Waves

Part I: Sound waves

1. Motion of inviscid composite fluid

Consider a fluid: inviscid ($\nu = 0$), and compressible.

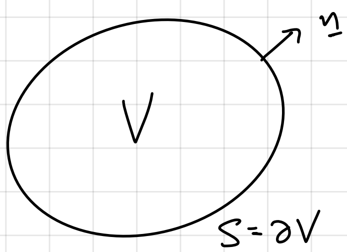
Now we have eqns for

$$\left.\begin{array}{ll} \underline{u}(\underline{x}, t) & 3 \\ p(\underline{x}, t) & 1 \\ \rho(\underline{x}, t) & 1 \end{array}\right\} \text{5 eqns.}$$

1.1 Conservation of mass

Consider a fixed volume V, surface S, outward normal $\underline{n}$.

Mass in V is



$$m = \int_V \rho \, dV$$

Mass per unit time crossing S into V is

$$-\int_S \rho \underline{u} \cdot \underline{n} \, dS.$$

So

$$\frac{d}{dt} \int_V \rho \, dV = -\int_S \rho \underline{u} \cdot \underline{n} \, dS.$$

(mass cannot be created or destroyed)

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \nabla \cdot (\rho \underline{u}) \, dV$$

But V arbitrary, so

$\rho \underline{u}$: mass flux.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

$$\Rightarrow \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho = -\rho (\nabla \cdot \underline{u})$$

If $\rho = \text{const} \Rightarrow \nabla \cdot \underline{u} = 0$

There is always a need to determine BCs.

At a stationary rigid boundary,

$$\underline{u} \cdot \underline{n} = 0 \qquad \text{(kinematic BC)}$$

At a moving rigid boundary.

$$\underline{u} \cdot \underline{n} = \underline{U}_b \cdot \underline{n}.$$

Fluid is inviscous $\Rightarrow$ no conditions on $\underline{u} \wedge \underline{n}$.

Incompressible flow is a special case $\frac{D\rho}{Dt} = 0 \Rightarrow \nabla \cdot \underline{u} = 0$.

1.2 Momentum Conservation

Momentum in V is

$$\int_V \rho \underline{u} \, dV$$

Change with time due to:

(i) flux (strictly transport) into V

$$-\int_S \rho \underline{u} (\underline{u} \cdot \underline{n}) \, dS$$

(ii) Surface forces on S

$$-\int_S p \underline{n} \, dS.$$

inviscid so tangential stress has no effect.

(iii) Any body forces in V

$$\int_V \underline{F} \, dV \quad \left(\text{or } \int_V \rho \underline{f} \, dV \right)$$

$$\Rightarrow \frac{d}{dt} \int_V \rho \underline{u} \, dV = -\int_S \rho \underline{u} (\underline{u} \cdot \underline{n}) \, dS - \int_S p \underline{n} \, dS + \int_V \underline{F} \, dV.$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho \underline{u}) + \nabla \cdot (\rho \underline{u} \underline{u}) = -\nabla p + \underline{F}$$

Apply conservation of mass

$$\Rightarrow \rho \frac{D\underline{u}}{Dt} = -\nabla p + \underline{F} \qquad \text{(Euler's eqn)}$$

BCs: At a (stress) free boundary, $p = 0$ (or $p = p_a$)

At interface between 2 fluids,

$$[p]_-^+ = \begin{cases} 0 & \text{if there is no surface tension.} \\ T\kappa & \text{if there is surface tension.} \end{cases}$$

$$[T] = \frac{\text{Force}}{\text{length}} = \frac{\text{Energy}}{\text{Area}}$$

1.3 Equation of state

Assume $p = p(\rho)$. In general, $p = p(\rho, T)$.

For a perfect gas,

$$p = \rho R T.$$

We would need another eqn for $T(\underline{x}, t)$.

We assume that the timescale of motion is much less than the timescale of thermal diffusion, so that the "heat content" (entropy) does not change.

Entropy

The classical interpretation

S: Heat content divided by instantaneous temp.

(unit: $J K^{-1}$, $kg\ m\ s^{-2} K^{-1}$)

Isentropic motion: entropy does not change, ie. reversible, adiabatic (no transfer of heat or mass) motions.

$$\frac{DS}{Dt} = 0.$$

A further special case: homentropic motion, ie. S const everywhere and for such motion.

$$p(\rho, S) = p(\rho)$$

Homentropic : the req'd approximation is line for sound $f \sim 1\,\mathrm{kHz}$, period $\sim 10^{-3}\,\mathrm{s}$, $\lambda = 30\,\mathrm{Dm}$,

$$t_{\mathrm{Diff}} = \frac{\lambda^2}{\kappa} \approx 10^4\,\mathrm{s}$$

diffusivity

For homentropic motion of a perfect gas,

$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}.$$

where ρ_0, p_0 are reference values.

$$\gamma = \frac{c_p}{c_v} = \frac{\text{s.h.c. at constant } p}{\text{s.h.c at constant } V}.$$

$$= \begin{cases} \frac{5}{3} & \text{for monoatomic gases} \\ \frac{7}{3} & \text{diatomic} \end{cases}$$

1.4 Energy equation for a reversible adiabatic motion

$$de = -p\, dV = \frac{p}{\rho^2} d\rho.$$

Since $V = \frac{1}{\rho}$ specific volume.

e internal energy per unit mass.

For a perfect gas,

$$\left(\frac{\partial e}{\partial \rho}\right)_{\substack{\text{reversible} \\ \text{adiabatic}}} = \frac{p}{\rho^2} = \frac{p_0}{\rho^2}\left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\Rightarrow \quad e = \frac{1}{\gamma - 1}\frac{p}{\rho} + \text{const. (chosen to be 0)}.$$

Also for Part V, enthalpy is

$$H := e + pV = e + \frac{p}{\rho} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}.$$

2. Linear Sound Waves

2.1 Linear wave eqn

Assume $\underline{F} = 0$. Consider a base (rest) state:

$$\underline{u} = 0, \quad p = p_0, \quad \rho = \rho_0 \quad \text{(spatially + temporally const.)}$$

Consider a small perturbation

$$\underline{u} = 0 + \underline{u}(\underline{x}, t)$$

$$p = p_0 + \tilde{p}(\underline{x}, t) \qquad , \ |\tilde{p}| \ll |p_0| \quad \text{smooth}$$

$$\rho = \rho_0 + \tilde{\rho}(\underline{x}, t) \qquad , \ |\tilde{\rho}| \ll \rho_0$$

$S = S_0$ const.

Since we consider homentropic flow $p(\rho, S) = p(\rho)$

$$[p] = \frac{\text{Force}}{\text{Area}} = ML^{-1}T^{-2}. \qquad [\rho] = ML^{-3}.$$

$$\Rightarrow \quad [p/\rho] = L^2 T^{-2}$$

$$\left.\frac{\partial p}{\partial \rho}\right|_{\rho_0} = c_0^2 . \Rightarrow \quad \tilde{p} = c_0^2 \tilde{\rho} .$$

from Taylor expansion about p_0 and ρ_0.

Conservation of mass and then linearise $\quad \frac{\partial \rho}{\partial t} + \underline{u} \nabla \rho = -\rho \nabla \cdot \underline{u}$

if smooth enough.

$$\frac{\partial}{\partial t}(\cancel{\rho_0} + \tilde{\rho}) - \underline{u} \cdot \nabla(\cancel{\rho_0} + \cancel{\tilde{\rho}}) = -(\rho_0 + \cancel{\tilde{\rho}}) \nabla \cdot \underline{u}$$

$$\Rightarrow \quad \frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \underline{u} = 0$$

Conservation of mom.

quad.

$$(\rho_0 + \cancel{\tilde{\rho}}) \frac{\partial \underline{u}}{\partial t} + \cancel{(\rho_0 + \tilde{\rho}) \underline{u} \cdot \nabla \underline{u}} = -\nabla(\cancel{p_0} + \tilde{p})$$

$$\Rightarrow \rho_0 \frac{\partial \underline{u}}{\partial t} = -\nabla \tilde{p} = -c_0^2 \nabla \tilde{\rho} .$$

$$\Rightarrow -\rho_0 \frac{\partial}{\partial t}(\nabla \cdot \underline{u}) = \frac{\partial^2 \tilde{\rho}}{\partial t^2} = c_0^2 \nabla^2 \tilde{\rho}$$

Also,
$$\frac{\partial^2 \tilde{p}}{\partial t^2} = c_0^2 \nabla^2 \tilde{p} ,$$

where $c_0^2 = \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0}$.

For adiabatic motion in a perfect gas, $p = p_0 \left(\frac{\rho}{\rho_0}\right)^\gamma$.

$$\Rightarrow c_0^2 = \frac{\gamma p_0}{\rho_0} .$$

- air: $c_0^2 \approx 340 \text{ m s}^{-1}$
- water: $c_0^2 \approx 1500 \text{ m s}^{-1}$.

2.2 The acoustic velocity potential ϕ

Consider the curl of the Euler eqn.

$$\nabla \wedge \left(\rho_0 \frac{\partial \underline{u}}{\partial t}\right) = \rho_0 \frac{\partial}{\partial t}(\nabla \wedge \underline{u}) = -\nabla \wedge (\nabla \tilde{p}) = \underline{0}$$

$$\Rightarrow \frac{\partial}{\partial t}(\nabla \wedge \underline{u}) = 0$$

So if $\nabla \wedge \underline{u} = 0$ at $t=0$. $\nabla \wedge \underline{u} = 0 \ \forall t$. $\Rightarrow \boxed{\underline{u} = \nabla \phi.}$

The velocity remains irrotational $\Rightarrow$ velocity potential

From
$$\rho_0 \frac{\partial}{\partial t} \nabla \phi = -\nabla \tilde{p} \Rightarrow \rho_0 \frac{\partial \phi}{\partial t} = -\tilde{p} + \beta(t).$$

but $\phi \to \phi - \int^t \beta \, dt$ doesn't change $\nabla \phi$, so WLOG $\beta = 0$.

$$\Rightarrow \frac{\partial \phi}{\partial t} = -\frac{\tilde{p}}{\rho_0} = -\frac{c_0^2 \tilde{\rho}}{\rho_0}$$

$$\Rightarrow \boxed{\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi.}$$

$$\tilde{p} = \rho_0 \frac{\partial \phi}{\partial t}, \quad \underline{u} = \nabla \phi, \quad \tilde{\rho} = -\frac{\rho_0}{c_0^2} \frac{\partial \phi}{\partial t}.$$

<u>Note</u> eqns for $\tilde{p}$, $\tilde{\rho}$, $\underline{u}$, ϕ are linear, so sol^n can be superposed.
ϕ_1, ϕ_2 sol^n, then $\lambda \phi_1 + \mu \phi_2$ for scaler λ, μ is also a sol^n.

When we linearised, kept $\rho_0 \frac{\partial \underline{u}}{\partial t}$, threw away $\rho_0 \underline{u} \cdot \nabla \underline{u}$,

$|\underline{u}| \sim U$ varies over some length L, $[\nabla] \sim 1/L$, and from wave eqn,

$\frac{1}{L} \sim \frac{1}{c_0} \frac{\partial}{\partial t}$, so

$$\frac{|\rho_0 \underline{u} \cdot \nabla \underline{u}|}{|\rho_0 \partial \underline{u}/\partial t|} \sim \frac{U^2/L}{c_0 U/L} \sim \frac{U}{c_0} \ll 1,$$

ie. fluid speed is much smaller than c_0.

$$\frac{U}{c_0} = M$$

is the Mach number.

NB In steady flow, from Bernoulli, $p \sim \frac{1}{2} \rho U^2$.

$\Rightarrow$ density variation scales like $\frac{\rho_0 U^2}{c_0^2}$, i.e. $|\tilde{\rho}| \ll \rho_0$.

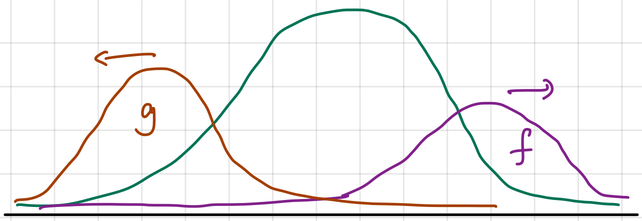
2.3 Plane Waves

Recall $\phi(\underline{x}, t) = \phi(x, t)$, i.e. $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial z} = 0$.

$$\Rightarrow \quad \frac{\partial^2 \phi}{\partial t^2} = c_0^2 \frac{\partial^2 \phi}{\partial x^2}.$$

has general sol^n

$$\phi = f(x - c_0 t) + g(x - c_0 t).$$



More generally, if $\hat{\underline{k}}$ is a unit vector, then

$$\phi = f(\hat{\underline{k}} \cdot \underline{x} - c_0 t)$$

satisfies the wave eqn. This is a plane wave.

It propagates in $\hat{\underline{k}}$ direction at speed c_0 independent of f.

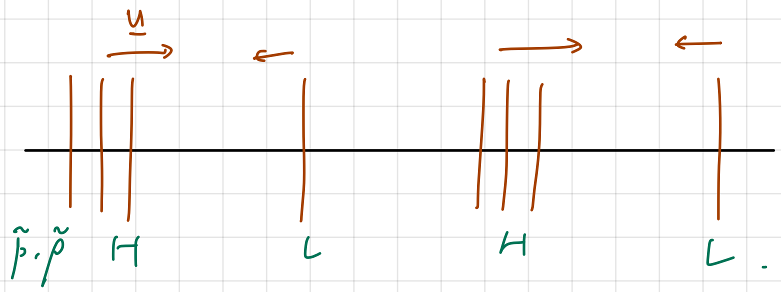
Note $\underline{u} = \nabla\phi = \hat{\underline{k}} f'$. $\Rightarrow$ $\underline{u} \parallel \hat{\underline{k}}$

$$\tilde{p} = -\rho_0 \frac{\partial\phi}{\partial t} = \rho_0 c_0 f' = \rho_0 c_0 \, \underline{u}\cdot\hat{\underline{k}}.$$

and so pressure and velocity fluctuations are in phase.

$\tilde{p}/|\underline{u}|$ is the <u>impedance</u> (here it is $\rho_0 c_0$).

These waves are called <u>longitudinal</u> (unlike a wave in a string)



2.4 Harmonic Plane Waves

The special case $f = A\exp(i(\underline{k}\cdot\underline{x} - \omega t))$. (real part understood).

is a <u>harmonic</u> plane waves.

- A : complex amplitude
- ω : frequency
- $\underline{k}$: wavevector
- $|\underline{k}|$: wavenumber.
- $\underline{k}\cdot\underline{x} - \omega t$: phase
- $c = \omega/|\underline{k}|$: phase speed

- $T = \dfrac{2\pi}{\omega}$: period
- $\nu = \dfrac{2\pi}{|\underline{k}|}$: wavelength

For the plane wave to satisfy the wave eqn, $\omega, \underline{k}$ must satisfy the <u>dispersion relation</u>

$$\omega^2 = c_0^2 \, \underline{k}\cdot\underline{k}.$$

$$\omega = \pm c_0 |\underline{k}|.$$

For acoustic wave in a stationary medium,

(1) The dispersion is <u>isotropic</u>. ω depends only on $|\underline{k}|$, not the direction

(2) The waves are <u>non-dispersive</u>. The phase speed $c = \frac{\omega}{|\underline{k}|} = c_0$ is indpt. of freq (or equivalently wavelength / wavenumber from dispersion relation).

(3) By Fourier decomposition in $\underline{k}$, <u>any</u> disturbance can be written as the sum of harmonic plane waves

(4) Two roots / eqn has ∂_t^2 $\Leftrightarrow$ IVP needs two ICs, e.g. ϕ, $\partial_t \phi$ as ICs.

<u>3. Energetics</u>

<u>3.1 Energy density and energy flux</u>

Total energy density of the system (per unit volume) is

$$E = \rho \Big[\underbrace{\tfrac{1}{2}|\underline{u}|^2}_{K \text{ (KE density)}} + \underbrace{\int^{\rho} \frac{p(\hat{\rho}, s)}{\hat{\rho}^2}\, d\hat{\rho}}_{W \text{ (internal energy density due to pressure)}} \Big]$$

$$\begin{aligned}
\frac{\partial E}{\partial t} + \nabla \cdot (\underline{u}E + p\underline{u}) &= \rho \underline{u} \cdot \frac{\partial \underline{u}}{\partial t} + \frac{p}{\rho}\frac{\partial \rho}{\partial t} + \cancel{\frac{E}{\rho}\frac{\partial \rho}{\partial t}} && \Big\}\ \frac{\partial E}{\partial t} \\
&\quad - \cancel{\frac{E}{\rho}\frac{\partial \rho}{\partial t}} - \cancel{E\,\frac{\underline{u}}{\rho} \cdot \nabla \rho} && \Big\}\ E(\nabla \cdot \underline{u}) \\
&\quad + \rho \underline{u} \cdot \nabla\left(\tfrac{1}{2}\rho|\underline{u}|^2\right) + \frac{p}{\rho}\underline{u} \cdot \nabla \rho + \cancel{\frac{E}{\rho}\underline{u} \cdot \nabla \rho} && \Big|\ \underline{u} \cdot \nabla E \\
&\quad + \underline{u} \cdot \nabla p + p\, \nabla \cdot \underline{u} && \Big|\ \nabla \cdot (p\underline{u}).
\end{aligned}$$

$$= \underbrace{\underline{u} \cdot \left(\rho \frac{\partial \underline{u}}{\partial t} + \rho \underline{u} \nabla \underline{u} + \nabla p\right)}_{\substack{\text{conservation of mom.} \\ \to 0}} + p \underbrace{\left(\nabla \cdot \underline{u} + \frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{\underline{u}}{\rho} \cdot \nabla \rho\right)}_{\substack{\text{conservation of mass} \\ \to 0}}$$

$$= 0$$

Therefore, in a fixed vol. V, exact result:

$$\underbrace{\frac{d}{dt}\int_V E \, dV}_{\substack{\text{change in energy} \\ \text{in } V}} = \underbrace{-\int_S E \underline{u} \cdot \underline{n} \, dS}_{\substack{\text{advection of} \\ \text{energy into } V}} \underbrace{- \int_S p \, \underline{u} \cdot \underline{n} \, dS}_{\substack{\text{rate of working} \\ \text{of pressure on} \\ \text{the surface of } V.}}$$

3.2 Acoustic energy eqn for linear sound waves

For such waves, the energy eqn is more straightforward to derive. because we can use the linearised mass and mom. eqn.

Remember

$$c_0^2 \tilde{\rho} = \tilde{p} \quad \Rightarrow \quad \frac{c_0^2 \tilde{\rho}^2}{\rho_0} = \frac{\tilde{p}^2}{c_0^2 \rho_0}$$

and

$$\underline{u} \cdot \underbrace{\left(\rho_0 \frac{\partial \underline{u}}{\partial t} + \nabla \tilde{p}\right)}_{\substack{\text{linear conservation} \\ \text{of mom}}} + \frac{c_0^2 \tilde{\rho}}{\rho_0} \underbrace{\left(\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \underline{u}\right)}_{\substack{\text{linear conservation} \\ \text{of mass}}} = 0.$$

$$\Rightarrow \frac{\partial}{\partial t}\Big(\underbrace{\tfrac{1}{2}\rho|\underline{u}|^2}_{\text{KE}} + \underbrace{\tfrac{1}{2}\frac{c_0^2 \tilde{\rho}^2}{\rho_0}}_{\substack{\text{W (PE due to} \\ \text{compression)}}}\Big) + \nabla \cdot \underbrace{(\tilde{p}\,\underline{u})}_{\substack{\text{acoustic wave} \\ \text{energy flux,} \\ \text{or acoustic intensity } \underline{I}}} = 0.$$

<u>Note</u>: $\underline{u} \cdot \nabla E$ is third order

$$\Rightarrow \frac{\partial}{\partial t}(\underbrace{K+W}_{\text{Acoustic energy density}}) + \nabla\cdot \underline{I} = 0$$

Comments:

(1) $\underline{I} = \tilde{p}\,\underline{u}$ has units Wm^{-2}. (power per unit area)

- Loudness in decibels is defined

$$120 + 10\log_{10}|\underline{I}|$$

$\underline{I} = 10^{-12}\ Wm^{-2}$ (0 dB) : limit of hearing

$\underline{I} = 1\ Wm^{-2}$ (120 dB) : threshold of pain.

Acoustic intensity of voice in conversation $\sim 10^{-5}\,W$.

- depend on distance like $1/r^2$

(2) Remember $\underline{u} = \nabla\phi$, $\tilde{p} = -\rho_0 \frac{\partial\phi}{\partial t}$, $\tilde{\rho} = -\frac{\rho_0}{c_0^2}\frac{\partial\phi}{\partial t}$ in terms of ϕ.

$$K = \tfrac{1}{2}\rho_0|\nabla\phi|^2, \quad W = \frac{1}{2}\frac{\rho_0}{c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2, \quad \underline{I} = -\rho_0\frac{\partial\phi}{\partial t}\nabla\phi.$$

(3) For any plane wave, $\phi = f(\hat{\underline{k}}\cdot\underline{x} - c_0 t)$

$$\Rightarrow \underline{u} = \nabla\phi = \hat{\underline{k}}f', \quad c_0^2\tilde{\rho} = \tilde{p} = -\rho_0\frac{\partial\phi}{\partial t} = \rho_0 c_0 f' = \rho_0 c_0(\underline{u}\cdot\hat{\underline{k}})$$

$$\Rightarrow K = \tfrac{1}{2}\rho_0|\underline{u}|^2 = \tfrac{1}{2}\rho_0 f'^2 = \frac{1}{2}\frac{c_0^2\tilde{\rho}^2}{\rho_0} = W.$$

Instantaneously, there is equipartion of energy between KE and PE.

$$\underline{I} = \tilde{p}\,\underline{u} = \rho_0 c_0 (f')^2\hat{\underline{k}} = (K+W)\,c_0\hat{\underline{k}}$$

ie. acoustic energy is transported in the direction of the wave at velocity $\underline{c}_g = c_0\hat{\underline{k}}$.

The group velocity is the velocity at which energy propagates.

Note $|c_g| \neq c_0$ for all kinds of waves; i.e. group and phase velocity are not always the same.

(4) For Harmonic plane waves, $f = A\exp(i\underline{k}\cdot\underline{x} - \omega t)$

* Real part must be taken before calculating quadratic quantities

Define a time avg. over one period of the wave $T = \frac{2\pi}{\omega}$.

$$\langle \cdot \rangle := \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \cdot \, dt$$

Note $\langle \cos^2\omega t\rangle = \langle \sin^2\omega t\rangle = \frac{1}{2}$, $\langle \sin\omega t \cos\omega t\rangle = 0$

$$\langle K\rangle = \frac{1}{4}\rho_0 |A|^2 k^2.$$

Consider $Ae^{i\omega t}$, $Be^{i\omega t}$,

$$A = A_r + iA_i, \qquad B = B_r + iB_i$$

$$\langle \underbrace{(A_r\cos\omega t - A_i\sin\omega t)}_{Re(Ae^{i\omega t})}\underbrace{(B_r\cos\omega t - B_i\sin\omega t)}_{Re(Be^{i\omega t})}\rangle$$

$$= \frac{1}{2}A_rB_r + \frac{1}{2}A_iB_i$$

$$= Re\left(\frac{1}{2}(A_r + iA_i)(A_i - iB_i)\right),$$

i.e.

$$\langle Re(Ae^{i\omega t})\, Re(Be^{i\omega t})\rangle = \frac{1}{2}Re(AB^*).$$

Similarly,

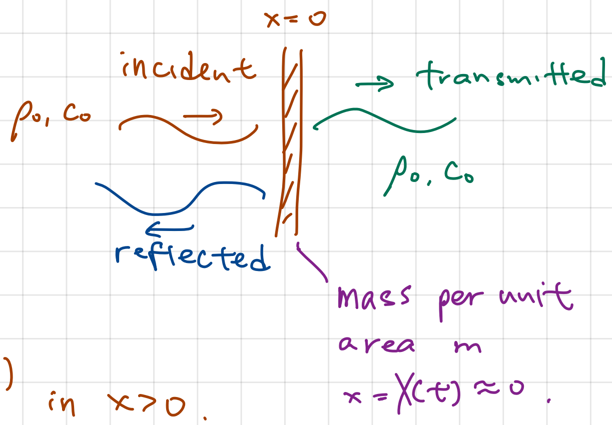
$$\langle W\rangle = \frac{1}{4}\rho_0 k^2|A|^2.$$

(5) Derivation of acoustic energy eqn from the <u>full</u> energy eqn of 3.1 is tricky because internal energy has to be calculated to 2^{nd} order, and in particular, eqm energy needs to be shown to be conserved.

4. Examples and applications : transmission + sph. sym.

4.1 Transmission

<u>Example</u> Sound bouncing off a "thin" wall ($\ll \lambda$) between two rooms. WLOG normal incidence



In <u>$x<0$</u> :

$$\tilde{p} = A e^{i\omega(t - x/c_0)} + R e^{i\omega(t + x/c_0)}$$

<u>$x>0$</u> :

$$\tilde{p} = T e^{i\omega(t - x/c_0)}$$

We assume no waves $\propto e^{i\omega(t + x/c_0)}$ in $x>0$.

This is a 'radiation condition' due to causality.

"A" amplitude of the incoming wave is given, and exponentials are in a convenient form for the BVP. Note

$$\tilde{p} = -\rho_0 \frac{\partial \phi}{\partial t}, \quad \underline{u} = \nabla \phi .$$

For example in $x>0$,

$$\phi = -\frac{1}{i\omega\rho_0} T e^{i\omega(t - x/c_0)}$$

and

$$u = \frac{1}{\rho_0 c_0} \left[A e^{i\omega(t - x/c_0)} - R e^{i\omega(t + x/c_0)} \right] \qquad x<0$$

$$u = \frac{1}{\rho_0 c_0} T e^{i\omega(t - x/c_0)} \qquad x>0.$$

Kinematic b.c. : $\dot{X}(t) = u(x_+, t) = u(x_-, t)$.

$$\Rightarrow \boxed{u(0_+, t) = u(0_-, t)} \quad \text{by linearisation}$$

Dynamic b.c. : $m\ddot{X} \simeq \tilde{p}(0_-,t) - \tilde{p}(0_+,t) = -[\tilde{p}]_-^+$

Assume $X(t) = X_0 e^{i\omega t}$. Then

$$i\omega \rho_0 c_0 X_0 = A - R = T \qquad \text{(KBC)}$$

$$-m\omega^2 X_0 = A + R - T \qquad \text{(DBC)}$$

$$\Rightarrow \quad R = \frac{A}{1-2i\alpha}, \qquad T = \frac{-2i\alpha A}{1-2i\alpha}.$$

where $\alpha = \dfrac{\rho_0 c_0}{\omega m}$, $= \dfrac{\rho_0/k}{m} \sim \dfrac{\text{mass of gas per } \lambda}{\text{mass of wall}}$.

Check $m = 0 \Rightarrow \alpha \to \infty$, $R \to 0$, $T \to A$

$m \to \infty \Rightarrow \alpha \to 0$, $R \to A$, $T \to 0$.

low freq (small α) $\Rightarrow$ large α $\Rightarrow$ good transmission

high freq (large α) $\Rightarrow$ small α $\Rightarrow$ weak transmission

Exercise Show that

$$\langle \tilde{p} u \rangle = \begin{cases} \dfrac{1}{2\rho_0 c_0}\left(|A|^2 - |R|^2\right) & \text{in } x<0 \\ \dfrac{1}{2\rho_0 c_0}|T|^2 & \text{in } x>0 \end{cases}$$

Note :

- "$-$" sign is because reflected waves send energy to the left
- Energy flux indpt of x.

4.2 Spherically Symmetric Waves

In spherical polar coordinates,

$$\nabla^2 \tilde{p} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\tilde{p})$$

Then

$$\frac{\partial^2}{\partial t^2}\tilde{p} = c_0^2 \nabla^2 \tilde{p} \Rightarrow \frac{\partial^2}{\partial t^2}(r\tilde{p}) = c_0^2 \frac{\partial^2}{\partial r^2}(r\tilde{p})$$

1D wave eqn for $r\tilde{p}$

$$\Rightarrow \quad \tilde{p} = \underbrace{\frac{1}{r} f(r - c_0 t)}_{\text{outgoing}} + \underbrace{\frac{1}{r} g(r + c_0 t)}_{\text{incoming}}$$

There is <u>often</u> a radiation condition requiring outgoing waves

$$\Rightarrow \quad g = 0.$$

Conventionally,

$$\tilde{p} = \frac{\dot{q}(t - r/c_0)}{4\pi r} \quad \Leftrightarrow \quad \tilde{\rho} = \frac{\dot{q}(t - r/c_0)}{4\pi c_0^2 r}$$

$$\Rightarrow \quad \phi = -\frac{q(t - r/c_0)}{4\pi \rho_0 r}$$

Recall $\underline{u} = \nabla\phi = \frac{\partial \phi}{\partial r}\hat{\underline{r}}$.

$$\underline{u} = \frac{\hat{\underline{r}}}{4\pi\rho_0}\left(\underbrace{\frac{\dot{q}(t - r/c_0)}{c_0 r}}_{\substack{\text{dominate for} \\ \text{large } r \text{ in} \\ \text{"far field"}}} + \underbrace{\frac{q(t - r/c_0)}{r^2}}_{\substack{\text{dominate for} \\ \text{small } r \text{ in} \\ \text{the near field.}}} \right)$$

r is large or small relative to wavelength.

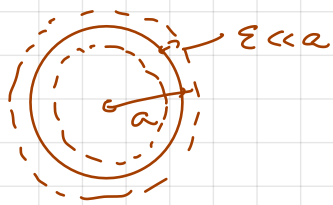
Mass transport out of a sphere of radius r is

$$4\pi r^2 \rho_0 (u_r(r,t)) = \frac{r}{c_0}\dot{q}(t - r/c_0) + q(t - r/c_0)$$

$r \to 0$, then mass transport tends to $q(t)$, so $q(t)$ is the mass flux from a point source at the origin.

4.3 Example: solution for a pulsating sphere.

Consider a sphere performing small amplitude radial oscillation on the surface of the sphere



$$r = a + \varepsilon e^{i\omega t}$$

The radial velocity

$$u = \frac{\partial \phi}{\partial r} = i\omega \varepsilon e^{i\omega t}$$

Taylor :

$$u(a+\varepsilon e^{i\omega t}, t) \approx \underbrace{u(a,t)}_{O(\varepsilon)} + \underbrace{\cancel{\varepsilon e^{i\omega t} \frac{\partial u}{\partial r}(a,t)}}_{O(\varepsilon^2)}$$

Apply B.C on $r=a$. Expect wave like sol^n and the forcing dependence

$$\phi = \frac{A e^{i\omega(t-(r-a)/c_0)}}{4\pi \rho_0 r}$$

Easy to apply B.C.

$$u = -\frac{A}{4\pi\rho_0}\left(\frac{1}{r^2} + \frac{i\omega}{c_0 r}\right) e^{i\omega(t-(r-a)/c_0)}$$

At $r=a$,

$$i\omega\varepsilon = -\frac{A}{4\pi\rho_0 a^2}\left(1 + \frac{i\omega a}{c_0}\right)$$

$$\Rightarrow A = -\frac{4\pi i \rho_0 a^2 \omega \varepsilon}{1 + \frac{i\omega a}{c_0}}$$

$$\tilde{p} = i\omega\rho_0 \phi = -\frac{\varepsilon a^2 \omega^2 \rho_0}{1 + \frac{i\omega a}{c_0}} \left(\frac{e^{i\omega(t-(r-a)/c_0)}}{r}\right)$$

Noting sound has wavelength $\lambda = \frac{2\pi c_0}{\omega}$.

Sphere is large if $a \gg \frac{2\pi c_0}{\omega}$, i.e. $\frac{\omega a}{c_0} = ka \gg 1$

Small if $a \ll \frac{2\pi c_0}{\omega}$, ie. $ka \ll 1$.

"Small" sphere are called <u>compact sources</u>. They are inefficient. (<u>Ex</u>. Show $\tilde{p}$ and $\tilde{u}$ are close to out of phase and so I is small $\Rightarrow$ can't propagate much energy into far field.).

Part II. Elastic waves (in solids)

5. Equations of linear elasticity

5.1 Deformation of solid

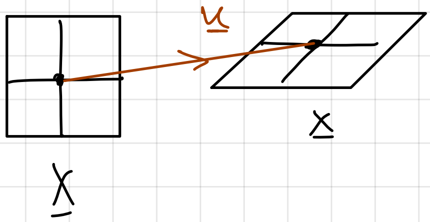
We make the continuum assumption average over volumes large enough to contain many molecules but are volumes smaller than the scale of interest. We can this define $\rho(\underline{x},t)$. As a body is deformed, a particle move from its reference position $\underline{X}$ to a new position $\underline{x}(t)$.

Elastic solids remember the original configuration $\underline{X}$.

<u>Conventional notation.</u>

Particle displacement $\underline{u}(\underline{x},t) = \underline{x}(t) - \underline{X}$

velocity $\underline{v}(\underline{x},t) = \frac{d\underline{u}}{dt}$



<u>N.B</u> In solid, $\quad \frac{d\underline{u}}{dt} = \left.\frac{\partial \underline{u}}{\partial t}\right|_{\underline{X}}$

liquid, $\quad \frac{D}{Dt} = \left.\frac{\partial}{\partial t}\right|_{\underline{x}} + \underline{v}\cdot\nabla$.

acceleration $\underline{a}(\underline{x},t) = \frac{d\underline{v}}{dt}$

It is still convenient to use Eulerian notation / position $\underline{x}$ not the initial position $\underline{X}$.

<u>Note</u> In solid, $\underline{u}$ is displacement not velocity.

5.2 The Stress Tensor

The forces acting on a material can be either

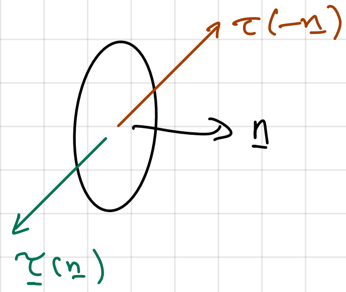
(1) <u>Body force</u> acts on and is proportional to volume,

$$\underline{F} \text{ or } \rho \underline{f}$$

(2) Surface forces (traction / stress) acting on and proportional to surfaces.

The force exerted by the outside on the solid inside of the element is assumed to be surface force $\underline{\tau}(\underline{x}, t, \underline{n})$

"Clearly" by N3, $\underline{\tau}(-\underline{n}) = -\underline{\tau}(\underline{n})$.



The traction (or stress) $\underline{\tau}$ acting on $\underline{n}\,dS$ depends on the orientation $\underline{n}$

Note $[\underline{\tau}]$ = Force / area

In an inviscid fluid, $\underline{\tau} = -p(\underline{n})$.

In general, $\underline{\tau}(\underline{x}, t, \underline{n})$ is linearly related to $\underline{n}$ via a second rank tensor $\underline{\sigma}$. Consider a small material tetrahedron with 3 faces alligned parallel to Cartesian coords with area ε^2 and normal $\underline{n}$.

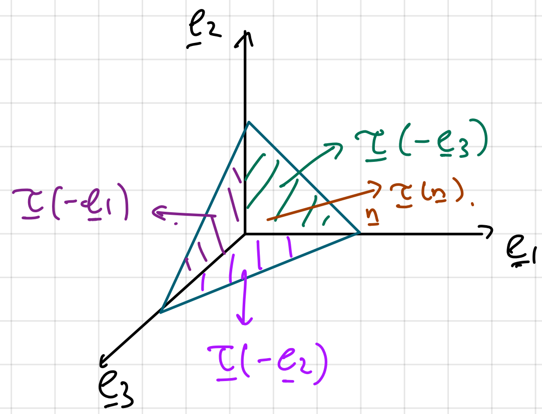
Top "sloping face" of area ε^2.

The other faces have area $\varepsilon^2 n_i$.

The surface forces are $O(\varepsilon^2)$.

Body forces and inertia $O(\varepsilon^3)$.

So surface forces must balance as $\varepsilon \to 0$.



$$\varepsilon^2\left(\underline{\tau}(\underline{n}) - n_1 \underline{\tau}_1(\underline{e}_1) - n_2 \underline{\tau}_2(\underline{e}_2) - n_3 \underline{\tau}_3(\underline{e}_3)\right) = O(\varepsilon^3).$$

$$\Rightarrow \tau_i(\underline{x}, t, \underline{n}) = \sigma_{ij} n_j .$$

where $\sigma_{ij}(\underline{x}, t) = \tau_i(\underline{x}, t, \underline{e}_j)$.

The traction $\underline{\tau}$ exerted by the outside on the inside of a surface with unit normal $\underline{n}$ is given by

$$\underset{\sim}{\tau} = \underset{=}{\sigma} \cdot \underset{\sim}{n} ,$$

$\underset{=}{\sigma}$ is the Cauchy stress tensor.

5.3 Conservation Laws

5.3.1 Momentum conservation

An arbitrary material volume $V(t)$, with surface S, outward normal $\underset{\sim}{n}$. Integrate $\underset{\sim}{F} = m\underset{\sim}{a}$ over all particles in V.

$$\int_V \rho(\underset{\sim}{x},t)\, \underset{\sim}{a}(\underset{\sim}{x},t)\, dV = \int_V \underset{\sim}{F}(\underset{\sim}{x},t)\, dV + \int_S \underset{=}{\sigma}(\underset{\sim}{x},t) \cdot \underset{\sim}{n}\, dS \qquad (**)$$

$\sigma_{ij} n_j$

Use div. thm. and arbitrariness of V, mom eqn

$$\boxed{\rho a_i = F_i + \frac{\partial}{\partial x_j}(\sigma_{ij})} \qquad (*)$$

Exercise Check this is consistent with mom. budget for a fixed control vol.

If there are no long-range forces which may impose body couples on the material (counter-ex: magnetic field), Conservation of angular momentum proves $\underset{=}{\sigma}$ symmetric.

5.3.2 Angula momentum

Take moment of $(**)$ about $\underset{\sim}{0}$.

$$\int_V \rho\, \underset{\sim}{x} \wedge \underset{\sim}{a}\, dV = \int_V \underset{\sim}{x} \wedge \underset{\sim}{F}\, dV + \int_S \underset{\sim}{x} \wedge \underset{=}{\sigma} \cdot \underset{\sim}{n}\, dS$$

p-th component:

$$\varepsilon_{pqi} x_q (\rho a_i - F_i) = \frac{\partial}{\partial x_j}\left(\varepsilon_{pqi} x_q \sigma_{ij}\right)$$

$$= \varepsilon_{pqi} x_q \frac{\partial \sigma_{ij}}{\partial x_j} + \varepsilon_{pji} \sigma_{ij}$$

Use $(*)$ for LHS

$$\varepsilon_{pqi} x_q \frac{\partial \sigma_{ij}}{\partial x_j} = \varepsilon_{pqi} x_q \frac{\partial \sigma_{ij}}{\partial x_j} + \varepsilon_{pji} \sigma_{ij}$$

$$\Rightarrow \varepsilon_{pji} \sigma_{ij} = 0$$

$$\Rightarrow \quad \sigma_{ij} = \sigma_{ji} \,,$$

ie. $\underline{\underline{\sigma}}$ symmetric.

5.3.3 Energy

As usual, contract mom. eqn (*) with v_i.

$$v_i a_i = \frac{d}{dt}\left(\frac{1}{2}|\underline{v}|^2\right) \quad . \quad \frac{d}{dt}(\rho V) = 0$$

RHS :

$$v_i \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(v_i \sigma_{ij}) - \sigma_{ij}\frac{\partial v_i}{\partial x_j}$$

$$= \frac{\partial}{\partial x_j}(v_i \sigma_{ij}) - \sigma_{ij}\left(\frac{1}{2}\frac{\partial v_i}{\partial x_j} + \frac{1}{2}\frac{\partial v_j}{\partial x_i}\right)$$

So change in KE:

rate of work by surface forces

$$\frac{d}{dt}\int_{V(t)} \frac{1}{2}\rho|\underline{v}|^2 dV = \int_V \underline{v}\cdot\underline{F}\, dV + \int_S \underline{v}\cdot\underline{\underline{\sigma}}\cdot\underline{n}\, dS$$

rate of working by body forces

$$- \int_V \sigma_{ij} \cdot \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dV$$

rate of release of internal energy

N.B. We will return to the 3rd term but remember for simple case of a gas of part I,

$$\sigma_{ij} = -p\,\delta_{ij}$$

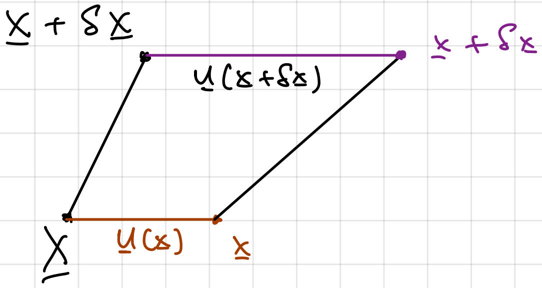
so this term is just $+\int_V p\,\nabla\cdot\underline{v}\, dV$ adiabatic decompression of a gas.

6. Stress

Stress: restoring force due to strain (deformation of material).

6.1 Strain and the infintestimal strain assumption

Elastic stresses result from the change in separation between material particles.



$$[\delta \underline{x} - \delta \underline{X}]_i = [\underline{u}(\underline{x}+\delta\underline{x}) - \underline{u}(\underline{x})]_i$$
$$= [(\delta\underline{x}\cdot\nabla)\underline{u}]_i + HOT$$
$$\approx \frac{\partial u_i}{\partial x_j}\,\delta x_j$$

From now on, assume deformation is small (infinitestimal strain),

i.e. $|\delta\underline{x} - \delta\underline{X}| \ll |\delta\underline{x}|$, so $\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$.

For hard solids (metal, rock) a small deformation produces a very large restoring force, e.g. $|D\underline{u}| \sim 10^{-5} - 10^{-2}$ (not true for rubber)

Write $$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{e_{ij}} + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$
$$= \frac{1}{2}e_{ij} + \frac{1}{2}\varepsilon_{ijk}\omega_k$$

where e_{ij}: strain tensor, $\underline{\omega} = \nabla \wedge \underline{u}$ rotation

For small deformation, the 2nd term only gives a rigid rotation, so

$$\delta x \simeq \delta\underline{X} + \frac{1}{2}\underline{\omega}\wedge\delta\underline{X}$$

and hence no elastic stresses.

Note Difference between solids and fluids:

- Solids: $\underline{u}, \underline{\omega}, \underline{\underline{e}}$ displacement, rotation, strain
- Fluids: $\underline{u}, \underline{\omega}, \dot{\underline{\underline{e}}}$: velocity, vorticity, rate of strain.

$\underline{\underline{e}}$ is the (Cauchy) strain tensor which gives the stress.

Note From def^n, $\underline{\underline{e}}$ symmetric

Exercise Show that with small deformation,

$$|\delta \underline{x}|^2 - |\delta \underline{X}|^2 \simeq 2\, \delta \underline{X} \cdot \underline{\underline{e}} \cdot \delta \underline{X}.$$

Consequence of infinitesimal strain

$|D\underline{u}| \ll 1 \Rightarrow$ (1) $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{du_j}{dt}\frac{\partial}{\partial x_j} \approx \frac{\partial}{\partial t}$ and we can neglect convective derivative

$$\Rightarrow V_i := \frac{du_i}{dt} \approx \frac{\partial u_i}{\partial t} \quad \text{and} \quad a_i := \frac{dV_i}{dt} \approx \frac{\partial^2 u_i}{\partial t}$$

(2) $\dfrac{\text{Volume change}}{\text{Initial volume}} \approx \dfrac{\int_S \underline{u}\cdot\underline{n}\, dS}{V} = \frac{1}{V}\int_V \nabla\cdot\underline{u}\, dV \approx |\nabla\cdot\underline{u}| \ll 1.$

(3) $\tilde{\rho} = -\rho\, \nabla\cdot\underline{u} \ll \rho \Rightarrow$ density changes can be ignored in the mom. eqn.

$$\theta := \nabla\cdot\underline{u} = \operatorname{Tr}\underline{\underline{e}}$$

is the dilitation (in general $\neq 0$)

$$\underline{\underline{e}} := \frac{1}{2}\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)$$

is the Cauchy strain tensor

6.2 Constitutive eqn for a linear elastic solid

This is the relationship between stress and strain.

For a linear elastic solid, we assume that the const. eqn is

(1) Local + instantaneous (ie. $\underline{\sigma}$ does not depend on $\nabla \underline{\underline{e}}$, $\frac{\partial \underline{\underline{e}}}{\partial t}$)

(2) <u>linear</u> : since $|\underline{\underline{e}}| \ll 1$,

$$\sigma_{ij}(x,t) = C_{ijkl}\, e_{kl}(x,t)$$

where $\underline{\underline{\underline{\underline{C}}}}$ is a property of material.

C_{ijkl} sym in i,j and k,l $\Rightarrow$ 36 independent parameters in a general anistropic material.

(3) We will further assume material is <u>uniform</u> and <u>isotropic</u>.

Then

$$C_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu\, \delta_{ik}\delta_{jl} + \mu'\, \delta_{il}\delta_{jk}. \quad \text{(isotropic)}$$

uniform $\Rightarrow$ λ, μ, μ' indpt. of $\underline{x}, t$.

Finally, symmetry of $\underline{\sigma}$ $\Rightarrow$ $\mu = \mu'$, hence

$$\boxed{\sigma_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij}}$$

where λ, μ are called the Lamé moduli.

Note
$$\sigma_{kk} = (3\lambda + 2\mu)\, e_{kk}.$$

$$\Rightarrow \boxed{e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\,\delta_{ij}\,\sigma_{kk}\right)}$$

6.3 Simple deformations

(a) Can define pressure

$$p := -\frac{1}{3}\sigma_{kk}.$$

In general,

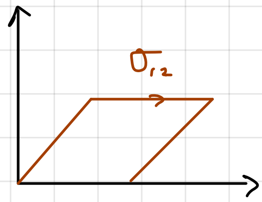
$$\sigma_{ij} = \underbrace{-p\,\delta_{ij}}_{\substack{\text{hydrostatic}\\ \text{stress}\\ \text{(changes volume)}}} + \underbrace{\bar{\sigma}_{ij}}_{\substack{\text{deviatoric stress}\\ \text{(changes shape)}}}$$

In "hydrostatic" situations ($\bar{\sigma}_{ij} = 0$),

$$p = -K\theta \quad ,$$

with $K = \lambda + \frac{2}{3}\mu$ the <u>bulk modulus</u> $(K > 0)$. This corresponds to compression for $p > 0$

(b) Simple shear $\quad \underline{u} = (\gamma y, 0, 0)$

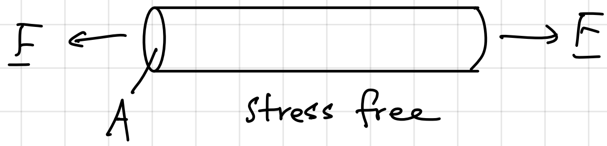


$$\underline{\underline{e}} = \gamma \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{\sigma}} = \mu\gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where μ is the <u>shear modulus</u>. $(\mu \geq 0)$.

In an elastic <u>inviscid</u> fluid, $\mu = 0$.

(c) Uniaxial extension



$$\underline{\underline{\sigma}} = \frac{F}{A} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

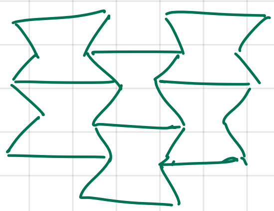
$$\Rightarrow \underline{\underline{e}} = \frac{F}{A} \frac{1}{2\mu(3\lambda + 2\mu)} \begin{pmatrix} 2(\lambda+\mu) & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = \frac{F}{EA} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{pmatrix}.$$

where $E = \dfrac{(3\lambda + 2\mu)\mu}{\lambda + \mu}$ is the Young's modulus,

$\nu = \dfrac{\lambda}{2(\lambda + \mu)}$ is Poisson's ratio.

Young's modulus is a measure of tensile stiffness / strength.

<u>Note</u> $\mu < 0$ in auxetic materials (e.g. open crystaline), i.e. materials that <u>expands</u> in the direction perpendicular to the load.



7. Wave equations for elastic solids

7.1 Equations

Consider $\underline{F} = \underline{0}$, or consider perturbations about static eqm.

Remember momentum eqn

$$\rho a_i = \frac{\partial \sigma_{ij}}{\partial x_j}, \quad \sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}, \quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\underline{\underline{\sigma}} = \lambda \nabla \underline{u}\, \underline{\underline{I}} + 2\mu \underline{\underline{e}}, \quad \underline{\underline{e}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right).$$

$$\Rightarrow \quad \rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} = (\lambda + \mu)\, \nabla(\nabla \cdot \underline{u}) + \mu \nabla^2 \underline{u} \qquad (A).$$

Recall for a general $\underline{q}$,

$$\nabla^2 \underline{q} = \nabla(\nabla \cdot \underline{q}) - \nabla \wedge (\nabla \wedge \underline{q}).$$

Then

$$\rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} = (\lambda + 2\mu)\, \nabla(\nabla \cdot \underline{u}) - \mu \nabla \wedge (\nabla \wedge \underline{u}). \qquad (B)$$

$\nabla \cdot (A)$, $\theta = \nabla \cdot \underline{u}$ $\Rightarrow$ $\boxed{\dfrac{\partial^2 \theta}{\partial t^2} = \dfrac{\lambda + 2\mu}{\rho_0} \nabla^2 \theta = c_p^2 \nabla^2 \theta.}$

Exercise $\nabla \cdot (B)$ shows same result.

Writing $\underline{\omega} = \nabla \wedge \underline{u}$. $\nabla \wedge (B)$

$$\Rightarrow \quad \rho_0 \frac{\partial^2}{\partial t^2} \underline{\omega} = -\mu \nabla \wedge (\nabla \wedge \underline{\omega})$$

Because $\nabla \wedge (\nabla(\nabla \cdot \underline{u})) = 0$.

Note that $(\nabla \wedge (\nabla \wedge \underline{\omega}))_i = \varepsilon_{ijk}\, \partial_j\, \varepsilon_{klm} \partial_l \omega_m$

$$= (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})\, \partial_j \partial_l \omega_m$$

$$= (\underbrace{\nabla(\nabla \cdot \underline{\omega})}_{=0})_i - (\nabla^2 \underline{\omega})_i = -(\nabla^2 \underline{\omega})_i$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 \underline{\omega}}{\partial t^2} = \frac{\mu}{\rho_0} \nabla^2 \underline{\omega} = c_s^2 \nabla^2 \underline{\omega}.}$$

Exercise Show with $\nabla \times (A)$.

7.2 Boundary Conditions

There at <u>least</u> 4 different natural choices of BCs.

(1) Rigid boundary $\underline{u} = \underline{0}$

(2) (Stress) free boundary $\underline{\tau} = \underline{\sigma} \cdot \underline{n} = \underline{0}$

(3) Solid - solid boundary :

- cts displacement $[\underline{u}]_-^+ = \underline{0}$ → 3 conditions
- cts stress / traction $[\underline{\sigma} \cdot \underline{n}]_-^+ = \underline{0}$ → 3 conditions

(4). Solid - <u>inviscid fluid</u>

- no tangential stress in fluid ($\mu_f = 0$) $\Rightarrow [\underline{u} \cdot \underline{n}]_-^+ = 0$ → 1 condition
- cty of normal stress / traction $\underline{n} \cdot \underline{\sigma} \cdot \underline{n} = -p$ ← 1 condition on normal stress.
- $\underline{n} \wedge \underline{\sigma} \cdot \underline{n} = \underline{0}$ ← 2 conditions

7.3 Energy

Assume $\underline{F} = \underline{0}$, $\underline{v} \simeq \frac{\partial \underline{u}}{\partial t} = \dot{\underline{u}}$

Remember from § 5.3,

$$\frac{d}{dt} \underbrace{\int_V \frac{1}{2} \rho_0 \dot{\underline{u}}^2 \, dV}_{K} = \int_S \dot{\underline{u}} \cdot \underline{\sigma} \cdot \underline{n} \, dS - \int_V \sigma_{ij} \dot{e}_{ij} \, dV$$

$$\sigma_{ij} \dot{e}_{ij} = (\lambda \delta_{ij} e_{kk} + 2\mu e_{ij}) \dot{e}_{ij}$$

$$= \lambda e_{kk} \dot{e}_{ll} + 2\mu e_{ij} \dot{e}_{ij} = \dot{\sigma}_{ij} e_{ij} = \frac{1}{2} \frac{\partial}{\partial t} (\sigma_{ij} e_{ij})$$

$$\sigma_{ij} e_{ij} = \frac{1}{2} \lambda e_{kk} e_{ll} + 2\mu e_{ij} e_{ij} = W \quad \left(\text{elastic potential energy (density)}\right).$$

Since volume arbitrary,

$$\frac{d}{dt} \int_V (K + W) = \int_S \dot{\underline{u}} \cdot \underline{\sigma} \cdot \underline{n} \, dS$$

$$\Rightarrow \frac{\partial}{\partial t}(K+W) + \nabla \cdot \underline{I} = 0$$

energy flux, not identity matrix

where $K = \frac{1}{2}\rho_0 |\underline{\dot{u}}|^2$. (KE)

$W = \frac{1}{2}\lambda\, e_{kk} e_{ll} + 2\mu\, e_{ij} e_{ij} = \frac{1}{2}\sigma_{ij} e_{ij}$ (elastic PE)

$\underline{I} = -\underline{\dot{u}} \cdot \underline{\underline{\sigma}}$ (energy flux vector)

Exercise let $e'_{ij} = e_{ij} - \frac{1}{3} e_{kk} \delta_{ij}$. Show $W \geq 0$ iff

$$\kappa = \lambda + \frac{2}{3}\mu \geq 0 \quad \text{and} \quad \mu \geq 0.$$

(Remember $p = -\kappa\theta$, $\theta = \nabla \cdot \underline{u}$).

7.4 Elastic Waves

2 different classes

$$\frac{\partial^2 \theta}{\partial t^2} = c_p^2 \nabla^2 \theta \,. \qquad c_p = \left(\frac{\lambda + 2\mu}{\rho_0}\right)^{1/2}, \quad \theta = \nabla \cdot \underline{u}$$

Primary waves $c_p > c_s$. Also called pressure / compressional waves.

$$\frac{\partial^2 \underline{\omega}}{\partial t^2} = c_s^2 \nabla^2 \underline{\omega} \,, \qquad c_s = \left(\frac{\mu}{\rho_0}\right)^{1/2}, \qquad \underline{\omega} = \nabla \wedge \underline{u}$$

Secondary waves / Sheer waves

- P-wave speed > S-wave speed
- In an inviscid fluid, $\mu = 0 \Rightarrow$ S-wave don't exist
- P-waves are sound waves

$$\tilde{p} = -\lambda \nabla \cdot \underline{u} \,.$$

In mantle, $\lambda = 450$ GPa, $\mu = 250$ GPa, $\rho \simeq 5 \times 10^3$ kg m^{-3}

$$\Rightarrow c_p \sim 13 \text{ km s}^{-1}, \quad c_s \sim 7 \text{ km s}^{-1}.$$

7.5 Plane Waves

Look for sol^n $\underline{u} = \underline{f}(\hat{\underline{k}} \cdot \underline{x} - ct)$, $\hat{\underline{k}}$ unit vector $\parallel \underline{k}$.

Substitute into (B)

$$\frac{\partial^2 \underline{u}}{\partial t^2} = \left(\overbrace{\frac{\lambda+2\mu}{\rho_0}}^{c_p^2}\right)\nabla(\nabla\cdot\underline{u}) - \overbrace{\frac{\mu}{\rho_0}}^{c_s^2}\nabla\wedge(\nabla\wedge\underline{u})$$

$$\Rightarrow c^2\underline{f}'' = c_p^2\,\hat{\underline{k}}\,\hat{\underline{k}}\cdot\underline{f}'' - c_s^2\,\hat{\underline{k}}\wedge(\hat{\underline{k}}\wedge\underline{f}'') \qquad (*).$$

Sub into (A).

$$\rho_0\frac{\partial^2 \underline{u}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\underline{u}) + \mu\nabla^2\underline{u}$$

$$\Rightarrow \rho_0 c^2\underline{f}'' = (\lambda+\mu)\,\hat{\underline{k}}\,\hat{\underline{k}}\cdot\underline{f}'' + \rho_0 c_s^2\,\underline{f}''. \qquad (**).$$

$\hat{\underline{k}}\cdot(*) \Rightarrow \quad c^2\hat{\underline{k}}\cdot\underline{f}'' = c_p^2\,\hat{\underline{k}}\cdot\underline{f}'' \qquad (I)$

$\hat{\underline{k}}\wedge(**) \Rightarrow \quad c^2\hat{\underline{k}}\wedge\underline{f}'' = c_s^2\,\hat{\underline{k}}\wedge\underline{f}'' \qquad (II)$

Exercise Show analogous result to $\hat{\underline{k}}\wedge(*)$, $\hat{\underline{k}}\cdot(**)$.

(I): If $c^2 = c_p^2$, $\hat{\underline{k}}\wedge\underline{f}'' = 0 \Rightarrow \underline{f}'' \parallel \hat{\underline{k}}$.

WLOG $\underline{u} = \hat{\underline{k}}\,f(\hat{\underline{k}}\cdot\underline{x} - c_p t)$, $\theta = f'$, $\underline{\omega} = 0$

↖ scaler

Plane P-wave is longitudinal.

(II): if $c^2 = c_s^2$, $\hat{\underline{k}}\cdot\underline{f}'' = 0 \Rightarrow \underline{f}'' \perp \hat{\underline{k}}$.

WLOG $\underline{u} = \underline{g}(\hat{\underline{k}}\cdot\underline{x} - c_s t)$, $\hat{\underline{k}}\cdot\underline{g} = 0$, $\theta = 0$, $\underline{\omega} = \hat{\underline{k}}\wedge\underline{g}'$.

A plane S-wave of arbitrary shape is transverse.

If the direction of $\underline{g}$ is fixed, waves are polarised.

7.6 Energies of Plane Waves

Consider P-waves. $\dot{\underline{u}} = -c_p\hat{\underline{k}}f'$

$$K = \frac{1}{2}\rho|\dot{\underline{u}}|^2 = \frac{1}{2}\rho c_p (f')^2.$$

$$\underline{\underline{e}} = \hat{\underline{k}}\,\hat{\underline{k}}\,f', \quad \underline{\underline{\sigma}} = (\lambda\underline{\underline{I}} + 2\mu\,\hat{\underline{k}}\,\hat{\underline{k}})\,f'$$

$$W = \frac{1}{2}(\lambda e_{kk}e_{ll} + 2\mu e_{ij}e_{ij}) = \frac{1}{2}(\lambda+2\mu)(f')^2 = \frac{1}{2}\rho c_p^2 (f')^2$$

$$\underline{I} = -\dot{\underline{u}} \cdot \underline{\underline{\sigma}} = c_p(\lambda + 2\mu)\,\hat{\underline{k}}\,(f')^2 = c_p\,(\rho c_p^2)\,\hat{\underline{k}}\,(f')^2$$

So equipartition, i.e. $K = W$. $\underline{I} = \hat{\underline{k}}\, c_p\,(K+W)$. Energy propagates in direction of wave at speed c_p.

Exercise Check for S waves. $K = W$, $\underline{I} = \hat{\underline{k}}\, c_s (K+W)$

Both S & P waves are non-dispersive.

8. Harmonic Plane Waves + Polarisation

Harmonic waves are a special case of plane waves.

P: $\underline{u} = \underline{A}\, e^{i(\underline{k}\cdot\underline{x} - \omega t)}$, $\underline{A} \parallel \underline{k}$, $\omega / |\underline{k}| = c_p$.

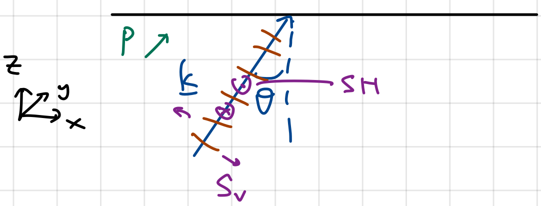
S: $\underline{u} = \underline{B}\, e^{i(\underline{k}\cdot\underline{x} - \omega t)}$, $\underline{B} \perp \underline{k}$, $\omega / |\underline{k}| = c_s$.

8.1 Polarisation of S-waves

Suppose we have a plane boundary at $z = 0$ (horizontal)

WLOG, assume $\underline{k}$ in x-z plane.

$$\underline{k} = k(\sin\theta, 0, \cos\theta)$$



θ is the angle of incidence, $k = |\underline{k}|$.

P-wave: $\underline{u} = A(\sin\theta, 0, \cos\theta)\, e^{i[k(x\sin\theta + z\cos\theta) - \omega t]}$, $A \in \mathbb{C}$.

S-wave: $\underline{B} \perp \underline{k}$. So $\underline{B}$ gives direction of polarisation. Two diff. types.

(i) Purely horizontal.

$$\underline{u} = B_H(0, 1, 0)\, e^{i(\quad)}, \quad B_H \in \mathbb{C}.$$

S_H waves are called "horizontally polarised"

(ii) A part that lies in the same plane as $\underline{k}$ and it's "vertical", i.e. $\perp$ direction with non-zero z-cpt.

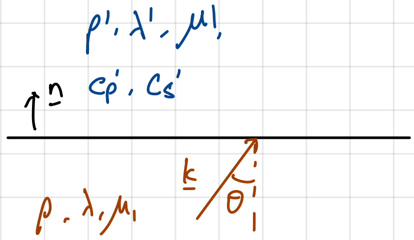
$$\underline{u} = B_V(\cos\theta, 0, -\sin\theta)\, e^{i(\quad)}. \quad B_V \in \mathbb{C}.$$

S_V waves "vertically polarised".

8.2 Reflection and Refraction of harmonic plane waves

A general disturbance contains P, SV, SH waves.

Consider a wave $\underline{k} = k(\sin\theta, 0, \cos\theta)$ incident on an interface at $z=0$



Cty of displacement $[\underline{u}]_-^+ = 0$

cty of traction $[\underline{\sigma} \cdot \underline{n}]_-^+ = 0$

(i) In general we have 6 conditions to determine 3 transmitted and 3 reflected amplitudes.

(ii) Upper layer is inviscid elastic fluid,

- only P wave in $z>0$
- 4 conditions, so still ok.

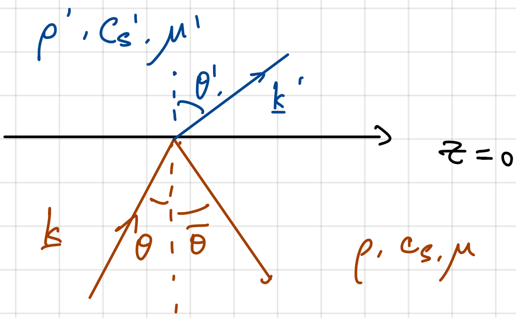
(iii) For free or rigid boundaries, 3 conditions for 3 reflected amplitudes.

Note Only SH waves contain u_y, hence $\sigma_{yz} \propto \frac{\partial u_y}{\partial z}$. $\Rightarrow$ SH waves are decoupled from P/SV waves, ie. SH waves only excite SH waves, P/SV waves only excite P/SV waves.

9. Examples on reflection and transmission

3 important examples of non-dispersive waves in elastic solids

9.1 SH waves



In $z<0$

$$\underline{u} = (0,1,0)\begin{bmatrix} B\exp(ik(x\sin\theta + z\cos\theta) - i\omega t) \\ + R\exp(i\bar{k}(x\sin\bar\theta - z\cos\bar\theta) - i\bar\omega t) \end{bmatrix}$$

In $z>0$,

$$\underline{u} = (0,1,0)\left[T\exp(ik'(x\sin\theta' + z\cos\theta') - i\omega' t)\right]$$

9.1.1 Angles

At $z=0$, $[u_y]_-^+ = 0 = [\sigma_{yz}]_-^+ = 0 \quad \forall x, t.$

$\Rightarrow$ the factor $e^{ikx\sin\theta - i\omega t}$ must be the same for all waves

$\Rightarrow \omega = \bar{\omega} = \omega'$ and $k\sin\theta = k'\sin\theta' = \bar{k}\sin\bar{\theta}$.

So in the lower half plane, $\omega = \bar{\omega} \Rightarrow k = \bar{k} \Rightarrow \theta = \bar{\theta}$

Considering upper half plane, $kc_s = k'c_s'$

$$\Rightarrow \frac{\sin\theta}{c_s} = \frac{\sin\theta'}{c_s'} \qquad \text{(Snell's law)}$$

So we have

$$k'\cos\theta' = \frac{kc_s}{c_s'}\left(1 - \frac{c_s'^2}{c_s^2}\sin^2\theta\right)^{1/2} = k\left(\frac{c_s^2}{c_s'^2} - \sin^2\theta\right)^{1/2} = m'$$

If $c_s' > c_s$, possible that $\sin\theta > c_s/c_s' \Rightarrow$ no real sol^n for θ':

$$m' = i\hat{m}', \quad \hat{m}' > 0, \quad \hat{m}' \in \mathbb{R}.$$

Then the "transmitted wave" is

$$(0,1,0)\, T \exp(i(kx\sin\theta - \omega t)) \exp(-\hat{m}' z),$$

ie the wave is <u>evanescent</u> in $z>0$. Amplitude decays so that the wave only penetrates $O(1/\hat{m}')$.

<u>Total internal reflection</u> for $\theta > \theta_c$. Critical angle

$$\sin\theta_c = \frac{c_s}{c_s'}, \quad \frac{c_s}{c_s'} < 1.$$

9.1.2 Amplitudes

$$[u_y]_-^+ = 0 \Rightarrow B + R = T$$

$$[\sigma_{yz}]_-^+ = 0 = \left[\mu \frac{\partial u_y}{\partial z}\right]_-^+ \Rightarrow i\mu k\cos\theta(B-R) = i\mu' k'\cos\theta'\, T$$

$$\Rightarrow B - R = ZT. \qquad Z = \frac{\mu' k'\cos\theta'}{\mu k\cos\theta} = \frac{\rho' c_s'\cos\theta'}{\rho c_s\cos\theta}$$

$$\Rightarrow \quad R = \frac{1-Z}{1+Z} B, \qquad T = \frac{2}{1+Z} B.$$

(c.f. acoustic waves impedence $\rho_1 c_1$ etc: when impedence is matched $\Leftrightarrow$ $Z=1$ $\Leftrightarrow$ $B=0$, i.e. Total transmission).

9.1.3 Energy Flux

Expect $\langle I_z \rangle$ to be const. with z.

$$I_z = -\dot{u}_y \sigma_{yz} = -\mu \dot{u}_y \frac{\partial u_y}{\partial z}$$

$$\langle I_z \rangle_I = -\frac{1}{2} \mathrm{Re}\left(\dot{u}_y^* \mu \frac{\partial u_y}{\partial z} \right)_I = \frac{1}{2} \omega \mu k \cos\theta \, |B|^2$$

incoming

$$\langle I_z \rangle_R = -\frac{1}{2} \mathrm{Re}\left(\dot{u}_y \mu \frac{\partial u_y}{\partial z} \right)_R = -\frac{1}{2} \omega \mu k \cos\theta \, |R|^2$$

reflected

$$\langle I_z \rangle_T = -\frac{1}{2} \mathrm{Re}\left(\dot{u}_y \mu \frac{\partial u_y}{\partial z} \right)_R$$

$$= \begin{cases} \frac{1}{2} \omega \mu' k' \cos\theta' \, |T|^2 & \cos\theta' \text{ real} \\ 0 & \cos\theta' \text{ imaginary} \end{cases}$$

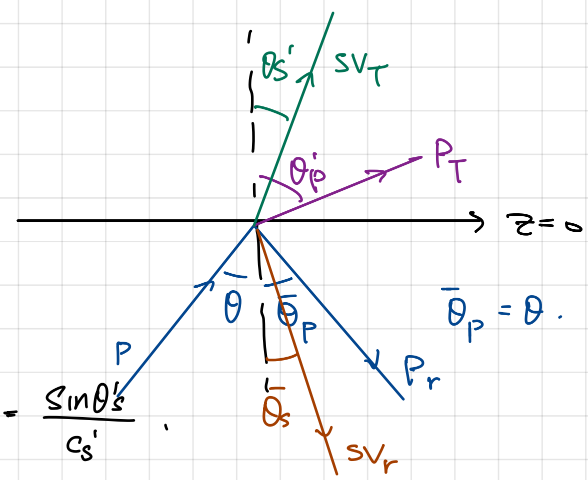
$\therefore$ If $\cos\theta'$ real, $|B|^2 - |R|^2 = Z|T|^2$.

$\cos\theta'$ imaginary, $|R| = |B|$

9.2 P/SV wave

Note $\theta_s < \theta_p$

9.2.1 Angles



BCs true $\forall t$ $\Rightarrow$ ω same for all modes

$\forall x$ $\Rightarrow$ Snell's law, ie.

$$\frac{\sin\theta}{c_p} = \frac{\sin\bar{\theta}_p}{c_p} = \frac{\sin\bar{\theta}_s}{c_s} = \frac{\sin\theta_p'}{c_p'} = \frac{\sin\theta_s'}{c_s'}.$$

Many possibilities for evanescent waves / critical angle etc.

9.2.2 Amplitudes

4 BCs $\Rightarrow$ 4×4 matrix for the 4 amplitudes.

- (stress) free / rigid boundary $\Leftrightarrow$ no waves in $z>0$
- $\mu=0$ and/or $\mu'=0$ make SV waves disappear
- Both μ and μ' are zero, recover acoustic wave.

9.3 Note on Evanescent waves

$\omega \in \mathbb{R}$, $\underline{k}$ can be complex, write $\underline{k} = \underline{k}_r + i\underline{k}_i$

$$\Rightarrow \underline{u} \propto e^{-\underline{k}_i \cdot \underline{x}} e^{i(\underline{k}_r \cdot \underline{x} - \omega t)}$$

Propagation in $\underline{k}_r$ direction. Attenuation in $\underline{k}_i$ direction.

Dispersion relation $\omega^2 = c_{P/S}^2 \, \underline{k}\cdot\underline{k} = c_{P/S}^2 (\underline{k}_r\cdot\underline{k}_r - \underline{k}_i\cdot\underline{k}_i + 2i\underline{k}_r\cdot\underline{k}_i)$

$$\omega \in \mathbb{R} \Rightarrow \underline{k}_r\cdot\underline{k}_i = 0 \Rightarrow c := \frac{\omega}{|\underline{k}_r|} = c_{P/S}\left(1 - \frac{|\underline{k}_i|^2}{|\underline{k}_r|^2}\right) < c_{P/S}.$$

ie. propagation along a boundary is slower than through the interior (important in sensitivity)

SH waves require either forcing (as discussed in 9.1) or actually 2 boundaries (see part II) to be sustained near a boundary. A particular combination of P/SV waves can combine to be sustained near a single boundary.

9.4 Rayleigh Waves (Actually an eigenmode)

Consider a particular combination of evanescent P wave and evanescent SV wave in the vacinity of a stress free boundary, and evanescent in solid so $\to 0$ as $z \to -\infty$.

$\underline{k}_p = k(1,0,-ia)$. $c^2 = \frac{\omega^2}{k^2} = c_p^2(1-a^2)$ Vacuum

$\underline{k}_s = k(1,0,-ib)$. $c^2 = c_s^2(1-b^2)$.

$\underline{\underline{\sigma}} \cdot \underline{n} = 0$ $\qquad z = 0$

solid $\lambda, \mu, \rho, c_p, c_s$

k, a, b all > 0.

$$\underline{u} = \underbrace{A(i,0,a)\, e^{ik(x-ct)+kaz}}_{\text{P wave } \parallel\, \underline{k}_p} + \underbrace{B(b,0,-i)\, e^{ik(x-ct)+kbz}}_{\text{SV wave } \perp\, \underline{k}_s}$$

BCs:

- $\sigma_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0$ at $z=0$.

$$\Rightarrow \mu k\,(2iaA + (b^2+1)B) = 0.$$

- $\sigma_{zz} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + 2\mu \frac{\partial u_z}{\partial z} = \underset{\rho c_p^2}{(\lambda+2\mu)}\, \nabla\cdot\underline{u} - 2\underset{\rho c_s^2}{\mu} \frac{\partial u_x}{\partial x} = 0$

recall $\nabla\cdot\underline{u} = 0$ for SV wave (exercise).

$$\Rightarrow \rho k\left[\underbrace{c_p^2(a^2-1)}_{=c_s^2(b^2-1)} A - 2c_s^2(-A + ibB)\right] = 0.$$

$$\Rightarrow \begin{pmatrix} 2ia & b^2+1 \\ b^2+1 & -2ib \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$

Note $\quad a = \left(1 - \frac{c^2}{c_p^2}\right)^{1/2}, \quad b = \left(1 - \frac{c^2}{c_s^2}\right)^{1/2} \Leftrightarrow b^2 + 1 = 2 - \frac{c^2}{c_s^2}$

$$\Rightarrow \quad 4ab = b^2 + 1 \quad \text{for a sol}^n.$$

$$4\left(1 - \frac{c^2}{c_p^2}\right)^{1/2}\left(1 - \frac{c^2}{c_s^2}\right)^{1/2} = \left(2 - \frac{c^2}{c_s^2}\right)^{1/2}$$

A cubic with one real root in $0 < c(\nu) < c_s < c_p$,

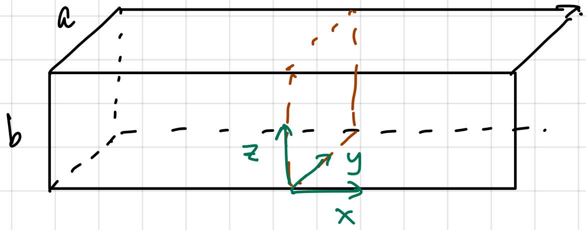
$\nu = \frac{\lambda}{2(\lambda+\mu)}$ Poisson's ratio

Note $c = \omega/k$ is indpt. of k. wave is non-dispersive.

Part III: Dispersive waves

10. Examples

10.1 Acoustic Waveguide



Rectangular duct. Consider linear acoustic wave eqn in part I.

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi \quad . \ \underline{u} = \nabla \phi .$$

$$\left.\begin{aligned} \frac{\partial \phi}{\partial z} &= 0 \quad \text{on } z = 0, b \\ \frac{\partial \phi}{\partial y} &= 0 \quad \text{on } y = 0, a \end{aligned}\right| \Rightarrow \frac{\partial \phi}{\partial \underline{n}} = 0 \text{ on solid boundary}$$

Try $\phi = e^{i(kx - \omega t)} f(y,z)$.

$$\Rightarrow -\omega^2 f = c_0^2 \left(-k^2 f + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right).$$

Eval problem

$$f_{mn}(y,z) = A_{mn} \cos\left(\frac{m\pi y}{a}\right) \cos\left(\frac{n\pi z}{b}\right) \quad , m,n \in \mathbb{Z}.$$

and frequencies ω_{mn} must satisfy

$$\omega_{mn}^2(k) = c_0^2 \left(k^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)$$

This is dispersion relation. Except for the case $m = n = 0$,

$$\frac{\omega}{k} \neq \text{const}.$$

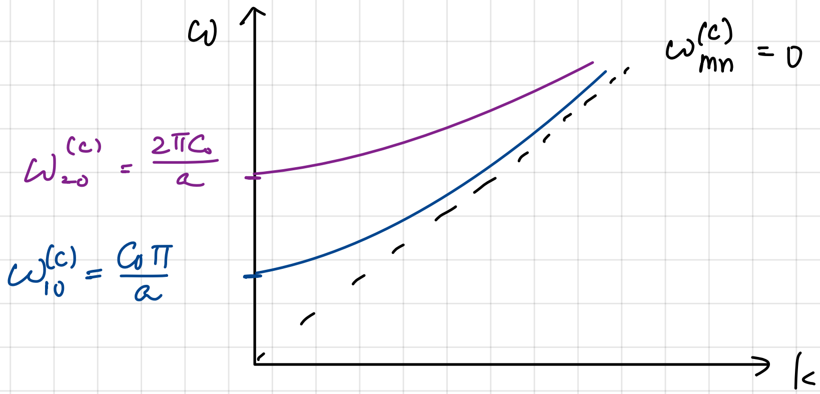
and waves are dispersive.

Cut-off freq. are since $\exists \min_k \omega = \omega|_{k=0}$

$$\omega_{mn}^{(c)} = c_0 \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right).$$

If we attempt to excite m,n modes with freq. $\omega < \omega_{mn}^{(c)}$

$\Rightarrow k$ imaginary so exponential decay, as $x \to \infty$ (cf evanescent)



10.1.1 Energetics

Consider cross-sectional averages

$$\overline{\cdot} = \frac{1}{ab}\int_0^a dy \int_0^b dz \;\cdot$$

$$\langle \bar{K} \rangle = \frac{1}{2}\rho_0 \langle \overline{|\nabla \phi|^2} \rangle$$

$$= \frac{1}{2}\rho_0 \left[\overline{\frac{1}{2} \operatorname{Re}(\nabla\phi \cdot \nabla\phi^*)} \right]$$

$$= \frac{1}{4\chi_{mn}} \rho_0 |A_{mn}|^2 \Big(\underbrace{l^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}_{\omega^2/c_0^2} \Big),$$

where $\chi_{mn} = (2-\delta_{m0})(2-\delta_{n0})$,

$$\langle \bar{W} \rangle = \frac{1}{2}\frac{c_0^2}{\rho_0} \langle \overline{\tilde{\rho}^2} \rangle$$

$$= \frac{1}{2}\frac{\rho_0}{c_0^2} \langle \overline{\left|\frac{\partial \phi}{\partial t}\right|^2} \rangle$$

$$= \frac{\rho_0 \omega^2 |A_{mn}|^2}{4\chi_{mn} c_0^2} = \langle \bar{K} \rangle.$$

$$\langle \bar{I}_x \rangle = \langle \overline{\tilde{p}\, u_x} \rangle$$

$$= \frac{1}{2\chi_{mn}} \rho_0 \omega k |A_{mn}|^2.$$

$$= \frac{c_0^2 k}{\omega} \langle \bar{K} + \bar{W} \rangle.$$

Energy is transported at speed

$$c_{gx} = \frac{c_0^2 k}{\omega} = \frac{\partial \omega}{\partial k}.$$

10.1.2 Superposition of plane waves

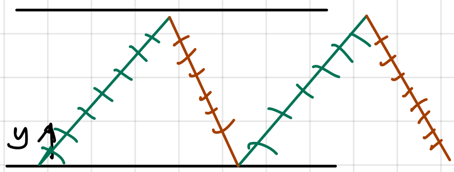
Simple case when $m \neq 0,\ n = 0$. Dispersion relation is

$$1 = \underbrace{\frac{c_0^2 k^2}{\omega^2}}_{\cos^2\theta} + \underbrace{\frac{m^2 \pi^2 c_0^2}{\omega^2 a^2}}_{\sin^2\theta}$$

$$\Rightarrow \phi \propto \exp\left[\frac{i\omega}{c_0}\left(x\cos\theta + y\sin\theta - c_0 t\right)\right] + \exp\left[\frac{i\omega}{c_0}\left(x\cos\theta - y\sin\theta - c_0 t\right)\right]$$

Note that apparent phase speed in x-dir, i.e.

$$\frac{\omega}{k} = c = \frac{c_0}{\cos\theta} > c_0$$



but

$$c_{gx} = \frac{c_0^2 k}{\omega} = c_0 \cos\theta < c_0 .$$

$$c = \frac{\omega}{k} = \frac{c_0}{k}\left(k^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^{1/2}$$

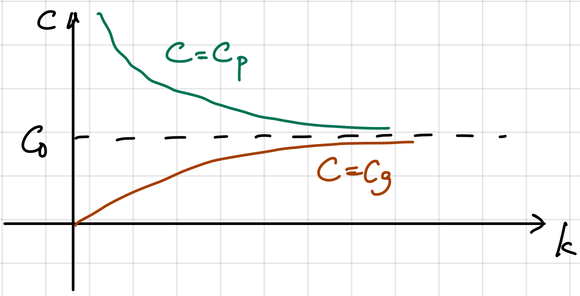
$$= c_0\left(1 + \frac{m^2\pi^2}{a^2k^2} + \frac{n^2\pi^2}{b^2k^2}\right)^{1/2}$$

$$c_g = \frac{c_0 k}{\left(k^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^{1/2}}$$

So if $m, n \neq 0$,

$c \to \infty$ as $k \to 0$

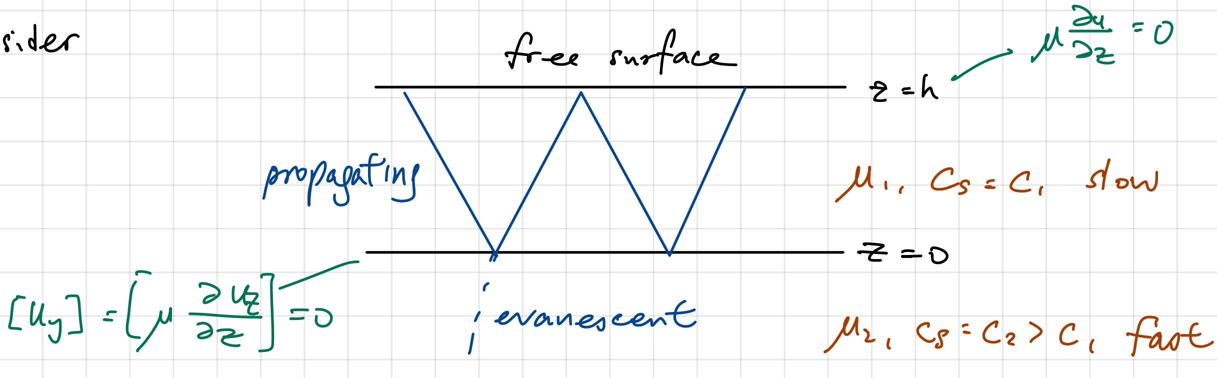
$c_g = 0$ at $k = 0$



10.2 Love Waves

Elastic SH waves trapped in a layer, wave guide

Consider



Look for SH waves $u = \left(0, f(z)\, e^{ik(x-ct)}, 0\right)$ trapped in low velocity layer $0<z<h$.

Wave eqn

For $0<z<h$,

$$-c^2k^2 f = c_1^2\left(f'' - k^2 f\right) \quad \therefore \frac{\partial u}{\partial z} = 0 \text{ on } z=h$$

$$\Rightarrow u_1 = A\, e^{ik(x-ct)} \cos\left(m_1(h-z)\right), \quad m_1 = k\left(\frac{c^2}{c_1^2} - 1\right)^{1/2}.$$

For $z<0$,

$$-c^2k^2 f = c_2^2\left(f'' - k^2 f\right)$$

$$\Rightarrow u_2 = B e^{ik(x-ct) + \underbrace{m_2 z}_{\to 0 \text{ as } z \to -\infty}}, \quad m_2 = k\left(1 - \frac{c^2}{c_2^2}\right)^{1/2}$$

For the req'd form of the sol^n must have $c_1 < c < c_2$, i.e. horizontal wave speed in between the layer speed.

BCs at $z=0$ yields the dispersion relation.

$$[u] = 0 \Rightarrow \quad A\cos m_1 h = B$$

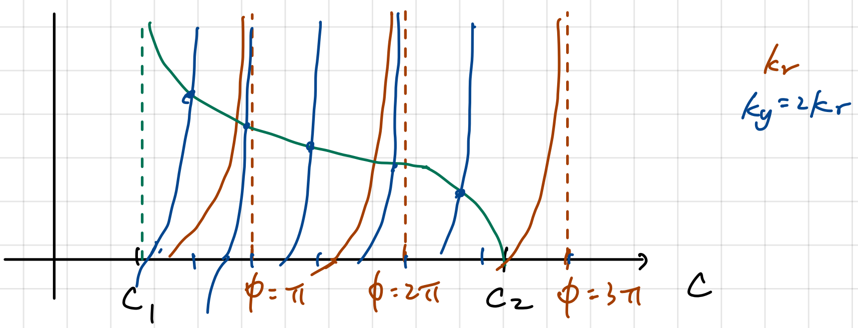
$$\left[\mu \frac{\partial u}{\partial z}\right] = 0 \Rightarrow \quad A\mu_1 m_1 \sin m_1 h = B\mu_2 m_2$$

Combining,

$$\mu_1 m_1 \tan m_1 h = \mu_2 m_2$$

$$\Rightarrow \tan\left(\underbrace{\left(\frac{c^2}{c_1^2} - 1\right)^{1/2} kh}_{\phi}\right) = \frac{\mu_2}{\mu_1}\left(\frac{1 - \frac{c^2}{c_2^2}}{\frac{c^2}{c_1^2} - 1}\right)^{1/2}$$

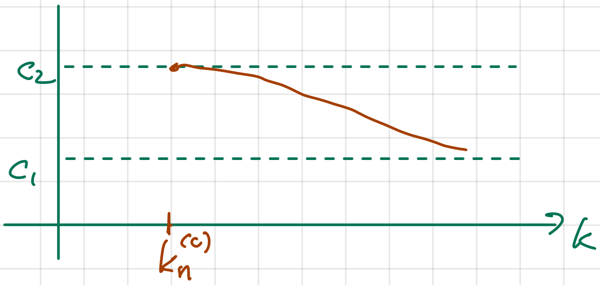
Implicit f^n $c(k) = \omega(k)/k$.



Note: (1) There is always at least one sol^n

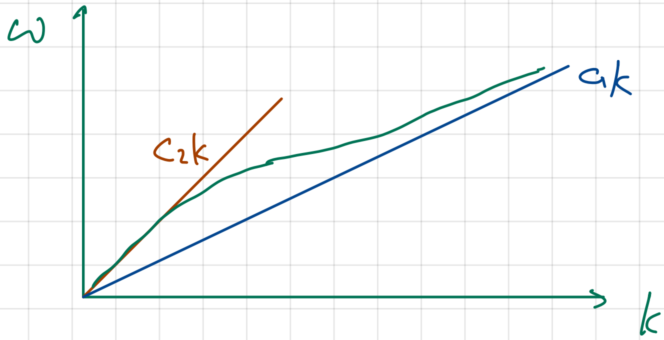
(2) Get n sol^n if $(n-1)\pi < kh\left(\frac{c_2^2}{c_1^2}-1\right)^{1/2} < n\pi$

(3) As $k\uparrow$, tan curves squash to left $\Rightarrow$ each sol^n has $c(k)\downarrow$ appears at c_2, i.e. for $n>1$, and decreases c_1.



$$\omega_n^{(c)} = k_n^{(c)} c_2 = \frac{n\pi}{h}\left(\frac{c_2^2}{c_1^2}-1\right)^{1/2} c_2$$

(4) Freq. is in a range between $\omega_n^{(c)}$ and $c_1 k$ as $k\to\infty$.



Exercise $c_g(k) = \partial\omega/\partial k$, $\omega = ck$.

11. Dispersive Wavepackets

11.1 Beats and modulation

Suppose a linear problem. Suppose gives sol^n of the form $e^{i(kx-\omega(k)t)}$

Consider two sol^{n}s of superpositions of two waves of equal amplitude, and nearly equal k, ω.

$$\phi = \cos(k_1x - \omega_1 t) + \cos(k_2 x - \omega_2 t),$$

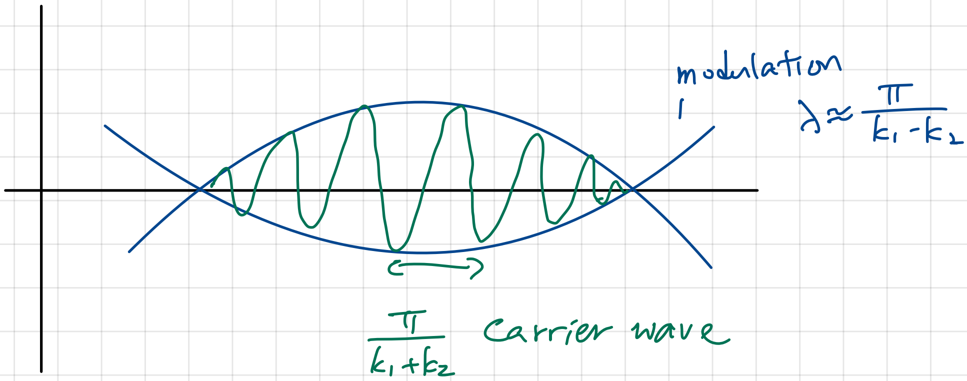
where $k\approx k_1 \approx k_2$, $\omega \approx \omega_1(k_1) \approx \omega_2(k_2)$. Could write

$$k_1 x - \omega_1 t = \frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t + \frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t$$

$$k_2 x - \omega_2 t = \frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t - \frac{k_1-k_2}{2}x + \frac{\omega_1-\omega_2}{2}t$$

Then

$$\phi = 2\cos\left(\frac{(k_1+k_2)x}{2} - \frac{(\omega_1+\omega_2)t}{2}\right)\cos\left(\frac{(k_1-k_2)x}{2} - \frac{(\omega_1-\omega_2)t}{2}\right).$$



So we have a <u>modulated</u> wave in which individual wave crests move at the <u>phase speed</u>

$$c = \frac{\omega_1+\omega_2}{k_1+k_2} \approx \frac{\omega}{k}.$$

The envelope and each pulse within the envelop moves at <u>group velocity</u>

$$c_g = \frac{\omega_1-\omega_2}{k_1-k_2} \approx \frac{d\omega}{dk}.$$

11.2 Sol^n to IVPs by FT

Consider a system on $-\infty < x < \infty$ with $\omega(k)$. If there is no disturbance at or arriving from $\pm\infty$, how does an initial disturbance evolve? Key idea is to use superposition based around FTs.

Let $\phi(x,t) = \int_{-\infty}^{\infty} \hat{\phi}(k,t)\, e^{ikx}\, dk \Leftrightarrow \hat{\phi}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x,t)\, e^{-ikx}\, dx$.

<u>Note</u> (1) If ϕ real $\Rightarrow \hat{\phi}(-k) = \hat{\phi}^*(k)$.

(2) If ϕ real, even in x, ie. $\phi(x) = \phi(-x) \Rightarrow \hat{\phi}(k)$ also real.

Thus, ϕ can be split into many waves of the form e^{ikx} with varying k. Each of these waves can evolve in time like $e^{-i\omega(k)t}$.

So by superposition, can obtain the time evolution of ϕ if have enough BC.

(1) First order in time, i.e. $\frac{\partial \phi}{\partial t} = \frac{\partial^n \phi}{\partial x^n}$, we only need 1 initial condition, i.e. $\phi(x,0)$.

At $t=0$,

$$\hat{\phi}(k,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x,0)\, e^{-ikx}\, dx$$

At time t,

$$\hat{\phi}(k,t) = \hat{\phi}(k,0)\, e^{-i\omega(k)t}$$

$$\Rightarrow \phi(x,t) = \int_{-\infty}^{\infty} \hat{\phi}(k,t)\, e^{i(kx-\omega(k)t)}\, dk.$$

(2) Second order in time.

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^n \phi}{\partial x^n}$$

Need 2 ICs, $\phi(x,0)$, $\frac{\partial \phi}{\partial t}(x,0)$.

The dispersion relation has the form

$$\omega^2 = f(k) \Rightarrow \omega = \pm [f(k)]^{1/2}.$$

So a disturbance e^{ikx} gives rise to waves

$$e^{i(kx \mp \omega t)}$$

where $\omega = \sqrt{f}$. Hence decompose

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega(k)t)} + B(k)\, e^{i(kx+\omega(k)t)}\, dk$$

amplitudes

ICs $\Rightarrow$

$$A + B = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x,0)\, e^{-ikx}\, dx$$

$$-i\omega A + i\omega B = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial t}(x,0)\, e^{-ikx}\, dx$$

If $\frac{\partial \phi}{\partial t}(x,0) = 0 \Rightarrow A = B = \frac{1}{2}\hat{\phi}(k,0)$.

$\phi(x,0) = 0 \Rightarrow B = -A = \frac{1}{2i\omega} \frac{\partial \hat{\phi}}{\partial t}$.

12. Method of Stationary Phase

12.1 Stationary phase

Suppose

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ikx - i\omega(k)t}\, dk.$$

What is seen by an observer moving at speed V at large t?

Consider the limit as $t \to \infty$ with $x = Vt$ for fixed V $(V = O(1))$.

Then

$$\phi(Vt, t) = \int_{-\infty}^{\infty} A(k)\, e^{i\theta}\, dk,$$

where $\theta = \theta(k) = [kV - \omega(k)]\, t$ is the phase.

Notice

$$\frac{d\theta}{dk} = \left(V - \frac{d\omega}{dk}\right) t$$

and remember large t.

(1) If $d\omega/dk \neq V$, then phase changes rapidly. The integrand oscillates rapidly. There is so much cancellation that the integrand is exponentially small.

(2) If $\omega' = V$, $k = k_0(V)$, we have a point of stationary phase and the cancellation is much weaker near $k = k_0$.

(3) Near $k = k_0$,

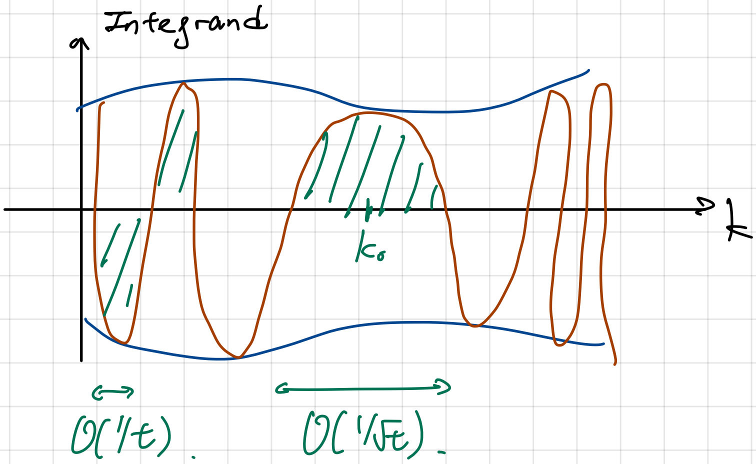
$$\phi = [k_0 V - \omega(k_0)]\, t + \cancelto{0}{(k-k_0)(V - \omega(k_0))\, t} - \frac{1}{2}(k-k_0)^2 \omega''(k_0)\, t + \cdots$$

$\Rightarrow \theta$ changes rapidly with k again for $|k - k_0| \gg (\omega'' t)^{1/2}$.

The dominant contribution only comes from

$$|k - k_0| = O(1/\sqrt{\omega'' t})$$

as $t \to \infty$.



Suggests:

$$\phi \sim A(k_0)\, e^{i(k_0 V - \omega(k_0))t} \int_{k-k_0 \to \text{"}-\infty\text{"}}^{k+k_0 = \text{"}\infty\text{"}} \exp\left(-\frac{i\omega''(k_0)}{2}(k-k_0)^2 t\right) d(k-k_0)$$

The upper and lower limit on the integral are initially somewhere $\gg \pm 1/\sqrt{\omega'' t}$ but are then replaced by $\pm\infty$. The best proof uses steepest descent in the complex k-plane.

Now from complex methods that

$$\int_{-\infty}^{\infty} e^{-a\xi^2}\, d\xi = \sqrt{\pi/a} \qquad \forall a \in \mathbb{C},\ Re(a) \geq 0.$$

for cty for the conditionally converging case. with $Re(a) = 0$.

$$\Rightarrow \phi(Vt, t) \sim A(k_0) \sqrt{\frac{2\pi}{i\omega''(k_0)t}}\, e^{i(k_0 V - \omega(k_0))t}, \quad \omega'(k_0) = V$$

as $t \to \infty$

12.2 Observations

(1) We have assumed there is a single point of S.P. k_0. If there are more, they each give contributions, which must be added together. If there are no points of S.P., ϕ is exponentially small.

(2) For 2^{nd} order eqn, $\frac{\partial^2\phi}{\partial t^2}$ need to add contributions from waves $e^{i(kx+\omega t)}$ from dispersion relation involving ω^2.

(3) If $\omega''(k_0)=0$, need to include HOT in Taylor Series.

If $\omega''\neq 0$, then HOT give $O(t^{-3/2})$ correction.

(4)

$$\frac{1}{\sqrt{\pm i}} = e^{\mp i\pi/4}.$$

$$\Rightarrow \frac{1}{\sqrt{i\omega''}} = \frac{1}{\sqrt{|\omega''|}} e^{-\frac{\pi i}{4}\operatorname{sgn}(\omega'')}.$$

$$\Rightarrow \phi(Vt,t) \overset{t\to\infty}{\sim} A(k_0)\sqrt{\frac{2\pi}{|\omega''(k_0)|t}} \exp\left(i(k_0V-\omega(k_0))t - \frac{i\pi}{4}\operatorname{sgn}(\omega'')\right) \quad (**)$$

(5) Similar ideas apply in 2D / 3D.

$$\phi(\underline{x},t) = \int A(\underline{k})\, e^{i\underline{k}\cdot\underline{x}}\, d\underline{k}$$

then at $\underline{x}=\underline{V}t$, $t\to\infty$, see waves

$$\frac{\partial}{\partial \underline{k}}(\underline{k}\cdot\underline{V} - \omega(\underline{k})) = \underline{0}, \quad \underline{V} = \underline{c}_g(\underline{k}), \quad \underline{c}_g = \frac{\partial\omega}{\partial\underline{k}} = \nabla_{\underline{k}}\omega$$

(6) The radiation condition is that energy propagates outwards from a source, NOT that the crests do. E.g. if a source at $\underline{0}$ starts emitting waves at $t=0$, with wave vec $\underline{k}$ and group velocity $\underline{c}_g(\underline{k})$, the disturbance reaches $\underline{x}=\underline{c}_g(\underline{k})t$ at time t.

12-3 Interpretation of (**)

An observer moving with speed V will eventually see only those waves in the initial spectrum $A(k)$ with waveno. (k_0) s.t.

$$c_g(k_0) = \left.\frac{\partial\omega}{\partial k}\right|_{k_0} = V.$$

The local amplitude decreases like $1/\sqrt{|\omega''|t}$ due to dispersion of nearby waveno. and conservation of energy. Waveno. in the range $[k_0, k_0+\delta k]$ have group velocities in the range

$$[\omega'(k_0), \omega'(k_0+\delta k)] \approx [\omega'(k_0), \omega'(k_0)+\delta k\,\omega''(k_0)]$$

$\Rightarrow$ separation like $\delta k\,\omega''(t)$.

$$\text{Energy} \propto \int_{Vt}^{(V+\delta k\,\omega'')t} \phi^2\,dx \Rightarrow \text{const.}$$

$$\Rightarrow \quad \phi \propto \frac{1}{\sqrt{\omega'' t}}, \qquad \text{WLOG } \omega''>0.$$

12.4 Elastic Beam equation (e.g. twanged ruler)

$$\frac{\partial^2\phi}{\partial t^2} + \gamma\frac{\partial^4\phi}{\partial x^4} = 0$$

$$\frac{\partial\phi}{\partial x}(x,0) = 0$$

$$\phi(x,0) = f(x).$$

Sol^n: dispersion relation : $\omega^2 = \gamma^2 k^4$

$$\Rightarrow \quad \phi = \int_{-\infty}^{\infty} \tfrac{1}{2}\hat{f}(k)\Big[\underbrace{e^{i(kx-\gamma k^2 t)}}_{\phi_+} + \underbrace{e^{i(kx+\gamma k^2 t)}}_{\phi_-}\Big]\,dk.$$

$$= \phi_+ + \phi_-$$

Put $x = Vt$

- ϕ_+ has SP at k_0, where $V - 2\gamma k = 0 \Rightarrow k_0 = \frac{V}{2\gamma}$.

$$k_0 V - \gamma k_0^2 = \frac{V^2}{4\gamma}, \qquad \omega''(k_0) = 2\gamma.$$

- ϕ_- has SP at $-k_0$, and since ϕ real, $[\hat{f}(k)]^* = \hat{f}(-k)$

$\Rightarrow$ ϕ_- has complex conjugate contribution to ϕ_+ to the overall integral.

$$\phi \sim \frac{1}{2}\hat{f}\left(\frac{V}{2\gamma}\right)\left(\frac{2\pi}{2\gamma t}\right)^{1/2} \underbrace{\exp\left(\frac{iV^2 t}{4\gamma} - \frac{i\pi}{4}\right)}_{\phi_+} + \underbrace{c.c.}_{\phi_-}$$

Note If $f(x) = f(-x) \Rightarrow \hat{f}(k) = \hat{f}(-k) = [\hat{f}(k)]^*$,

$$\phi \sim \hat{f}\left(\frac{V}{2\gamma}\right)\left(\frac{\pi}{\gamma t}\right)^{1/2} \cos\left(\frac{V^2 t}{4\gamma} - \frac{\pi}{4}\right)$$

as $x, t \to \infty$ with x/t fixed.

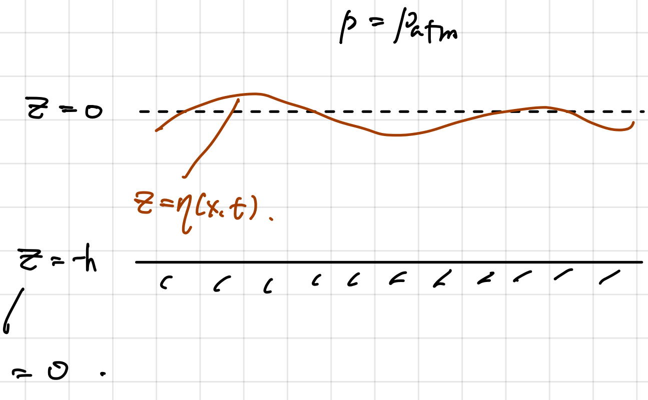
13. Linearised Water waves

13.1 Problem description

inviscid, irrot. fluid $\Rightarrow \underline{u} = \nabla\phi$

incomp. $\nabla \cdot \underline{u} = 0$

$\Rightarrow \nabla^2 \phi = 0$



Surface tension in the interface

(1) DBC: $$p|_{z=\eta_-} = p_{atm} - T \underbrace{\frac{\eta_{xx}}{(1+\eta_x)^{3/2}}}_{\text{curvature}}$$

T: Coeff. of surface tension — Force / unit length, or energy / unit area

(2) KBC: $$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w \quad \text{at} \quad z = \eta.$$

Unsteady Bernoulli

$$\Rightarrow \rho\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right) + p - p_{atm} + \rho g z = f(t).$$

WLOG $f(t) = 0$, $\phi \mapsto \phi + \int f(\hat{t})\, d\hat{t}$, $\underline{F} = -\nabla(\rho g z)$.

13.2 Linearized Problem

$$\nabla^2 \phi = 0$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -h$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at} \quad z = 0 \qquad (\text{KBC})$$

$$\rho \frac{\partial \phi}{\partial t} + \rho g \eta - T\eta_{xx} \quad \text{indpt of } x \; (=0) \qquad (\text{DBC})$$

Try a sol^n $\eta = A e^{i(kx - \omega t)}$, $\phi = B e^{i(kx - \omega t)} f(z)$.

$$\nabla^2 = 0, \quad \left.\frac{\partial \phi}{\partial z}\right|_{z=-h} = 0 \Rightarrow f(z) = \cosh k(z+h)$$

BCs at $z = 0$:

KBC: $\quad -i\omega A = kB \sinh kh.$

DBC: $\quad -i\omega(\cosh kh) B + \left(g + \frac{Tk^2}{\rho}\right) A = 0.$

$$\Rightarrow \boxed{\omega^2 = gk\left[1 + \frac{Tk^2}{g\rho}\right] \tanh kh}$$

This is disp. relation for capillary surface gravity waves

Note $\quad l_c = \sqrt{T/g\rho}$ is capillary length. For water/air interface, at $20°C$, $l_c \approx 2.7\text{mm}$

13.3 Limiting Cases

Rewrite

$$\omega^2 = gk \tanh kh \left(1 + l_c^2 k^2\right)$$

Exercise Show that

$$c_g = \frac{\omega}{2k}\left(\frac{1 + 3l_c^2k^2}{1 + l_c^2k^2} + \frac{2kh}{\sinh 2kh}\right)$$

(a) <u>Shallow water</u> ($|kh| \ll 1$)

$$\tanh kh \approx \sinh kh \approx kh.$$

(i) Difficult to also have $l_c^2 k^2 \gg 1$, since this means

$$h \ll \lambda \ll l_c.$$

$\Rightarrow h \ll 1\,\text{mm}$

(ii) Much more common to also have $l_c^2 k^2 \ll 1$. i.e. that surface tension doesn't matter $\lambda \gg l_c$. This is a long (surface) gravity wave. Non-dispersive with phase speed

$$c = \pm\sqrt{gh} = c_g$$

e.g. tides / tsunami, $\lambda \sim 100\,\text{km}$, $h \sim 1\,\text{km} \Rightarrow c \sim 100\,\text{m s}^{-1}$

(b) <u>Deep water</u> ($|kh| \gg 1$)

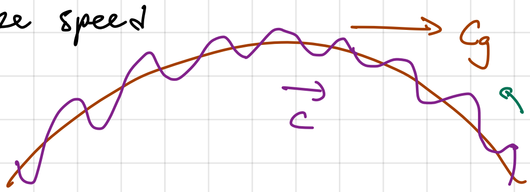
$$\tanh kh \approx \pm 1, \quad \frac{2kh}{\sinh 2kh} \to 0$$

(i) $l_c^2 k^2 \gg 1$ ($\lambda \ll 2\,\text{cm}$) short capillary wave

$$\Rightarrow \omega = \pm\left(\frac{T|k|^3}{\rho}\right)^{1/2} = \pm\left(g l_c |k|^3\right)^{1/2}$$

$$\Rightarrow c_g = \frac{3}{2}\frac{\omega}{k} = \frac{3}{2}c.$$

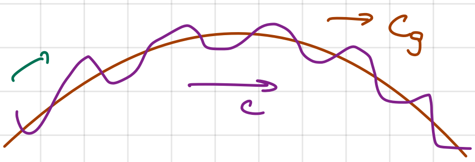
Crests would appear at the <u>front</u> of wavepacket. Shorter waves have a higher phase speed than longer waves.

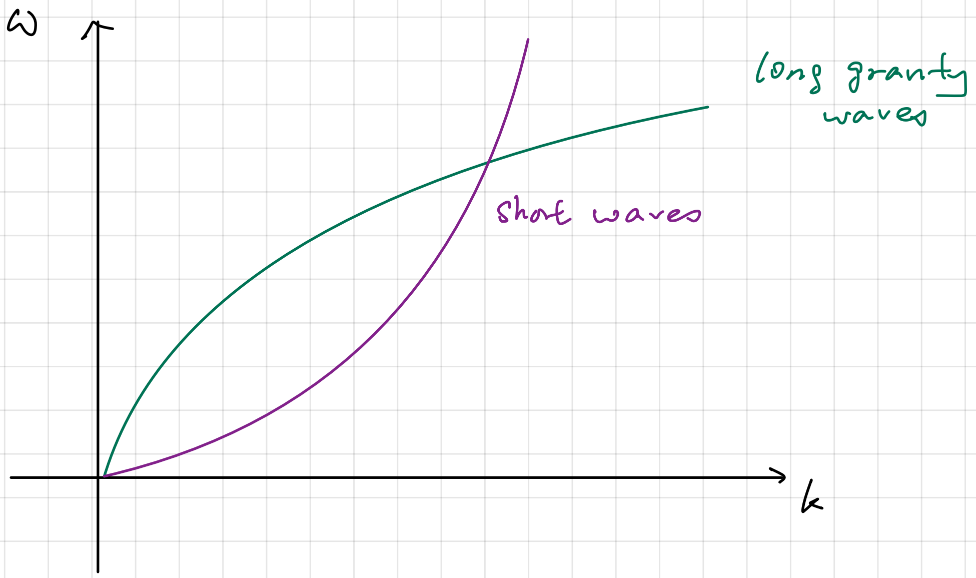


(ii) $l_c^2 k^2 \ll 1$

$$\omega \approx \pm\sqrt{g|k|}, \quad c_g = \frac{\omega}{2k} = \frac{1}{2}c$$

Deep ocean waves — crests appear at the <u>back</u> of wavepacket. Shorter waves has smaller phase speed





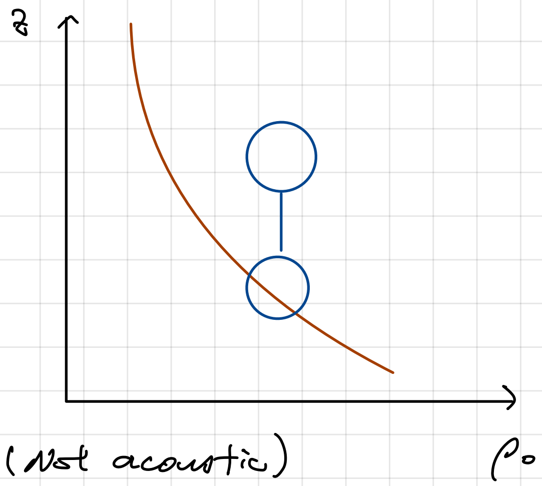
(?)

- $\exists\, k_c$ s.t. c_g minimum, so $\omega'' = 0$, so stationary phase breaks down
- $\exists\, k_2 > k_c$ s.t. phase speed minimised.

14. Internal Gravity Waves

A fluid that is lifted up above its eqm position will fall back under $\underline{g} = -g\hat{\underline{z}}$, and overshoot due to inertia

$\Rightarrow$ wave mechanism



14.1 Linear Theory for Incompressible Fluids (Not acoustic)

e.g. Ocean/lakes. In the atmosphere, we need to take into account pressure effect. They can be dealt with straightforwardly using e.g. potential temperature:

So 3 eqns:

$$\frac{D\rho}{Dt} = 0 \qquad \text{(cons. of mass)}$$

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g} \qquad \text{(cons. of mom.)}$$

$$\nabla \cdot \underline{u} = 0 \qquad \text{(incomp.)}$$

linearize the system about a state of rest $\underline{u}=\underline{0}$.

Background density distribution $\rho=\rho_0(z)$.

Hydrostatic pressure $p_0 = p_{ref} - \int_0^z \rho_0 g \, dz$, ie.

$$\frac{dp_0}{dz} = -\rho_0(z) g.$$

So we get 5 eqn:

① $$\frac{\partial \tilde{\rho}}{\partial t} + w \frac{d\rho_0}{dz} = 0.$$

② $$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial \tilde{p}}{\partial x}$$

③ $$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial \tilde{p}}{\partial y}$$

④ $$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial \tilde{p}}{\partial z} - g\tilde{\rho},$$

⑤ $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\left[\begin{array}{l} \rho = \tilde{\rho} + \rho_0(z) \\ p = \tilde{p} + p_0 . \end{array}\right]$$

$\frac{\partial}{\partial x}$② + $\frac{\partial}{\partial y}$③

$$\Rightarrow \rho_0 \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\rho_0 \frac{\partial}{\partial t}\frac{\partial w}{\partial z} = -\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{p}\right] = -\nabla_h^2 \tilde{p}. \quad ⑥$$

$\frac{\partial}{\partial z}$⑥ + $\nabla_h^2$④

$$\Rightarrow \frac{\partial}{\partial z}\left[\rho_0 \frac{\partial^2 w}{\partial t \partial z}\right] = -\nabla_h^2 \frac{\partial \tilde{p}}{\partial z} = -\rho_0 \frac{\partial}{\partial t} \nabla_h^2 w - g \nabla_h^2 \tilde{\rho}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\left[\rho_0 \frac{\partial^2}{\partial z^2} + \rho_0\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right] w + \frac{\partial \rho_0}{\partial z}\frac{\partial w}{\partial z}\right) = -g\nabla_h^2 \tilde{\rho} \quad ⑦$$

$\frac{\partial}{\partial t}$⑦ using ①

$$\Rightarrow \boxed{\frac{\partial^2}{\partial t^2}\left[\nabla^2 w + \frac{1}{\rho_0}\left(\frac{d\rho_0}{dz}\right)\frac{\partial w}{\partial z}\right] = \frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z} \nabla_h^2 w} \quad ⑧$$

Remember $d\rho_0/dz < 0$.

Consider a situation where we have motion s.t. w varies on a scale L_w, ρ_0 varies on scale L_{ρ_0}.

We want $\frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial w}{\partial z} \ll \frac{\partial^2 w}{\partial z^2}$ and that occurs if

$$\frac{1}{L_{\rho_0}}\frac{1}{L_w} \ll \frac{1}{L_w^2} \Rightarrow L_w \ll L_{\rho_0}$$

i.e. velocity varies over much shorter distances than the density field — Boussinesq approximation. (BA).

BA assumes that density variations are only important in the "buoyancy term". i.e. when multiplied by g. Formally, equivalent to $|\rho_{ref} - \rho| \to 0$, $g \to \infty$ s.t. $-\frac{g}{\rho_{ref}}\frac{d\rho_0}{dz}$ and $g\rho'$ remain finite.

Then (8) reduces to

$$\frac{\partial^2}{\partial t^2}\nabla^2 w - \frac{g}{\rho_0}\frac{d\rho_0}{dz}\nabla_h^2 w = 0$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2 \nabla_h^2 w = 0} \qquad (9)$$

Note $N \in \mathbb{R}$ since $d\rho_0/dz < 0$. N is the Buoyancy frequency (frequency of vertical oscillation of fluid particles in a linearly stratified fluid) . a.k.a. Brunt-Väisälä frequency.

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}.$$

Typical period of oscillation in the atmosphere + ocean ~ minutes

Note The derivation allowed ρ_0 premultiplying $\frac{\partial u}{\partial t}$ etc to be f^n of z. BA equivalent to making ρ_0 on LHS of (2) – (4) const. so

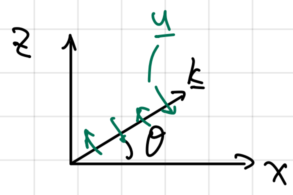
$$N^2 = -\frac{g}{\rho_{ref}}\frac{d\rho_0}{dz}.$$

14.2 Plane Waves Sol^n of ⑨

Such sol^n exists for w (and by back substitution for all variables) for const. N. $\underline{k} = (k, l, m)$, $\kappa = |\underline{k}|$ with $w \propto e^{i(\underline{k}\cdot\underline{x} - \omega t)}$

We then get dispersion relation

$$\omega^2 = \frac{N^2(k^2+l^2)}{k^2+l^2+m^2}$$



or

$$\omega = \pm N \cos\theta$$

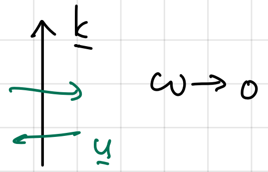
Note for const. N

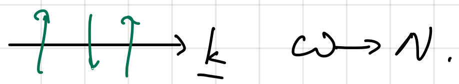
Comments :

(1) From ⑤, $\nabla\cdot\underline{u} = 0 \Rightarrow \underline{k}\cdot\underline{u} = 0 \Rightarrow$ Waves are transverse

$\underline{u}\cdot\nabla\underline{u} = \underline{0} \Rightarrow$ monochromatic waves are exact non-linear sol^n.

(2) For real $\underline{k}$, $|\omega| \le N$. As $\omega \to 0$, $\underline{k}$ point straight up, and $\underline{u}$ purely horizontal. As $\omega \to N$, $\underline{k}$ horizontal





Forced waves with $w > N$ are evanescent.

(3) Sharp change in environment : $[W]$, $[\tilde{p}] = 0$, and so $[\partial w/\partial z] = 0$ also

(4) ω is independent of $|\underline{k}| = \kappa$ but depend on direction of $\underline{k}$.

Exercise $\quad c_g = \frac{\partial \omega}{\partial \underline{k}} = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right) = \frac{N^2}{\omega}\frac{m}{\kappa^4}\left(km, lm, -k^2-l^2\right)$

$$\left(= \pm\frac{N}{\kappa}\sin\theta\,(\sin\theta, 0, -\cos\theta)\right)$$

in 2D if $l = 0$

We have $\underline{k} \cdot \underline{c_g} = 0$. $\Rightarrow$ energy propagates $\perp$ phase

Also see phase velocity and group velocity $\underline{c_g}$ have <u>different</u> vertical direction.

<u>Note</u> Vertical cpt of $c = W/m$. Remember radiation condition energy propagates outwards.

We oscillate to source at fixed $\omega < N$. Disp. rel. fixes $\hat{\underline{k}}$ but not $|\underline{k}| = k$.

<u>Example</u> In 2D, $\underline{k} = k(\cos\theta, 0, \sin\theta)$, $\cos\theta = \pm \omega / N$.

$$\underline{c_g} = \pm \frac{N}{k} \sin\theta \, (\sin\theta, 0, -\cos\theta) \quad \propto \sin\theta_g$$

$\propto \cos\theta_g$

Take + root, then

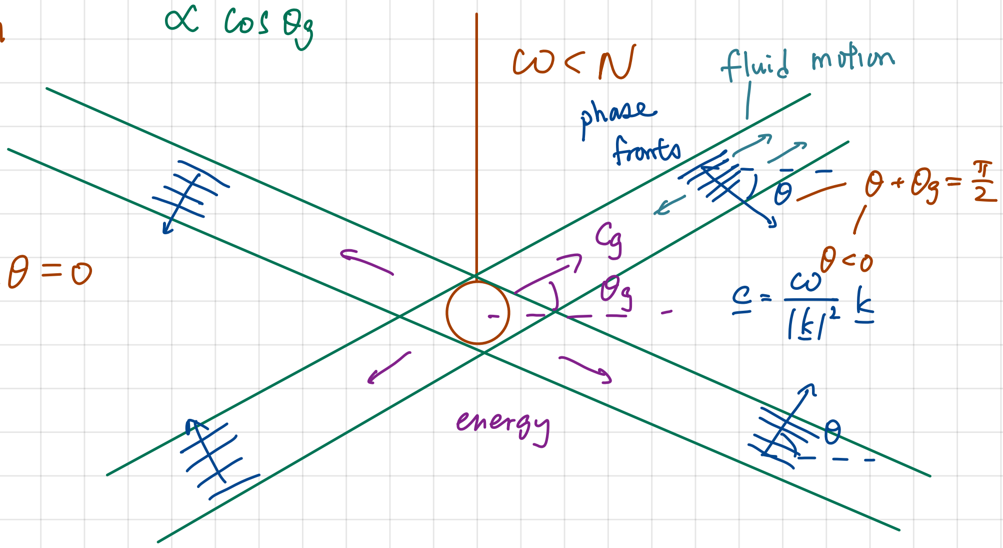
$$\cos\theta_g = \sin^2\theta$$

$$\sin\theta_g = -\sin\theta\cos\theta$$

$$\Rightarrow \cos\theta_g \cos\theta + \sin\theta_g \sin\theta = 0$$

$$\Rightarrow \cos(\theta_g - \theta) = 0$$

$$\Rightarrow \theta_g = \frac{\pi}{2} + \theta.$$



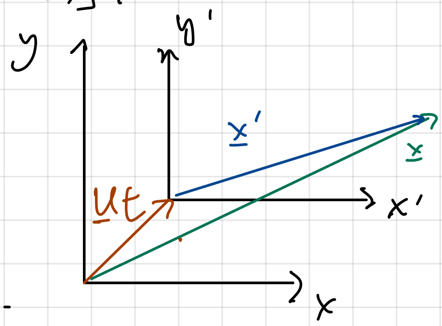
15. Moving Sources of Waves (Waves in a uniformly moving medium)

15.1 Dispersion relations and group velocity with moving source/media

Consider waves of form $\exp(i(\underline{k}\cdot\underline{x} - \omega t))$ in a uniform, stationary medium with pos. vector $\underline{x}$. Consider an observer or a source moving with const. velocity $\underline{U}$. In the observer's/source's frame of reference, the medium moves with velocity $-\underline{U}$ and the position vector is $\underline{x}' = \underline{x} - \underline{U}t$.

Thus the wave becomes $\exp[i(\underline{k}\cdot\underline{x}' + \underline{k}\cdot\underline{U}t - \omega t)]$.

So freq. in the frame of observer/frame is



$$\omega' = \omega - \underline{k}\cdot\underline{U}.$$

The dispersion relation for medium with vel. $-\underline{U}$ is

$$\omega = \Omega(\underline{k}) = \Omega_0(\underline{k}) - \underline{k}\cdot\underline{U}$$

drop prime

where $\Omega_0(\underline{k})$ is disp. rel. for the stationary medium. $\Omega_0(\underline{k})$ is often called the intrinsic frequency, $\Omega(\underline{k})$ is the extrinsic freq.

The group velocity is

$$\underline{c}_g(\underline{k}) = \nabla_{\underline{k}}(\Omega_0(\underline{k}) - \underline{k}\cdot\underline{U}) = \underline{c}_{g0} - \underline{U}$$

Notes

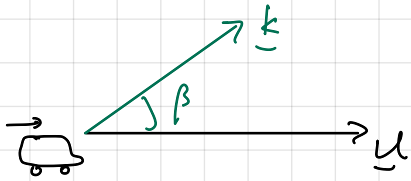
1. Even if $\Omega_0(\underline{k})$ isotropic $(f(|\underline{k}|))$, $\Omega(\underline{k})$ is not, because $\underline{U}$ introduces a direction.

2. Group velocities behave as expected under Galilean transformation, But phase "velocity" does not, as $\underline{c} = \frac{\omega}{|\underline{k}|}\hat{\underline{k}}$ is not a line velocity.

15.2 Doppler effect

Source freq. ω' and vel. $\underline{U}$ in a medium with sound speed c_0.

In the moving frame,



$$\omega' = c_0|\underline{k}| - \underline{U}\cdot\underline{k} = \omega(1 - M\cos\beta),$$

where $M = |\underline{U}|/c_0$ is the Mach no.

$$\Rightarrow \omega = \frac{\omega'}{1 - M\cos\beta}$$

Note: If source approaching fixed observer. $\cos\beta > 0 \Rightarrow \omega > \omega'$.

" receding $\cos\beta < 0 \Rightarrow \omega < \omega'$

16. Steady waves from moving sources

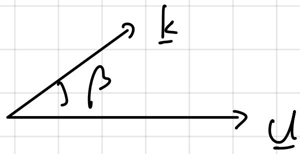
In general, waves are unsteady because of the $e^{-i\omega t}$ factor. However, they can be steady in the frame of reference of a moving source (or equivalently when the medium is moving uniformly).

Intrinsic freq. in a stationary medium

Since $\omega = \Omega_0(\underline{k}) - \underline{U}\cdot\underline{k} = 0$

extrinsic freq.

$$\Rightarrow c_{p0}(\underline{k}) = U\cos\beta.$$



16.1 Capillary waves in 1D

Consider a cm-size object moving at speed $\underline{U}$ in deep water. If $|\underline{U}|$ large enough, we will see a steady wavepacket in the frame of the object.

Consider 1D case for simplicity.

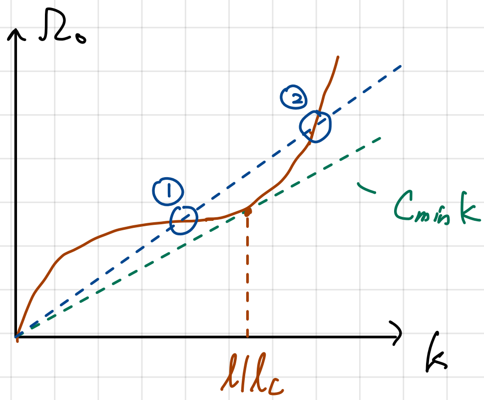
$$\omega(k) = \pm\left(g|k| + \frac{T}{\rho}|k|^3\right)^{1/2} - Uk$$

Look for $\omega(k) = 0$.

Note that $\frac{\partial c}{\partial k} = \frac{\partial}{\partial k}\left(\frac{\omega}{k}\right) = \frac{1}{k}(c_g - c_p)$,

So $c_{min} = c_g$,

$$c = \frac{\omega}{k} = \left(\frac{g}{k} + \frac{T}{\rho}k\right)^{1/2}$$



$$\Rightarrow \frac{\partial c}{\partial k} = \frac{1}{2}\left(\frac{g}{k} + \frac{T}{\rho}k\right)^{-1/2}\left(-\frac{g}{k^2} + \frac{T}{\rho}\right) = 0 \quad \text{when} \quad k^2 = \frac{g\rho}{T} = \frac{1}{l_c^2}.$$

So no sol^n if $U < \min_k \frac{\Omega_0}{k} = (2g\,l_c)^{1/2} = c_{min}$

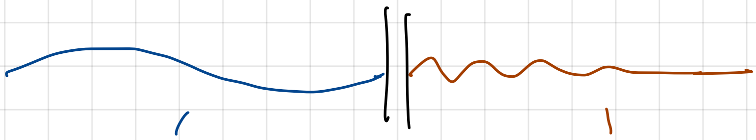
two sol^n if $U > c_{min}$ (with sol^n in all other quadrants)

$$c_g = \frac{\partial \Omega_0}{\partial k} - U$$

and information radiates outward.

For point ①, $c_g < 0 \Rightarrow$ no good in $x > 0 \Rightarrow$ long waves behind in $x < 0$.

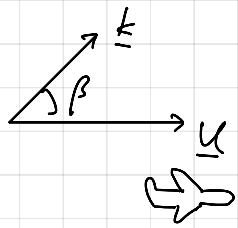
point ②, $c_g > 0 \Rightarrow$ no good in $x < 0 \Rightarrow$ short waves ahead in $x > 0$.



16.2 Supersonic boom and Mach cones ($M > 1$)

Consider an aircraft with velocity $\underline{U} = Mc_0(1, 0)$.

In the frame of ref. of aircraft,



$$\omega = c_0|\underline{k}| - \underline{U}\cdot\underline{k} = c_0|\underline{k}|(1 - M\cos\beta)$$

$\Rightarrow$ Steady waves if $\beta = \arccos(1/M) \rightarrow$ unique angle indpt of $|\underline{k}|$

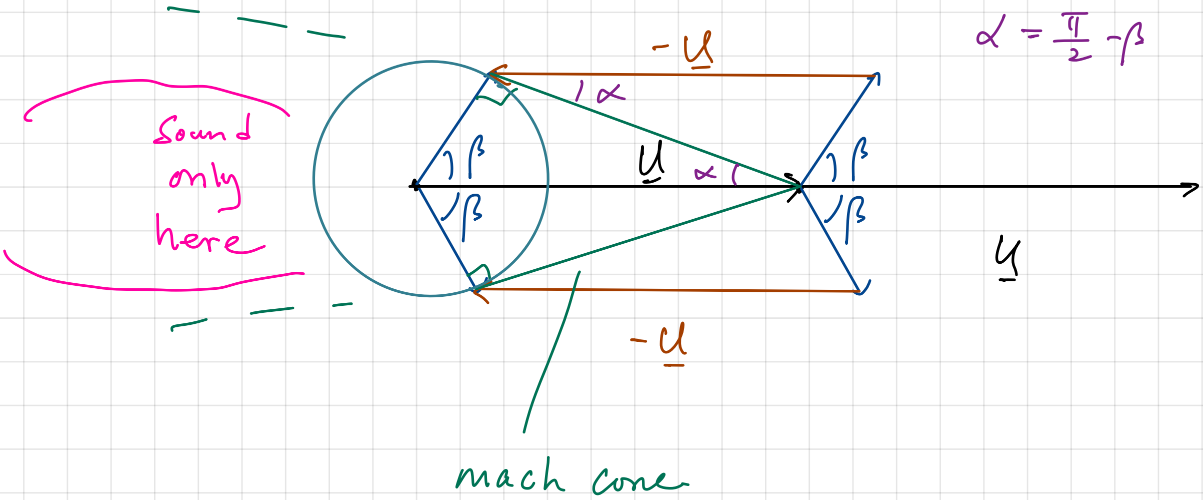
$$\Rightarrow \hat{\underline{k}} = \left(\frac{1}{M}, \left(1 - \frac{1}{M^2}\right)^{1/2}\right) = (\cos\beta, \sin\beta)$$

For these waves,

$$\underline{c}_g = c_0\underline{k} - \underline{U}$$

$$\Rightarrow \underline{c}_g = \left(\frac{c_0}{M} - Mc_0, \; c_0\left(1 - \frac{1}{M^2}\right)^{1/2}\right)$$

$$= \frac{c_0}{M}\left(1 - M^2, (M^2 - 1)^{1/2}\right) = c_0(M^2-1)^{1/2}(-\sin\beta, \cos\beta).$$



Hence, steady waves (boom!) on Mach cone (green lines) behind aircraft of semi-angle

$$\alpha = \arcsin(1/M)$$

Unsteady waves (sound) are found within the Mach cone (pink region) because

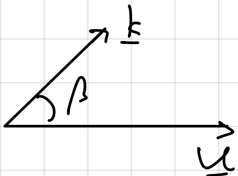
$$\underline{c}_g = c_0 \hat{\underline{k}} - \underline{U} \quad \forall \underline{k}$$

Note For real sonic booms, non-linearity matters

16.3 Kelvin Ship Waves

Consider a ship / duck with steady velocity $\underline{U}$, with $\underline{U} = U(1,0)$ in deep water (ignore surface tension).

In the frame of the ship,



$$\omega = \sqrt{g|\underline{k}|} - \underline{U}\cdot\underline{k}$$

Steady waves if

$$\boxed{\cos\beta = \frac{1}{U}\cdot\sqrt{\frac{g}{|\underline{k}|}}}$$

Note $c_{p0} = (g/|\underline{k}|)^{1/2}$ in stationary fluid

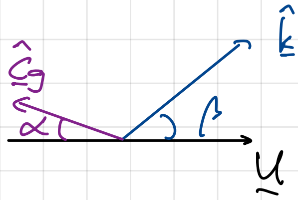
For these waves,

$$\underline{c_g} = \frac{1}{2}\sqrt{\frac{g}{|\underline{k}|}}\,\hat{\underline{k}} - \underline{U} = \frac{1}{2} U \cos\beta(\cos\beta, \sin\beta) - U(1,0)$$

$$= \frac{U}{4}(\cos 2\beta, \sin 2\beta) - \frac{3U}{4}(1,0).$$

Therefore, energy propagates in a direction

$$\hat{\underline{c}}_g = (-\cos\alpha, \sin\alpha), \quad \tan\alpha = \frac{\sin 2\beta}{3-\cos 2\beta}.$$



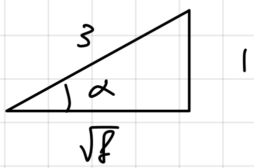
Locus of $\underline{c_g}$ lies on a circle of radius $\frac{U}{4}$ centred at $-\left(\frac{3U}{4}, 0\right)$.

Max. over α:

$$\sec^2\alpha \frac{d\alpha}{d\beta} = \frac{2\cos 2\beta}{3-\cos 2\beta} - \frac{2\sin^2 2\beta}{(3-\cos 2\beta)^2} = \frac{2}{(3-\cos 2\beta)^2}(3\cos 2\beta - 1)$$

max when $\cos 2\beta = 1/3$, $\sin 2\beta = \sqrt{8}/3$

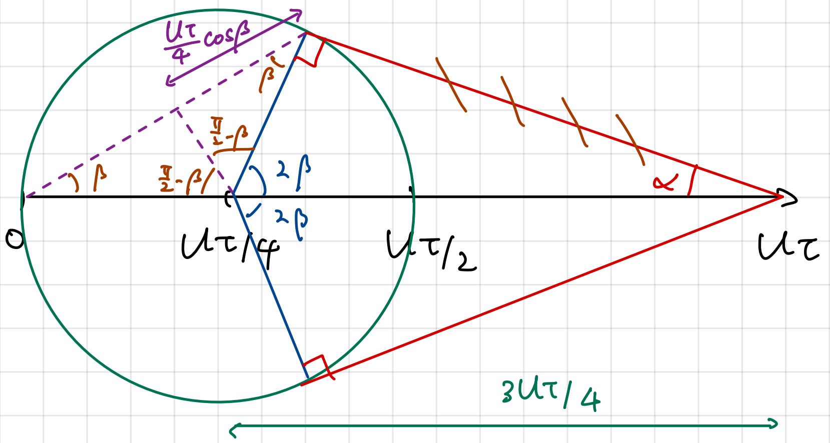
$$\Rightarrow \tan\alpha = \frac{1}{\sqrt{8}}$$



$$\Rightarrow \alpha = \arcsin\left(\frac{1}{3}\right) \approx 19°.$$

Interpretation: while a ship / duck moves a distance $U\tau$, waves emitted at $t=0$ travelled a distance $c_{g0}\tau$, where

$$c_{g0} = \frac{c_{p0}}{2} = \frac{U\cos\beta}{2}$$



Shape of wave crests

Idea: at position $\underline{x} = (x, y) = r(-\cos\alpha, \sin\alpha)$.

We know $\alpha \Rightarrow$ know $\beta \Rightarrow$ know $|\underline{k}| \Rightarrow$ phase. $\theta = \underline{k} \cdot \underline{x}$. (because $\omega = 0$)

$$\theta = \underline{k} \cdot \underline{x} = |\underline{k}| r \cos(\pi - \alpha - \beta)$$

Recall $\cos\beta = \frac{1}{U}\sqrt{\frac{g}{|\underline{k}|}} \Rightarrow |\underline{k}| = \frac{g}{(U\cos\beta)^2}$.

$$\Rightarrow \theta = -\frac{gr}{U^2\cos^2\beta} \cos\alpha \cos\beta \left(1 - \tan\alpha \tan\beta\right)$$

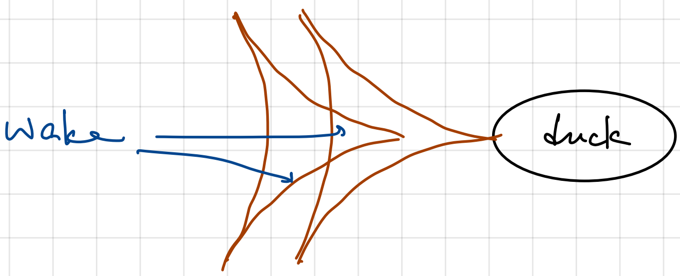
$$= \frac{gx}{U^2\cos\beta}\left(1 - \frac{\sin 2\beta \tan\beta}{3 - \cos 2\beta}\right)$$

$$= \frac{gx}{U^2\cos\beta} \cdot \frac{3 - \cos^2\beta + \sin^2\beta - 2\sin^2\beta}{3 - \cos 2\beta}$$

$$= \frac{2gx}{U^2\cos\beta(3 - \cos 2\beta)}$$

$$\Rightarrow x = \frac{U^2\theta}{2g} \cos\beta \, (3 - 2\cos 2\beta),$$

$$y = -x\tan\alpha = -\frac{U^2\theta}{2g}\cos\beta \sin 2\beta.$$



Unsteady waves overlying this pattern.

Part IV : Ray Theory

17. Ray Theory

17.1 Introduction

Ray theory (aka geometric optics).

WKBJ method is an asymptotic approximation for long distance/time propagation through slowly varying media, i.e. media which change on length scales $\gg 1/k$ and/or timescales $\gg 1/\omega$.

Motivation is waves often transport energy/momentum over distances $\gg 1/k$ and times $\gg 1/\omega$ into regions of different properties.

e.g. waves on beaches generated originally by storms in deep ocean.

clear air turbulence due to breaking waves or shear instabilities associated with IGW from low levels.

Q: How can we use local sol^n (on the wavescale) to find behaviour on much larger scales (over which medium changes)

17.2 Multiple Scales

Assume medium varies on $O(1)$ scale (length and/or time)

wavelength (period) $O(\varepsilon)$ scale, $\varepsilon \ll 1$.

The issue is that wavelength varies across medium, so can't just assume $e^{i\underline{k}\cdot\underline{x}}$ are $O(1)$ scales. Key idea is to focus on phase, i.e. look for sol^n

$$\phi(\underline{x},t) = A(\underline{x},t)\, e^{i\theta(\underline{x},t)/\varepsilon}.$$

(1) Variation in phase θ/ε rapid

(2) Variation in A slow.

Remember we assumed $e^{i(\underline{k}\cdot\underline{x}-\omega t)}$ for uniform media.

Defn $\underline{k}(\underline{x},t) := \frac{1}{\varepsilon}\frac{\partial\theta}{\partial\underline{x}} = \frac{1}{\varepsilon}\nabla_{\underline{x}}\theta$

$\omega(\underline{x},t) := -\frac{1}{\varepsilon}\frac{\partial\theta}{\partial t}$.

Note Both $\underline{k}$, ω large, and may still vary slowly with $\underline{x}, t$.

$$\frac{\partial\phi}{\partial t} = \phi\Big(\underbrace{\cancel{\frac{1}{A}\frac{\partial A}{\partial t}}}_{O(1)} - \underbrace{\frac{i}{\varepsilon}\frac{\partial\theta}{\partial t}}_{O(1/\varepsilon)}\Big) \approx -i\omega\phi$$

Similarly,
$$\frac{\partial\phi}{\partial\underline{x}} \approx i\underline{k}\phi$$

Both at $O(1/\varepsilon)$

So locally, i.e. on $O(\varepsilon)$ scales, $\phi \approx Ae^{i(\underline{k}\cdot\underline{x}-\omega t)}$ with const. values of A, $\underline{k}$, ω, where $\underline{k},\omega$ are local wave vector and frequency.

Similarly,
$$\frac{\partial^n}{\partial t^n}\phi \approx (-i\omega)^n\phi.$$

and so for ϕ to satisfy wave eqn, a local dispersion relation must apply
$$\omega = \Omega(\underline{k};\underline{x},t)$$

with local properties $N(z)$, $h(x)$, etc.

buoyancy freq.

17.3 Phase and wave crests

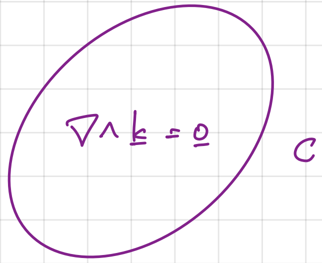
From defns,
$$\nabla\wedge\underline{k} = 0 \Rightarrow \frac{\partial k_j}{\partial x_i} = \frac{\partial x_i}{\partial k_j} \qquad (1)$$

$$\frac{\partial\underline{k}}{\partial t} = -\nabla_x\omega(\underline{x},t) \qquad (2)$$

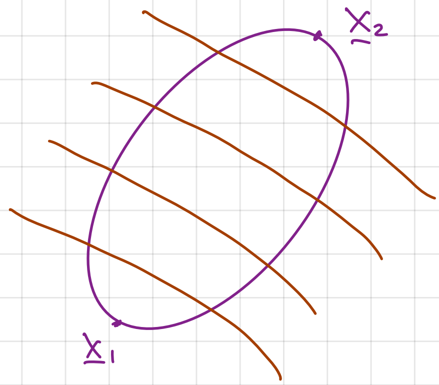
natural generalisation of $e^{i\underline{k}\cdot\underline{x}}$ dependence

$d\theta = \varepsilon(\underline{k}\cdot d\underline{x} - \omega\, dt) \Rightarrow$ at fixed t, $\phi \propto e^{i\int k(x)\,dx}$ over an $O(1)$ lengthscale.

These eqns lead to an interpretation in terms of conservation from (1) and Stoke's thm.

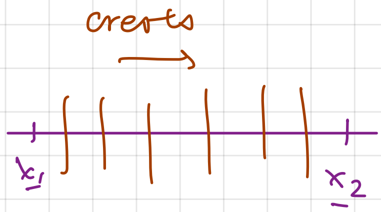


$$\Rightarrow \int_C \underline{k} \cdot d\underline{x} = 0 .$$



$\theta(\underline{x}_2) - \theta(\underline{x}_1) = \varepsilon \int_{\underline{x}_1}^{\underline{x}_2} \underline{k} \cdot d\underline{x}$ indpt of path $\Rightarrow$ # crests between $\underline{x}_1$, $\underline{x}_2$ also indpt of path.

From (2), $\quad \dfrac{d}{dt} \displaystyle\int_{\underline{x}_1}^{\underline{x}_2} \underline{k} \cdot d\underline{x} = \omega(\underline{x}_1) - \omega(\underline{x}_2)$



Change in # of crests in the region

= # entering the region − # leaving the region

18. Ray Tracing Equations

18.1 Evolution of $\underline{k}$, ω

As a wavepacket moves through medium, $\underline{k}$, ω must vary smoothly and slowly in order to satisfy the local dispersion relation.

$$\boxed{\omega(\underline{x}, t) = \Omega(\underline{k}(\underline{x}, t) \, ; \underline{x}, t).}$$

Apply chain rule,

$$\frac{\partial \omega}{\partial x_i} = \frac{\partial \Omega}{\partial k_j} \cdot \frac{\partial k_j}{\partial x_i} + \frac{\partial \Omega}{\partial x_i}$$

$$\overset{(1),(2)}{\Rightarrow} \quad -\frac{\partial k_i}{\partial t} = \frac{\partial \Omega}{\partial k_j} \frac{\partial k_i}{\partial x_j} + \frac{\partial \Omega}{\partial x_i} .$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + (\underset{\sim}{c}_g)_i \frac{\partial}{\partial x_j} \right) k_i = - \frac{\partial \Omega}{\partial x_i}$$

$$\Rightarrow \boxed{\frac{\partial \underset{\sim}{k}}{\partial t} + \underset{\sim}{c}_g \cdot \nabla_x \underset{\sim}{k} = - \nabla_x \Omega .}$$

$\therefore$ Moving with the group velocity $\underset{\sim}{c}_g$, $\underset{\sim}{k}$ of the wavepacket changes iff medium varies with $\underset{\sim}{x}$.

Similarly,

$$\frac{\partial \omega}{\partial t} = \frac{\partial \Omega}{\partial k_j} \frac{\partial k_j}{\partial t} + \frac{\partial \Omega}{\partial t}$$

$$\overset{(2)}{\Rightarrow} \boxed{\frac{\partial \omega}{\partial t} + \underset{\sim}{c}_g \cdot \nabla_x \omega = \frac{\partial \Omega}{\partial t}}$$

$\therefore$ Moving with group velocity $\underset{\sim}{c}_g$, ω of wavepacket changes iff medium depends on <u>time</u>.

<u>18.2 Ray tracing</u>

We define a <u>ray</u> as a trajectory $\underset{\sim}{x} = \underset{\sim}{x}(t)$ of a wave particle which satisfies

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{c}_g .$$

We then obtain the <u>ray tracing equations</u>.

(1): $\dfrac{d\underset{\sim}{x}}{dt} = \underset{\sim}{c}_g = \dfrac{\partial \Omega}{\partial \underset{\sim}{k}} = \nabla_{\underset{\sim}{k}} \Omega$

Let $\left.\dfrac{d}{dt}\right|_g := \dfrac{\partial}{\partial t} + \underset{\sim}{c}_g \cdot \nabla_x$, ie the time derivative moving along a ray.

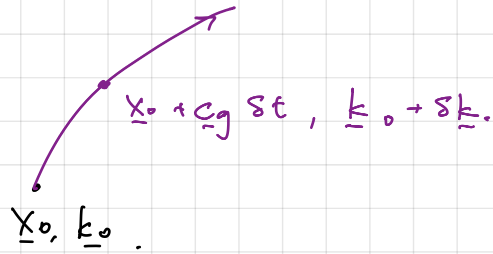
(2): $\left.\dfrac{d\omega}{dt}\right|_g = \dfrac{\partial \Omega}{\partial t}$

(3) : $\dfrac{d\underset{\sim}{k}}{dt} = - \dfrac{\partial \Omega}{\partial \underset{\sim}{x}} = - \nabla_x \Omega$

(4): $\dfrac{1}{\varepsilon} \left.\dfrac{d\theta}{dt}\right|_g = -\omega + \underset{\sim}{c}_g \cdot \underset{\sim}{k}$

Given Ω and IC $\underline{k}_0$, ω_0 and θ_0 at position $\underline{x}_0$, we can integrate along a ray to calculate the evolution.

Do this process for every ray (i.e. lots of initial positions and $\underline{k}_0$ $\Rightarrow$ had $\underline{k}$, ω everywhere).



18.3 Comments

1) For a uniform, constant medium (Part III),

$$\frac{\partial \Omega}{\partial t} = 0, \quad \frac{\partial \Omega}{\partial \underline{x}} = 0 \Rightarrow \omega, \underline{k} \text{ are const. on rays.}$$

$$\Rightarrow \underline{c}_g = \frac{\partial \Omega}{\partial \underline{k}} \text{ const. on rays}$$

$$\Rightarrow \text{rays are straight lines}$$

2) RTE have 1st integral $\omega = \Omega(\underline{k}, \underline{x}, t)$, can be used to solving one of (1) to (3).

3) Rays in a wind (moving medium). Recall if $\omega = \Omega(\underline{k})$ for a stationary medium, then for a medium with velocity $-\underline{U}$

$$\Rightarrow \omega = \Omega_0(\underline{k}) - \underline{k} \cdot \underline{U}.$$

If $\underline{U}(\underline{x})$ is slowly varying, eqn for a ray (1) is now

$$\frac{d\underline{x}}{dt} = \frac{\partial}{\partial \underline{k}}\left(\Omega_0 - \underline{U} \cdot \underline{k}\right) = \underbrace{\frac{\partial \Omega_0}{\partial \underline{k}}}_{\text{intrinsic } \underline{c}_g} - \underbrace{\underline{U}(\underline{x})}_{\text{wind velocity}}.$$

* 4) The eqns (2), (3) for $\underline{k}$, ω are Hamilton's eqn for a dynamical sys. with generalised coords $\underline{q} = \underline{x}$, $\underline{p} = \underline{k}$. Wave packets behave like particles with Hamiltonian $H = \Omega(\underline{k}; \underline{x}, t)$.

Action is $S = \theta/t$ (i.e. phase is action).

H-J eqn: $$\frac{\partial S}{\partial t} + \Omega(\nabla S; \underline{x}, t) = 0$$

* 5) In general, the <u>energy density</u> is not conserved. Instead, the <u>wave action</u>
$$I = E/\omega$$
obeys the conservation law.
$$\frac{\partial I}{\partial t} + \nabla \cdot (\underline{c}_g I) = 0$$
In special case where $\partial\Omega/\partial t = 0$
$$\Rightarrow \left.\frac{d\omega}{dt}\right|_g = 0$$
then conservation of I eqv. to conservation of wave energy E.
$$E \propto |A|^2 \omega^2 \longleftrightarrow I \propto |A|^2 \omega.$$

19. (Isotropic) Ray tracing examples

19.1 Isotropic (local) dispersion relation

A dispersion relation is isotropic iff it is a f^n of the waveno.
$$\kappa = (k_1^2 + k_2^2 + k_3^2)^{1/2},$$
ie. $\omega = \Omega(\kappa)$ and so does not depend on $\hat{\underline{k}} = \underline{k}/\kappa$, ie. direction.

Rays are defined by
$$\frac{d\underline{x}}{dt} = \underline{c}_g = \nabla_{\underline{k}} \Omega = \frac{\partial\Omega}{\partial \underline{k}}.$$

For isotropic relations,
$$\frac{\partial\Omega}{\partial k_i} = \frac{\partial\Omega}{\partial\kappa} \cdot \underbrace{\frac{\partial\kappa}{\partial k_i}}_{= k_i/\kappa \, = \hat{k}_i}$$

$$\Rightarrow (\underline{c}_g)_i = \frac{\partial\Omega}{\partial k_i} = \frac{\partial\Omega}{\partial\kappa} \hat{k}_i.$$

For isotropic dispersion relations, rays $\parallel$ to $\underline{k}$.

Actually, the eqn for a ray:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{k_2/\kappa \ \frac{\partial \Omega}{\partial \kappa}}{k_1/\kappa \ \frac{\partial \Omega}{\partial \kappa}} = \frac{k_2}{k_1}.$$

Wavecrests

Remember lines of constant phase (WLOG x-y plane),

$$\theta_x \, dx + \theta_y \, dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\theta_x}{\theta_y} = -\frac{\varepsilon k_1}{\varepsilon k_2} = -\frac{k_1}{k_2}.$$

ie. in general, $\underline{k}$ always $\perp$ crests.

Since crests are lines of const. phase, so normal is given by $\nabla_x \theta \propto \underline{k}$.

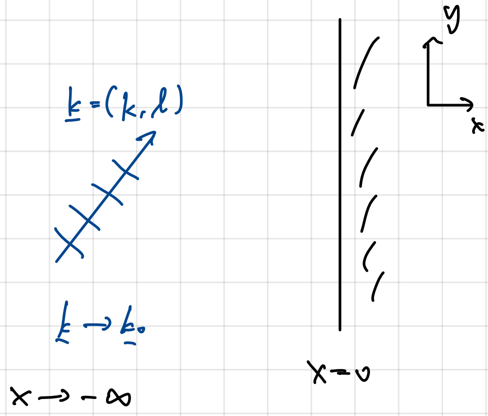
For isotropic dispersion relations. Since rays $\parallel$ $\underline{k}$, rays $\perp$ crests.

19.2 Waves on a Beach

Stationary water $\underline{U} = \underline{0}$., slowly varying depth $h(x)$ prescribed.

Neglect surface tension.

$$\Rightarrow \Omega^2 = g\,|\underline{k}| \tanh\left(|\underline{k}|\, h(x)\right).$$



Far from the shore, assume $h \to \infty$.

$$\Rightarrow \kappa = |\underline{k}|, \quad \kappa_\infty = (k_\infty^2 + l_\infty^2)^{1/2} \text{ as } x \to -\infty$$

$$\Rightarrow \omega \to \omega_\infty = (g\,\kappa_\infty)^{1/2}.$$

$$\left.\frac{d\omega}{dt}\right|_g = \left(\frac{\partial}{\partial t} + \underline{c}_g \cdot \nabla\right)\omega = \frac{\partial \Omega}{\partial t} = 0 \Rightarrow \omega \text{ const. on rays,}$$

$$\Rightarrow \omega = \omega_\infty \text{ everywhere}$$

$$\left.\frac{dl}{dt}\right|_g = -\frac{\partial \Omega}{\partial y} = 0 \Rightarrow l = l_\infty \text{ everywhere}$$

$$\left.\frac{dk}{dt}\right|_g = -\frac{\partial \Omega}{\partial x} \neq 0 \Rightarrow k \text{ will vary on rays.}$$

Can avoid solving this by using the first integral $\omega = \Omega$

$$\Rightarrow \omega_\infty^2 = g\left(k_\infty^2 + l_\infty^2\right)^{1/2} = g\left(k^2(x) + l_\infty^2\right)^{1/2} \tanh\left[\left(k(x)^2 + l_\infty^2\right)^{1/2} h(x)\right].$$

Note If we know $k(x)$, we can find eqns for rays, crests, etc.

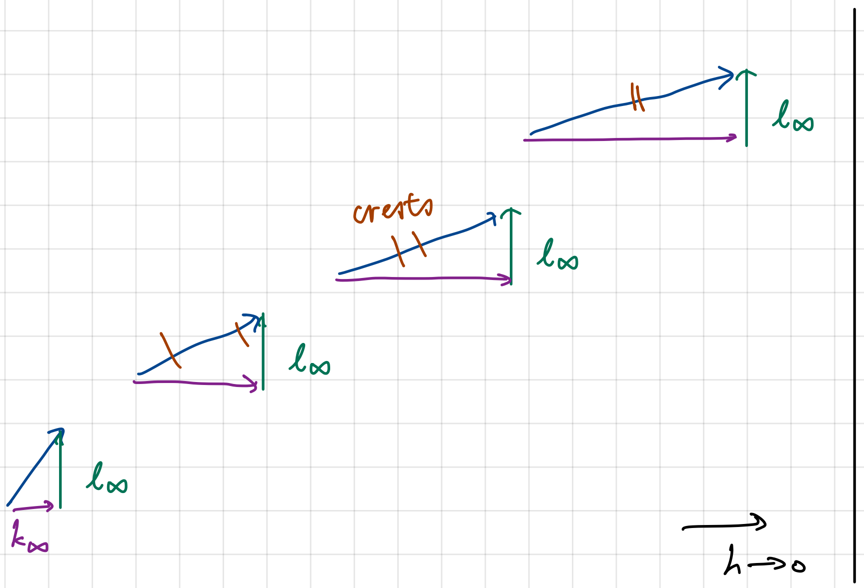
As $h \to 0$, $[\;\;] \to 0$, then

$$g\left(k_\infty^2 + l_\infty^2\right)^{1/2} \simeq g\left(k(x)^2 + l_\infty^2\right) h(x).$$

$$\Rightarrow \quad k(x)^2 + l_\infty^2 \approx \frac{\left(k_\infty^2 + l_\infty^2\right)^{1/2}}{h(x)} \simeq k(x)^2.$$

Self consistent since implication is $k \sim O(h^{-1/2})$ as $h \to 0$

$[\;\;] \sim O(h^{1/2}) \to 0$ as $h \to 0$.



19.3 Snell's law

Here, and for § 19.4, restrict attention to steady, slowly varying media with non-dispersive waves (locally a special case of isotropy), i.e.

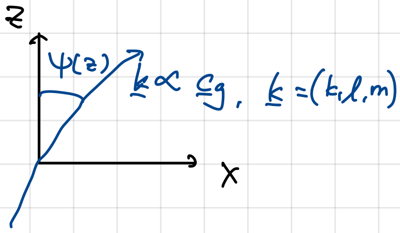
$$\omega = c(\underline{x})\,|\underline{k}| = \Omega(|\underline{k}|, \underline{x})$$

Non-dispersive $\underline{c}_g = c\hat{\underline{k}} \Rightarrow$ rays parallel to $\hat{\underline{k}}$, and there is only one speed at each $\underline{x}$.

Steady $\Rightarrow \frac{\partial \Omega}{\partial t} = 0 \Rightarrow \omega$ const. on rays $\Rightarrow c$ indpt. of t.

Snell's law then follows. In a planar stratified medium where $c = c(z)$, the angle $\psi(z)$ between a ray and the z-axis obeys Snell's law.

$$\frac{\sin \psi(z)}{c(z)} = \text{const.}$$



Pf: Since $\underline{k} \parallel$ ray.

$$\frac{\sin \psi(z)}{c(z)} = \frac{(k^2+l^2)^{1/2}}{c|\underline{k}|} = \frac{(k^2+l^2)}{\omega} = \text{const.}$$

Since $\frac{\partial \Omega}{\partial x} = 0$, k const. on a ray.

$\frac{\partial \Omega}{\partial y} = 0$, l const. on a ray.

$\frac{\partial \Omega}{\partial t} = 0$, ω const. on a ray $\square$.

To find the eqn of a ray in a stratified medium. WLOG, let us assume $\underline{k} = (k, 0, m)$

$$\frac{d\underline{x}}{dt} = \underline{c}_g = c\hat{\underline{k}} = \frac{dz/dt}{dx/dt}$$

$$= \frac{dz}{dx}$$

$$= \frac{cm/\kappa}{ck/\kappa} = \pm\sqrt{\frac{\kappa^2 - k^2}{k^2}} = \pm\sqrt{\frac{\omega^2}{k^2 c^2(z)} - 1} \Rightarrow x(z)$$

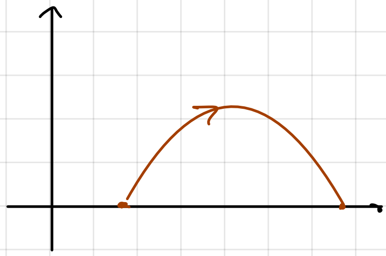
Classic example

Summer nights, temp. inversion leads to sound speed increasing with height.

$$c(z) = C_0(z + l)$$

$\Rightarrow$ rays are semicircles.

$\Rightarrow$ Sounds travel far on summer nights.



19.4 Fermat's Principle

The time of travel between fixed points A, B,

$$\int_A^B \frac{ds}{c(\underline{x})}$$

is stationary (usually minimum) along a ray with respect to variations in the path.

Pf: (for non-dispersive) $\quad \frac{ds}{c} = \frac{|\dot{\underline{x}}|\,dt}{c} \Rightarrow \mathcal{L}(\underline{x}, \dot{\underline{x}}) = \frac{(\dot{x}_j \dot{x}_j)^{1/2}}{c(\underline{x})}.$

E-L: $\quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i} \Rightarrow \frac{d}{dt}\left(\frac{\dot{x}_i}{c|\dot{\underline{x}}|}\right) = |\dot{\underline{x}}| \frac{\partial}{\partial x_i}\left(\frac{1}{c}\right).$

On a ray, $\quad \dot{\underline{x}} = c\hat{\underline{k}}. \Rightarrow |\dot{\underline{x}}| = c. \Rightarrow \frac{\dot{\underline{x}}}{c|\dot{\underline{x}}|} = \hat{\underline{k}}/c. = \underline{k}/\omega.$

On a ray, ω indpt of time

$\Omega = c|\underline{k}|$

$$\Rightarrow LHS = \frac{1}{\omega}\frac{d\underline{k}}{dt} = -\frac{1}{\omega}\frac{\partial \Omega}{\partial \underline{x}}. = -\frac{|\underline{k}|}{\omega}\frac{\partial c}{\partial \underline{x}}. = -\frac{1}{c}\frac{\partial c}{\partial \underline{x}}.$$

at fixed $\underline{k}$

$$RHS = c\frac{\partial}{\partial \underline{x}}\left(\frac{1}{c}\right) = -\frac{c}{c^2}\frac{\partial c}{\partial \underline{x}} = -\frac{1}{c}\frac{\partial c}{\partial \underline{x}} = LHS. \qquad \square$$

Exercise Show that Fermat's principle in a stratified medium implies Snell's law as RHS of E-L eqn $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = 0.$

Part V : Non-linear 1D waves

20. 1D Waves (in a perfect gas)

Before, considered linear waves – small amplitude and smooth.

Now, consider <u>non-linear</u> waves in a perfect compressible gas or in shallow water.

For the simplest 1D wave,

$$\underline{u}(\underline{x},t) = (u(x,t), 0, 0)$$

Non-linear: no superposition / no Fourier / no dispersion relation / no stationary phase / no rays.

20.1 1D waves in a perfect gas

Recall conservation of mass $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0$

mom $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p$ $\quad \frac{c^2}{\rho}\frac{\partial \rho}{\partial x}.$

In 1D, homentropic,

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\cdot\frac{\partial \rho}{\partial x} = c^2(\rho)\frac{\partial \rho}{\partial x}.$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \qquad (1)$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} = 0 \qquad (2).$$

(2) + λ(1):

$$\left(\frac{\partial}{\partial t} + (u+\lambda\rho)\frac{\partial}{\partial x}\right)u + \lambda\left(\frac{\partial}{\partial t} + \left(u+\frac{c^2}{\lambda\rho}\right)\frac{\partial}{\partial x}\right)\rho = 0.$$

For these to have same operator, $\lambda\rho = \frac{c^2}{\lambda\rho} \Rightarrow \lambda = \pm c/\rho.$

$$\Rightarrow \left(\frac{\partial}{\partial t} + (u\pm c)\frac{\partial}{\partial x}\right)u \pm \frac{c}{\rho}\left(\frac{\partial}{\partial t} + (u\pm c)\frac{\partial}{\partial x}\right)\rho = 0$$

Define $Q = \int_{\rho_0}^{\rho}\frac{c(\hat{\rho})}{\hat{\rho}}\,d\hat{\rho} \Rightarrow \frac{\partial Q}{\partial t} = \frac{c}{\rho}\frac{\partial \rho}{\partial t}. \quad \frac{\partial Q}{\partial x} = \frac{c}{\rho}\frac{\partial \rho}{\partial x}$

$$\Rightarrow \left(\frac{\partial}{\partial t} + (u\pm c)\frac{\partial}{\partial x}\right)(u\pm Q) = 0$$

on any path satisfying $\frac{dx}{dt} = u \pm c$ ← $C_\pm$ characteristics

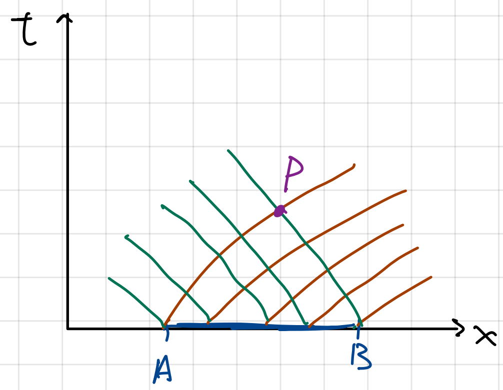
$$R_\pm = u \pm Q \quad \text{const.},$$

Riemann invariants

$$\frac{d}{dt} R_+ = \left(\frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x}\right) R_+ = \left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right) R_+ = 0.$$

20.2 Method of Characteristics

This forms the basis for the method of characteristics. If we can find which C_+ and which C_- char. pass through a given point P, and if we know $u+Q$ on C_+ and $u-Q$ on C_- (e.g. by following the char. back to initial / boundary conditions at $[A, B]$), then we will know $u, Q(\rho)$ at P, and hence we know $\rho, p(\rho)$, $c(\rho)$ and so we know $u \pm c$, and then we can in principle extend char. farther.



20.3 Issues

(1) Char. are not straight lines in (x, t) plane.

e.g. on the C_- from B to P, $u-Q$ const., but u, Q and hence $u-c$ will vary if the C_+ bring different values of $u+Q$ for initial conditions in $[A, B]$.

(2) Hence, in general, simultaneous sol^n. along C_+ and C_- char. is a difficult non-linear problems. Solvable problems usually rely on a simple sol^n for one set of char.

(3) The value of P depends on values in the finite interval $[A, B]$.

(4) Difficulties arise if two C_+ intersect with contradictory $u + Q$ values $\Rightarrow$ shocks

20.4 Perfect Gas

For adiabatic changes in a perfect gas,

$$p = p_0 \left(\rho/\rho_0 \right)^{\gamma}$$

c_0 reference sound speed at p_0, ρ_0

$$\Rightarrow c^2 = \left.\frac{dp}{d\rho}\right|_\rho = \frac{\gamma p_0}{\rho_0} (\rho/\rho_0)^{\gamma-1} = c_0^2 (\rho/\rho_0)^{\gamma-1}$$

$$Q = \int_{\rho_0}^{\rho} \frac{c(\hat{\rho})}{\hat{\rho}} d\hat{\rho} = \int_{\rho_0}^{\rho} \frac{c_0}{\rho_0^{(\gamma-1)/2}} \hat{\rho}^{(\gamma-3)/2} d\hat{\rho}$$

$$= \frac{2}{\gamma-1} c_0 \left((\rho/\rho_0)^{(\gamma-1)/2} - 1 \right).$$

$$\Rightarrow \boxed{Q = \frac{2}{\gamma-1}(c - c_0), \quad \rho = \rho_0 \left(c/c_0 \right)^{2/\gamma-1}, \quad p = p_0 \left(c/c_0 \right)^{2\gamma/\gamma-1}}$$

20.5 Simple Waves

A simple wave is a wave which is simple, where one of R_I is uniformly const. WLOG $R_I = 0$, i.e. where $u = Q$ or where $u = -Q$.

Consider the sol^n for a perfect gas. Suppose the ICs satisfy

$$\underbrace{u(x,0) = Q(x,0)}_{\text{i.e. } R_- = 0} = \frac{2}{\gamma-1}[c(x,0) - c_0] \quad \forall x.$$

Note This includes the special case of undisturbed fluid $u=0$, $c=c_0$.

$u - Q = 0 \quad \forall x$ initially $\Rightarrow$ along every c_-, $u = Q$ everywhere $\forall x \; \forall t$.

$$\Rightarrow \quad c = c_0 + \left(\frac{\gamma - 1}{2}\right) u$$

Along any C_+, $u + Q = \text{const.} \Rightarrow u + Q = 2u = \text{const.}$

$$\Rightarrow u + c = c_0 + \frac{\gamma+1}{2} u = \text{const.}$$

$\Rightarrow$ C_+ char. are straight lines of the form

$$x = x_0 + \left[c_0 + \frac{\gamma+1}{2} u(x_0, 0) \right] t.$$

So given (x, t), we solve this non-linear eqn for $x_0(x,t)$ and $u(x_0,0)$

Const. on C_+ $\Rightarrow$ $u(x,t) = u(x_0, 0)$.

A sol^n with $u = Q$ $(R_- = 0)$ is a <u>right-going simple wave</u> (C_+ char.)

" $u = -Q$ $(R_+ = 0)$ " left-going " (C_- char.)

Note We sometimes speak of a simple wave within a certain region rather than $\forall x$.

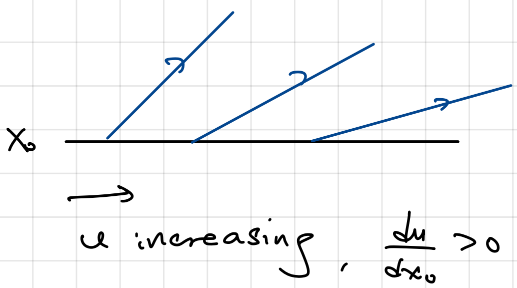
21. Shocks, Pistons and Fans

21.1 Shock formation (in simple waves)

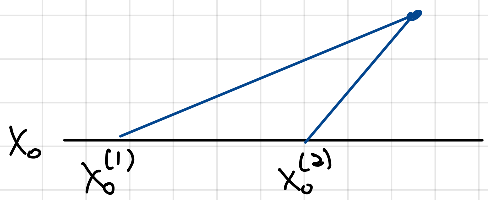
C_+ char.:

$$\frac{dx}{dt} = c_0 + \frac{\gamma+1}{2} u,$$

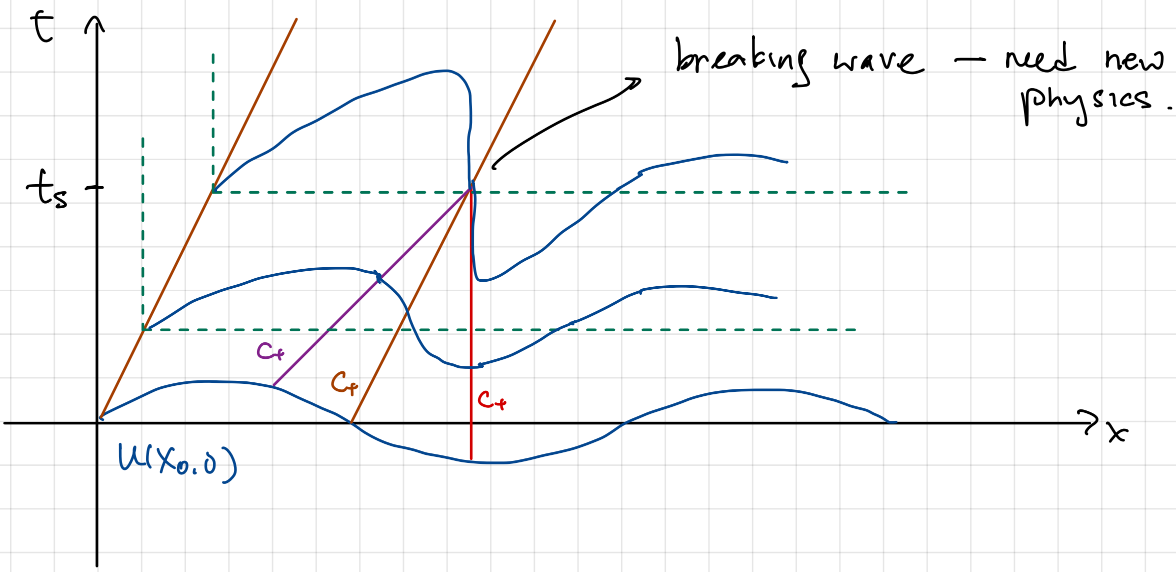
so larger values of $u(x,0)$ propagate faster.



But if $\frac{du}{dx_0} < 0$ anywhere, then C_+ will inevitably cross, giving contradictory predictions for $u(x,t)$. This is unacceptable: a shock (discty in soln) must form (at shock we need new physics)



u decreasing. $\frac{du}{dx_0} < 0$.



At t_s,

$$\frac{\partial u}{\partial x} = \frac{\partial u/\partial x_0}{\underbrace{\partial x/\partial x_0}_{\to 0}} \to -\infty$$

$$\Rightarrow \frac{\partial x}{\partial u} = \frac{\partial x/\partial x_0}{\partial u/\partial x_0} \to 0$$

Recall eqn for C_+ char. is

$$x = x_0 + \left[c_0 + \frac{\gamma+1}{2} u(x_0,0)\right] t$$

So x is no longer monotonically increasing in x_0 when

$$\left(\frac{\partial x}{\partial x_0}\right)_t = 0 \text{ first}, \qquad \frac{\partial x}{\partial x_0} = 1 + \frac{\gamma+1}{2} \frac{du}{dx_0} t = 0$$

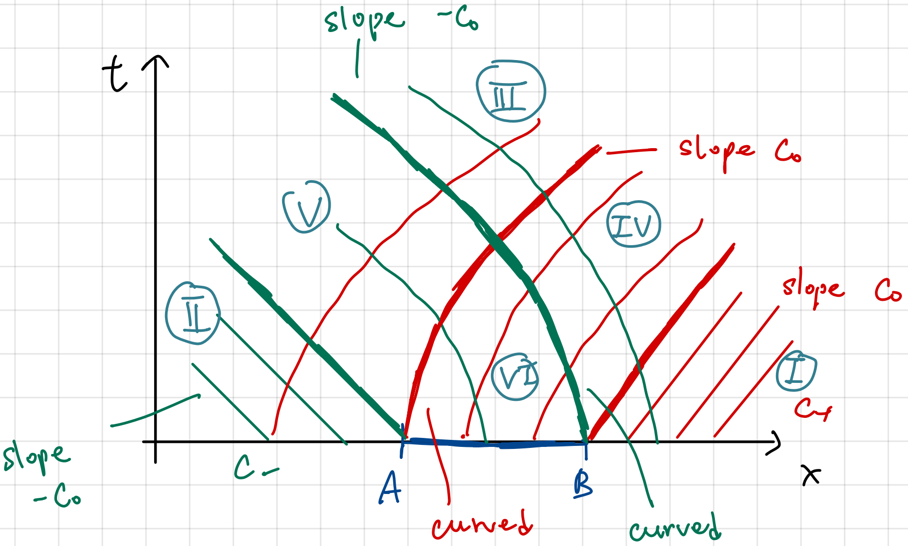
So a shock wave forms at

$$t = t_s = \frac{2}{\gamma+1} \frac{1}{\max_{x_0}\left(-\frac{du}{dx_0}(x_0,0)\right)}$$

at the $x(x_0,t)$ corresponding to the maximising x_0, t_s

21.2 Simple wave regions in IVP

As an example, suppose at $t=0$, $u=0$, $c=c_0$, except in a finite region $[A,B]$. The pattern of char. defines 6 regions



In regions I, II, III, all C_+ and C_- originate in the undisturbed fluid, i.e. $u=Q=0$. In these regions, $u=0$, $c=c_0$, and $C_\pm$ are straight lines with slope $= \pm c_0$.

In region IV, every C_- originates in undisturbed fluid to the right of B $\Rightarrow$ $u=Q$. This region contains a simple wave

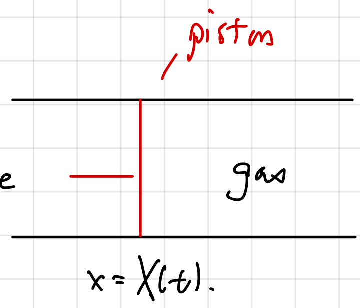
$$u = u(x_0,0), \quad c = c_0 + \frac{\gamma+1}{2} u(x_0,0)$$

for some $x_0(x,t)$. The C_+ char. are straight, but not necessarily parallel, with slope in general $\neq c_0$.

Region V contains left-going simple wave, $u=-Q$, with straight C_- with slope not necessarily $-c_0$.

Region VI is complicated.

21.3 Pistons



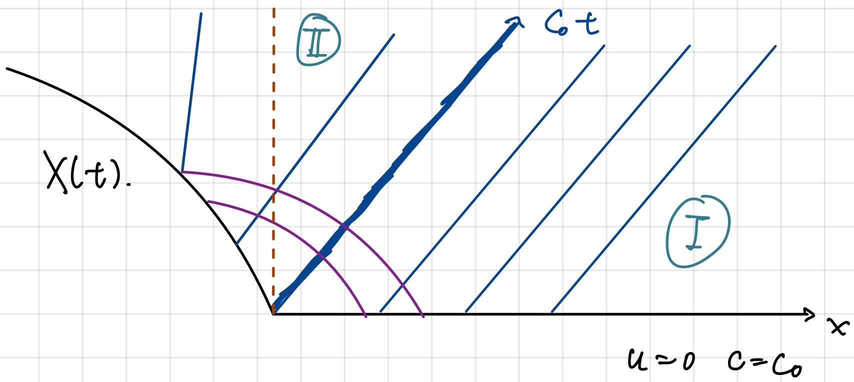
Consider a piston with $u=0$, $c=c_0$ everywhere

at $t=0$, piston $X(0)=0$. $X(t)$ is

prescribed for $t>0$.

Sol^n : All C_- char. originate from $x>0$, $t=0$, where $u=0$, $c=c_0$.

Assume they reach the piston.



In region I, C_+ come from $t=0$, $x>0$, $u=0 \Rightarrow c=c_0 \quad \forall t>0$.

$u=0$ remains at rest.

In region II, C_+ come from piston. At $t=\tau$, $x=X(\tau)$, C_+ has

$$u = \dot{X}(\tau).$$

$$\Rightarrow x = \left[c_0 + \frac{\gamma+1}{2}\dot{X}(\tau)\right](t-\tau) + X(\tau). \quad \text{for } t \geq \tau.$$

Given (x,t), this is an implicit eqn for $\tau(x,t)$ can be solved

(in general)

Exercise Find the exact sol^n for uniform acceleration $X(\tau) = \frac{1}{2}a\tau^2$.

(select root $0<\tau<t$).

21.4 Rarefraction (Expansion) fans

Consider $\ddot{X} \le 0 \ \forall t$, i.e. piston moving away from gas and $\dot{X}$ is non-increasing, so

$$c = c_0 + \frac{\gamma-1}{2}\dot{X}(\tau) < c_0, \quad u + c = c_0 + \frac{\gamma+1}{2}\dot{X}(\tau).$$

(1) dx/dt on C_+ decreases with τ, so char. diverge $\Rightarrow$ no shocks.

$dx/dt = 0$ if $\dot{X}$ can reach $-\frac{2c_0}{\gamma+1}$

(2) c decreases with τ,

$$\rho = \rho_0 (c/c_0)^{2/\gamma-1}, \quad p = p_0 (c/c_0)^{2\gamma/\gamma-1},$$

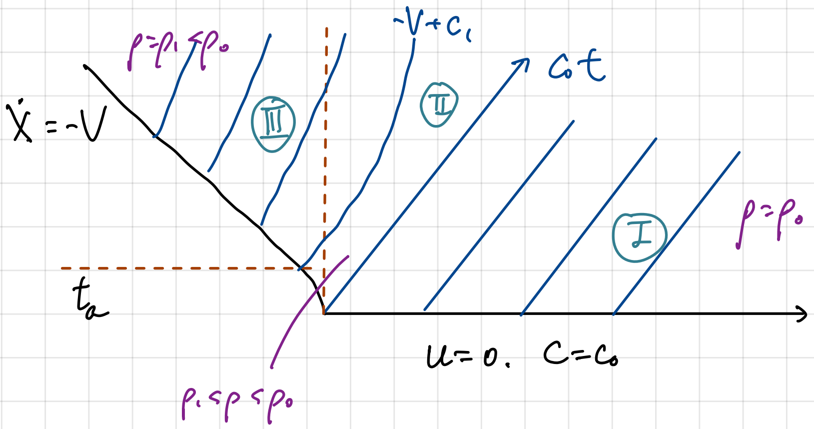
$$c = c_0 (\rho/\rho_0)^{(\gamma-1)/2} \Leftrightarrow \rho \text{ decreasing with } \tau.$$

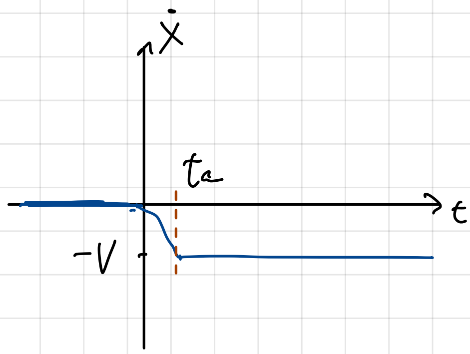
$\Rightarrow$ rarefaction wave as gas expands.

(3) At $\dot{X} = -\frac{2c_0}{\gamma-1} \Rightarrow c = 0 = \rho$. $u \pm c = \dot{X}$. C_+ and C_- are parallel if $\dot{X} \le -\frac{2c}{\gamma-1}$, the gas loses contact with the piston, and C_- do not reach the piston.

This occurs at $M \gtrsim 5$. ($\gamma = 1.4$ for air). This is hypersonic speed.

(4) Consider rapid acceleration up to a const. speed $-V$.



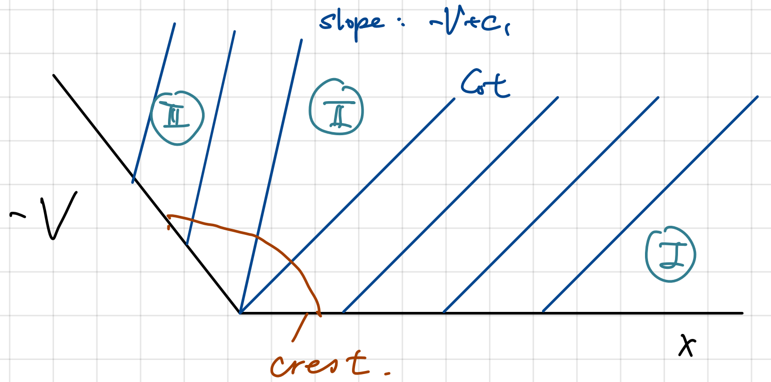


C_+ slopes are uniform in each of I and III, and range over intermediate values in region II.

We wish to take the limit $t_a \to 0$, i.e. an impulsive start.

$\Rightarrow$ region II reduces to a wedge called an <u>expansion fan</u>

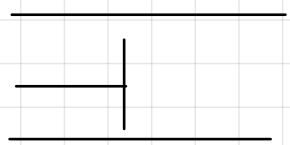
with many C_+ coming out of origin



The C_+ slopes are const. because we are considering simple waves, and must be given by x/t in region II to go through the origin $\Rightarrow u(x,t),\ c(x,t)$ calculate like C_-.

C_- straight in I, III, and curved in II.

<u>21.5 Shock formation</u>



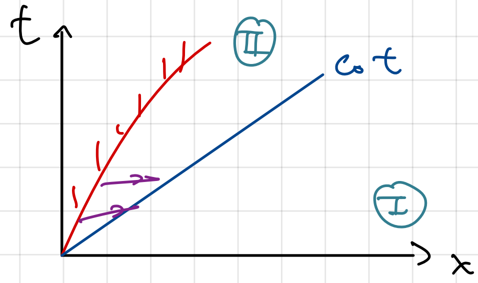
Shocks will form if piston moves towards the gas whatever the form of $X(t)$.

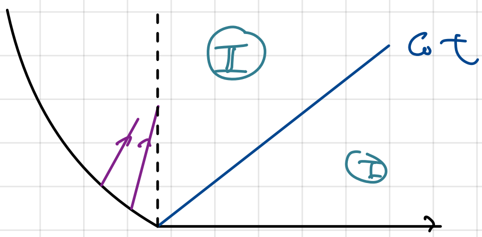
$\ddot{X} \leq 0 \Rightarrow$ no shock.

C_+ Char. have slopes c_0 in region I (i.e. to right of $x = c_0 t$) and slopes $c_0 + \frac{\gamma+1}{2}\dot{X}(t) > c_0$ in region II.

$\Rightarrow$ inevitably get a crossing.

In fact, a similar argument for successive C_+ shows that a shock forms if $\dot{X}(t) > 0$ for any t.





Char. start to cross when $\left.\frac{\partial x}{\partial \tau}\right|_t = 0$, but

$$x = \left(c_0 + \frac{\gamma+1}{2}\dot{X}(\tau)\right)(t-\tau) + X(\tau).$$

$$\Rightarrow \frac{\gamma+1}{2}\ddot{X}(\tau)(t-\tau) - \left(c_0 + \frac{\gamma+1}{2}\dot{X}(\tau)\right) + \dot{X}(\tau) = 0$$

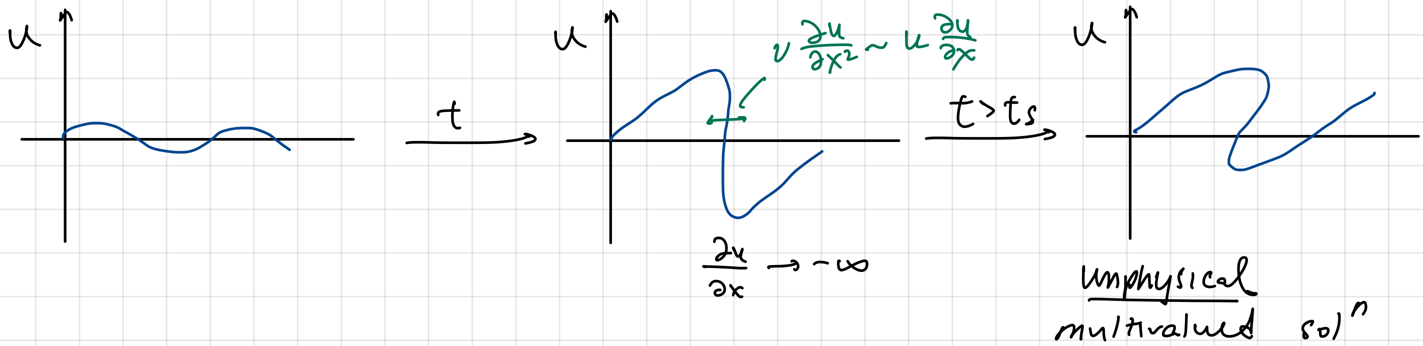
$\therefore$ the shock forms at

$$t = t_c = \min_{\tau}\left(\tau + \frac{2c_0 + (\gamma-1)\dot{X}(\tau)}{(\gamma+1)\ddot{X}(\tau)}\right).$$

at the x corresponding to the minimising τ.

22. Rankine-Hugoniot relations for shocks

We have seen non-linear waves can form shocks.



Solve by "regularise" the problem by adding more physics.

We have neglected viscosity (internal friction), which gives an extra term $\nu\frac{\partial^2 u}{\partial x^2}$, where ν is kinematic viscosity.

ν of air $\sim 10^{-5}\ m^2 s^{-1}$.

In general, $\nu \ll \mathcal{U} L$.

typical velocity $\mathcal{O}(10^{1-2}\,\mathrm{m\,s^{-1}})$.

length $\sim\mathcal{O}(10^{2}\,\mathrm{m})$

Then virtually everywhere

$$\nu \frac{\partial^2 u}{\partial x^2} \sim \frac{\nu \mathcal{U}}{L^2} \ll u\frac{\partial u}{\partial x} \sim \frac{\mathcal{U}^2}{L}.$$

But as the shock forms, a region with very large local gradient appears, where $\frac{\partial}{\partial x} \gg \frac{1}{L}$ $\Rightarrow$ viscosity (thermal conduction) regularise the shock when $u \sim c$. For air $\nu u_{xx} \sim u u_x$ when $\delta x \sim 10^{-8}$ m.

$\Rightarrow$ entropy not constant (usually increases due to dissipation radiation / diffusion). How can we understand?

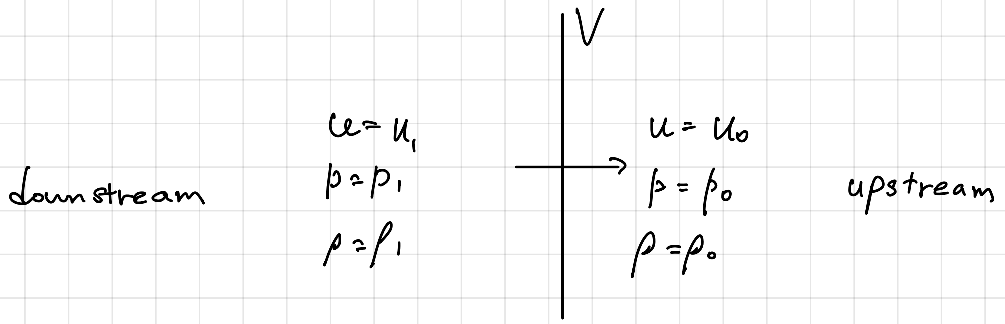
22.1 R-H relations

It is still possible to understand how properties of interest change (or "jump") across a shock without knowing the internal details of the shock. 2 key attributes:

(1) p, ρ, u, T jump almost discretely across shock

(2) Shocks in general move.

Consider a shock separating to uniform regions (the shock width δx is very much smaller than these regions in horizontal extent).



Mass, mom, total energy obey eqns of the form

$$\frac{\partial}{\partial t}(\quad) + \nabla \cdot (\text{flux}) = 0$$

if there is no external forces.

Switch to a frame of reference where the shock is at rest.

0	
$u = U_1 = u_1 - V$	$u = U_0 = u_0 - V$
$p = p_1$	$p = p_0$
$\rho = \rho_1$	$\rho = \rho_0$,

Then fluxes of mass, mom (and total energy to leading order) must be the same on both sides of the shock.

Conservation of mass (more precisely the matching of mass flux across the shock):

$$\boxed{\rho_1 U_1 = \rho_0 U_0} \quad \left(\begin{array}{c}\text{mass flux}\\ \text{balance}\end{array}\right) \quad (1)$$

$$\boxed{p_1 + \rho_1 U_1^2 = p_0 + \rho_0 U_0^2} \quad \left(\begin{array}{c}\text{mom. flux}\\ \text{balance}\end{array}\right) \quad (2).$$

Recall in §1.4, internal energy per unit mass for an ideal / perfect gas is

$$e = \frac{1}{\gamma - 1} \frac{p}{\rho}.$$

$$U_1 \left(\frac{1}{2}\rho_1 U_1^2 + \rho_1 e_1 + p_1 \right) = U_0 \left(\frac{1}{2}\rho_0 U_0^2 + \underbrace{\rho_0 e_0 + p_0}_{\text{enthalpy } H.} \right)$$

$$\Rightarrow \rho_1 U_1 \left(\frac{1}{2} U_1^2 + \frac{1}{\gamma-1}\frac{p_1}{\rho_1} + \frac{p_1}{\rho_1} \right) = \rho_0 U_0 \left(\frac{1}{2} U_0^2 + \frac{1}{\gamma-1}\frac{p_0}{\rho_0} + \frac{p_0}{\rho_0} \right).$$

$$\Rightarrow \boxed{\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}u_0^2 + \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}} \qquad (3)$$

(leading order total energy flux balance)

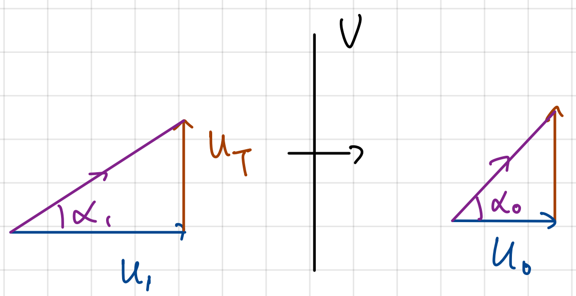
Eqn (1) — (3) are R-H relations connecting the variables across the shock: $u_0, \rho_0, p_0, u_1, \rho_1, p_1, V \Rightarrow 7$ quantities of interest, so need 4 further pieces of info to solve the problem, e.g. from IC or BC.

Note Across the shock, it is not proved that

$$\frac{p_1}{p_0} = \left(\frac{\rho_1}{\rho_0}\right)^\gamma.$$

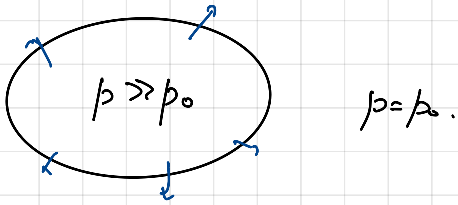
This relation hold for perfect / ideal gas in homentropic flow, i.e. uniform and const. entropy. But entropy also jumps across a shock, i.e. shocks not adiabatic. There is typically both thermal conduction because of large ∇T and also heat generation due to viscous dissipation because of large values of ∇u.

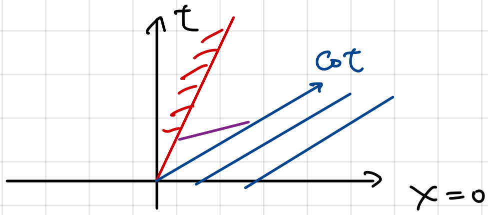
Note velocity parallel to the shock is unchanged on either side of a shock. (if $u_T \neq 0$, oblique shock).



23. Shock example calculation

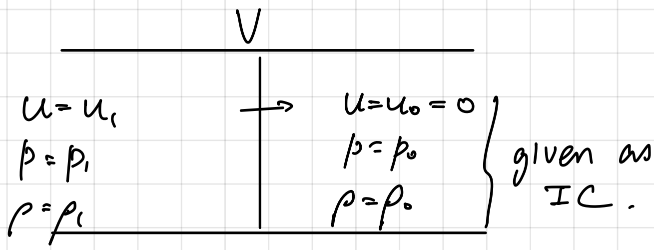
23.1 Shock invading a fluid at rest.





We have the RH relations, so to solve,

we can for example consider



$$p_1 = p_0 (1+\beta).$$

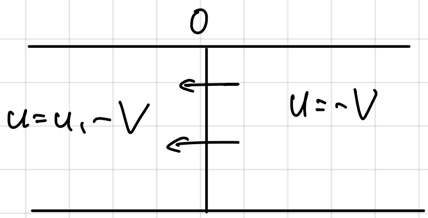
β is the shock strength.

- $\beta \ll 1$: weak shock
- $\beta \gg 1$: strong shock,

Now can calculate u_1, V, ρ_1.

General advice: consider objective before calculation. Aim for $\rho_1(\rho_0, p_0, \beta)$

As before, transform to frame where wave is stationary



RH relations:

$$\rho_1 (u_1 - V) = \rho_0 (-V)$$

Note $u_1 < V$,

$$p_1 + \rho_1 (u_1 - V)^2 = p_0 + \rho_0 V^2.$$

$$\Rightarrow p_1 - p_0 = \rho_0 V^2 - \rho_1 \left(\frac{\rho_0 (-V)}{\rho_1} \right)^2$$

$$= \left(\rho_0 - \frac{\rho_0^2}{\rho_1}\right) V^2 = \frac{\rho_0}{\rho_1} (\rho_1 - \rho_0) V^2$$

$$\Rightarrow V^2 = \left(\frac{p_1 - p_0}{\rho_1 - \rho_0} \right) \cdot \frac{\rho_1}{\rho_0}, \quad (V - u_1)^2 = \left(\frac{p_1 - p_0}{\rho_1 - \rho_0} \right) \frac{\rho_0}{\rho_1}$$

Total energy:

$$\frac{\gamma}{\gamma-1}\underbrace{\left(\frac{p_1}{\rho_1}-\frac{p_0}{\rho_0}\right)}_{H_1-H_0} = \frac{1}{2}\left(V^2-(V-u_1)^2\right).$$

$$= \frac{1}{2}\left(\frac{p_1-p_0}{\rho_1-\rho_0}\right)\left(\frac{\rho_1}{\rho_0}-\frac{\rho_0}{\rho_1}\right)$$

$$= \frac{1}{2}\frac{p_1-p_0}{\rho_1-\rho_0}\left(\frac{\rho_1^2-\rho_0^2}{\rho_0\rho_1}\right) = \frac{1}{2}(p_1-p_0)\cdot\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right).$$

So we have the Hugoniot adiabatic:

$$\boxed{\frac{\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1}-\frac{p_0}{\rho_0}\right) = \frac{1}{2}(p_1-p_0)\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right)} \quad (HA).$$

23.2 Comments

(1) RH relations are unaffected by the velocity flipped in sign.

but it is physically "obvious" that we can assume $V>0$ if $p_1>p_0$.

ie. the shock compresses the gas as it crosses the shock.

This is true because

- entropy $S = C_v \log(p/\rho^\gamma) + C$ (const.) must increase across a shock
- The full eqn for the shocks internal structure has no sol^n o/w.
- This makes sense of the char. as an IVP develops.

(2) In HA, let $p_1 = (1+\beta)p_0$,

$$\frac{\gamma}{\gamma-1}\left(\frac{1+\beta}{\rho_1}-\frac{1}{\rho_0}\right)\cancel{p_0} = \frac{\beta}{2}\cancel{p_0}\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right)$$

$$\overset{\times \rho_1}{\Rightarrow} \quad \frac{\rho_1}{\rho_0}\left(\frac{\beta}{2}+\frac{\gamma}{\gamma-1}\right) = (1+\beta)\frac{\gamma}{\gamma-1}-\frac{\beta}{2}.$$

$$\Rightarrow \quad \frac{\rho_1}{\rho_0}\left(1+\frac{\gamma-1}{2\gamma}\cdot\beta\right) = 1+\beta-\frac{\gamma-1}{2\gamma}\beta. = 1+\frac{(\gamma+1)\beta}{2\gamma}.$$

$$\Rightarrow \boxed{\frac{\rho_1}{\rho_0} = \frac{1 + \frac{\gamma+1}{2\gamma}\beta}{1 + \frac{\gamma-1}{2\gamma}\beta}} \qquad (*).$$

Note Entropy $S = C_v \log(p/\rho^\gamma) + C$.

$$\Rightarrow \frac{S_1 - S_0}{C_v} = \log\left(\frac{p_1}{\rho_1^\gamma}\right) - \log\left(\frac{p_0}{\rho_0^\gamma}\right)$$

$$= \log\left(\frac{p_1}{p_0}\right) - \gamma \log\left(\frac{\rho_1}{\rho_0}\right).$$

$$= \log(1+\beta) - \gamma \log\left(\frac{2\gamma + (\gamma+1)\beta}{2\gamma + (\gamma-1)\beta}\right)$$

It can be established that $\frac{S_1 - S_0}{C_v} > 0$, ie. entropy of fluid increases as it crosses the shock. Eqv. to showing

$$\frac{\rho_1}{\rho_0} < (1+\beta)^{1/\gamma}.$$

For weak shocks $(\beta \ll 1)$,

$$\frac{S_1 - S_0}{C_v} = \frac{(\gamma^2 - 1)\beta^3}{12\gamma^2}.$$

Small compared to $p_1 - p_0$, $\rho_1 - \rho_0$, etc. This justifies approximations underlying the RH relation for (total) energy.

For strong shocks $(\beta \gg 1)$, $\frac{\rho_1}{\rho_0} \lesssim \frac{\gamma+1}{\gamma-1}$ remains finite.

(3) Consider velocities

$$V^2 = \frac{p_1 - p_0}{\rho_1 - \rho_0} \cdot \frac{\rho_1}{\rho_0} = \frac{\beta p_0}{\rho_0} \cdot \frac{\rho_1/\rho_0}{\frac{\rho_1}{\rho_0} - 1}$$

Note: $C_0^2 = \gamma p_0/\rho_0$.

$$\frac{\rho_1}{\rho_0} - 1 = \frac{1 + \frac{\gamma+1}{2\gamma}\beta - 1 - \frac{\gamma-1}{2\gamma}\beta}{1 + \frac{\gamma-1}{2\gamma}\beta} = \frac{\beta/\gamma}{1 + \frac{\gamma-1}{2\gamma}\beta} \qquad (**)$$

$$\Rightarrow V^2 = \frac{\frac{c_0^2}{\gamma}\beta\left(1 + \frac{\gamma+1}{2\gamma}\beta\right)}{\beta/\gamma} = c_0^2\left(1 + \frac{\gamma+1}{2\gamma}\beta\right) > c_0^2 .$$

Fluids enter the shock from the undisturbed shock from the undisturbed state <u>supersonically</u>

$$(V-u_1)^2 = \frac{p_1 - p_0}{\rho_1 - \rho_0}\cdot\frac{\rho_0}{\rho_1}$$

$$= \frac{\frac{p_1}{\rho_1}\left(1 - \frac{1}{1+\beta}\right)\frac{\rho_0}{\rho_1}}{1 - \frac{\rho_0}{\rho_1}} = \frac{\frac{c_1^2}{\gamma}\left(\frac{\beta}{1+\beta}\right)}{\frac{\rho_1}{\rho_0} - 1} .$$

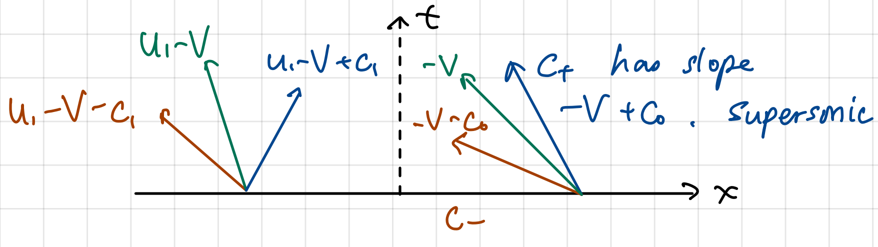
$$\Rightarrow (V-u_1)^2 = \frac{\frac{c_1^2}{\gamma(1+\beta)}\beta\cdot\left(1 + \frac{\gamma-1}{2\gamma}\beta\right)}{\beta/\gamma} = \frac{c_1^2(2\gamma + (\gamma-1)\beta)}{2\gamma(\beta+1)} .$$

$$\Rightarrow (V-u_1)^2 = c_1^2\left(\frac{2\gamma + 2\gamma\beta - 2\gamma\beta + \gamma\beta - \beta}{2\gamma(\beta+1)}\right)$$

$$= c_1^2\left(1 - \frac{(\gamma+1)\beta}{2\gamma(\beta+1)}\right) < c_1^2$$

Fluid emerges from the shock from downstream with a relative velocity $V - u_1$, that is <u>subsonic</u> with c_1.

* (4) Entropy is transported with the fluid velocity in adiabatic flow. Adding this to C_+ and C_- char. gives



Note How the no. of characteristics approaching the shock matches the no. of parameters (i.e. info) we were given ($u_0, \rho_0, p_0, p_1 = (1+\beta)p_0$).

24. Non-linear 1D Shallow water waves

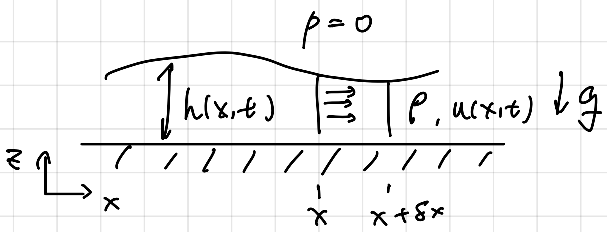
24.1 Shallow water waves

"Shallow water" means wavelength $\gg$ depth.

Flow is almost horizontal, so u indpt. of z. (effectively, u is a depth-averaged quantity.)

Hydraulics / river flow : geophysical flow (in IB) ($\Omega = 0$).

Mass conservation :



$$\boxed{\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho h u) = 0}$$

Vertical accel. are negligible $\Rightarrow$ p hydrostatic

$$\Rightarrow \quad p = \rho g (h - z)$$

Sideway pressure force is

$$F = \int_0^h p \, dz = \tfrac{1}{2}\rho g h^2 .$$

Horizontal mom :

$$\frac{\partial}{\partial t}(\rho h u \, \delta x) = \rho h u^2\big|_x - \rho h u^2\big|_{x+\delta x} + F(x) - F(x+\delta x)$$

transport in — out

$$\Rightarrow \quad \frac{\partial}{\partial t}(\rho h u) + \frac{\partial}{\partial x}(\rho h u^2) = -\frac{\partial}{\partial x}\left(\tfrac{1}{2}\rho g h^2\right) = -\rho g h \frac{\partial h}{\partial x}$$

$$\Rightarrow \quad \rho h \frac{\partial u}{\partial t} + \rho h u \frac{\partial u}{\partial x} + \underbrace{\rho u \left(\frac{\partial h}{\partial t} + h \frac{\partial u}{\partial x} + u \frac{\partial h}{\partial x}\right)}_{=0} = -\rho g h \frac{\partial h}{\partial x} .$$

mass $\Rightarrow$ $$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$$

mom. $\rightarrow$ $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + h\frac{\partial h}{\partial x} = 0$$

Remember $\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} = 0$.

So shallow water eqns are identical to ideal perfect gas eqns if

Shallow water		Perfect gas
h		ρ
u	$\Longleftrightarrow$	u
g		c^2/ρ

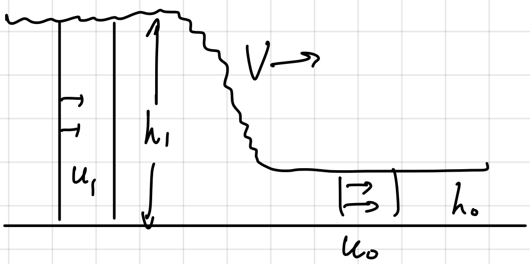
This requires $c = \sqrt{g\rho}$ rather than $c = c_0 (\rho/\rho_0)^{\frac{\gamma-1}{2}}$, eqv. to rarefied gas dynamics with $\gamma = 2$.

So SWE can be solved with C_+, C_- char. and can have shocks due to wave steepening.

* 24.2 Hydraulic Jump

NL SW waves can form shocks called <u>hydraulic jumps</u> or <u>bores</u>.

The extra physics controlling a hydraulic jump is different across a gas shocked : ρ, p, u jump so that mom., mass, total energy are conserved. In hydraulic jump, only h and u can jump, and so energy not conserved.



Jump in the frame where the jump is stationary.

$U_i = u_i - V \longrightarrow h_1 U_1 = h_0 U_0 \quad (\frac{1}{\rho}$ mass flux$)$.

$$\frac{1}{2} g h_1^2 + h_1 U_1^2 = \frac{1}{2} g h_0^2 + h_0 U_0^2 \quad \left(\frac{1}{\rho} \text{ mom. flux}\right)$$

force due to the integrated hydrostatic pressure

mom. flux.

Example Assume h_0, u_0 given and also one of the h_1, u_1, i.e. $h_1 = h_0(1+\beta)$. Calculate $V + u_1$. Take the simple example of $u_0 = 0$.

$$h_1(-V + u_1) = -h_0 V \Rightarrow V - u_1 = \frac{h_0}{h_1} V.$$

$$\frac{1}{2} g (h_1^2 - h_0^2) = h_0 V^2 - h_1 (V - u_1)^2 = V^2 \left(h_0 - \frac{h_0^2}{h_1}\right) = \frac{V^2 h_0}{h_1}(h_1 - h_0).$$

$$\Rightarrow \boxed{V^2 = \frac{1}{2} g (h_0 + h_1) \frac{h_1}{h_0}, \quad (V - u_1)^2 = \frac{1}{2} g (h_0 + h_1) \frac{h_0}{h_1}}$$

$$h_1 = h_0(1+\beta) \Rightarrow \frac{V^2}{g h_0} = \frac{1}{2}\left(1 + \frac{h_1}{h_0}\right)\left(\frac{h_1}{h_0}\right) = \left(1 + \frac{\beta}{2}\right)(1+\beta) > 1.$$

ie. bore/jump is faster than the long wave speed $\sqrt{g h_0}$. The jump is called supercritical

$$\frac{V}{\sqrt{g h_0}} = Fr \quad (\text{Froude no.})$$

On the other side,

$$\frac{(V - u_1)^2}{g h_1} = \frac{1}{2}\left(\frac{h_0}{h_1} + 1\right)\left(\frac{h_0}{h_1}\right) = \frac{1}{2}\left(1 + \frac{1}{1+\beta}\right)\left(\frac{1}{1+\beta}\right)$$

$$= \frac{1 + \beta/2}{(1+\beta)^2} < 1.$$

ie. fluid behind jump has fluid velocity less than long wave speed.

$Fr < 1$ flow is subcritical.

* 24.3 Energy

The energy flux is

$$\underbrace{\tfrac{1}{2}(\rho g h^2)u}_{\text{pressure work}} + uh\Big(\underbrace{\tfrac{1}{2}\rho u^2}_{\text{KE}} + \underbrace{\tfrac{1}{2}\rho g h}_{\text{GPE}}\Big) = \rho u h\left(\tfrac{1}{2}u^2 + gh\right).$$

By conservation of mass, $\rho u h$ cts, but $\frac{1}{2}u^2 + gh$ discts since steady Bernoulli does not apply.

In fact, energy flux into the jump (upstream h_0) − energy flux out of the jump (downstream h_1):

$$\rho V h_0\left(\left(\tfrac{1}{2}V^2 + g h_0\right) - \left(\tfrac{1}{2}(V-u_1)^2 + g h_1\right)\right) = \frac{\rho V h_0^2 g \beta^3}{4(1+\beta)} > 0$$

i.e. dissipation at jump

Two possibilities for energy loss:

- $\beta \gtrsim 0.5$ — a turbulent bore: energy is lost to small scale motion / viscous dissipation.
- $\beta \lesssim 0.5$ — undular bore: waves carry energy away.