where Ω_0 is the intrinsic frequency from the dispersion relation for capillary-gravity waves in a stationary fluid layer. Wlog, lets suppose $U > 0$: i.e. the punter is actually making the punt go forwards, so probably not a tourist...

Therefore, we look for solutions to

$$\begin{aligned} \omega(k) &= \pm(g|k|)^{1/2}\left(1 + l_c^2 k^2\right)^{1/2} - Uk; l_c^2 = \frac{T}{\rho g}, & (16.1) \\ &= \Omega_0(k) - Uk & (16.2) \end{aligned}$$

remembering the definition of the capillary length l_c, and the (intrinsic) frequency Ω_0. We wish to look for solutions to the equation $\omega(k) = 0$.

There can "clearly" be no solutions if U is less than the minimum value $c_{\min}$ of the phase speed for such waves in stationary media, i.e. (fixing ideas by thinking about the first quadrant where ω and k are both positive). But we can see that

$$\begin{aligned} c &= \left(\frac{g}{k}\right)^{1/2}\left(1 + l_c^2 k^2\right)^{1/2}, \\ \rightarrow \frac{\partial c}{\partial k} &= \frac{1}{2}\left(\frac{g}{k^3[1 + l_c^2 k^2]}\right)^{1/2}(l_c^2 k^2 - 1), \end{aligned}$$

and so $c_{\min} = \sqrt{2gl_c}$, which occurs at $kl_c = 1$. So, if $U < (2gl_c)^{1/2}$ (about 23 cm/s in water), then there can be no stationary waves.

From the structure of the dispersion relation, there are two solutions for positive k and positive Ω_0 (and two complex conjugate solutions when k is negative and Ω_0 is negative in the third quadrant, since we have taken $U > 0$).

1. One solution has low wavenumber $kl_c < 1$ and (inevitably) intrinsic group velocity $c_{g0} < U$, and so

$$c_g = \frac{\partial \Omega_0}{\partial k} - U < 0.$$

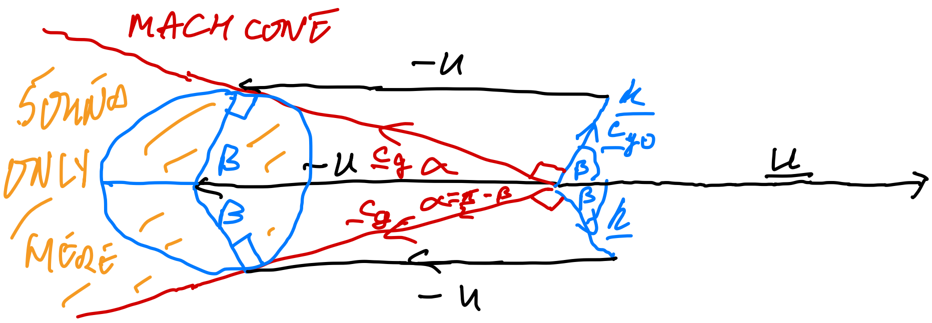

Figure 16.1: Schematic showing Mach cone.

Therefore, from the radiation condition that information must radiate outwards, these long waves can't appear in $x > 0$, and so they appear *behind* the obstacle in $x < 0$.

2. Conversely, the other solution has high wavenumber $kl_c > 1$ and (inevitably) intrinsic group velocity $c_{g0} > U$, and so $c_g = c_{g0} - U > 0$. Therefore, from the radiation condition that information must radiate outwards, these short waves can't appear in $x < 0$, and so they appear *in front of* the obstacle in $x > 0$.

16.2 Supersonic Boom & Mach Cones

Now let us consider an aircraft (or even a train with an effervescent passenger) with velocity (the vehicle, not Elsa) $\boldsymbol{U} = Mc_0(1, 0)$ where $M > 1$ and so the aircraft is most definitely feeling supersonic. Now consider a sound wave, with wave vector $\boldsymbol{k}$, which is at an angle β to $\boldsymbol{U}$.

Therefore, in the frame of the aircraft

$$\omega = c_0|\boldsymbol{k}| - \boldsymbol{U}.\boldsymbol{k} = c_0\kappa(1 - M\cos\beta).$$

The requirement that the waves are steady in this frame (and so $\omega = 0$) thus requires the waves to be at a *unique* angle independent of κ:

$$\begin{aligned} \beta &= \arccos\left(\frac{1}{M}\right), \\ \rightarrow \hat{\boldsymbol{k}} &= \left[\frac{1}{M}, \left(1 - \frac{1}{M^2}\right)^{1/2}\right]. \end{aligned}$$

For these waves, the group velocity $\mathbf{c}_g$ is given by

$$\begin{aligned}\mathbf{c}_g &= c_0\hat{\boldsymbol{k}} - \boldsymbol{U}, \\ &= \left[\frac{c_0}{M} - Mc_0, c_0\left(1 - \frac{1}{M^2}\right)^{1/2}\right] \\ &= \frac{c_0}{M}\left[1 - M^2, \left(M^2 - 1\right)^{1/2}\right], \\ &= c_0\left(M^2 - 1\right)^{1/2}[-\sin\beta, \cos\beta].\end{aligned}$$

Therefore, $\mathbf{c}_g$ is perpendicular to $\hat{\boldsymbol{k}}$.

Hence, *steady* waves are found on the *Mach cone* **behind** the aircraft, with semi-angle α where

$$\alpha = \frac{\pi}{2} - \beta = \arcsin\left(\frac{1}{M}\right).$$

Since the expression for the group velocity

$$\mathbf{c}_g = c_0\hat{\boldsymbol{k}} - \boldsymbol{U},$$

is true for all wavenumbers, unsteady waves (i.e. sound) are only found within the Mach cone, (i.e. *inside* the spherical surfaces defined by the steady waves) as shown in figure 16.1 in orange. Of course, this has little relevance to real supersonic flight, as real sonic booms are typically nonlinear.

16.3 Kelvin Ship Waves

As the final example for the a steady wave pattern for a moving source, let's consider deep water waves where (unlike the first punting example) surface tension is not significant. Define the coordinate system so that the ship has steady velocity $\boldsymbol{U} = U(1,0)$ in the positive $x-$direction. Consider also surface deep water waves with wave vector $\boldsymbol{k}$ at an angle β to the $x-$direction.

Therefore, when surface tension is not important, the frequency of such waves in the frame moving with the ship is

$$\omega = \sqrt{g|k|} - \boldsymbol{u}.\boldsymbol{k} = \sqrt{g|k|} - U|\boldsymbol{k}|\cos\beta.$$

Therefore there are steady waves if

$$\cos\beta = \frac{1}{U}\sqrt{\frac{g}{|\boldsymbol{k}|}},$$

and for these waves, the group velocity $\mathbf{c}_g$ is given by

$$\begin{aligned}
\mathbf{c}_g &= \frac{\hat{\boldsymbol{k}}}{2}\sqrt{\frac{g}{|\boldsymbol{k}|}} - \boldsymbol{u}, \\
&= \frac{U}{2}\cos\beta(\cos\beta, \sin\beta) - U(1,0); \\
&= \frac{U}{4}(\cos 2\beta, \sin 2\beta) - \frac{3U}{4}(1,0).
\end{aligned}$$

Therefore, the energy from the waves propagates in the direction

$$\hat{\mathbf{c}}_g = (-\cos\alpha, \sin\alpha); \;\; \tan\alpha = \frac{\sin 2\beta}{3 - \cos 2\beta},$$

and the locus of $\mathbf{c}_g$ lies on a circle of radius $U/4$. Maximising this expression over α, (thus identifying in a very real sense the "biggest" circle) occurs when $\cos 2\beta = 1/3$, $\sin 2\beta = \sqrt{8}/3$, and so

$$\alpha_{\max} = \arcsin\left(\frac{1}{3}\right) \simeq 19.5^\circ.$$

This behaviour can be interpreted by realising that a packet of waves emitted at $t = 0$, will have travelled a distance $c_{g0}\tau$ in the direction of the wave vector $\boldsymbol{k}$ (and hence at an angle β to the horizontal) when the ship has moved $U\tau$ horizontally, and c_{g0} is the group velocity associated with the intrinsic frequency:

$$c_{g0} = \frac{c_{p0}}{2} = \frac{U\cos\beta}{2}.$$

This is at a point on a circle centred at $(U\tau/4, 0)$ with radius $U\tau/4$, at the end of a radius at an angle 2β to the horizontal. This situation is shown schematically in figure 16.2, where the group velocity $\mathbf{c}_g$ in the frame of the duck (at the right end of the black line of total length Ut) is shown with a green line, arriving at the same point on the edge of the circle. How cool is that!

Shape of Wave Crests

We can actually derive a parametric equation for the essentially V-shaped wave crests (behind the duck/ship). The key idea is to consider the position vector at some point on one of the crests:

$$\boldsymbol{x} = (x, y) = r(-\cos\alpha, \sin\alpha),$$

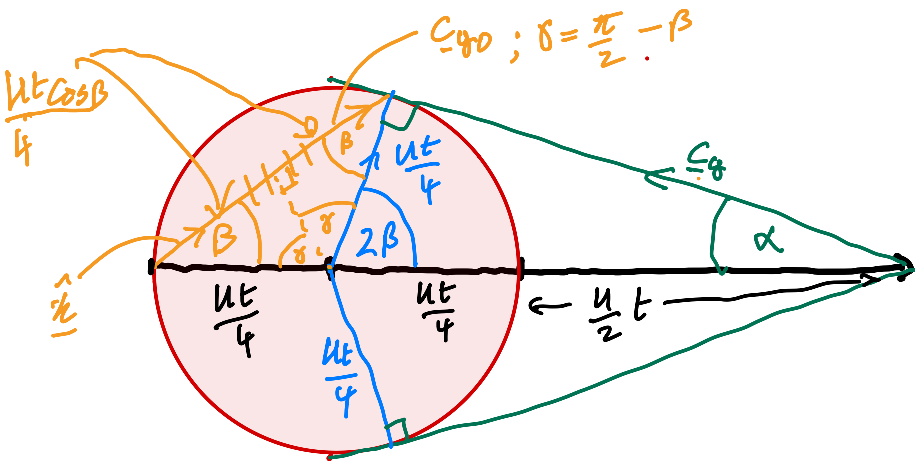

Figure 16.2: Schematic of the steady Kelvin waves (emitted at time 0 in direction $\hat{\boldsymbol{k}}$ along the orange line) in the wake of a ship (or duck) travelling at speed U when the ship/duck has arrived at the right end of the straight black line at time t.

defining the duck as the origin. Knowing α, we can associate this point with a particular β as above, and hence $|\boldsymbol{k}|$.

The real trick (I think) is that we know that the phase $\theta = \boldsymbol{k}.\boldsymbol{x}$, precisely because in this frame of reference $\omega = 0$ and the waves are steady. Therefore

$$\begin{aligned}
\theta = \boldsymbol{k}.\boldsymbol{x} &= |\boldsymbol{k}| r \cos(\pi - \alpha - \beta), \\
&= -\frac{g}{U^2 \cos^2 \beta} r \cos\alpha \cos\beta (1 - \tan\alpha \tan\beta), \\
&= \frac{gx}{U^2 \cos\beta} \left(1 - \frac{\sin 2\beta \tan\beta}{3 - \cos 2\beta}\right), \\
&= \frac{2gx}{U^2 \cos\beta (3 - \cos 2\beta)},
\end{aligned}$$

after various heroic algebraic manipulations and substitutions both from above and from our beloved addition formulae.

The last trick is to appreciate that this is actually a parametric equation for $x(\beta)$ (and $y(\beta)$) for a given value of the phase, which wlog can be chosen to be associated with the crest:

$$\begin{aligned}
x &= \frac{U^2 \theta}{2g} \cos\beta (3 - \cos 2\beta), \\
y = -x \tan\alpha &= -\frac{U^2\theta}{2g} \cos\beta \sin 2\beta,
\end{aligned}$$

which as we vary β actually traces out a dart shape behind the duck/ship.

Part IV

Ray Theory

Chapter 17

Introduction to Ray Theory

I watch the ripples change their size, but never leave the stream
David Bowie

17.1 Motivation

Ray theory (also known as geometric optics, or WKB method after Wentzel-Kramers-Brillouin, although Harold Jeffreys of St John's developed it 3 years before...) is an asymptotic approximation for long distance and/or time propagation through **slowly varying** media. Here this slow variation refers to changes which occur on length scales $L \gg 1/\kappa$ and/or timescales $T \gg 1/\omega$ for waves with characteristic wave number κ and frequency ω.

The motivation for development and use of this theory is that waves often transport energy and/or momentum over such long distances and times into regions with different properties. For example, oceanic surface waves are often formed in deep water by storms, yet propagate all the way to shoaling beaches where they break. Similarly, clear air turbulence is associated with breaking waves (where the breaking may be associated with enhanced velocity shear locally) at high altitudes, yet the waves have often be generated by flow over topography at the ground. The real challenge is to identify a self-consistent way in which we can use **local** solutions (i.e. on the scales of the waves) to find the behaviour of the system on much larger scales (i.e. on the scales of the variations in the medium).

17.2 Multiple Scales

To rise to this challenge, we need to introduce (formally) the powerful concept of *multiple scales*. Therefore, we assume that the medium varies on an

$O(1)$ scale (in length and/or time) while the wavelength and/or period of the waves vary on a much smaller and faster scale $O(1/\epsilon)$ where $\epsilon \ll 1$. The important issue which we need to address is that the wavelength (or equivalently the wave vector $\boldsymbol{k}$) varies as the waves propagate across the (changing) medium and so we can't just simply assume that the waves are proportional to $\exp(i\boldsymbol{k}.\boldsymbol{x})$ over $O(1)$ scales.

Phase

The key idea is to generalise our previous approach by focussing on a (rapidly varying) *phase* $\theta(\boldsymbol{x},t)/\epsilon$ dependence for the waves of interest, i.e. we look for solutions of the form

$$\phi(\boldsymbol{x},t) = A(\boldsymbol{x},t)e^{i\theta(\boldsymbol{x},t)/\epsilon}. \tag{17.1}$$

Here, variations in the phase θ/ϵ are assumed to be rapid, while the variations in A are assumed to be slow.

We now generalise the concepts of wave vector and frequency to allow them to vary in space and time by **defining** these quantities as:

$$\boldsymbol{k}(\boldsymbol{x},t) \;:=\; \frac{1}{\epsilon}\frac{\partial\theta}{\partial\boldsymbol{x}} = \frac{1}{\epsilon}\boldsymbol{\nabla}_x\theta, \tag{17.2}$$

$$\omega(\boldsymbol{x},t) \;:=\; -\frac{1}{\epsilon}\frac{\partial\theta}{\partial t}, \tag{17.3}$$

where the spatial gradient is made explicit. Both these locally defined quantities are (by construction) large, and yet may possibly vary slowly with $\boldsymbol{x}$ and/or t.

Returning to (17.1)

$$\frac{\partial\phi}{\partial t} = \left[\frac{1}{A}\frac{\partial A}{\partial t} + \frac{i}{\epsilon}\frac{\partial\theta}{\partial t}\right]\phi \simeq -i\omega\phi, \tag{17.4}$$

at $O(1/\epsilon)$ as the first term in the square bracket is $O(1)$ and so can be ignored at this order. Similarly, to leading order $O(1/\epsilon)$,

$$\frac{\partial\phi}{\partial\boldsymbol{x}} \simeq i\boldsymbol{k}\phi. \tag{17.5}$$

Therefore, **locally**, i.e. on $O(\epsilon)$ scales,

$$\phi \simeq Ae^{i(\boldsymbol{k}.\boldsymbol{x}-\omega t)},$$

with **constant** values of A, $\boldsymbol{k}$ and ω, where $\boldsymbol{k}$ and ω are the **local** wave vector and frequency. Since (for example)

$$\frac{\partial^n}{\partial t^n}\phi \simeq (-i\omega)^n\phi,$$

for ϕ to satisfy a wave equation, a **local** dispersion relation must apply where

$$\omega = \Omega(\mathbf{k}; \mathbf{x}, \mathbf{t}).$$

Ω involves local properties, such as layer depth $h(x)$ or buoyancy frequency $N(z)$ which vary in space and/or time.

17.3 Phase and Wave Crests

From the definitions (17.2) and (17.3), and the application of our memories of vector calculus and the chain rule, the following three expressions can be deduced:

$$\begin{aligned}
\boldsymbol{\nabla}_x \times \boldsymbol{k}(\boldsymbol{x}, t) &= \mathbf{0}, &(17.6)\\
\frac{\partial \boldsymbol{k}}{\partial t} &= -\boldsymbol{\nabla}_x \omega(\boldsymbol{x}, t), &(17.7)\\
d\theta &= \epsilon(\boldsymbol{k}.\mathbf{dx} - \omega dt). &(17.8)
\end{aligned}$$

These equations lead to three interesting conclusions.

1: Conservation of Crests

Since the curl of $\boldsymbol{k}$ is zero as in (17.6), Stoke's Theorem tells us that the line integral of $\boldsymbol{k}$ around any closed curve is zero. This in turn implies that, for any two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$:

$$\theta(\boldsymbol{x}_2) - \theta(\boldsymbol{x}_1) = \epsilon \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \boldsymbol{k}.\mathbf{dx},$$

independently of the path taken between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. (Think about going from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ on one path, then back from $\boldsymbol{x}_2$ to $\boldsymbol{x}_1$ on the other path, and bring Stokes to the rescue.) But the left hand side can be interpreted as counting the number of wave crests between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and so the number of crests (or indeed troughs) is independent of path.

2: Flux of Crests

Furthermore, integrating (17.7) in space, we obtain, for two arbitrary points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$:

$$\frac{d}{dt} \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \boldsymbol{k}.\mathbf{dx} = \omega(\boldsymbol{x}_1) - \omega(\boldsymbol{x}_2).$$

Therefore, the change in the number of crests in $[\boldsymbol{x}_1, \boldsymbol{x}_2]$ is equal to the difference between the number of crests entering and the number of crests leaving.

3: Generalization of Wave Vector

Finally, at fixed time, (17.8) implies that, on an $O(1)$ length scale,

$$\phi \propto \exp\left[i \int \boldsymbol{k}(\boldsymbol{x}).\mathbf{dx}\right],$$

which is the appropriate generalization of $\exp(i\boldsymbol{k}.\boldsymbol{x})$ for slowly varying ambient media.

Chapter 18

Ray Tracing Equations

Voodoo voodoo ray ray hey-yah
Gerald Rydel Simpson

We now have all the tools we need to follow wave packets in slowly changing media, provided we follow special (characteristic) paths, called *rays*. It's all a bit magical, but it works.

18.1 Evolution of $\boldsymbol{k}$ and ω

As a wave packet moves through the medium, $\boldsymbol{k}$ and ω must vary smoothly and slowly in order to satisfy the local dispersion relation:

$$\omega(\boldsymbol{x}, t) = \Omega[\boldsymbol{k}(\boldsymbol{x}, t), \boldsymbol{x}, t]. \tag{18.1}$$

Apply the chain rule, like a proper mathematician.

Chain rule for $\partial/\partial \boldsymbol{x}$

With the vertical bars showing explicitly what is being kept constant:

$$\left.\frac{\partial \omega}{\partial x_i}\right|_t = \left.\frac{\partial \Omega}{\partial k_j}\right|_{\boldsymbol{x},t} \left.\frac{\partial k_j}{\partial x_i}\right|_t + \left.\frac{\partial \Omega}{\partial x_i}\right|_{\boldsymbol{k},t}. \tag{18.2}$$

Since, as described above in (17.6),

$$\boldsymbol{\nabla} \times \boldsymbol{k} = \boldsymbol{0} \rightarrow \frac{\partial k_j}{\partial x_i} = \frac{\partial k_i}{\partial x_j},$$

(which also can be established by assuming that partial derivatives of the phase commute, which seems reasonable to this applied mathematician...) Also, from (17.7),

$$\frac{\partial \omega}{\partial x_i} = -\frac{\partial k_i}{\partial t},$$

while, by definition, the *local* group velocity

$$\mathbf{c}_g \equiv \boldsymbol{\nabla}_{\boldsymbol{k}}\Omega, \ \ (\mathbf{c}_g)_j \equiv \frac{\partial \Omega}{\partial k_j} \ .$$

Substituting these three expressions into (18.2) and rearranging, we obtain

$$\begin{aligned} \left[\frac{\partial}{\partial t} + (\mathbf{c}_g)_j \frac{\partial}{\partial x_j}\right] k_i &= -\frac{\partial \Omega}{\partial x_i}, \\ \frac{\partial \mathbf{k}}{\partial t} + (\mathbf{c}_g \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})\, \mathbf{k} &= -\boldsymbol{\nabla}_{\mathbf{x}}\Omega. \end{aligned}$$

Therefore, when moving with velocity $\mathbf{c}_g$, the wave vector $\boldsymbol{k}$ of a wave packet changes if and only if the medium varies with $\boldsymbol{x}$, i.e. *spatially.* Such changes are called *refraction*, leading for example (as we shall see) to surface ocean waves' tendency to break parallel to a shore, provided of course that there isn't either an old and wise reef or a young and dumb headland or "point" to trigger breaking...

Chain rule for $\partial/\partial t$

With the vertical bars again showing explicitly what is being kept constant:

$$\left.\frac{\partial \omega}{\partial t}\right|_{\boldsymbol{x}} = \left.\frac{\partial \Omega}{\partial k_i}\right|_{\boldsymbol{x},t} \left.\frac{\partial k_i}{\partial t}\right|_{\boldsymbol{x}} + \left.\frac{\partial \Omega}{\partial t}\right|_{\boldsymbol{k},\boldsymbol{x}} . \tag{18.3}$$

Again using (17.7), and so

$$\frac{\partial \omega}{\partial x_j} = -\frac{\partial k_j}{\partial t},$$

and doing a bit of rearranging, we obtain the analogous expression

$$\frac{\partial \omega}{\partial t} + (\mathbf{c}_g \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})\, \omega = \frac{\partial \Omega}{\partial t} \, .$$

Therefore, when moving with velocity $\mathbf{c}_g$, the frequency ω of a wave packet changes if and only if the medium varies with t, i.e. *temporally.*

18.2 Ray Tracing

As we can now understand how wave vectors and frequencies change as the wave packet travels at velocity $\mathbf{c}_g$, we now want to make sure that we stay on such paths. These are (of course) *characteristics*, and we define these particular paths to be *rays*. Specifically, a *ray* is a trajectory $\boldsymbol{x} = \boldsymbol{x}(t)$ for a wave packet which satisfies the equation:

$$\frac{d\mathbf{x}}{dt} = \mathbf{c}_g(\boldsymbol{k}(\boldsymbol{x}, t); \boldsymbol{x}, t)\,.$$

The Ray Tracing Equations

Understanding this concept, we now have the *Ray Tracing Equations.* Rays are defined as solutions to:

$$\frac{d\mathbf{x}}{dt} = \mathbf{c}_g = \frac{\partial\Omega}{\partial\boldsymbol{k}} = \boldsymbol{\nabla}_{\boldsymbol{k}}\Omega. \tag{18.4}$$

The time derivative moving along a ray (at the group velocity) is defined as

$$\left.\frac{d}{dt}\right|_g = \frac{\partial}{\partial t} + \mathbf{c}_g \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}\,. \tag{18.5}$$

Using this definition, we then have:

$$\left.\frac{d\omega}{dt}\right|_g = \frac{\partial\Omega}{\partial t}\,, \tag{18.6}$$

$$\left.\frac{d\mathbf{k}}{dt}\right|_g = -\boldsymbol{\nabla}_{\mathbf{x}}\Omega\,, \tag{18.7}$$

$$\frac{1}{\epsilon}\left.\frac{d\theta}{dt}\right|_g = -\omega + \mathbf{c}_g.\boldsymbol{k}\,. \tag{18.8}$$

Therefore, given Ω and initial conditions $\boldsymbol{k}_0$, ω_0 and θ_0 at position $\boldsymbol{x}_0$, we can integrate (in principle) along a ray to calculate the evolution of $\boldsymbol{k}$, ω and θ. If we repeat this processes for many rays (with lots of different initial data) we can find (once again in principle) how $\boldsymbol{k}$ and ω vary everywhere in the domain of interest. How cool is that!

18.3 Comments

1. In general, rays are curved. For uniform (in space), constant (in time) media (such as those we considered in Part III)

$$\frac{\partial\Omega}{\partial\boldsymbol{x}} = \mathbf{0},\ \frac{\partial\Omega}{\partial t} = 0.$$

Therefore, $\boldsymbol{k}$ and ω are constant on rays. Therefore the group velocity $\boldsymbol{c}_g$ is constant on rays. Therefore, for such media, the rays are straight lines (as our good friend D'Alembert knew).

2. The ray tracing equations actually have a first integral $\omega = \Omega(\boldsymbol{k}, \boldsymbol{x}, t)$, which can be exploited to simplify the solution of problems.

3. *** Non-Examinable Alert*** The equations for $\boldsymbol{k}$ and ω are Hamilton's (William Rowan not Alexander) Equations for a dynamical system, where the generalised coordinates $\boldsymbol{q} = \boldsymbol{x}$ and the conjugate momenta $\boldsymbol{p} = \boldsymbol{k}$, and the Hamiltonian $H = \Omega(\boldsymbol{k}, \boldsymbol{x}, t)$ (further reinforcing the point that there is a first integral of the system). Wave packets behave like particles travelling with the group velocity. Further, if you ask for a little more action please we identify the action with the phase, i.e. $S = \theta/\epsilon$, and can then obtain the *Hamilton-Jacobi equation*

$$\frac{\partial S}{\partial t} + \Omega\left(\boldsymbol{\nabla} S; \mathbf{x}, t\right) = 0\,.$$

4. *** Non-Examinable Alert*** In general, the local *energy density*, E, is *not* necessarily conserved for a slowly-varying wavetrain, but the *wave action*, I, is (trust me I'm a doctor, and if you don't look in Whitham's book):

$$\frac{\partial I}{\partial t} + \boldsymbol{\nabla}.(\mathbf{c}_g I) = 0\,,$$

where

$$I = \frac{E}{\omega}\,.$$

In the "special" case when the medium is time-independent, and so the frequency ω is constant along a ray from (18.6), then conservation of wave action is indeed equivalent to conservation of wave energy. Since energy density is proportional to $|A|^2\omega^2$ where A is an amplitude of displacement, and ω is frequency, $I \propto |A|^2\omega$.

5. (Examinable toggled back on...) It is also possible to consider rays in a moving medium, which is very important for atmospheric problems for example, as the wind blows. Recall that if the dispersion relation for a stationary medium is $\omega = \Omega_0(\boldsymbol{k}, t)$, then the dispersion relation for a medium moving with velocity $-\boldsymbol{U}$ is

$$\omega = \Omega_0(\boldsymbol{k}) - \boldsymbol{U}.\boldsymbol{k}.$$

If $\boldsymbol{U}$ is *slowly varying* (i.e. relative to the characteristic wavelengths of the waves) $\boldsymbol{U}(\boldsymbol{x})$, then the equation for the ray (18.4) becomes

$$\frac{d\mathbf{x}}{dt} = \frac{\partial}{\partial \boldsymbol{k}}\left(\Omega_0 - \boldsymbol{U}.\boldsymbol{k}\right) = \frac{\partial \Omega_0}{\partial \boldsymbol{k}} - \boldsymbol{U}(\boldsymbol{x}). \tag{18.9}$$

In this circumstance, totally unsurprisingly the ray evolves due both to the (intrinsic) group velocity **and** the wind velocity.

Chapter 19

(Isotropic) Ray Tracing Examples

Waves make the only sound on Echo Beach
Mark Gane

To come to the end of this part of the course, we are going to consider some examples, in particular the beautiful example of why waves break parallel to beaches. Ray tracing is indeed particularly straightforward for cases where the (local) dispersion relation is *isotropic*).

19.1 Isotropic Dispersion Relations

A dispersion relation is *isotropic* if it is a function of wavenumber $\kappa = (k_1^2 + k_2^2 + k_3^2)^{1/2} = |\boldsymbol{k}|$, and **not** wave vector direction $\hat{\boldsymbol{k}} = (k_1/\kappa, k_2/\kappa, k_3/\kappa)$. For such (local) dispersion relations, there are (at least) two useful properties.

Rays

For isotropic dispersion relations $\Omega(\kappa)$,

$$\begin{aligned} (\boldsymbol{c}_g)_i &= \frac{\partial \Omega}{\partial k_i} = \frac{d\Omega}{d\kappa}\frac{\partial \kappa}{\partial k_i}, \\ &= \frac{d\Omega}{d\kappa}\frac{k_i}{\kappa} = \frac{d\Omega}{d\kappa}\hat{k}_i, \end{aligned}$$

and so, for *isotropic* dispersion relations, rays, defined by

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{c}_g = \frac{d\Omega}{d\kappa}\hat{\boldsymbol{k}},$$

are parallel to $\boldsymbol{k}$.

Therefore, for a two-dimensional situation, the equation for the ray $y(x)$ is the solution to

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{k_2}{k_1}. \tag{19.1}$$

Crests

In general, (not just for isotropic dispersion relations) wave crests (and indeed troughs) are lines of constant phase. So the normal is (of course) $\boldsymbol{\nabla}_x\theta \propto \boldsymbol{k}$, i.e. the wave vector is normal to wave crests. Restricting attention to two-dimensional situations, the crests are defined by constant phase, and so

$$\theta_x dx + \theta_y dy = 0,$$

i.e. solutions to the equation

$$\frac{dy}{dx} = -\frac{\theta_x}{\theta_y} = -\frac{k_1}{k_2}. \tag{19.2}$$

Therefore, for *isotropic* dispersion relations, rays are parallel to $\boldsymbol{k}$, and **perpendicular** to crests. We now consider three interesting and illustrative examples.

19.2 Example 1: Waves on a Beach

Let us consider (surface) waves in stationary water $\boldsymbol{U} = \mathbf{0}$ of depth $h(x)$ such that $h \to \infty$ as $x \to \infty$, and $h \to 0$ as $x \to 0$ "slowly" (compared to waves) such that it arrives at a beach with $h(0) = 0$. We ignore surface tension, and so the (local) dispersion relation is isotropic:

$$\Omega^2 = g\kappa \tanh\left[\kappa h(x)\right], \ \ \kappa = |\boldsymbol{k}| = \sqrt{k^2 + l^2}.$$

Now turning attention to rays approaching the shore, far out to sea (where $h \to \infty$)

$$\omega \to \omega_\infty = \sqrt{g\kappa_\infty}, \ \ \kappa_\infty = \sqrt{k_\infty^2 + l_\infty^2}.$$

We now think about the ray-tracing equations.

1. Since Ω does not depend on time, $\omega = \omega_\infty$ everywhere.

2. Since Ω does not depend on y, $l = l_\infty$ everywhere.

3. On a ray

$$\left.\frac{dk}{dt}\right|_g = -\frac{d\Omega}{dx} \neq 0,$$

and so k will vary on waves.

- Since the waves are coming on-shore, and so from negative x, $k_\infty > 0$.
- $d\Omega/dx \propto dh/dx < 0$ since the "sea" shallows from left to right.
- Therefore we expect k to increase.

However, we do not need to solve the differential equation for k, as we can exploit the "first integral" relating $\omega = \omega_\infty = \Omega$ and so $k(x)$ can be determined from

$$\omega_\infty^2 = g\kappa_\infty = g\sqrt{k(x)^2 + l_\infty^2}\tanh\left[\sqrt{k(x)^2 + l_\infty^2}h(x)\right].$$

Once we know $k(x)$, equations for the shapes of rays and crests can be constructed as in the previous section.

Specificallly, as $h \to \infty$, we expect the argument of the hyperbolic tangent to get very small, and so:

$$\tanh\left[\sqrt{k(x)^2 + l_\infty^2}h(x)\right] \simeq \sqrt{k(x)^2 + l_\infty^2}h(x).$$

Therefore,

$$g\left(k(x)^2 + l_\infty^2\right)h(x) \simeq g\sqrt{k_\infty^2 + l_\infty^2}.$$

As the right hand side is finite, and $h(x) \to 0$, k must be getting large and so $\kappa \simeq k$. Therefore

$$k^2 \simeq \frac{\sqrt{k_\infty^2 + l_\infty^2}}{h(x)}.$$

Note that this is indeed consistent with the assumption that the argument of tanh is getting very small as $h(x) \to \infty$, even though $k \to \infty$ too, since $k \sim O(h^{-1/2})$ as $x \to 0$.

Therefore, whatever the orientation of the wave vector is in the far field, (provided it is coming onshore with $k_\infty > 0$) eventually $k \gg |l_\infty|$ and so the wave vector will be close to horizontal, pointing in the (positive) $x-$direction, and so wave crests will be parallel to the shore $x = 0$. Known by every kid who has ever gone to the sea side, but now you know the underlying mathematics: think how your friends will be impressed!

Non-dispersive case

The situation is particularly straightforward if the waves are (locally) *non-dispersive* i.e. the phase speed c of the waves does not depend on $|\boldsymbol{k}|$:

$$\omega = c(\boldsymbol{x})|\boldsymbol{k}| = \Omega(\boldsymbol{k}, \boldsymbol{x}), \tag{19.3}$$

This is (of course!) a special case of a (locally) *isotropic* dispersion relation, and so the rays are again parallel to $\boldsymbol{k}$. Furthermore, there is only one speed c at each $\boldsymbol{x}$, and, if the medium is not time-dependent, ω is constant on rays, c is independent of time, and $\boldsymbol{c}_g = c\hat{\boldsymbol{k}}$.

19.3 Example 2: Snell's Law

Consider a stratified medium where the (local) phase speed is defined to be $c = \Omega(\kappa, z)/\kappa$, and the waves are (locally) non-dispersive. Then the angle $\Psi(z)$ between a ray and the $z-$axis obeys *Snell's Law*:

$$\frac{\sin\Psi(z)}{c} = A, \tag{19.4}$$

where A is a constant (and c is clearly also a function of z).

The proof (which applies for isotropic dispersion relations) goes as follows. Since the waves are non-dispersive, rays are parallel to $\boldsymbol{k} = (k, l, m)$, and so

$$\frac{\sin\Psi(z)}{c} = \frac{\sqrt{k^2+l^2}}{c|\boldsymbol{k}|} = \frac{\sqrt{k^2+l^2}}{\omega},$$

where ω is the frequency on the ray, given locally by the dispersion relation $\Omega \equiv c\kappa$ by definition. But now from the ray-tracing equations *on a ray*:

- $\partial\Omega/\partial x = 0 \to k$ is constant;
- $\partial\Omega/\partial y = 0 \to l$ is constant;
- $\partial\Omega/\partial t = 0 \to \omega$ is constant;

and so the right hand side is a constant, and the law is proved.

Equation for a Ray in a Stratified Medium

Without loss of generality, one can restrict the problem to two dimensions (since k and l are constant, define the horizontal projection of the wave vector to be the $x-$direction). So, we let $\boldsymbol{k} = (k, 0, m), \kappa = |\boldsymbol{k}|$. For an isotropic dispersion relation (for which a non-dispersive dispersion relation is a special case), the ray is defined as the solution of the two equations:

$$\begin{aligned}\frac{dx}{dt} &= \frac{d\Omega}{d\kappa}\frac{k}{\kappa},\\ \frac{dz}{dt} &= \frac{d\Omega}{d\kappa}\frac{m}{\kappa},\end{aligned}$$

and so the ray is given by

$$\frac{dz}{dx} = \frac{m}{k} = \pm\sqrt{\frac{\kappa^2 - k^2}{k^2}} = \pm\sqrt{\frac{\omega^2}{k^2c^2} - 1} = f(z), \tag{19.5}$$

for some function $f(z)$ (since $\Omega(z)$ and so c is a function of z) and so (in principle) can be integrated to obtain $x(z)$.

Exercise: Midsummer's Night Sound

Sound waves (as in Part I) are just such a non-dispersive system. On (still) summer nights, since the ground can cool rapidly (due to radiation) it is possible for the air temperature (and hence sound speed) to actually increase with height, e.g. as $c(z) = c_0(z+1)$ for appropriately scaled z. Use (19.3) to show that the rays are (arcs) of semi-circles, thus allowing for the possibility of long-range sound propagation through such refraction.

This phenomenon is also a cause of major annoyance near airports, as at particular times of the year, the first plane of the day (near dawn) can appear particularly noisy in the surrounding environs.

19.4 Example 3: Fermat's Principle

Fermat's Principle is not to avoid giving proofs of difficult theorems, but rather the statement that the time of travel

$$\tau(A,B) \equiv \int_A^B \frac{ds}{c(\boldsymbol{x})}, \tag{19.6}$$

between fixed points A and B is *stationary* (usually minimum) along a *ray* with respect to variations in the path $\boldsymbol{x}(t)$ for waves with phase speed c. Though this is a very general principle, applicable even to non-isotropic dispersion relations when posed correctly, the above form is easiest to establish when the dispersion relation is indeed locally non-dispersive with $\Omega = c|\boldsymbol{k}|$, and the medium does not depend on time and so ω is constant on rays.

The proof goes as follows. The quantity of interest is

$$\frac{ds}{c} = \frac{|\dot{\boldsymbol{x}}|dt}{c}.$$

Therefore, the *Lagrangian* of interest $L(\boldsymbol{x}, \dot{\boldsymbol{x}})$ is

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{(\dot{x}_j \dot{x}_j)^{1/2}}{c(\boldsymbol{x})}.$$

For a particular path to be stationary, we need the Euler-Lagrange equations to be satisfied:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) &= \frac{\partial \mathcal{L}}{\partial x_i}, \\ \rightarrow \frac{d}{dt}\left(\frac{\dot{x}_i}{c|\dot{\boldsymbol{x}}|}\right) &= |\dot{\boldsymbol{x}}|\frac{\partial}{\partial x_i}\left(\frac{1}{c}\right), \end{aligned}$$

where the (total) time derivative on the left hand side should be understood to be $d/dt|_g$, i.e. travelling along the ray at the group velocity.

For a non-dispersive dispersion relation, on a ray

$$\dot{\boldsymbol{x}} = \frac{\partial \Omega}{\partial \kappa}\hat{\boldsymbol{k}} = c\hat{\boldsymbol{k}}.$$

Therefore

$$\frac{\dot{\boldsymbol{x}}}{c|\dot{\boldsymbol{x}}|} = \frac{\hat{\boldsymbol{k}}}{c} = \frac{\boldsymbol{k}}{\omega}.$$

Therefore, the Euler-Lagrange equations reduce to

$$\frac{d}{dt}\left(\frac{\boldsymbol{k}}{\omega}\right) = -\frac{1}{c}\frac{\partial c}{\partial \boldsymbol{x}}.$$

On a ray, the left hand side has constant frequency, and so

$$\begin{aligned} \frac{d}{dt}\left(\frac{\boldsymbol{k}}{\omega}\right) &= \frac{1}{\omega}\frac{d\boldsymbol{k}}{dt}, \\ &= -\frac{1}{\omega}\left.\frac{\partial \Omega}{\partial \boldsymbol{x}}\right|_{\boldsymbol{k},t}, \\ &= -\frac{|\boldsymbol{k}|}{\omega}\frac{\partial c}{\partial \boldsymbol{x}}, \end{aligned}$$

which reduces to the right hand side using the dispersion relation.

Exercise

Show that Fermat's Principle implies Snell's Law (as defined in the previous section) in a stratified medium where $\mathcal{L}$ is independent of both x and y.

Part V

Nonlinear (1D) Waves

Chapter 20

1D Waves (In a Perfect Gas)

It's not my function, nothing ever change
Lynval Golding

Everything we have seen so far has been for *linear* waves, i.e. smooth and low-amplitude waves. In this last part we will consider *nonlinear* waves in compressible (and perfect) gases, and in shallow water (though the approach is applicable to other systems, such as traffic flow, blood flow in flexible vessels etc). We focus on the simplest 1D (in space) case, where the velocity $\boldsymbol{u}(\boldsymbol{x}, t) = [u(x, t), 0, 0]$. Since the waves are nonlinear, many of the approaches we have used before are no longer applicable: we don't have *superposition* (in general) of solutions; specifically *Fourier decomposition* is not possible; waves don't satisfy *dispersion relations*; and there is no *stationary phase* and no *rays*. What can we do?

20.1 Governing Equations

We still (thank goodness) have conservation of mass, which in 1D becomes

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0\,. \tag{20.1}$$

We also (thank Isaac) have conservation of momentum, in the form of the 1D Euler equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,. \tag{20.2}$$

Since $p(\rho, S)$, and we assume that the flow is homentropic:

$$\frac{\partial p}{\partial x} = \frac{dp}{d\rho}\frac{\partial \rho}{\partial x} = c^2\frac{\partial \rho}{\partial x}\,,$$

where the sound speed (squared) is

$$c^2(\rho) = \frac{dp}{d\rho}(\rho)\,. \tag{20.3}$$

Therefore (20.2) can also be written as a PDE for u and ρ:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} = 0. \tag{20.4}$$

We can simplify (20.1) and (20.4) by a (cunning) change of coordinates.

First add (20.4) and $\lambda(x,t)\times$(20.1) where $\lambda(x,t)$ is to be determined, and collect derivatives of u and ρ to obtain

$$\left[\frac{\partial}{\partial t} + (u+\lambda\rho)\,\frac{\partial}{\partial x}\right]u + \lambda\left[\frac{\partial}{\partial t} + \left(u + \frac{c^2}{\lambda\rho}\right)\frac{\partial}{\partial x}\right]\rho = 0\,.$$

We fix λ by requiring the operators in the square brackets to be the same, which requires

$$u + \lambda\rho = u + \frac{c^2}{\lambda\rho}\,, \quad \text{i.e.} \quad \lambda = \pm\frac{c}{\rho}\,.$$

Therefore, we obtain

$$\left[\frac{\partial}{\partial t} + (u\pm c)\frac{\partial}{\partial x}\right]u \pm \frac{c}{\rho}\left[\frac{\partial}{\partial t} + (u\pm c)\frac{\partial}{\partial x}\right]\rho = 0\,.$$

To make this even neater, we define

$$Q(\rho) = \int_{\rho_0}^{\rho}\frac{c(\hat\rho)}{\hat\rho}d\hat\rho\,, \tag{20.5}$$

and so

$$\frac{\partial Q}{\partial t} = \frac{c}{\rho}\frac{\partial \rho}{\partial t}\,, \quad \frac{\partial Q}{\partial x} = \frac{c}{\rho}\frac{\partial \rho}{\partial x}\,,$$

and so

$$\left(\frac{\partial}{\partial t} + (u\pm c)\frac{\partial}{\partial x}\right)(u\pm Q) = 0\,. \tag{20.6}$$

The *Riemann invariants*, $R_\pm(x,t)$ are defined by

$$R_\pm = u \pm Q\,. \tag{20.7}$$

R_+ is constant on any path satisfying

$$\frac{dx}{dt} = u + c, \tag{20.8}$$

and such paths are called C_+ *characteristics*. Similarly, R_- is constant on any path satisfying

$$\frac{dx}{dt} = u - c, \tag{20.9}$$

and such paths are called C_- *characteristics*. Physically, waves carrying constant values of $R_\pm$ propagate at speeds $\pm c$ relative to the local flow speed u.

20.2 Method of Characteristics

This manipulation forms the basis for the *method of characteristics*.

- Suppose we can find which C_+ characteristic and which C_- characteristic pass through a given point P .
- Also suppose we know $u + Q$ on the C_+ characteristic and $u - Q$ on the C_- characteristic, for example through following the characteristics back to initial/boundary data.
- Then we can determine u, $Q(\rho)$ at P.
- Hence we can determine ρ, $p(\rho)$ and $c(\rho)$ at P.
- Hence we can determine $u \pm c$ at P, and so (in principle) we can extend the characteristics further.

20.3 Issues

Unfortunately, there is no such thing as a free α.

1. In general, characteristics are not straight lines in the (x, t) plane. For example, on the C_- (characteristic) from $B = (x_B, 0)$ to $P = (x_p, t_p)$ $u - Q$ is constant. However, if the C_+ (characteristic) brings different values of $u + Q$ from initial conditions $x_0 \in [x_A, x_B]$ at $t = 0$, u and Q and hence $u - c$ will vary.
2. Therefore, in general simultaneous solution along C_+ and C_- characteristics is a difficult, nonlinear problem.
3. Solvable problems usually rely on a simple solution for one set of characteristics.

4. The value at $[x_p, t]$ typically depends on values in the finite interval $[x_A, x_B]$.

5. Difficulties will arise if two C_+ (or two C_-) characteristics intersect with contradictory $u + Q$ (or $u - Q$) values, leading to *shocks*. Sounds very exciting.

20.4 Perfect Gas

We will often consider the behaviour of nonlinear waves in a perfect gas. For such a gas, remember (from Part I) that $p(\rho)$ is given by

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma},$$

where γ is the ratio of specific heats. Therefore,

$$c^2 = \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \quad \text{and} \quad \frac{c}{c_0} = \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} \tag{20.10}$$

and so

$$Q = \frac{2}{\gamma - 1}(c - c_0), \quad \rho = \rho_0 \left(\frac{c}{c_0}\right)^{\frac{2}{\gamma-1}}, \quad p = p_0 \left(\frac{c}{c_0}\right)^{\frac{2\gamma}{\gamma-1}}. \tag{20.11}$$

20.5 Simple Waves

A *simple wave* is a wave where one of the Riemann invariants $R_\pm$ is uniformly constant (wlog zero), i.e. where $u = Q$ (equivalent to $R_- = 0$) or $u = -Q$ (equivalent to $R_+ = 0$).

As an instructive example, let us consider a perfect gas (as in the previous section).

- Assume that the initial conditions satisfy, for all x, that $R_- = 0$, i.e.

$$u(x, 0) = Q(x, 0) = \frac{2}{\gamma - 1}\left[c(x, 0) - c_0\right].$$

- Note that this includes the special case of an undisturbed fluid $u = 0$, $c = c_0$ as $u - Q = 0$ for all x initially.

- Now, as $u = Q$ along every C_- everywhere, i.e. for all x and for all t, this implies that
$$c = c_0 + \left(\frac{\gamma - 1}{2}\right) u.$$

- Furthermore, along any C_+ characteristic, $u + Q = 2u$ is a constant.

- Therefore
$$u + c = c_0 + \left(\frac{\gamma + 1}{2}\right) u,$$
is also a constant.

- Hence the C_+ characteristics are straight lines of the form
$$x = x_0 + \left[c_0 + \left(\frac{\gamma + 1}{2}\right) u(x_0, 0)\right] t. \tag{20.12}$$

- Therefore, given (x, t), we (in principle) can solve this nonlinear equation to determine $x_0(x, t)$.

- As u is constant on the C_+ characteristic, we can then simply solve $u(x, t) = u(x_0, 0)$.

- A solution with $u = Q$, $(R_- = 0)$ (as in this example) is a *right-going simple wave*, propagating along C_+ characteristics.

- Conversely, a solution with $u = -Q$, $(R_+ = 0)$ is a *left-going simple* wave, propagating along C_- characteristics.

Chapter 21

Shocks, Pistons and Fans

Constant elevation causes expansion
Rakim Allah

In this chapter we will consider how shocks might form in simple waves, and discuss piston problems, in particular considering how expansion signals propagate when a piston moves in a way to increase the volume occupied by the gas.

21.1 Shock formation in simple waves

In the previous lecture, we calculated that the C_+ characteristics are defined by

$$\frac{dx}{dt} = c_0 + \left(\frac{\gamma+1}{2}\right) u,$$

so, inevitably, larger values of $u(x, 0)$ propagate faster. Therefore, if $du/dx_0 < 0$ anywhere, then C_+ characteristics will start to cross, giving contradictory predictions for $u(x, t)$. We want the world to remain well-posed, so a *shock* (i.e. a discontinuity in the solution) forms and is *regularized* by new physics, which we will discuss further in the following chapters. Recall (20.12), the equation for the C_+ characteristics:

$$x = x_0 + \left[c_0 + \left(\frac{\gamma+1}{2}\right) u(x_0, 0)\right] t.$$

Therefore x is no longer monotonically increasing in x_0 at a time t when

$$\left.\left(\frac{\partial x}{\partial x_0}\right)\right|_t = 0$$

for the first time. But from (20.12),

$$\frac{\partial x}{\partial x_0} = 1 + \left(\frac{\gamma+1}{2}\right)\frac{du}{dx_0}t, \tag{21.1}$$

which is zero for the first time (and hence a shock forms) at $t = t_s$, where

$$t_s = \left(\frac{2}{\gamma+1}\right)\frac{1}{\max_{x_0}\left[-\frac{du}{dx_0}(x_0,0)\right]}. \tag{21.2}$$

The shock appears at the $x(x_0,t)$ corresponding to the maximising x_0 (and hence t_s).

Equivalent interpretations

There are two alternative, but entirely equivalent interpretations of what is happening at a shock.

1. That $\partial x/\partial x_0 \to 0$ at some t is a mathematical statement that C_+ characteristics cross, as varying x_0 (which can be thought of as a label of different characteristics) does not vary the point in space time (x,t) of interest, i.e. two (or more) characteristics must pass through (x,t).

2. Alternatively, from (21.2) it is clear that, for a shock to form $du/dx_0 < 0$, and for $t_s > 0$ the maximum magnitude of this derivative must be finite. Therefore, at the time of the shock formation $t \to t_s$:

$$\frac{\partial u}{\partial x} = \frac{\frac{du}{dx_0}}{\frac{\partial x}{\partial x_0}} \to -\infty, \tag{21.3}$$

   and so the wave front becomes a vertical cliff, with a discontinuity in properties (e.g. of u) across it. This is an inevitable consequence of the nonlinear self-advection of faster fluid.

21.2 Initial Value Problems for Simple Waves

Initial value problems (for simple waves) can be interpreted in terms of propagation of information by the various characteristics into (and out of) various regions. Consider the example shown in figure 21.1. Suppose at $t = 0$, $u = 0$, $c = c_0$ everywhere except in the finite region $[A, B]$ marked with a green line. The pattern of characteristics then define 6 regions. Dividing characteristics (C_+ in red, C_- in blue) are marked with thicker lines.

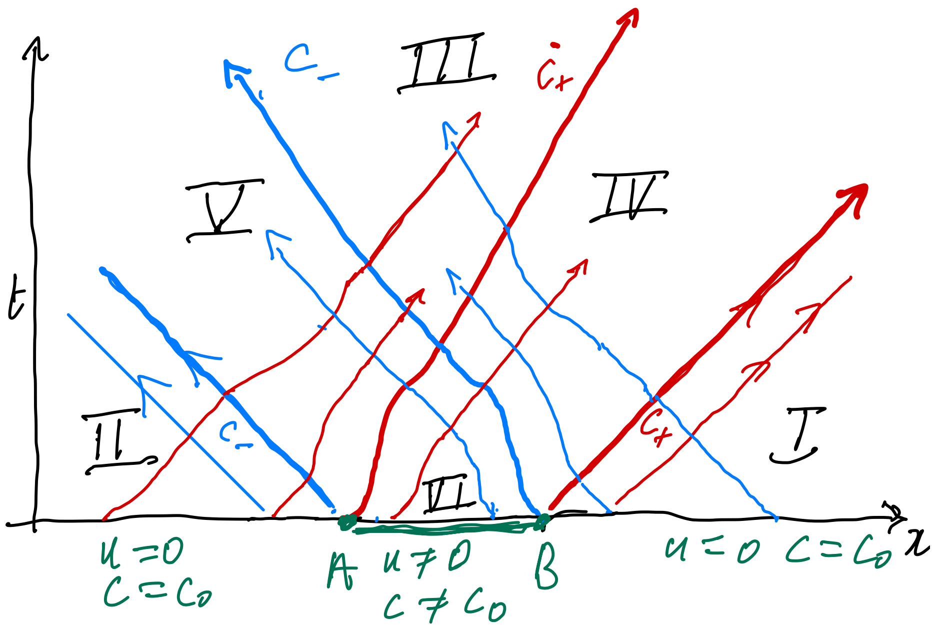

Figure 21.1: Schematic of space time showing six different regions for an IVP where the initial data are non-trivial only in $[A, B]$ C_+ characteristics are plotted in red, while C_- characteristics are plotted in blue.

- In regions I, II (and III!) all C_+ and C_- originate in undisturbed fluid with $u = Q = 0$. Therefore in these regions $u = 0$, $c = c_0$. C_+ characteristics are straight lines with slope c_0, while C_- characteristics are straight lines with slope $-c_0$. Notice how region III is "above" (i.e. at later times) $[A, B]$, yet the information associated with the non-zero initial conditions there has propagated away.

- That propagation is occuring in regions IV and V. In region IV, every C_- characteristic originates in undisturbed fluid to the right of B. This implies that (here) $u = Q$, and so this region contains a simple wave:

$$u = u(x_0, 0), \; c = c_0 + \left(\frac{\gamma - 1}{2}\right) u(x_0, 0),$$

  for some $x_0(x, t)$, carrying the initial value "information" from $x_0 \in (A, B)$. The C_+ characteristics are straight, but not necessarily parallel.

- Similarly, region V contains a left-going simple wave $u = -Q$ with straight C_- characteristics.

- Region VI is complicated...

21.3 Piston Problems

Another class of problem of interest concern pistons. These problems are partially boundary value problems as information from the piston leads to simple waves. Without loss of generality, consider a semi-infinite region of gas at rest (hence with $u = 0$ $c = c_0$ everywhere), with a barrier at $x = 0$, $t = 0$. This barrier is actually a piston, and the position $X(t)$ of the piston is prescribed for all time (with $X(0) = 0$ of course).

We may construct the solution as follows. All C_- characteristics originate from $x > 0$, $t = 0$ where $u = 0$, $c = c_0$. Generically, we assume that the piston either has $\dot{X} > 0$, or at least has sufficiently small magnitude so that all C_- reach the piston. (We consider what happens when this assumption fails below.)

There are then two regions of interest.

1. For points in space time (x, t) such that $x > c_0 t$ all C_+ characteristics come from $t = 0$, $x > 0$ and so $u = 0$, $c = c_0$. Therefore, in this region for all $t > 0$, $u = 0$ and the fluid remains at rest. Notice that the "information" that the piston has moved travels away from the piston at the finite sound speed c_0.

2. In the other region with (x, t) such that $x < c_0 t$ (i.e. to the left of the dividing C_+ characteristic $x = c_0 t$ emanating from $x = 0$, $t = 0$) all the C_+ characteristics come from the piston. Specifically, the C_+ characteristic at $t = \tau$, $x = X(\tau)$ has $u = \dot{X}(\tau)$ (since the gas remains in contact with the piston remember) and so is defined (for $t > \tau$) by

$$x = X(\tau) + \left[c_0 + \left(\frac{\gamma + 1}{2}\right)\dot{X}(\tau)\right](t - \tau), \ t > \tau. \tag{21.4}$$

   Given (x, t) this is an implicit equation for $\tau(x, t)$ which can be solved (in general).

Exercise

Find the exact solution for uniform acceleration $X(\tau)$, selecting the physical root $0 < \tau < t$.

21.4 Rarefaction & Expansion Fans

Now consider the specific situation with $\ddot{X} \leq 0$ for all time, with the piston moving away (to negative x) from the gas, and $\dot{X}$ non increasing. Therefore, in the region II with (x, t) such that $x < c_0 t$ to the left of the dividing C_+ characteristic $x = c_0 t$, the sound speed

$$c = c_0 + \left(\frac{\gamma - 1}{2}\right) \dot{X}(\tau) < c_0. \tag{21.5}$$

Therefore, the equation for the C_+ characteristics starting from the piston is

$$\frac{dx}{dt} = u + c = c_0 + \left(\frac{\gamma + 1}{2}\right) \dot{X}(\tau). \tag{21.6}$$

There are three key observations which can now be made.

1: Rarefaction and Shock Absence

From (21.6), dx/dt on the C_+ characteristics decreases with τ, and indeed becomes zero if $\dot{X} = -2c_0/(\gamma + 1)$. Clearly, the characteristics diverge, and so, for such piston movements, there are no shocks.

From (21.5), c decreases with τ and so ρ decreases with τ. Therefore, the dividing characteristic $x = c_0 t$ is the leading edge of a *rarefaction wave*. As the gas expands to fill the extra space due to the piston receding to the left, the signal of reduced density propagates also to the right.

2: Vacuum Formation

Indeed, at $\dot{X} = -2c_0/(\gamma - 1)$, from (21.5) $c = 0$. Therefore $\rho = 0$, $u \pm c = \dot{X}$, and so the C_+ and C_- characteristics are actually parallel, and tangent to the space time curve of the piston location. If $\dot{X} < -2c_0/(\gamma - 1)$ (as will inevitably occur as the piston continues to accelerate) the gas actually loses contact with the piston, and the C_- characteristics do not reach it. A vacuum appears between the limiting characteristics and the piston. Since (for air) $\gamma \simeq 1.4$, this occurs at Mach number $M \geq 5$ (i.e. for *hypersonic* flow, as opposed to merely *supersonic* flow with $M \geq 1$). A related observation (known to space travellers, though not always to film directors) is that if a container of pressurised gas (a spaceship or space suit) in a vacuum is pierced instantaneously, the gas can flow out no faster than this speed (which is finite, although really quite significant...)

3: Expansion Fans

Now consider rapid acceleration over a relatively short (in some appropriate sense) time t_a from zero to a constant (terminal) velocity $\dot{X} = -V < 0$. There are now three different regions.

- As usual, to the right of the dividing C_+ characteristic, $x \geq c_0 t$, the gas remains at rest in region I, with constant sound speed c_0, and all the C_+ characteristics have the same slope c_0.
- As the ultimate piston speed is also constant, there is another region III for $t > t_a$ to the **left** of another dividing C_+ characteristic, defined as
$$x = \left[c_0 - \left(\frac{\gamma+1}{2}\right)V\right](t - t_a) + X(t_a),\ t > t_a, \tag{21.7}$$
In this region, the sound speed is constant
$$c_1 = c_0 - \left(\frac{\gamma-1}{2}\right)V < c_0,$$
the density is also constant $\rho_1 < \rho_0$, and all the C_+ characteristics are parallel to the dividing characteristic defined by (21.7).
- There is then an intermediate region II, where the density drops from right to left (on the space time plot) from ρ_0 to ρ_1, the sound speed drops from c_0 to c_1, and the slope of the C_+ characteristics drops from c_0 to $c_0 - (\gamma+1)V/2$.

Being good applied mathematicians, we can imagine taking the limit as $t_a \to 0$, and requiring an impulsive start. This means that region II reduces to a wedge emanating from $x = 0$, $t = 0$. This wedge is called an *expansion fan*, as the gas is indeed expanding to fill the ever increasing volume as the piston recedes to the right. Note that if the piston is receding at a constant speed, the density in region III remains constant, balanced by a wider and wider region occupied by the expansion fan where the density $\rho < \rho_0$, the initial equilibrium value.

All C_- characteristics start from $x > 0$, $t = 0$, and since we consider a simple wave (and we are not pulling the piston back too fast) $u = Q$ everywhere. In the expansion fan (region II) all the C_+ characteristics are still straight. They must go through the origin, and so are all proportional to x/t. On any individual C_+ characteristic u, Q and hence c is constant, but they all vary across the fan. Therefore, in regions I and III, C_- characteristics are also straight, but they are curved (as worked through on the example sheet) in region II.

21.5 Shock Formation

Conversely, shocks will form if the piston moves towards the gas, whatever the form of the $X(t)$. As the C_+ characteristics have slope c_0 in the (undisturbed) region I, while they have slope

$$c_0 + \left(\frac{\gamma+1}{2}\right)\dot{X}(\tau) > c_0,$$

to the left of the dividing C_+ characteristic $x = c_0 t$, and so crossings are inevitable. Indeed, a similar argument for successive C_+ characteristics shows that a shock forms for a receding piston with $\dot{X} < 0$, if it decelerates so that $\ddot{X}(t) > 0$ for some t.

The characteristics start to cross at the first time t when

$$\left.\frac{\partial x}{\partial \tau}\right|_t = 0,$$

where the equation for x on such C_+ characteristics is given by (21.4). Differentiating with respect to τ, the shock forms at

$$t = t_s = \min_\tau \left[\tau + \frac{2c_0 + (\gamma-1)\dot{X}(\tau)}{(\gamma+1)\ddot{X}(\tau)}\right], \tag{21.8}$$

at the x corresponding to the minimising τ.

Regularization

We thus have seen that nonlinear waves can form shocks by wave-steepening, and in general $\partial u/\partial x \to -\infty$ at some critical time t_s. For subsequent times, the model predicts the unphysical situation that u becomes multivalued, with three different values for u at a given x for some range. "Clearly" some extra physical process(es) need to be considered to regularize the wave through this "breaking".

One key physical process that we have neglected is viscosity (the internal "friction" resisting flow in a real fluid). This would add an extra term on the right hand side of (for example) the equation for the evolution of the R_+ Riemann invariant, and so

$$\left[\frac{\partial}{\partial t} + \left(c_0 + \left[\frac{\gamma+1}{2}\right]u\right)\frac{\partial}{\partial x}\right](2u) = \nu\frac{\partial^2}{\partial x^2}u.$$

where ν is the kinematic viscosity $O(10^{-5}\mathrm{m}^2\mathrm{s}^{-1})$ for air.

The conventional approach to ignore this term is to suppose that there is typical velocity scale U and length scale L of the flow, and to argue that

$$\nu\frac{\partial^2}{\partial x^2}u \sim \frac{\nu U}{L^2},\ u\frac{\partial u}{\partial x} \sim \frac{U^2}{L},$$

and so we can ignore the viscous term as $\nu \ll UL$ (i.e. the flow Reynold's number is high, because who in their right mind would want to study slow viscous flow...perhaps don't answer that...)

This scaling argument is well-satisfied almost everywhere. However, as waves steepen towards forming shocks, inevitably very large local gradients are formed with $\partial/\partial x \gg 1/L$, and hence viscosity (and/or thermal conduction) can *regularize* or "smooth out" (incipient) shocks. The characteristic scale for this can be tiny: for *transonic* speeds $u \sim c_0$, the "width" of the boundary layer of the shock $\delta_x \sim \nu/u \sim 10^{-8}$m.

Chapter 22

Shock Equations

Don't give up the game until your heart stops beating
Bernard Sumner

In this chapter, appropriate equations are "derived" to describe fluxes (i.e. transports per unit area) across shocks which are continuous (at least to leading order) and hence can allow the determination of key properties (and/or the answering of examination questions...) The particular balance equations are commonly referred to as the *Rankine-Hugoniot relations* (or jump conditions, although the quantities don't jump across the jump...do keep up...)

22.1 Rankine-Hugoniot Relations

Consider a shock separating two uniform regions (where the shock width is very small compared to other dimensions). To fix ideas, choose $V > 0$ and have region "0" to the right and region "1" to the left. In general the shock would then be travelling at a speed $V > u_1$, the speed *downstream* or "behind" the shock (where the pressure is p_1 and the density is ρ_1) while upstream or "in front of" the shock, the relevant quantities are u_0, p_0 and ρ_0, in general all different from the downstream quantites. (The region to the right is called "upstream" as the shock has not yet passed through that fluid.) It is most convenient to switch to a frame of reference in which the shock is stationary, so that the (relative) velocities are then $U_1 = u_1 - V$ (downstream/to the left) and $U_0 = u_0 - V$ (upstream/to the right) respectively.

Formally, mass, momentum and total energy all satisfy equations in the general form (when there are no external sources)

$$\frac{\partial}{\partial t}\left(\text{quantity}\right) = -\boldsymbol{\nabla}\cdot\left(\text{flux}\right).$$

Therefore, applying the divergence theorem and the stationarity, fluxes of mass, momentum and total energy must be the same either side of the shock. (The regularization processes mentioned above mean that there is typically some heating local to the shock, and indeed the entropy inevitably increases when crossing a shock, i.e. going from upstream to downstream. However, provided the shock is sufficiently weak, these effects are higher order, and so it is still formally appropriate to proceed under the assumption that total energy flux across the shock is continuous. Don't stress about internal stresses...)

The mass flux balance and momentum flux balance (aka conservation of mass and conservation of momentum) are thus respectively

$$\rho_1 U_1 = \rho_1(u_1 - V) = \rho_0(u_0 - V) = \rho_0 U_0, \tag{22.1}$$
$$p_1 + \rho_1 U_1^2 = p_0 + \rho_0 U_0^2, \tag{22.2}$$

remembering the appropriate rate of working by the pressure.

For the (total) energy equation, we need to cast our minds back to the very first chapter (section 1.4, seems a long time ago) where we considered the internal energy *per unit mass* for an ideal/perfect gas:

$$e_i = \frac{1}{\gamma - 1}\frac{p_i}{\rho_i}, \; i = 0, 1;$$

and so the flux of total energy *per unit volume* either side of the shock is

$$U_1\left[\frac{1}{2}\rho_1 U_1^2 + \rho_1 e_1 + p_1\right] = U_0\left[\frac{1}{2}\rho_0 U_0^2 + \rho_0 e_0 + p_0\right].$$

Combining terms and using (22.1), we obtain the third Rankine-Hugoniot relation, the (strictly leading order for a "real" shock) total energy flux balance:

$$\frac{1}{2}U_1^2 + \left(\frac{\gamma}{\gamma - 1}\right)\frac{p_1}{\rho_1} = \frac{1}{2}U_0^2 + \left(\frac{\gamma}{\gamma - 1}\right)\frac{p_0}{\rho_0}. \tag{22.3}$$

Equations (22.1)-(22.3) are the *Rankine-Hugoniot* relations. They yield three equations connecting the seven quantities of interest at the shock, i.e. its velocity V and the two sets of three variables either side: u_0, ρ_0, p_0 upstream; and u_1, ρ_1, p_1 downstream. Therefore, we need 4 further pieces of information to solve a problem, for example from initial or boundary conditions.

Note, velocity parallel to a shock is unchanged on either side of an *oblique* shock, as the gas velocity is in that case no longer parallel to the shock velocity. As in general u_1 and u_0 are different, the angle that the shock makes with the gas velocity is different either side of a shock.

Across a shock, note it is not in fact true that

$$\frac{p_1}{p_0} = \left(\frac{\rho_1}{\rho_0}\right)^\gamma,$$

as this relationship only holds for perfect/ideal gases in homentropic flow, i.e. for constant, uniform entropy. Because of the associated compression, shocks are not adiabatic. There is both thermal conduction, because of large $\boldsymbol{\nabla} T$, and also heat generation through viscous dissipation because of large values of $\boldsymbol{\nabla} u$.

22.2 * Non-Examinable Full Equations *

For a proper treatment of the full equations for compressible flow in shocks, see G. K. Batchelor's book *Introduction to Fluid Dynamics* (CUP) or G. B. Whitham's book *Linear and Nonlinear Waves* (Wiley). Here is a brief and superficial (indeed sketchy) sketch of some of the key ideas.

In a fluid *at rest*, the stress is isotropic, and given by

$$\sigma_{ij} = -P\delta_{ij},$$

where $P(\rho, S)$ is the thermodynamic pressure determined by an equation of state. However, a fluid *in motion* is (inevitably) not quite in thermodynamic equilibrium. Therefore, the *fluid-dynamic* pressure p, **defined** as the isotropic part of the stress tensor, is slightly different from P and the stress tensor is not itself isotropic, and is given by

$$\sigma_{ij} = -p\delta_{ij} + d_{ij},$$

where d_{ij} is the deviatoric stress tensor.

In a *Newtonian* fluid, the deviatoric stress is a local, linear, instantaneous and isotropic function of the *strain-rate* tensor

$$\mathbf{e} = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right],$$

which should be compared with the linear elasticity of Part II for the *strain* tensor, while remembering that it must be traceless.

For such fluids,

$$d_{ij} = 2\mu\left(e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}\right)$$

where μ is the *dynamic viscosity* $= \rho\nu$, ν being the *kinematic viscosity*. This viscosity is also sometimes called the *shear viscosity* as it quantifies the resistance of a real fluid to flow under shearing stresses.

Real fluids also don't like being pushed around by normal stresses either, such that when compression or expansion occurs, there is a mismatch between the fluid dynamic pressure p and the (thermodynamic) pressure P such that $p = P - \kappa \boldsymbol{\nabla} . \boldsymbol{u}$, where κ is the *bulk viscosity*. (Note this phenomenon is not relevant to incompressible fluids, and then $p = P$, just as in static equilibrium, and p is a Lagrange multiplier remember...)

Momentum balance then yields the (full) Navier-Stokes equations:

$$\rho \frac{D\boldsymbol{u}}{dt} = \mathbf{F} - \boldsymbol{\nabla} P + \mu \nabla^2 \boldsymbol{u} + \left(\kappa + \frac{1}{3}\mu \right) \boldsymbol{\nabla}(\boldsymbol{\nabla} . \boldsymbol{u}), \tag{22.4}$$

where $\mathbf{F}$ is some body force, and I have used the thermodynamic pressure P to show the role of κ.

The *mechanical energy equation* is obtained by taking the dot product $\boldsymbol{u}\cdot$ (22.4).

The *total energy equation*, with no body forces is

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}|\boldsymbol{u}|^2 + e\right)\right] + \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{u}\left(\frac{1}{2}|\boldsymbol{u}|^2 + e + \frac{P}{\rho}\right)\right] = -\boldsymbol{\nabla} \cdot \boldsymbol{q} + \boldsymbol{\nabla} \cdot (\boldsymbol{d} \cdot \boldsymbol{u}), \tag{22.5}$$

where e is (here, sorry about the shocking notation) the internal energy per unit mass, and $\boldsymbol{q}$ is the heat flux. Subtracting this equation from the mechanical energy equation yields the *internal energy equation*, which can actually be expressed as an equation for the entropy S:

$$\rho T \frac{DS}{Dt} = 2\mu \left(e_{ij} - \frac{1}{3}\delta_{ij} e_{kk} \right)^2 + \kappa (\boldsymbol{\nabla} \cdot \boldsymbol{u})^2 - \boldsymbol{\nabla} \cdot \boldsymbol{q},$$

where T is the (absolute) temperature, and we have used the second law of thermodynamics to state

$$de = TdS - PdV.$$

For steady shocks, the right hand side of (22.5) is a divergence of something that tends to zero far from the shock. Therefore, the total energy flux

$$\rho \boldsymbol{u} \left(\frac{1}{2}|\boldsymbol{u}|^2 + e + \frac{P}{\rho} \right)$$

can be taken to be continuous across the shock (though it varies within it). Furthermore, the entropy S does vary (and of course increases) across the shock, and so the flow is **not** homentropic. So, we now switch back on our examinability.

Chapter 23

Example Shock Calculation

You need to find out, 'cos no one's gonna tell you what I'm on about
Noel Gallagher

In this chapter, an example calculation will be worked through of a shock "invading" a fluid at rest, so the full glory of the required mathematical keepy-uppy can be appreciated, as well as the interesting fact that the flow is supersonic and subsonic either side of such a shock. This type of shock is what would be expected for a piston with positive $\dot{X}$, or as the far-field approximation of the spherical blast-wave from an explosion.

23.1 Shock invading a fluid at rest

Consider a situation where the pressure is p_0 in a stationary fluid ($u_0 = 0$) of density ρ_0. A shock is "invading" this fluid at velocity V and downstream (behind) the shock, the pressure is p_1, the velocity u_1 and density ρ_1. As we have the Rankine-Hugoniot relations (22.1)-(22.3), we need one further relationship to solve this problem. Here, we assume that we know the *shock strength* β relating p_1 to p_0:

$$p_1 = (1 + \beta)p_0, \tag{23.1}$$

so $\beta \ll 1$ describes a *weak shock*, while $\beta \gg 1$ describes a *strong shock*. Given this relationship, the objective is to calculate u_1, V and ρ_1.

The general advice for solution using the R-H relations is to stop and think what the question asks for before embarking on (potentially) messy algebra to eliminate variables. Here we aim for $\rho_1(\rho_0, p_0, p_1)$ first. As before, we transform to a frame where the shock is stationary so, using (22.1)-(22.3):

$$\rho_1(u_1 - V) = \rho_0(-V), \tag{23.2}$$

$$p_1 + \rho_1(u_1 - V)^2 = p_0 + \rho_0 V^2. \tag{23.3}$$

Substituting (23.2) into (23.3) and rearranging:

$$p_1 - p_0 = \frac{\rho_0}{\rho_1}(\rho_1 - \rho_0)V^2. \qquad (23.4)$$

Note that this equation has a (trivial) solution of $\rho_1 = \rho_0$ and $p_1 = p_0$. Henceforth therefore, we search for a solution where $\rho_1 \neq \rho_0$. Therefore,

$$V^2 = \frac{(p_1 - p_0)}{(\rho_1 - \rho_0)}\frac{\rho_1}{\rho_0}, \qquad (23.5)$$

$$(V - u_1)^2 = \frac{(p_1 - p_0)}{(\rho_1 - \rho_0)}\frac{\rho_0}{\rho_1}. \qquad (23.6)$$

We have to be careful with the choice of signs, as (23.2) implies that $V > u_1$.

Now considering the total energy flux equation (22.3)

$$\begin{aligned} \frac{\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_0}{\rho_0}\right) &= \frac{1}{2}\left[V^2 - (V - u_1)^2\right], \\ &= \frac{1}{2}\left(\frac{p_1 - p_0}{\rho_1 - \rho_0}\right)\left[\frac{\rho_1}{\rho_0} - \frac{\rho_0}{\rho_1}\right], \end{aligned}$$

using (23.5)-(23.6), and remembering that the left hand side is the difference of the two enthalpies. Cancelling the factor of $\rho_1 - \rho_0$ on the right hand side, we arrive at the key expression, originally due to Rankine (1870) and (posthumously) to Hugoniot (1889):

$$\frac{\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_0}{\rho_0}\right) = \frac{1}{2}(p_1 - p_0)\left(\frac{1}{\rho_0} + \frac{1}{\rho_1}\right). \qquad (23.7)$$

This expression can now be rearranged to express $\rho_1(\rho_0, p_0, p_1)$ and then back-substitute for V and u_1.

23.2 Comments

1: The shock is compressive

The R-H relations are unaffected if the velocities are flipped in sign, but it is still physically "obvious" that we can assume $V > 0$ if $p_1 > p_0$ and vice versa, i.e. the shock compresses the gas passing through it. This "obvious" statement is actually true because:

- (* non-examinable guff about entropy again *) the entropy S, defined as

$$S = c_v \ln\left(\frac{p}{\rho^\gamma}\right) + c, \qquad (23.8)$$

where c is a constant, * **must** * (the "must" being the non-examinable bit) increase across the shock;

- the full equations for the shock's internal structure have no solution otherwise;
- (examinable toggled back on) this makes sense of the characteristics (such as in formulation as a piston problem) as an IVP develops.

2: Density ratio implies entropy increases

Substituting (23.1), i.e. $p_1 = (1+\beta)p_0$ into (23.7), multiplying across by $\rho_1(\gamma-1)/\gamma$ and collecting terms in ρ_1/ρ_0, we obtain

$$\frac{\rho_1}{\rho_0} = \frac{1+\left(\frac{\gamma+1}{2\gamma}\right)\beta}{1+\left(\frac{\gamma-1}{2\gamma}\right)\beta}. \tag{23.9}$$

Therefore, using (23.8),

$$\begin{aligned} \frac{S_1-S_0}{c_v} &= \ln\left(\frac{p_1}{\rho_1^\gamma}\right) - \ln\left(\frac{p_0}{\rho_0^\gamma}\right), \\ &= \ln\left(\frac{p_1}{p_0}\right) - \ln\left(\frac{\rho_1^\gamma}{\rho_0^\gamma}\right), \\ &= \ln(1+\beta) - \gamma\ln\left[\frac{2\gamma+(\gamma+1)\beta}{2\gamma+(\gamma-1)\beta}\right]. \end{aligned}$$

It can be established that $(S_1-S_0)/c_v > 0$, i.e. the entropy of the gas increases as it crosses the shock, which is equivalent to showing that

$$\frac{\rho_1}{\rho_0} < (1+\beta)^{1/\gamma}.$$

- Indeed, for weak shocks with $\beta \ll 1$,

  $$\frac{S_1-S_0}{c_v} = \frac{(\gamma^2-1)\beta^3}{12\gamma^2},$$

  showing that the change in entropy is small compared to the change of pressure and density (both $O(\beta)$) formally justifying the approximation underlying the R-H relation (22.3) for the total energy flux.

- For strong shocks with $\beta \gg 1$,

  $$\frac{\rho_1}{\rho_0} \lesssim \frac{\gamma+1}{\gamma-1},$$

  and so, crucially, the density either side of the shock remains positive and finite.

3: Supersonic upstream/subsonic downstream

Now consider the velocity *upstream* of the shock, i.e. in the unperturbed fluid at rest from (23.5), and remember that the sound speed c_0 is given by:

$$c_0^2 = \gamma \frac{p_0}{\rho_0}.$$

Substituting this and (23.1) into (23.5), we obtain

$$\begin{aligned} V^2 &= \frac{(p_1 - p_0)}{(\rho_1 - \rho_0)} \frac{\rho_1}{\rho_0}, \\ &= \frac{\beta}{\gamma} c_0^2 \frac{\rho_1/\rho_0}{[\rho_1/\rho_0 - 1]}. \end{aligned}$$

From (23.9),

$$\frac{\rho_1}{\rho_0} - 1 = \frac{\beta/\gamma}{1 + \left(\frac{\gamma-1}{2\gamma}\right)\beta},$$

and so

$$V^2 = c_0^2 \left[1 + \left(\frac{\gamma - 1}{2\gamma}\right)\beta\right] > c_0^2, \tag{23.10}$$

and so fluid enters the shock from the unperturbed upstream side *supersonically.* Perhaps it's feeling gin and tonically.

Similarly, on the downstream side of the shock,

$$c_1^2 = \gamma \frac{p_1}{\rho_1}.$$

Therefore from (23.6), using (23.1) again,

$$\begin{aligned} (V - u_1)^2 &= \frac{(p_1 - p_0)}{(\rho_1 - \rho_0)} \frac{\rho_0}{\rho_1}, \\ &= \frac{p_1}{\rho_1}\left(1 - \frac{1}{\beta + 1}\right)\left[\frac{\rho_1}{\rho_0} - 1\right]^{-1}, \\ &= \frac{c_1^2}{\gamma}\left(\frac{\beta}{\beta + 1}\right)\left(\frac{\gamma}{\beta}\right)\left(1 + \left[\frac{\gamma - 1}{2\gamma}\right]\beta\right), \end{aligned}$$

and so, after some appropriately heroic rearranging (this is Part II Waves after all):

$$(V - u_1)^2 = c_1^2 \left[1 - \frac{\beta(\gamma + 1)}{2\gamma(\beta + 1)}\right] < c_1^2, \tag{23.11}$$

demonstrating that the fluid emerges on the downstream side with a relative velocity $V - u_1$ that is *subsonic* with respect to c_1.

*** Non-examinable Information Flow**

Entropy is actually transported with the fluid velocity in adiabatic flow.

- On the *upstream* side (to the *right* of the shock):
  - the C_+ characteristic has slope $-V + c_0 < 0$ because the fluid crosses the shock *supersonically*;
  - the entropy characteristic has slope $-V < 0$;
  - the C_- characteristic has slope $-V - c_0 < 0$.
- On the *downstream* side (to the *left* of the shock):
  - the C_+ characteristic has slope $u_1 - V + c_1 > 0$ since the relative flow is *subsonic*;
  - the entropy characteristic has slope $u_1 - V < 0$;
  - the C_- characteristic has slope $u_1 - V - c_1 < 0$.

Note how there are 4 characteristics approaching the shock, matching the number of parameters (a.k.a. the *information*) we were given: p_1, u_0, ρ_0 and p_0; and if you don't find that cool I would respectfully suggest you are reading the wrong course...

Chapter 24

Nonlinear (1D) Shallow Water Waves

You got to roll with the punches and get to what's real
David Lee Roth

In this (final) chapter, the analogy between ideal/perfect gases and "shallow" layers of water is explored, and the (sadly non-examinable) phenomenon of the hydraulic jump, beloved of civil engineers and 80's hair metal bands, is introduced.

24.1 Shallow Water Equations in 1D

In this context *shallow water* means that characteristic wave lengths are very much bigger than the depth of the water. Conventionally, the equations we are about to derive are known as *shallow water equations*, although they apply for any (inviscid constant-density) fluid, or alternatively they are referred to as a special case of the *Saint-Venant equations*. (An "interesting" side note is that Adhémar Jean Claude Barré de Saint-Venant actually was the first to derive the Navier-Stokes equations correctly, but being neither the Lucasian Professor nor Sligo's greatest fluid dynamicist, he did not get appropriate credit...)

Whatever the history, if the water layer is appropriately shallow, the flow may be modelled as being almost horizontal and the (for simplicity) unidirectional fluid velocity u may be assumed to be independent of z. (Effectively, u is a depth-averaged quantity). This is actually quite a good model for river flows, or indeed leading order motions in the ocean, once the non-inertial frame effects of rotation, quantified through the Coriolis force, are taken into effect. For example, the ratio between the horizontal extent and depth of the

Pacific Ocean is rather similar to the ratio between the length or width of an A4 sheet of paper and its thickness, and so its entirely appropriate to think of an entire ocean basin as a shallow layer of fluid. (An even more "interesting" side-note about credit is that Saint-Venant succeeded both Coriolis and Poncelet at various stages of his career, both of whom are listed on the Eiffel Tower, while Saint-Venant is not. On the other hand, 72 men and zero women are listed, so it wasn't exactly a fair and transparent process...)

Once again whatever the history, the unidirectional flow of a shallow layer of fluid of constant density ρ can be modelled by considering the evolution of the layer depth $h(x,t)$ and the velocity $u(x,t)$. By consideration of an appropriate control volume, mass conservation is thus, pretty straightforwardly,

$$\frac{\partial}{\partial t}(\rho h)\frac{\partial}{\partial x}(\rho h u) = 0. \tag{24.1}$$

As vertical accelerations are negligible, the pressure distribution may be assumed to be (very close to) hydrostatic, and so, if (without loss of generality) we choose $p(h) = 0$,

$$p(z) = \rho g(h - z). \tag{24.2}$$

Therefore, within the (shallow) water layer, the (along stream) pressure force F is given by

$$F = \int_0^h p dz = \frac{1}{2}\rho g h^2, \tag{24.3}$$

Considering then a thin control "volume" $[x, x + \delta x]$, the local change of momentum must satisfy

$$\frac{\partial}{\partial t}(\rho h u \delta x) = \rho h u^2\big|_x - \rho h u^2\big|_{x+\delta x} + F(x) - F(x + \delta x),$$

and so, taking the appropriate limit

$$\frac{\partial}{\partial t}(\rho h u) + \frac{\partial}{\partial x}\left(\rho h u^2\right) = \frac{\partial}{\partial x}\left(-\frac{1}{2}\rho g h^2\right). \tag{24.4}$$

Dividing across by (the constant) density, applying the product rule, and applying (24.1) on the left hand side of (24.4) to eliminate some terms, mass and momentum conservation reduce to the coupled set of equations:

$$\begin{aligned} \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} &= 0\,, & (24.5)\\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} &= 0. & (24.6)\end{aligned}$$

These have **exactly** the same form as the equations (20.1) and (20.4) for 1D flow in a perfect (compressible) gas derived in Chapter 20:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0\,, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} &= 0,\end{aligned}$$

if we identify (h, u, g) in shallow water with $(\rho, u, c^2/\rho)$ in a perfect gas.

In particular, the "sound speed" in the shallow water equations is

$$c = \sqrt{g\rho},$$

rather than

$$c = c_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2},$$

and so a shallow water layer behaves like a (non-physical) perfect gas with $\gamma = 2$. We can thus use all the mathematical tools of characteristics, expansion fans etc as discussed in the preceding chapters to understand how such shallow layers of fluid can involve. In particular, shocks can form, due to the inevitable wave steepening implied by the advective term in (24.6). Such shocks are called *hydraulic jumps* or, specifically in rivers, *bores*: beautiful and important phenomena, which are nevertheless non-examinable.

24.2 * Non-examinable Hydraulic Jumps *

The analogy between perfect gases and shallow water is not after all perfect:

- across a (gas) shock ρ, p and u jump, so that mass, momentum and (total) energy are all conserved;
- across a hydraulic jump, only h and u can jump, and energy is not even vaguely conserved.

Consider a situation where:

- upstream of the jump $h = h_0$ and $u = u_0$;

- downstream of the jump $h = h_1$ and $u = u_1$;
- the shock is travelling at velocity V into the upstream, unperturbed fluid.

Therefore, in the frame where the jump is stationary, balancing $1/\rho$ times the mass flux and $1/\rho$ times the momentum flux either side of the jump, we obtain

$$\begin{aligned} h_1(u_1 - V) &= h_0(u_0 - V), && (24.7) \\ \frac{1}{2}gh_1^2 + h_1(u_1 - V)^2 &= \frac{1}{2}gh_0^2 + h_0(u_0 - V)^2, && (24.8) \end{aligned}$$

when we remember to include the force due the integrated hydrostatic pressure (24.3).

24.3 Example of Jump entering a Layer at Rest

There are now five quantities of interest (if we are still interested in something that is non-examinable): u_1, h_1, u_0, h_0 and V; and only two equations. Similarly to before, we are given the upstream conditions h_0 and u_0 (with $u_0 = 0$ for simplicity). We then need to make the further assumption that the *strength* of the jump β is known, i.e. $h_1 = (1 + \beta)h_0$ so that we can determine u_1 and V as follows.

Conservation of mass (flux) becomes:

$$h_1(u_1 - V) = -h_0 V.$$

Substituting this expression into (24.8) and collecting terms involving g and V:

$$\frac{1}{2}g(h_1^2 - h_0^2) = V^2\left(h_0 - \frac{h_0^2}{h_1^2}\right) = V^2\frac{h_0}{h_1}(h_1 - h_0).$$

Therefore, upon cancelling the common factor $h_1 - h_0 > 0$, we obtain

$$\begin{aligned} V^2 &= \frac{g}{2}(h_0 + h_1)\frac{h_1}{h_0}, && (24.9) \\ (V - u_1)^2 &= \frac{g}{2}(h_0 + h_1)\frac{h_0}{h_1}. && (24.10) \end{aligned}$$

Upstream flow is Supercritical

If $h_1 = (1+\beta)h_0$, where $\beta > 0$, then (24.9) implies

$$\frac{V^2}{gh_0} = \left(1 + \frac{\beta}{2}\right)(1+\beta) > 1, \tag{24.11}$$

and so the jump moves faster than the long wave speed into the upstream (unperturbed) flow. Such flow is called *supercritical* as the appropriate *Froude number* Fr, defined as

$$Fr \equiv \frac{V}{\sqrt{gh_0}} > 1. \tag{24.12}$$

Fr is the ratio of the flow speed to the (fastest) linear wave speed, and so is "clearly" analogous to the Mach number for compressible gas flow. (It is named after William Froude, an English engineer, who made several brilliant research break throughs despite being educated on the edge of the Cotswolds.)

Downstream flow is Subcritical

Conversely, when $h_1 = (1+\beta)h_0$, (24.10) implies

$$\frac{(V-u_1)^2}{gh_1} = \frac{\left(1+\frac{\beta}{2}\right)}{(1+\beta)^2} < 1, \tag{24.13}$$

and so the fluid behind (or downstream) of the jump has relative velocity less than the appropriate long wave speed $\sqrt{gh_1}$. Such a flow is called *subcritical* as

$$Fr \equiv \frac{|V-u_1|}{\sqrt{gh_1}} < 1. \tag{24.14}$$

Energy Loss across a Jump

The energy flux in such a shallow layer is the sum of the work done by the pressure, the transport of the kinetic energy, and the transport of the (gravitational) potential energy. i.e.

$$u\left(\frac{1}{2}\rho g h^2\right) + uh\left(\frac{1}{2}\rho u^2 + \frac{1}{2}\rho g h\right) = \rho u h\left[\frac{1}{2}u^2 + gh\right].$$

As mass is conserved, ρuh is continuous across the jump, and so the term in the square bracket is **not** conserved, as the steady Bernoulli equation does **not** apply.

Exercise

Show that the (positive) difference between the (upstream) energy flux *into* the jump and the (downstream) energy flux *out of* the jump is

$$\rho V h_0 \left(\frac{1}{2}V^2 + gh_0 - \left[\frac{1}{2}(V - u_1)^2 + gh_1 \right] \right) \quad = \quad \frac{\rho V h_0^2 g \beta^3}{4(1+\beta)}.$$

This quantity is not necessarily small, and there are two qualitatively different regimes worthy of (final) mention.

1. If $\beta \gtrsim 0.5$, a *turbulent bore* occurs, where the energy is lost to small scale motion and viscous dissipation.

2. If $\beta \lesssim 0.5$, typically an *undular bore* occurs, with finite amplitude waves on the downstream side carrying the energy away.

And now that the energy is being carried away, and we're comparing bores, it's a fine time to remember the heartfelt words of Mr Marc Almond:

Say hello, wave goodbye...