Symmetries, Fields and Particles

1. Introduction to Symmetries

Def^n (Symmetry) Invariance (of physical laws) under some set of transformations.

Example N2: $m\ddot{\underline{x}} = \underline{F}$ simplifies in cases $\underline{F}(\underline{x}) = F(r)\,\hat{\underline{r}}$, with $r = |\underline{x}|$, $\hat{\underline{r}} = \underline{x}/r$, then

$$\underline{x} \times \underline{p} = m\,\underline{x} \times \dot{\underline{x}}$$

is conserved.

Example Lagrangian mechanics $\left(L(q_i, \dot{q}_i; t) = T - V\right)$

Principle of least action: minimise

$$S = \int_{t_1}^{t_2} dt\; L(q_i, \dot{q}_i; t)$$

$$\Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad (E-L).$$

Noether's thm: invariance of L under some trans^n implies an associated conserved quantity. For example,

$$L = \frac{1}{2} m \dot{\underline{x}}^2 - U(\underline{x})$$

- $\frac{\partial L}{\partial t} = 0$, invariant under $t \mapsto t + \delta t$
  $\Rightarrow$ Hamiltonian H (total energy) conserved.

Recall: conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$$

$$H(x_i, p_i; t) = \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L \overset{E-L}{\Longrightarrow} \frac{\partial H}{\partial t} = 0$$

- L invariant under $x \mapsto x + \delta x$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}} = \text{const.} = p_x .$$

$\Rightarrow$ mom. in x-dir conserved.

- L invar. under rot^n about z-axis, then z-cpt of ang. mom., $x p_y - y p_x$, conserved.

In QM: states in Hilbert space $\mathcal{H}$.
Symmetry: $\exists$ invertible operator $U : \mathcal{H} \to \mathcal{H}$ which preserves inner products, up to an overall phase.

Let $|\Psi\rangle$, $|\Phi\rangle \in \mathcal{H}$, write $|U\Psi\rangle = U|\Psi\rangle$.
U is a sym. trans^n / operator iff $|\langle U\Phi | U\Psi\rangle| = |\langle \Phi | \Psi\rangle|$

Prop (Wigner's thm): Symmetry trans^n / operators are either (a) linear and unitary, (b) antilinear and antiunitary.

(a): $U(\alpha|\Psi\rangle + \beta|\Phi\rangle) = \alpha U|\Psi\rangle + \beta U|\Phi\rangle$.
$\langle U\Psi | U\Phi\rangle = \langle \Psi | \Phi\rangle$.

(b): $U(\alpha|\Psi\rangle + \beta|\Phi\rangle) = \alpha^* U|\Psi\rangle + \beta^* U|\Phi\rangle$.
$\langle U\Psi | U\Phi\rangle = \langle \Psi | \Phi\rangle^*$.

Suppose we have Hamiltonian H. (indpt. of t), then

Time evolution of state $\quad |\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle \qquad (\hbar = c = 1)$.

Suppose U is linear and unitary.

$$\langle U\Phi | U\Psi(t)\rangle = \langle \Phi | \Psi(t)\rangle$$
$$= \langle \Phi | e^{-iHt} | \Psi(0)\rangle.$$

Equivalently,

$$\langle U\Phi | U\Psi(t)\rangle = \langle U\Phi | e^{-iHt} | U\Psi(0)\rangle$$
$$= \langle \Phi | U^\dagger e^{-iHt} U | \Psi(0)\rangle.$$

$$\Rightarrow \quad e^{-iHt} = U^\dagger e^{-iHt} U$$

$$\Rightarrow \quad [H, U] = HU - UH = 0.$$

Example If $[H, \underline{p}] = 0$, then H indpt. of $\underline{x}$, i.e. invar. under $\underline{x} \mapsto \underline{x} + \underline{a}$, since $[x_j, p_k] = i\delta_{jk}$.

const.

Unitary operator: $e^{i\underline{a}\cdot\underline{p}}$.

ang. mom.

Example If H is rot^n invar., then $\underline{J}$ (or $\underline{L}$) commute with H.

- Set of U's form a group.
- Actions of U's on $|\Psi\rangle$'s form a group representations

2. Lie Groups and Lie Algebras

2.1 Lie Groups

Def^n A group is a set G with binary operation $\cdot$ s.t.

(i) Closure: $\forall g_1, g_2 \in G$, $g_1 \cdot g_2 \in G$.

(ii) Assoc.: $\forall g_1, g_2, g_3 \in G$, $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$.

(iii) identity : $\exists\, e \in G$ s.t. $\forall\, g \in G$, $e \cdot g = g \cdot e = g$

(iv) inverse : $\forall\, g \in G$, $\exists\, g^{-1} \in G$ s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Defⁿ A group G is abelian if $\forall g_1, g_2 \in G$, $g_1 g_2 = g_2 g_1$.

Defⁿ A manifold is a space which look like Euclidean space, like $\mathbb{R}^n$, on small scales.

Defⁿ A differentiable manifold satisfies certain smoothness conditions. Notions of differentiability in $\mathbb{R}^n$ extend to open subsets of our manifold.

Defⁿ A Lie Group consists of a differentiable manifold G along with a binary operator $\cdot$, s.t. $\cdot$ and the inverse operation $(\ )^{-1}$ are smooth.

Matrix Lie groups

The general linear group is

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

$$GL(n, \mathbb{F}) := \{ M \in \mathrm{Mat}_n(\mathbb{F}) \mid \det M \neq 0 \}$$

$\dim GL(n, \mathbb{R}) = n^2$, $\dim GL(n, \mathbb{C}) < \begin{matrix} 2n^2 & \text{real dim} \\ n^2 & \text{complex dim} \end{matrix}$

Important subgroups of $GL(n, \mathbb{F})$:

1. Special linear group

$$SL(n, \mathbb{F}) := \{ M \in GL(n, \mathbb{F}) \mid \det M = 1 \}.$$

2. Orthogonal group

$$O(n) := \{ M \in GL(n, \mathbb{R}) \mid M^T M = I \}$$

$\Rightarrow \det M = \pm 1$

$$SO(n) := \{ M \in O(n) \mid \det M = I \}$$

3. Pseudo orthogonal group: Define a $(n+m)\times(n+m)$ metric matrix

$$\eta := \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$$

$$O(n,m) := \{ M \in GL(n+m, \mathbb{R}) \mid M^T \eta M = \eta \}$$
$$SO(n,m) = \{ M \in O(n,m) \mid \det M = 1 \}.$$

4. Unitary group

$$U(n) := \{ M \in GL(n, \mathbb{C}) \mid M^\dagger M = I \}$$
$$SU(n) := \{ M \in U(n) \mid \det M = 1 \}$$

5. Pseudounitary group

$$U(n,m) := \{ M \in GL(n+m, \mathbb{C}) \mid M^\dagger \eta M = \eta \}$$
$$SU(n,m) := \{ M \in U(n,m) \mid \det M = 1 \}$$

6. Symplectic Group: Define a fix anti-sym $2n\times 2n$ matrix, e.g.

$$\Omega := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad \text{or} \quad J := \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

$$Sp(2n, \mathbb{R}) := \left\{ M \in GL(2n, \mathbb{R}) \;\middle|\; \begin{matrix} M^T \Omega M = \Omega \text{ or} \\ M^T J M = J \end{matrix} \right\}$$

Can show $\det M = 1$ using the Pfaffian.

Def^n Given a $2n\times 2n$ antisym matrix A, its Pfaffian, is given by

$$\mathrm{Pf}\, A := \frac{1}{2^n n!} \epsilon_{i_1\cdots i_{2n}} A_{i_1 i_2} A_{i_3 i_4} \cdots A_{i_{2n-1} i_{2n}}$$

antisym with $\varepsilon_{1,2,\dots,2n} = 1$

Group elements as transformations

We can define actions of group elts of G, $g \in G$, on a set X.

Def^n The left action of G on X is a map $L: G\times X \to X$ s.t. $L(e,x) = x \quad \forall x \in X$.

and

$$L(g_2, L(g_1, x)) = L(g_2 g_1, x)$$

$\forall x \in X$, $g_1, g_2 \in G$.

More usual notation: $\forall g \in G$, associate a map $g: X \to X$ s.t. $g(x) = gx$

Def^n The right action of G on X

$$g: X \to X$$
$$x \mapsto xg^{-1}.$$

$\forall x \in X$, $g \in G$.

$$g_2(g_1(x)) = x g_1^{-1} g_2^{-1} = (g_2 g_1)(x).$$

Def^n Conjugation by G on X is defined by

$$g(x) = gxg^{-1}$$

$\forall x \in X$, $g \in G$.

Def^n Given a group G and a set X, an <u>orbit</u> of an elt. $x \in X$ is the set of elt in X which are in the image of an action of G on x.

<u>Example</u> For left action, for $x \in X$,

$$Gx := \{ gx \mid g \in G \}.$$

Matrices $\longleftrightarrow$ linear operators on vectors.

The matrix group introduced above represent important operations in physics

- $O(n)$: rot^n / refl^n in $\mathbb{R}^n$

  Let $\underline{v}_1, \underline{v}_2 \in \mathbb{R}^n$. Define $(\underline{v}_2, \underline{v}_1) := \underline{v}_2^T \underline{v}_1$.

  For $R \in O(n)$,

$$(R\underline{v}_2, R\underline{v}_1) = \underline{v}_2^T R^T R \underline{v}_1 = \underline{v}_2^T \underline{v}_1 = (\underline{v}_2, \underline{v}_1)$$

- $U(n)$: preserves $\langle \underline{v}_2 | \underline{v}_1 \rangle = \underline{v}_2^\dagger \underline{v}_1$

<u>Example</u> $SO(2) = \left\{ R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in [0, 2\pi) \right\}$.

Can show $R(\theta_2) R(\theta_1) = R(\theta_2 + \theta_1) \Rightarrow$ 1-dim diff. manifold.

<u>Example</u> $SO(3)$: Axis of rot^n $\underline{n} \in S^2$ unit vec. in $\mathbb{R}^3$. and angle θ (magnitude of rot^n about $\underline{n}$). Manifold is a ball of radius π in $\mathbb{R}^3$ (with antipodal points identified, $\pi \underline{n} = -\pi \underline{n}$). Rotation by $\theta \in [-\pi, 0)$ about $\underline{n}$ is equiv. to rot^n by $\theta \in [0, \pi)$ about $-\underline{n}$, so take $\theta \in [0, \pi)$.

Depict manifold of $SO(3)$ as a ball of radius π in $\mathbb{R}^3$.
Each point $\theta \underline{n}$ correspond to an elt of $SO(3)$.

Note: The points on the surface must be identified as $\pi \underline{n} = -\pi \underline{n}$, ie. antipodal points are identical.

Let $R \in SO(3)$, have

$$R_{ij} = \cos\theta \, \delta_{ij} + (1-\cos\theta) n_i n_j - \sin\theta \, \varepsilon_{ijk} n_k .$$

<u>Def^n</u> A manifold (e.g. Lie Group) is <u>compact</u> if it is closed and bounded, ie. every limit point included in manifold and coord lie in bounded intervals. o/w, noncompact.

<u>Example</u> $SO(2)$ and $SO(3)$ are compact.

- $SO(n,m)$ act on vectors in $\mathbb{R}^{n,m}$ and preserves scaler product $V_2^T \eta V_1$ for $V_1, V_2 \in \mathbb{R}^{n,m}$, $\eta = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$.

<u>Example</u> $SO(1,1) = \left\{ \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \middle| \psi \in \mathbb{R} \right\}$ (ψ: rapidity).

$SO(1,3)$ or $SO(3,1)$ in SR. These are noncompact groups, since, ψ unbounded.

Parameterisation of Lie Groups

At least locally, we can assign a set of coords to group elts. Assign coords

$$x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$$

(dimension of manifold / group is $\dim(\mathbb{R}^n) = n$) s.t. $g(x) \in G$.

Closure : $g(y) g(x) = g(z)$

Smoothness : cpts of z are cts diff. f^n's of x and y.

$$z^r = \varphi^r(x, y), \qquad r = 1, \dots, n$$

Choose coords s.t. origin $\leftrightarrow$ group identity, $g(0) = e$.

Then $g(0)\, g(x) = g(x)$

$$\Rightarrow \varphi^r(x, 0) = x^r \quad \text{and} \quad \varphi^r(0, y) = y^r.$$

Inverse: $\exists\ \bar{x}$ s.t. $g(\bar{x}) = g(x)^{-1}$

$$\varphi^r(\bar{x}, x) = 0 = \varphi^r(x, \bar{x})$$

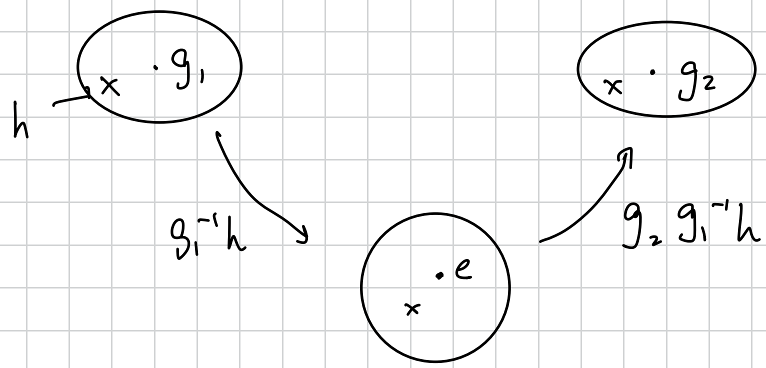
Assoc.: $\varphi^r(\varphi(x,y), z) = \varphi^r(x, \varphi(y,z))$

from $g(z)\,[g(y)\, g(x)] = [g(z)\, g(y)]\, g(x)$.

2.2 Lie Algebras

Lie groups can be complicated. Things are simpler if we "linearise" the Lie group looking at small nbds about any of its points.

A Lie group is homogeneous: every point looks like another. Take $g_1, g_2 \in G$, Take $h \in G$ in nbd of g_1



So we linearise the Lie group near its identity $\Rightarrow$ Lie Algebra.

Def^n (Lie Algebra) A Lie Algebra is a vector space V which additionally has as a vector product the Lie Bracket

$$[\cdot, \cdot] : V \times V \to V$$

possessing the following properties $\forall X, Y, Z \in V$

1) Antisymmetry $[X, Y] = [-Y, X]$
2) Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$
   (replaces assoc.)
3) Linearity: For $\alpha, \beta \in \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)

$$[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z]$$

Example Matrix commutator $[X,Y] = XY - YX$.

Let's choose a basis for V: $\{T_a\}$, $a = 1, \dots, \dim V$

The basis vectors are also called generators of the Lie algebra

Their Lie brackets

$$[T_a, T_b] = f^c{}_{ab} T_c \quad (\text{sum } c)$$

with $f^c{}_{ab} \in \mathbb{F}$ called structure constants.

Antisym of bracket $\Rightarrow$ $f^c{}_{ab} = -f^c{}_{ba}$

Jacobi with $X = T_a$, $Y = T_b$, $Z = T_c$ $\Rightarrow$ $f_{..}f_{..} + ff + ff = 0$

General elt of Lie Algebra

$$X = X^a T_a,$$

with $X^a \in \mathbb{F}$ coeff. Then

$$[X, Y] = X^a Y^b f^c{}_{ab} T_c$$

2.3 Lie Groups and their algebras

Claim: The Lie algebra of a Lie group G is the tangent space to G at its identity e.

Example $SO(2)$: $g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $g: \mathbb{R} \to SO(2)$.

The identity $g(0) = e = I_2$. Near the identity,

$$g(\theta) = I_2 + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{\theta^2}{2} I_2 + \dots$$

$$= e + \theta \left.\frac{dg}{d\theta}\right|_{\theta=0} + \dots$$

$\left.\frac{dg}{d\theta}\right|_{\theta=0}$ is tangent to the manifold at identity e.

$$\left.\frac{dg}{d\theta}\right|_{\theta=0} \in T_e(SO(2)) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\} = L(SO(2))$$

Take $SO(n)$: $\dim d = \frac{n(n-1)}{2}$, $M^T M = I$, $\det M = 1$.

Coords $\underline{x} = (x_1, \dots, x_d)$. Consider curve $\underline{x}(t) = (x_1(t), \dots, x_d(t))$

and $M(\underline{x}(t)) = M(t)$. $M: \mathbb{R}^d \to SO(n)$.

$$\frac{dM}{dt} = \sum_i \frac{\partial M}{\partial x_i} \frac{dx_i}{dt}.$$

Orthogonal $\Rightarrow M^T(t) M(t) = I \quad \forall t$

$$\Rightarrow \quad 0 = \frac{d}{dt}\left(M^T(t) M(t) \right)$$

$$= \dot{M}^T M + M^T \dot{M}$$

For $t = 0$, $M = I$, have

$$\left.\frac{dM^T}{dt}\right|_{t=0} = -\left.\frac{dM}{dt}\right|_{t=0}.$$

$\Rightarrow L(SO(n)) = T_e(SO(n)) \subseteq \mathrm{Skew}_n := \{ X \in \mathrm{Mat}_n(\mathbb{R}) \mid X^T = -X \}.$

Unitary Group (SU(n))

let $M(t)$ be a curve in $SU(n)$, with $M(0) = I$.

For small t, write

$$M(t) = I + tX + O(t^2), \quad \text{where } X = \frac{dM}{dt}\Big|_0$$

Have $I = M^\dagger M \ \forall t$

$$\Rightarrow I = (I + tX + \dots)^\dagger (I + tX + \dots)$$
$$= I + t(X + X^\dagger) + O(t^2)$$

$$\Rightarrow X^\dagger = -X \quad \text{antihermitian.}$$

Claim: $\mathrm{Tr}\, X = 0$.

Pf: Write

$$M(t) = \begin{pmatrix} 1 + tX_{11} & tX_{12} & \\ tX_{21} & 1 + tX_{22} & \\ & & \ddots \end{pmatrix}$$

$$\det M(t) = 1 = 1 + t\, \mathrm{tr}\, X + \dots$$

$$\Rightarrow \mathrm{tr}\, X = 0. \qquad \square$$

Note: Jacobi formula

$$\frac{d}{dt}\det(M(t)) = \det M(t)\, \mathrm{Tr}\left(M(t)^{-1} \frac{dM}{dt}\right).$$

For $t = 0$,

$$0 = \mathrm{Tr}(X).$$

So $L(SU(n)) = T_e(SU(n)) \subseteq \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid X^\dagger = -X,\ \mathrm{Tr}\, X = 0\}$.

For $U(n)$, $\det M$ can be a phase $e^{i\theta}$, $\mathrm{tr}\, X$ need not be 0.

Lie Algebra of a matrix Lie group

Consider 2 groups through the identity of G, $g_1(\underline{x}(t))$ and $g_2(\underline{y}(t))$. Product is

$$g_3(\underline{z}(t)) = g_2(\underline{y}(t))\, g_1(\underline{x}(t)).$$

with $X_1 = \dot{g}_1\big|_{t=0}$, $X_2 = \dot{g}_2\big|_{t=0}$

$$\Rightarrow \dot{g}_3\big|_{t=0} = \left(\dot{g}_2 g_1 + g_2 \dot{g}_1\right)\big|_{t=0} = X_2 + X_1 \in T_e(G)$$

Another vector in $T_e(G)$, hence closure.

Lie bracket from group commutator.

Def^n The <u>group commutator</u> of $g_1, g_2 \in G$ is

$$[g_1, g_2]_G := g_1^{-1} g_2^{-1} g_1 g_2 =: h.$$

let $g_1(t)$ and $g_2(t)$ be 2 curves through the identity. for $i = 1, 2$,

$$g_i(t) = e + tX_i + t^2 W_i + O(t^3).$$

Then

$$h(t) = [g_2(t)\, g_1(t)]^{-1} g_1(t)\, g_2(t)$$
$$= e + t^2 [X_1, X_2] + \dots$$

$$= X_1 X_2 - X_2 X_1$$

Can reparameterise $s = t^2$, then

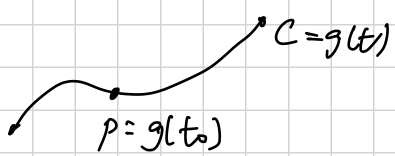
$$h(t) \in G \Rightarrow [X_1, X_2] \in L(G)$$

So also closure under Lie bracket.

Tangent space to matrix Lie group at general elt g.

Consider Curve $C = g(t)$.

Tangent space at point p.



Expand $g(t_0+\varepsilon) = g(t_0) + \varepsilon \dot{g}(t_0) + \mathcal{O}(\varepsilon^2)$

Given $g(t_0),\ g(t_0+\varepsilon) \in G$, thus $\exists\ h_p(\varepsilon) \in G$ s.t.

$$g(t_0+\varepsilon) = g(t_0)\, h_p(\varepsilon).$$

For small ε,

$$h_p(\varepsilon) = e + X_p \varepsilon + \dots$$

for some $X_p \in L(G) := T_e(G)$. Neglecting $\mathcal{O}(\varepsilon^2)$,

$$e + X_p \varepsilon = h_p(\varepsilon) = g(t_0)^{-1} g(t_0+\varepsilon)$$
$$= g(t_0)^{-1}\left(g(t_0) + \varepsilon \dot{g}(t_0)\right)$$

$$\Rightarrow \quad X_p \varepsilon = \varepsilon\, g(t_0)^{-1} \dot{g}(t_0)$$

$$\Rightarrow \quad X_p = g(t_0)^{-1} \dot{g}(t_0).$$

So we can map vectors $T_p(G)$ to vectors in $L(G) = T_e(G)$ by multiplying by $g(t_0)^{-1}$.

Conversely, for any $X \in L(G)$, $\exists$ a unique curve $g(t)$ s.t.

$$g(t)^{-1} \dot{g}(t) = X$$

and $g(t_0) = g_0$. Existence and uniqueness follow from diff. eqn.

Sol^n: $$g(t) = g(t_0) \exp(tX).$$

where $$\exp(tX) = \sum_{k=0}^{\infty} \frac{1}{k!} (tX)^k.$$

<u>One param. subgroups</u>

Given $X \in L(G)$, the curve $g_X(t) = e \exp(tX)$ forms an abelian subgroup of G "generated" by X.

Note $g_X(t_1)g_X(t_2) = g_X(t_1+t_2)$.

$g_X(t)$ isomorphic to $\begin{cases} (\mathbb{R},+) & \text{if only } g_X(0)=e \\ S^1 & \text{if } g_X(t_0)=e \text{ for some } t_0\neq 0. \end{cases}$

2.4 Lie groups from algebras

Given a Lie algebra $L(G)$ of a Lie group G, we can define the exponential map

$$\exp: L(G) \to G$$

for matrix Lie groups,

$$X \mapsto \exp X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

Locally, the map is bijection. Globally, the map is generally not bij.

Example $U(1) = \{ e^{i\theta} \mid \theta \in [0, 2\pi) \}$

$L(U(1)) = \{ ix \mid x \in \mathbb{R} \}$.

$\exp(ix)$ is not bij. since $e^{2\pi n i} = 1 \quad \forall n \in \mathbb{Z}$.

Example $G = O(n)$. Let $X = L(O(n)) \subseteq \mathrm{Skew}_n(\mathbb{R})$

Let $M = \exp tX$. Then

$$\begin{aligned} M^T &= (\exp(tX))^T \\ &= \exp(tX^T) \\ &= \exp(-tX) = M^{-1}. \end{aligned}$$

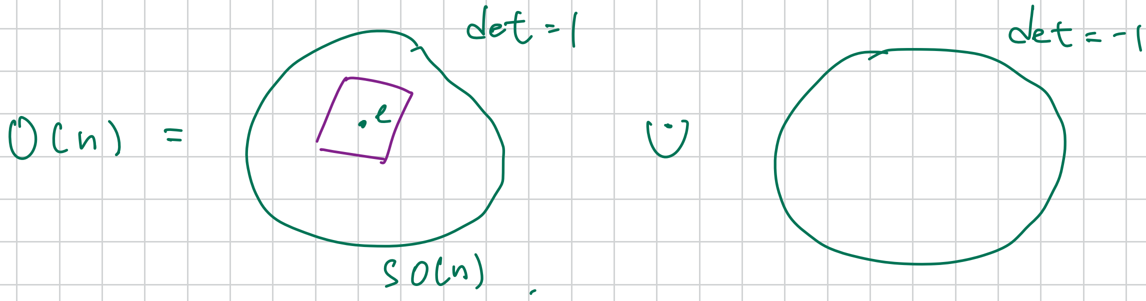
So $MM^T = MM^{-1} = I$, so $M \in O(n)$.

WTS: $\operatorname{Tr} X = 0 \Rightarrow \det M = 1$.

Let $\lambda_1, \dots, \lambda_n$ be eval of X. then $e^{t\lambda_1}, \dots, e^{t\lambda_n}$ are eval of $M = \exp tX \Rightarrow \det M = \prod_{i=1}^{n} e^{t\lambda_i} = 1$.

So $M \in SO(n)$.

$O(n)$ is <u>disconnected</u>.



<u>Claim</u> For $A \in \text{Skew}_n(\mathbb{R}) \Rightarrow A \in L(SO(n))$.

Pf: Define $\gamma(t) := \exp(tA)$ curve of matrices on some manifold. We have $(\gamma(t))^T \gamma(t) = I$ and $\det \gamma(t) = 1$.

$$\Rightarrow \gamma(t) \in SO(n).$$

By construction, $A = \dot{\gamma}(t)\big|_{t=0}$, tangent to the curve at identity, so $A \in L(SO(n))$. $\square$

$$d = \dim SO(n) = \dim L(SO(n)) = \dim \text{Skew}_n(\mathbb{R})$$
$$= \frac{n(n-1)}{2}.$$

Group product from Lie bracket:

- Baker-Campell-Hausdorff (BCH) formula:

<u>Claim</u>: For $X, Y \in L(G)$,

$$\exp tX \exp tY = \exp tZ \quad \text{for } Z \in L(G).$$

with $tZ = tX + tY + \frac{t^2}{2}[X,Y] + \frac{t^3}{12}\big([X,[X,Y]] + [Y,[Y,X]]\big)$
$+ \mathcal{O}(t^4)$.

3. Representations

3.1 Lie Group Representations

Group: transformations $\longrightarrow$ symmetries

Representations: How group actions transform vectors in vector spaces.

We've seen $GL(n, \mathbb{F})$ as a group of invertible matrices. These matrices are invertible linear maps on vector spaces $\mathbb{F}^n$

$$GL(n, \mathbb{F}): \mathbb{F}^n \to \mathbb{F}^n.$$

More generally, $GL(V)$ is a group of automorphisms on vector space V.

$$GL(V): V \to V$$

A representation D of G is a group homomorphism

$$D: G \to GL(V)$$

from G to the group automorphisms on some vec. space V, called the representation space associated with D.

that is, $\forall g \in G$, $D(g): V \to V$ is an invertible, linear map s.t.

$$v \mapsto D(g)v \quad \forall v \in V.$$

Linearity: $D(g)(\alpha v_1 + \beta v_2) = \alpha D(g) v_1 + \beta D(g) v_2$. $\forall \alpha, \beta \in \mathbb{F}$, $v_1, v_2 \in V$

Group hom^m: $D(g_2 g_1) = D(g_2) D(g_1)$.

$\Rightarrow D(e) = id_V$, $\quad id_V \, v = v \quad \forall v \in V$, and

$$D(g^{-1}) = D(g)^{-1} \quad \forall g \in G.$$

Def^n The dimension of a rep. is the dim of its rep. space (# basis vectors).

If V is finite-dim., say $\dim V = N$, then
$$GL(V) \cong GL(N, \mathbb{F})$$

Def^n A rep. is faithful if $D(g) = \mathrm{id}_V$ only for $g = e$, i.e. if $\ker D = \{e\}$.

Def^n The kernel of a map consists of all elts of G mapped to identity id.

Can show faithfulness $\Rightarrow$ D injective, i.e.
$$D(g_1) = D(g_2) \Rightarrow g_1 = g_2.$$

Examples With $G = (\mathbb{R}, +)$, $\alpha, \beta \in \mathbb{R}$.

(a) For some fixed $k \in \mathbb{R}$, $D(\alpha) := e^{k\alpha}$,

- $D(\alpha) D(\beta) = e^{k(\alpha+\beta)} = D(\alpha+\beta)$

If $k \neq 0$, this is faithful rep. $D(\alpha) = 1 \Rightarrow \alpha = 0$ only.

If $k = 0$ unfaithful — "trivial" rep.

(b) Another 1-dim example on G rep. $D(\alpha) := e^{ik\alpha}$.

- Unfaithful: $\ker D = \left\{ \frac{2\pi n}{k} \;\middle|\; n \in \mathbb{Z} \right\}$.

(c) 2d rep on $\mathbb{R}^2$
$$D(\alpha) := \begin{pmatrix} \cos k\alpha & -\sin k\alpha \\ \sin k\alpha & \cos k\alpha \end{pmatrix}.$$
Unfaithful.

(d) Infinite-dim rep: let $V = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$ and
$f \in \mathbb{R}^{\mathbb{R}}$
$$D(\alpha) f(x) = f(x - \alpha).$$

and $D(\alpha) f = f \quad \forall f \Rightarrow \alpha = 0 \Rightarrow \ker D = \{0\}$.

so faithful.

Def^n The trivial rep. D_0 is where $D_0(g) = 1 \ \forall g \in G$.

Quantities which are invar. under group transf^n transform in the trivial rep. In physics, call these singlets.

Def^n If G is a matrix Lie group, then (one) fundamental or the defining rep. D_f is

$$D_f(g) = g \qquad \forall g \in G.$$

Only $D_f(e) = e \Rightarrow$ faithful. If $G \subseteq GL(n, \mathbb{F})$, then $\dim D_f = n$.

Let G be a matrix Lie group and consider its Lie algebra as a vector space $V = L(G)$

Def^n The adjoint rep. $D^{adj} = Ad$ is map $Ad : G \to GL(L(G))$.

s.t. $\forall g \in G$, $Ad(g) \equiv Ad_g : L(G) \to L(G)$ with

$$Ad_g X := g X g^{-1} \qquad \forall X \in L(G).$$

Check adjoint rep. properties:

- Closure: For $X \in L(G)$, $\exists$ curve $g(t) = e + tX + \ldots$. For any $h \in G$, we have another curve

$$\tilde{g} = h g(t) h^{-1} = e + t h X h^{-1} + \ldots$$

So $Ad_h X = h X h^{-1} \in L(G)$

- Group operation:

$$Ad_{(g_2 g_1)} X = (g_2 g_1) X (g_2 g_1)^{-1} = Ad_{g_2}(Ad_{g_1} X).$$

- Lie bracket:

$$Ad_g [X, Y] = [Ad_g X, Ad_g Y].$$

3.2 Lie Algebra Representations

Def^n A representation d of a Lie algebra $L(G)$, (or $\mathfrak{g}$).
is a map from $L(G)$ to a set of linear maps within
$\mathfrak{gl}(V) = L(GL(V))$

$$d : L(G) \to \mathfrak{gl}(V)$$

where $\mathfrak{gl}(V) = L(GL(V))$ is the Lie algebra of $GL(V)$.
and where the Lie bracket is preserved., that is,
$\forall\, X \in L(G)$, $d(X): V \to V$. a linear map (not necessarily invertible) s.t. $v \mapsto d(X)\,v \quad \forall\, v \in V$.

- Linear
- Lie bracket preserved: $d([X,Y]) = [d(X), d(Y)]$

Note dim of rep = dim V.

Def^n (Trivial representation) $d_0(X) = 0 \quad \forall\, X \in L(G)$, i.e. $d_0(X)\underline{v} = \underline{0} \quad \forall \underline{v} \in V$.

Def^n (Fundamental rep.) For $G \leq GL(n, \mathbb{F})$, $\forall\, X \in L(G)$.

$$d_f : L(G) \to Mat_n(\mathbb{F})$$
$$X \mapsto d_f(X) = X$$

Def^n (Adjoint representation)

$$ad : L(G) \to \mathfrak{gl}(L(G))$$

$\forall\, X \in L(G)$, $ad_X : L(G) \to L(G)$
$$Y \mapsto ad_X(Y) = [X, Y].$$

Representations of $L(G)$ from repr. of G

Curve in group near identity

$$g(t) = e + tX + \ldots \in G.$$

Given representation $D(g(t)) = I + t\,d(X) + \ldots$ defining $d(X)$, Check that the Lie bracket is preserved.

$$D(g_1^{-1}g_2^{-1}g_1g_2) = D(g_1)^{-1}D(g_2)^{-1}D(g_1)D(g_2). \qquad (*)$$

Write $i=1,2$,

$$g_i(t) = e + tX_i + t^2 W_i + \dots$$
$$g_i(t)^{-1} = e - tX_i - t^2(W_i - X_i^2) + \dots$$

LHS of $(*)$ $= D(e - t^2[X_1,X_2] + \dots) = I + t^2 d([X_1,X_2]) + \dots$

RHS of $(*)$ $= I + t^2[d(X_1), d(X_2)] + \dots$

$$\Rightarrow \quad d([X_1,X_2]) = [d(X_1), d(X_2)] \text{ as req'd}$$

<u>Example</u> Adjoint rep(s): ad_X from Ad_g.

$$\begin{aligned} Ad_g Y &= gYg^{-1} \\ &= (I+tX)Y(I-tX) + O(t^2) \\ &= Y + t[X,Y] + O(t^2) \\ &= (I + t\,ad_X)Y + O(t^2). \end{aligned}$$

Consistent defn.

<u>Representations of G from repr. of $L(G)$?</u>

Given d a rep. of $L(G)$, let $g = \exp X$, $X \in L(G)$, $g \in G$, and let $\exp d(X) = D(g)$. Is D a rep. of G?

Can use BCH formula to confirm

$$D(g_2 g_1) = D(g_2)D(g_1)$$

However, this is not necessarily a repr. of the Lie group G, since exp may not be surjective.

Prop If G simply connected, then $D(g) = \exp d(X)$ is a rep. of G.

Def^n A Lie group is simply connected if
(i) Group G is path-connected, i.e. any 2 points can be joined by a cts path.
(ii) All closed curves can be shrunk to a point.

3.3 Useful Concepts

Def^n (Equivalent (or isomorphic) rep) D_1 and D_2 of G or d_1 and d_2 of $L(G)$ are equiv. if $\exists$ invertible linear maps R and S s.t.

$$D_2(g) = R\, D_1(g)\, R^{-1} \quad \forall g \in G$$
$$d_2(X) = S\, d_1(X)\, S^{-1} \quad \forall X \in L(G).$$

Def^n A rep. d of $L(G)$ with corresponding vector space V has an invariant subspace $W \subseteq V$ if $d(X)w \in W$ $\forall w \in W$, $\forall X \in L(G)$.

Example $V = \mathbb{R}^2$. If $d(X) = \begin{pmatrix} A(X) & B(X) \\ 0 & C(X) \end{pmatrix}$, then $W = \left\{ w = \begin{pmatrix} x \\ 0 \end{pmatrix} \right\}$ invar.

Example All rep. have 2 trivially invar. subspace, $W = \{0\}$, $W = V$.

Def^n An irreducible rep. ("irrep") is a rep. with no non-trivial invar. subspace. o/w. the rep. is reducible.

Def^n The direct sum of vec spaces U and W is

$$U \oplus W = \{ (u, w) \text{ or } u \oplus v \mid u \in U,\ w \in W \}.$$

and

- $(u_1 \oplus w_1) + (u_2 \oplus w_2) = (u_1 + u_2) \oplus (w_1 + w_2)$.
- $\lambda(u + w) = \lambda u \oplus \lambda w$
- $\dim U + \dim W = \dim U \oplus W$.

Def^n A totally reducible rep. d of $L(G)$ can be decomposed into irred. pieces, i.e., we can write its rep. space as a direct sum

$$V = W_1 \oplus \dots \oplus W_k$$

s.t. $d(X) w_i \in W_i$ $\forall X \in L(G)$, $w_i \in W_i$. Then $\exists$ basis s.t. $d(X)$ is block diag.

$$d(X) = \begin{pmatrix} d_1(X) & & 0 \\ & d_2(X) & \\ & & \ddots \\ 0 & & & d_k(X) \end{pmatrix}$$

Example $G = (\mathbb{R}, +)$. Rep. space $V = \{$all 2π periodic f^n $f: \mathbb{R} \to \mathbb{R}\}$.
with $(D(\alpha) f)(x) = f(x - \alpha)$, $x, \alpha \in \mathbb{R}$.
Not faithful since $\operatorname{Ker}(D) = \{2\pi k \mid k \in \mathbb{Z}\}$.
Invar. subspaces

$$W_n = \{ a_n \cos nx + b_n \sin nx \mid a_n, b_n \in \mathbb{R} \}.$$

$$V = W_0 \oplus W_1 \oplus \dots = \bigoplus_{n=0}^{\infty} W_n.$$

with $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

Unitarity of representations

Def^n Given a Lie group G and a rep. D whose representation space has inner product $\langle \cdot, \cdot \rangle$, D is unitary if

$$\langle D(g)v, D(g)w\rangle = \langle v, w\rangle$$

$\forall g \in G$. $u, v \in V$

Prop. Corresponding algebra rep. to $D(g)$ is antihermitian (skew-adjoint)

Note Adjoint operator $\dagger$ is defined w.r.t. inner product

$$\langle v, A^\dagger w\rangle = \langle Av, w\rangle$$

Prop If $\dim D = N$ is finite, then

$$D(g) \in U(N) \quad \forall g \in G$$

$$d(X) \in L(U(N)) \quad \forall X \in L(G)$$

Thm (Maschke) A finite-dim unitary operator is either irred. or totally reducible.

Pf (Sketch): For any invar. proper subspace $W \subset V$, its orthogonal complement $W_\perp$ is also invar. So $V = W \oplus W_\perp$

Similarly, reduce W and/or $W_\perp$ if either has invar. subspace.

$\dim V$ finite $\Rightarrow$ process terminates $\square$

Maschke's thm can be extended:

1. All finite-dim reps can be "made" unitary and are irred. or totally reducible.
2. If G compact Lie group, can find unitary finite-dim reps.

Tensor Product

Def^n Given vec. spaces V and W over $\mathbb{F}$, the tensor product space $V \otimes W$ is spanned by vectors $v \otimes w$ with $v \in V$, $w \in W$, with

(i) $(\alpha v) \otimes w = v \otimes (\alpha w) = \alpha (v \otimes w)$, $\alpha \in \mathbb{F}$

(ii) $(v_1 + v_2) \otimes w = (v_1 \otimes w) + (v_2 \otimes w)$

(iii) $v \otimes (w_1 + w_2) = (v \otimes w_1) + (v \otimes w_2)$

If $\{v_i\}$ and $\{w_j\}$ basis for V and W, then $\{v_i \otimes w_j\}$ a basis for $V \otimes W$

$$\Rightarrow \dim V \otimes W = (\dim V)(\dim W)$$

Def^n A <u>product vector / state</u> $\Phi \in V \otimes W$ can be written as $\Phi = v \otimes w$ for a single $v \in V$, $w \otimes W$. The cpts break down

$$\Phi_A := \Phi_{\alpha a} = V_\alpha W_a .$$

A linear combination of product states is <u>entangled state</u>.

Tensor product allow one to combine rep.

Let $D^{(1)}$ and $D^{(2)}$ be reps of G, with vec. spaces V, W.

$$D^{(1)}(g) : V_\alpha \mapsto D^{(1)}(g)_{\alpha\beta} V_\beta \qquad v \in V.$$

$$D^{(2)}(g) : W_a \mapsto D^{(2)}(g)_{ab} W_b \qquad w \in W$$

Tensor prod. rep. $D^{(1)} \otimes D^{(2)}$ acts on $V \otimes W$ s.t.

$$(D^{(1)} \times D^{(2)})(g)_{\underset{A}{\alpha a}, \underset{B}{\beta b}} = D^{(1)}(g)_{\alpha\beta} D^{(2)}(g)_{ab} .$$

$$\Phi_{\alpha a} \mapsto D^{(1)}(g)_{\alpha\beta} D^{(2)}(g)_{ab} \Phi_{\beta b}$$

If $\Phi_{\alpha a} = V_\alpha W_a$,

$$\Phi_{\alpha a} \xrightarrow{D^{(1)} D^{(2)}} (D^{(1)}_{\alpha\beta} V_\beta)(D^{(2)}_{ab} W_b)$$

Group $\to$ Algebra

Let $g(t) = g_t \in G$ be a curve in Lie group s.t. $g_0 = e$ and $\dot{g}_0 = X \in L(G)$. Find action of tensor prod. rep. of algebra representation $d^{(1)}(X)$, $d^{(2)}(X)$.

$$(D^{(1)} \otimes D^{(2)})(g_t)(v \otimes w) = \left(D^{(1)}(g_t)v\right) \otimes \left(D^{(2)}(g_t)w\right)$$

diff. $\Rightarrow$ $\frac{d}{dt}\left[(D^{(1)} \otimes D^{(2)})(g_t)(v \otimes w)\right]\Big|_{t=0}$

$$= \left[\frac{d}{dt} D^{(1)}(g_t)\, v\right]_{t=0} \otimes w + v \otimes \left[\frac{d}{dt} D^{(2)}(g_t)\, w\right]_{t=0},$$

$$\Rightarrow (d^{(1)} \otimes d^{(2)})(X) = \left(d^{(1)}(X) \otimes id_W\right) + \left(id_V \otimes d^{(2)}(X)\right), \quad \forall X \in L(G).$$

Important cor. to Maschke's thm.

Cor Rep. of $d^{(1)}$ and $d^{(2)}$ can, if finite, be written as a direct sum of irreps of $L(G)$

$$d^{(1)} \otimes d^{(2)} = \tilde{d}_1 \oplus \dots \oplus \tilde{d}_k =: \bigoplus_{i=1}^{k} \tilde{d}_i$$

irrep.

"Decomposing into irrep."

4. Angular Momentum

4.1 Relationship between $SO(3)$ and $SU(2)$

Lie algebras:

$$su(2) = L(SU(2)) = \{ X \in Mat_2(\mathbb{C}) \mid X^\dagger = -X, \ Tr X = 0 \}$$

Choose as a basis

$$T_a = -\frac{i}{2}\sigma_a, \quad \sigma_a \text{ Pauli matrices}$$

Using $\sigma_a \sigma_b = \delta_{ab} I + i \varepsilon_{abc} \sigma_c$

$$\Rightarrow [T_a, T_b] = \varepsilon_{abc} T_c,$$

i.e. $f^c{}_{ab} = \varepsilon_{abc}$ structure const.

$$\mathfrak{so}(3) = L(SO(3)) = \text{Skew}_3 = \{3\times 3 \text{ antisym matrices}\}$$

$$\tilde{T}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \tilde{T}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i.e.

$$(\tilde{T}_a)_{bc} = -\varepsilon_{abc}$$

$$\Rightarrow [\tilde{T}_a, \tilde{T}_b] = \varepsilon_{abc} \tilde{T}_c, \quad f^c{}_{ab} = \varepsilon_{abc}$$

So $L(SU(2)) \cong L(SO(3))$. isomorphic Lie Algebra.

Group Manifolds

- $SO(3)$ in §2.1: 3-ball, radius π, $\theta \underline{n}$ with $\underline{n}$ unit vec. with antipoles identified
- $SU(2)$: let $A \in SU(2)$ can be written $A = a_0 I + i \underline{a} \cdot \underline{\sigma}$. with $a_0, a_i \in \mathbb{R}$, $|a_0|^2 + |\underline{a}|^2 = 1$. So manifold is S^3.

Group Theory Aside

Def^n Let $H \leq G$, then $\forall g \in G$, can form a left coset of H as

$$gH := \{gh \mid h \in H\},$$

and right coset

$$Hg := \{hg \mid h \in H\}$$

Def^n $H \leq G$ is a normal subgroup if $gH = Hg \ \forall g \in G$.

Def^n For G and $H \triangleleft G$, define

$$G/H = \{ gH \mid g \in G \}$$

Define coset multiplication

$$(g_2 H)(g_1 H) = (g_2 g_1) H$$

Thm For $H \triangleleft G$, G/H is a group under coset multiplication with $H = eH$.

Def^n Such a group G/H is a quotient group.

Def^n The centre $Z(G)$ is $Z(G) = \{ x \in G \mid xg = gx \ \forall g \in G \}$.

Thm $Z(G) \trianglelefteq G$.

SU(2) manifold: S^3:

$$|a_0|^2 + |\underline{a}|^2 = 1$$

Have $Z(SU(2)) = \{ \pm I \}$.

Look at cosets $A\,Z(SU(2)) = \{ \pm A \}$ for any $A \in SU(2)$

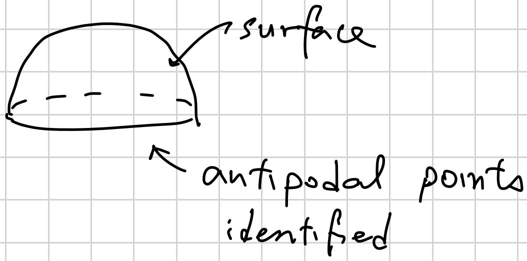
Set of all cosets form a group

$$SU(2) / \mathbb{Z}_2 .$$

$\{ \pm I \} \cong \mathbb{Z}_2$

Manifold of $SU(2)/\mathbb{Z}_2$ is $A = a_0 I + i\underline{a} \cdot \underline{\sigma}$ with antipoles identified.

We can draw the manifold of S^3 $(a_0 \geq 0)$ as upper hemisphere



Flatten to form a disc. Restore dimension, so it is a ball in 3-dim, so manifolds $SO(3)$ and $SU(2)/\mathbb{Z}_2$ are isomorphic

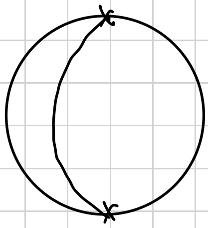
Explicit map:
$$\rho : SU(2) \longrightarrow SO(3).$$
$$A \longmapsto \rho(A) = R.$$

where $R_{ij} = \frac{1}{2} \operatorname{Tr}(\sigma_i A \sigma_j A^\dagger)$, $i,j = 1,2,3.$

This map is 2-to-1, with $\rho(A) = \rho(-A)$, is called a double covering of $SO(3)$ by $SU(2)$. $SU(2)$ is the double cover of $SO(3)$.

Prop Every Lie algebra is the Lie algebra of exactly one simply-connected Lie group.

$SU(2)$ is simply connected, while $SO(3)$ not simply connected.

SO(3):  curves between antipoles are closed but cannot be shrunk to a point

4.2 Representation of $L(SU(2))$

Let V be a real vector space
$$V = \{ \lambda^a T_a \mid \lambda^a \in \mathbb{R} \}$$
The complexification of V is the complex span of $\{T_a\}$
$$V_{\mathbb{C}} = \{ \lambda^a T_a \mid \lambda^a \in \mathbb{C} \}$$

$$su(n) = \{ X \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{Tr}\,X = 0,\ X^\dagger = -X \}$$

$$su(n)_{\mathbb{C}} = \{ X \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{Tr}(X) = 0 \} \cong sl(n, \mathbb{C}) = L(SL(n,\mathbb{C})).$$

Let $\mathfrak{g} = L(G)$ be a real Lie algebra and denote its complexification by $\mathfrak{g}_{\mathbb{C}} = L(G)_{\mathbb{C}}$.

Representations of $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$ have $d(x)$

$$\Rightarrow d(X+iY) = d(X) + i d(Y).$$

where $X, Y \in \mathfrak{g}$, $X + iY \in \mathfrak{g}_{\mathbb{C}}$.

Conversely, if have a rep. $d_{\mathbb{C}}$ of $L(G)_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$, then we can restrict it to $\mathfrak{g}$, e.g.

$$d(X) = d_{\mathbb{C}}(X), \quad X \in \mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}.$$

Def^n A <u>real form</u> of a complex Lie algebra $\mathfrak{h}$ is a real Lie algebra $\mathfrak{g}$ with $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}$

<u>Note</u> A complex Lie algebra may have >1 real form.

Now, $su(2)_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}} \{ \sigma_a \mid a = 1,2,3 \}$

More convenient basis (Cartan-Weyl basis)

$$H = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$E_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$E_- = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

These satisfy

$$[H, E_\pm] = \pm 2 E_\pm, \quad [E_+, E_-] = H.$$

Recall $\quad ad_X Y = [X, Y]$

$$[H, E_\pm] = ad_H E_\pm = \pm 2 E_\pm.$$

Also know $[H, H] = 0 = ad_H H.$, so $H, E_\pm$ are evec of ad_H with eval $0, \pm 2$. These $(0, \pm 2)$ are the roots of $SU(2)_{\mathbb{C}}$.

evals

Let d be a finite-dim irred. rep. of $SU(2)$ w/ rep. space V. Write an eval $d(H)$ as $v_\lambda \in V$, s.t.

$$d(H)\, v_\lambda = \lambda v_\lambda.$$

Def^n The eval λ are the weights of d.

Note Roots are weights of adjoint rep.

Claim: The operators $E_\pm$ are ladder (raising/lowering) operators (step), since

$$d(H)\, d(E_\pm)\, v_\lambda = \{ d(E_\pm) d(H) + [d(H), d(E_\pm)] \} v_\lambda$$

$$= \{ d(E_\pm)\lambda \pm 2 d(E_\pm) \} v_\lambda$$

$$= (\lambda \pm 2) \underbrace{d(E_\pm)\, v_\lambda}_{\substack{\text{evec of } d(H) \\ \text{with eval } \lambda \pm 2}}.$$

or $\quad d(E_\pm)\, v_\lambda = 0.$

Finite-dim space $\Rightarrow$ finite # evals and evecs.

There must be some Λ s.t.

$$d(H)\, v_\Lambda = \Lambda v_\Lambda$$

and

$$d(E_+)\, v_\Lambda = 0$$

We call Λ the highest weight of d.

Apply $d(E_-)$ n times

$$v_{\Lambda-2n} = (d(E_-))^n \; v_\Lambda$$

Process must terminate for some $n=N$. Suggests a basis for V

$$\{v_\Lambda, \; v_{\Lambda-2}, \; \dots, \; v_{\Lambda-2N}\}$$

Check that $d(E_+)$ does not give new LI vec.

$$d(E_+)\, v_{\Lambda-2n} = d(E_+)\, d(E_-)\, v_{\Lambda-2n+2}$$

$$= \left[d(E_-)\, d(E_+) + d(H)\right] v_{\Lambda-2n+2}$$

$$= \{d(E_-)\, d(E_+) + \Lambda-2n+2\}\; v_{\Lambda-2n+2}$$

Recurrence relation. Consider

$n=1$: $\quad d(E_+)\, v_{\Lambda-2} = 0 + \Lambda v_\Lambda$

$n=2$: $\quad d(E_+)\, v_{\Lambda-4} = (2\Lambda-2)\, v_{\Lambda-2}$.

In general,

$$d(E_+)\, v_{\Lambda-2n} = r_n \, v_{\Lambda-2n+2}$$

with

$$r_n = r_{n-1} + (\Lambda-2n+2) \quad , \quad r_1 = \Lambda$$

$$\Rightarrow r_n = (\Lambda+1-n)\, n.$$

Finite # of evals $\Rightarrow$ for $n=N$, we require

$$d(E_-)\, v_{\Lambda-2N} = 0$$

$$\Rightarrow r_{N+1} = 0$$

$$\Rightarrow (\Lambda+1-(N+1))(N+1) = (\Lambda-N)(N+1) = 0,$$

Then $N=\lambda$, a non-neg integer.

<u>Conclusion</u>: the finite-dim irreps of $SU(2)_{\mathbb{C}}$ are labelled by $\Lambda \in \mathbb{Z}_{\geq 0}$, d_Λ, with weights

$$S_\Lambda = \{-\Lambda, -\Lambda+2, \dots, \Lambda\}.$$

Non-degenerate weights

$$\dim d_\Lambda = \Lambda + 1$$

- d_0 is trivial rep.
- d_1 is fundamental rep.
- d_2 is adjoint rep.

In QM:

ang. mom. $\underline{J} = (J_1, J_2, J_3)$

estates $|j\, m\rangle$ with $2j \in \mathbb{Z}_{\geq 0}$, $2m \in \mathbb{Z}$,

$-j \leq m \leq j$.

Translation $d(H) = 2J_3$ $\qquad J^2 |jm\rangle = j(j+1)\,|jm\rangle$

$\Lambda = 2j$ $\qquad J_3 |jm\rangle = m\,|jm\rangle$

$\lambda = 2m$

$d(E_\pm) = J_1 \pm i J_2$

4.3 Representations of SU(2) and SO(3)

SU(2) is simply connected, rep d_Λ can give rep. D_Λ using exp. map.

For $SO(3) \cong SU(2)/\mathbb{Z}_2$, an elt. of $SO(3) \to$ pair of elts in $SU(2)$ $\{-A, A\}$.

Claim D_Λ is a rep. of $SO(3)$ iff it respects this identification of pairs A and $-A$.

Pf: Idea: need $D_\Lambda(A) = D_\Lambda(-A)$. find this iff Λ even.
Sufficient to check $D_\Lambda(I) = D_\Lambda(-I)$.
For $H = \sigma_3$ (C-W basis), we have

$$-I = \exp(i\pi H) \in SU(2)$$

then $D_\Lambda(-I) = \exp(i\pi\, d_\Lambda(H))$, $d_\Lambda(H)$ has evals (weights)
$\lambda \in \{-\Lambda, -\Lambda+2, \dots, \Lambda-2, \Lambda\}$.

Evals of $D_\Lambda(-I)$ are

$$e^{i\pi\lambda} = (-1)^\lambda = (-1)^\Lambda$$

So $D_\Lambda(-I) = D_\Lambda(I)$ only if λ even.

λ even $\Rightarrow SO(3)$. , λ odd $\Rightarrow$ "spinor representations" of $SU(2)$.

4.4 Tensor products of SU(2) irreps

Irreps: d_Λ and $d_{\Lambda'}$ with $\Lambda, \Lambda' \in \mathbb{Z}_{\geq 0}$.
Spaces: V_Λ and $V_{\Lambda'}$, $V_\Lambda \otimes V_{\Lambda'} = \mathrm{span}_{\mathbb{R}}\{v \otimes v' \mid v \in V_\Lambda,\ v' \in V_{\Lambda'}\}$.

For $X \in SU(2)$, the tensor product rep.

$$(d_\Lambda \otimes d_{\Lambda'})(X)(v \otimes v') = (d_\Lambda(X)\, v) \otimes v' + v \otimes (d_{\Lambda'}(X)\, v')$$

$$\dim(d_\Lambda \otimes d_{\Lambda'}) = (\Lambda+1)(\Lambda'+1).$$

Decomposition:

$$d_\Lambda \otimes d_{\Lambda'} = \bigoplus_{\Lambda'' \in \mathbb{Z}} \mathcal{L}^{\Lambda''}_{\Lambda\Lambda'}\, d_{\Lambda''},$$

Where $\mathcal{L}^{\Lambda''}_{\Lambda\Lambda'}$ (L-R coeff) are non-neg integers or multipliers.

For $SU(2)$, we have bases for V_Λ and $V_{\Lambda'}$.

$$\{v_\lambda\} : \lambda \in S_\Lambda = \{-\Lambda, -\Lambda+2, \dots, \Lambda\}$$

s.t.

$$d_\Lambda(H)\, v_\lambda = \lambda v_\lambda .$$

and

$$\{v_{\lambda'}\} : \lambda' \in S_{\Lambda'} = \{-\Lambda', -\Lambda'+2, \dots, \Lambda'\}.$$

$$d_{\Lambda'}(H)\, v_{\lambda'} = \lambda' v_{\lambda'} .$$

Basis for tensor prod. space is

$$\{v_\lambda \otimes v_{\lambda'} \mid \lambda \in S_\Lambda,\ \lambda' \in S_{\Lambda'}\}.$$

What is the result of H on tensor product basis vecs?

$$\begin{aligned}(d_\Lambda \otimes d_{\Lambda'})(H)(v_\lambda \otimes v_{\lambda'}) &= d_\Lambda(H)\, v_\lambda \otimes v_{\lambda'} + v_\lambda \otimes d_{\Lambda'}(H)\, v_{\lambda'} \\ &= (\lambda + \lambda')\, v_\lambda \otimes v_{\lambda'} .\end{aligned}$$

The new weights are sums of $\lambda \in \Lambda$, $\lambda' \in \Lambda'$.

The weight set is

$$S_{\Lambda\Lambda'} = \{\lambda + \lambda' \mid \lambda \in S_\Lambda,\ \lambda' \in S_{\Lambda'}\}.$$

The highest weight $\Lambda + \Lambda'$ has multiplicity 1. $\mathcal{L}^{\Lambda+\Lambda'}_{\Lambda\Lambda'} = 1$.

$$d_\Lambda \otimes d_{\Lambda'} = d_{\Lambda+\Lambda'} \oplus \underbrace{\tilde{d}_{\Lambda,\Lambda'}}_{\text{remainder}}$$

Remaining rep. $\tilde{d}_{\Lambda,\Lambda'}$ has weight set $\tilde{S}_{\Lambda\Lambda'}$ s.t.

$$S_{\Lambda,\Lambda'} = S_{\Lambda+\Lambda'} \cup \tilde{S}_{\Lambda,\Lambda'}$$

The highest weight in $\tilde{S}_{\Lambda\Lambda'}$ is $\Lambda + \Lambda' - 2$, with multiplicity 1.

Repeat...

$$d_\Lambda \otimes d_{\Lambda'} = d_{\Lambda+\Lambda'} \oplus d_{\Lambda+\Lambda'-2} \oplus \dots \oplus d_{|\Lambda-\Lambda'|} .$$

Example $\Lambda = \Lambda' = 1$ (corresponding to $j = j' = \frac{1}{2}$).

$S_1 = \{-1, 1\}$.

$S_{1,1} = \{-2, 0, 0, 2\}$

Highest weight is 2 : $S_{1,1} = \{-2, 0, 2\} \cup \{0\}$.

$= S_2 \cup S_0$.

$$\Rightarrow \quad d_1 \otimes d_1 = d_2 \oplus d_0$$

$$\left(j = \tfrac{1}{2} \otimes \tfrac{1}{2} \right) = (j=1) + (j=0)$$

use dimensionality to label irreps.

$$\underline{2} \otimes \underline{2} = \underline{3} \oplus \underline{1}.$$

5. Relativistic Symmetries

5.1 Lorentz Group.

$$x^\mu \longrightarrow x'^\mu, \quad \mu = 0, 1, 2, 3,$$

which leave scaler invariant $x^\mu \eta_{\mu\nu} x^\nu$ invar., where

$$\eta_{\mu\nu} := \mathrm{diag}(1, -1, -1, -1)$$

Let Λ denote such a transf^n : $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

Invariance of scaler product

$$\Rightarrow x^\mu \eta_{\mu\nu} x^\nu = \Lambda^\mu{}_\sigma x^\sigma \eta_{\mu\nu} \Lambda^\nu{}_\rho x^\rho.$$

$$\Rightarrow \eta_{\sigma\rho} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho \eta_{\mu\nu}, \quad \text{or} \quad \eta = \Lambda^T \eta \Lambda. \qquad (*)$$

So $\Lambda \in O(1,3)$ - Pseudoorthonormal group.

$(*) \Rightarrow$ 10 constraints. Λ is 4×4, so $16 - 10 = 6$ d.o.f.

The Lorentz group consists of 4 disjoint manifolds depending on $\det \Lambda$ and $\mathrm{sgn}(\Lambda^0{}_0)$.

From (*),

$$\det \Lambda^T \eta \Lambda = \det \eta$$

$$\Rightarrow \det \Lambda = \pm 1 .$$

Set $\rho = \sigma = 0$ in (*).

$$\Lambda^\mu{}_0 \eta_{\mu\nu} \Lambda^\nu{}_0 = \eta_{00} = 1 .$$

$$\Rightarrow (\Lambda^0{}_0)^2 - \sum_{i=1}^3 (\Lambda^i{}_0)^2 = 1$$

$$\Rightarrow \quad (\Lambda^0{}_0)^2 \geq 1$$

$$\Rightarrow \quad \Lambda^0{}_0 \geq 1 \text{ or } \Lambda^0{}_0 \leq -1 .$$

The case $\det \Lambda = 1$, $\Lambda^0{}_0 \geq 1$ contains the identity and forms $SO(1,3)^\uparrow$, the proper, orthochronous Lorentz group.

Other parts of $O(1,3)$ obtained by application of

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ and/or } P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

time-reversal parity.

Special cases of $SO(1,3)^\uparrow$

- Rotations : $[(\Lambda_R)^\mu{}_\nu] := \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$. $R \in SO(3)$
- Lorentz boost: $[(\Lambda_B)^\mu{}_\nu] := \begin{pmatrix} \cosh\psi & -\underline{n}^T \sinh\psi \\ \underline{n}\sinh\psi & I - \underline{n}\,\underline{n}^T(\cosh\psi - 1) \end{pmatrix}$. rapidity

  (not a subgroup ∵ not closed).

3 param. for rotn, 3 param for boost

- $SO(1,3)^\uparrow$ is not simply connected.
- $SO(1,3)^\uparrow \cong SL(2,\mathbb{C}) / \mathbb{Z}_2$.

5.2 Lie Algebra of the Lorentz group

Expand $\Lambda \in SO(1,3)^\uparrow$ about the identity $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$.

$$\Lambda^\mu{}_\nu(t) = \delta^\mu{}_\nu + t\omega^\mu{}_\nu + O(t^2)$$

Insert into $\Lambda^T \eta \Lambda = \eta$

$$\Rightarrow \quad \eta_{\mu\nu} + t(\omega_{\mu\nu} + \omega_{\nu\mu}) = \eta_{\mu\nu} + O(t^2)$$

$$\Rightarrow \quad [\omega_{\mu\nu}] = -[\omega_{\nu\mu}] = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

anti-sym matrix when indicies lowered.

$$[\omega^\sigma{}_\nu] = [\eta^{\sigma\mu}\omega_{\mu\nu}] = \begin{pmatrix} 0 & a & b & c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{pmatrix}.$$

Basis for $so(1,3)^\uparrow$ is

$$[K^\mu{}_\nu]: \; K_1 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & 0 \\ & & & \\ 0 & & & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ 1 & & 0 & \\ 0 & & & \end{pmatrix}, \quad K_3 = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix}$$

$$[J^\mu{}_\nu]: \; J_1 = \begin{pmatrix} 0 & & 0 & \\ & & & \\ 0 & & & 1 \\ & & -1 & \end{pmatrix}, \quad J_2, \quad J_3$$

K generates boosts, J generates rot's.

Lie Brackets:

$$[J_i, J_j] = \varepsilon_{ijk} J_k$$
$$[K_i, K_j] = -\varepsilon_{ijk} J_k$$
$$[J_i, K_j] = \varepsilon_{ijk} K_k.$$

It will be convenient to write

$$M^{0j} = K_j, \quad M^{ij} = \varepsilon_{ijk} J_k.$$

s.t.

$$(M^{\mu\nu})^\alpha{}_\beta = \eta^{\mu\alpha}\delta^\nu{}_\beta - \eta^{\nu\alpha}\delta^\mu{}_\beta,$$

and $\Lambda \in SO(1,3)^\uparrow$ can be written as

$$\Lambda = \exp\left(\tfrac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \overset{\text{or}}{=} \exp\left(\theta^i J_j + \varphi^i K_j\right).$$

where $\theta^i, \varphi^i \in \mathbb{R}$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Simplify by complexifying

$$L_i := \tfrac{1}{2}(J_i + iK_i)$$

$$R_i := \tfrac{1}{2}(J_i - iK_i)$$

The brackets will be

$[L_i, L_j] = \varepsilon_{ijk} L_k$ → $SU(2)$ structure const.

$[R_i, R_j] = \varepsilon_{ijk} R_k$ → 2 sets of $SU(2)$ algebra.

$[L_i, R_j] = 0$.

Look at linear combinations

$\theta^i J_j + \varphi^i K_j \in SO(1,3)^\uparrow$, $\theta^i, \varphi^i \in \mathbb{R}$

$\alpha^i L_i + \beta^i R_j \in SO(1,3)^\uparrow_{\mathbb{C}}$ $\alpha^i, \beta^i \in \mathbb{C}$.

$$SO(1,3)^\uparrow_{\mathbb{C}} \cong SL(2,\mathbb{C}) \oplus SL(2,\mathbb{C})$$

$$\cong SU(2)_{\mathbb{C}} \oplus SU(2)_{\mathbb{C}}.$$

Recall: Ang mom.

$$SU(2)_{\mathbb{C}} \cong SU(2) \oplus SU(2).$$

keep real — discard im part. — real form

Representations of $SO(1,3)^\uparrow_{\mathbb{C}}$:

$$\underset{SO(1,3)^\uparrow_{\mathbb{C}}\text{ rep.}}{d(\alpha L + \beta R)} = \alpha \underset{SU(2)_{\mathbb{C}}}{d(L)} \otimes I + \beta I \otimes \underset{SU(2)_{\mathbb{C}}}{d(R)}.$$

Since $[L, R] = 0$.

Real form : $J_i = L_i + R_i$, $K_i = -i(L_i - R_i)$.

$$d(J_i) = d(T_i) \otimes I + I \otimes d(T_i)$$
$$d(K_i) = -i\left(d(T_i) \otimes I - I \otimes d(T_i)\right).$$

Label these using highest SU(2) weights Λ on ang. mom. quantum numbers j_1, j_2. $2j_1$ and $2j_2$ are highest weights of SU(2).

Examples $(j_1, j_2) =$

(a) $(\frac{1}{2}, 0)$ "spinor", "Fundamental" rep. of $SO(1,3)^{\uparrow} \cong SL(2,\mathbb{C})$.

$\Rightarrow$ lefthanded Weyl fermion

(b) $(0, \frac{1}{2})$ conjugate to fund rep. "RH" Weyl fermion

(c) $(\frac{1}{2}, \frac{1}{2})$ 4-vec. rep. Under SO(3), this rep. is reducible $\underline{2} \otimes \underline{2} = \underline{1} \oplus \underline{3}$ ← dimensions

e.g. 4-vec decomposition $x^{\mu} = (x^0, \underline{x})$.

under full group, $(\frac{1}{2}, \frac{1}{2})$ irred.

Note Dirac spinor is $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

5.3 Poincaré group and algebra

Include translations with rot^{n}s and boosts $\Rightarrow$ isometry group of "distance" preserving trans^n – Lorentz scalars are invariant.

ISO(1,3)

- Semidirect product group : $ISO(1,3) = \underbrace{O(1,3)}_{\text{Lorentz}} \ltimes \underbrace{T^{1,3}}_{\text{translations}}$

Def^n (Semidirect product group) Let G be a group with $N \triangleleft G$ and $H \leq G$. (not necc. normal). G is a semi-direct product group $G = H \ltimes N = N \rtimes H$, provided the following equiv. conditions hold.

(i) N, H have trivial intersection and G is equal to the subgroup product $G = NH$ or HN.

(ii) Every el't of G can be written uniquely as nh (or hn) for some $n \in N$, $h \in H$.

Let φ be a group hom^m

$$\varphi : H \to \mathrm{Aut}(N),$$

i.e. $\forall\, h \in H$, we have

$$\varphi(h) = \varphi_h : N \to N$$

s.t. $\varphi_h(n) = hnh^{-1} \in N$ since N is normal.

Consider $G' = \{(n,h) \mid n \in N,\ h \in H\}$. with product

$$(n_2, h_2)(n_1, h_1) = (n_2 \varphi_{h_2}(n_1),\ h_2 h_1).$$

$G' \cong G$ with isom $f: G \to G'$, $f(g) = f(nh) = (n,h)$.

The direct product group is a special case when

$$\varphi_h(n) = n \quad \forall\, h, n.$$

For group el't $(\Lambda^\mu{}_\nu, a^\mu) \in ISO(1,3)$, with action on 4-vec.

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu$$

A convenient rep. in 5×5 matrices

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda x + a \\ 1 \end{pmatrix}.$$

Group mult:

$$\begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Lambda' & a' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda\Lambda' & \Lambda a' + a \\ 0 & 1 \end{pmatrix}$$

In not^n as before,

$$(n_1, h_1) = (a', \Lambda'), \qquad (n_2, h_2) = (a, \Lambda)$$

$$(a, \Lambda)(a', \Lambda') = (\Lambda a' + a, \Lambda\Lambda')$$
$$= (n_2\, \varphi_{h_2}(n_1), h_2 h_1).$$

with $\varphi_{h_2}(n_1) = \Lambda a'$.

Recall the generators of Lorentz algebra $M^{\mu\nu}$, with $M^{0j} = K_j$, $M^{ij} = \varepsilon_{ijk} J_k$. The generators for translations are $(P^\sigma)^\beta = \eta^{\sigma\beta}$.

5×5 rep.

$$\tilde{M}^{\mu\nu} = \begin{pmatrix} M^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{P}^\sigma = \begin{pmatrix} 0 & P^\sigma \\ 0 & 0 \end{pmatrix}$$

⇒ 6+4 = 10 generators, basis for Poincaré alg.

Then

$$\exp(\Lambda, a) = \exp(a_\sigma P^\sigma)\exp\left(\tfrac{1}{2}\omega_{\mu\nu} M^{\mu\nu}\right),$$

· Lie brackets :

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\sigma} M^{\mu\rho}.$$

$$[M^{\mu\nu}, P^\sigma] = \eta^{\nu\sigma} P^\mu - \eta^{\mu\sigma} P^\nu$$

$[P^\mu, P^\nu] = 0$ (transl^n commute – flat spacetime).

Later: proper def^n of "Casimir elts / operators / invariants".

These are objects which commute with all generators of the Lie alg.

Example $p^2 = p_\sigma p^\sigma$ is a quadratic Casimir.

Example Pauli – Lubanski pseudovector

$$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p^\sigma$$

$W^2 = W_\mu W^\mu$ is a quartic Casimir.

5.4 Representation of the Poincaré group

There are no finite-dim. unitary rep. of Poincaré.

Denote a unitary rep. of Poincaré group by $\mathcal{U}$.

$$\mathcal{U}(\Lambda, a) : V \to V$$

for any $(\Lambda, a) \in ISO(1,3)$.

Representations can be written

$$\mathcal{U}(\Lambda, a) = T(a)\, \mathcal{U}(\Lambda)$$

Defining shorthand

$$T(a) = \mathcal{U}(I, a)$$

$$\mathcal{U}(\Lambda) = \mathcal{U}(\Lambda, 0)$$

Note $$(\Lambda, \Lambda a) = (\Lambda, 0)(I, a) = (I, \Lambda a)(\Lambda, 0)$$

$$\Rightarrow \mathcal{U}(\Lambda)\, T(a) = T(\Lambda a)\, \mathcal{U}(\Lambda)$$

Translations generated by p^σ

$$(0, a) = \exp(a_\sigma p^\sigma) = e^{a \cdot p}$$

Let $|p,s\rangle \in V$ be evec. of $T(a)$ in a vec. space corresponding to a single-particle state.

$T(a)$ being a rep. of $(0,a)$, then

$$T(a)\ |p,s\rangle = e^{ia\cdot p}\ |p,s\rangle,$$

where ip^σ is an eval of P^σ.

s represents any internal discrete dof (e.g. spin)

Lorentz transf^n on evec :

$$T(a)\ \big(U(\Lambda)\ |p,s\rangle\big) = U(\Lambda)\ T(\Lambda^{-1}a)\ |p,s\rangle.$$
$$= e^{i(\Lambda^{-1}a)\cdot p}\ \big(U(\Lambda)\ |p,s\rangle\big)$$

commutes

$\Rightarrow$ eval $e^{ia\cdot(\Lambda p)}$, still an evec. with eval

$$p^\mu \mapsto p'^\mu = \Lambda^\mu{}_\nu\, p^\nu.$$

Lorentz transf^n preserves $(p')^2 = p^2$.

For any <u>fixed</u> p^2, we have an equivalence class of momenta. all related by Lorentz transf. Choose some standard / reference k^μ s.t. $k^2 = p^2$

$$p^\mu = L(p)^\mu{}_\nu\, k^\nu$$

for any p^μ with $p^2 = k^2$ is in the image of LT $L(p)^\mu{}_\nu$.

Evec

$$|p,s\rangle = U(L(p))\ |k,s\rangle.$$

Act with arbitrary LT Λ

$$U(\Lambda)\ |p,s\rangle = U(\Lambda L(p))\ |k,s\rangle$$

$$= U\left(L(\Lambda p) L^{-1}(\Lambda p)\right) U(\Lambda L(p)) |k,s\rangle$$
$$= U(L(\Lambda p))\, U(L^{-1}(\Lambda p)\, \Lambda\, L(p))\ |k,s\rangle \qquad (*).$$

Note $L(\Lambda p) k = \Lambda p \Rightarrow L^{-1}(\Lambda p)\Lambda p = k.$

Claim $W(\Lambda, p) := L^{-1}(\Lambda p)\Lambda L(p)$ is an elt. of the Lorentz group which leaves k^μ invar. Such elts form a subgroup, called a little group.

Let's assume we have a rep. of little group

$$U(W) |k,s\rangle = \sum_{s'} D_{s's}(W) |k,s'\rangle$$

The (scaler) coeff. $D_{s's}(W)$ define the little group rep. Then, we can insert into $(*)$ to induce rep. for Poincaré group.

$$\Rightarrow U(\Lambda) |p,s\rangle = U(L(\Lambda p))\, U(W(\Lambda, p)) |k,s\rangle$$
$$= \sum_{s'} D_{s's}(W(\Lambda, p))\, U(L(\Lambda p)) |k,s'\rangle$$
$$= \sum_{s'} D_{s's}(W(\Lambda, p)) |\Lambda p, s'\rangle$$

Remains to specify what are the physically interesting little groups.

6 possibilities for k^μ. 4 of these not suitable for single particle states.

- Spacelike 4-momenta $p^2 < 0$, e.g. $k^\mu = (0, -\underline{k})$
- Negative energy $p^0 < 0$, e.g. $k^\mu = (-k, 0)$ $\quad k^2 > 0$, or
  $(-|\underline{k}|, \underline{k})$ $\quad k^2 = 0$

- $p^\mu = k^\mu = 0$, vacuum.
- Massive states $p^2 = m^2 > 0$. Boost to frame where

$$k^\mu = (m, \underline{0})$$

This k^μ is invar. under 3-dim rot^n. So the little group is $SO(3)$. Irreps are those of $SU(2)$. States characterised by p^μ, j, j_3 .
(Λ, λ)

- Massless states $p^2 = 0$. Rotate to frame where

$$k^\mu = (\omega, 0, 0, \omega).$$

for fixed $\omega > 0$.

Claim Little group is $ISO(2) \cong SO(2) \ltimes T^2$
rot^n in plane   translations (not physical).

Find generators:

$$E_1 := K_1 - J_2 \qquad\qquad [J_3, E_1] = E_2$$
$$\Rightarrow \quad [E_2, J_3] = E_1$$
$$E_2 := K_2 + J_1 \qquad\qquad [E_1, E_2] = 0$$

6. Classification of Lie Algebra (a subset) (Cartan)

6.1 Definitions

Def^n A subalgebra of a Lie algebra $\mathfrak{g}$ is a vec. space which is also a Lie alg. under the Lie bracket.

Def^n An ideal (or invariant subalgebra) of Lie algebra is a subalg. $\mathfrak{h} \subset \mathfrak{g}$ s.t. $[X, Y] \in \mathfrak{h}$ $\forall$ $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$.
Trivial ideals are $\mathfrak{h} = \{0\}$, and $\{\mathfrak{g}\}$.

Def^n The <u>derived algebra</u> of a Lie alg $\mathfrak{g}$ is

$$i(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}_{\mathbb{F}}\{[X,Y] \mid X, Y \in \mathfrak{g}\}.$$

with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. This is an ideal of $\mathfrak{g}$.

Def^n The <u>centre</u> of $\mathfrak{g}$ is

$$J(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X,Y] = 0 \quad \forall\, Y \in \mathfrak{g}\}.$$

This is also an ideal of $\mathfrak{g}$.

Def^n A Lie alg is <u>Abelian</u> if $[X,Y] = 0 \quad \forall\, X, Y \in \mathfrak{g}$.

$$\Leftrightarrow J(\mathfrak{g}) = \mathfrak{g},$$

Note For $\mathfrak{g}$ abelian, $i(\mathfrak{g}) = 0$.

Def^n A Lie alg. is simple if it is non-abelian and has no nontrivial ideals, i.e.

$$\mathfrak{g} \text{ simple } \Leftrightarrow J(\mathfrak{g}) = 0 \quad , \quad i(\mathfrak{g}) = \mathfrak{g}.$$

Def^n A Lie alg. is semisimple if it has no Abelian, non-trivial ideal.

Prop If $\mathfrak{g}$ semisimple, $\mathfrak{g} = \mathfrak{g}_1 + \dots + \mathfrak{g}_n$, where $\mathfrak{g}_i$ simple

6.2 The Killing Form

Goal: Define an invar. bilinear form on a Lie algebra. Use this to define a 1-to-1 map from a vec space (Lie alg.) to its dual vec. space (roots and weights)

Def^n (Bilinear form). $B: V \times V \to \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) s.t. linear in both arguments (even for $\mathbb{C}$)

$$B(u, \alpha v + \beta w) = \alpha B(u,v) + \beta B(u,w),$$

where $\alpha, \beta \in \mathbb{F}$, $u, v, w \in V$. Sim for $B(\alpha u + \beta v, w)$.

Def^n A <u>symmetric</u> bilinear form $\Leftrightarrow$ $B(v,u) = B(u,v)$.

<u>Note</u> : no condition on $B(u,u)$ generally.

<u>Def^n</u> A bilinear form B is nondegenerate if $\forall v \in V$ $(v \neq 0)$, $\exists$ some $v \in V$ s.t. $B(v,w) \neq 0$.

Def^n The <u>Killing form</u> of Lie alg $\mathfrak{g}$ is the sym. bil. form.

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}.$$
$$(X, Y) \mapsto \kappa(X,Y) := \frac{1}{\mathcal{N}} \operatorname{Tr}(\mathrm{ad}_X \cdot \mathrm{ad}_Y).$$

for some normalisation factor $\mathcal{N}$ (used in physics appl.) Take $\mathcal{N} = 1$ here.

- ad_X linear $\Rightarrow$ κ bil.
- Tr cyclic $\Rightarrow$ κ sym
- Usually we are thinking of real $\mathfrak{g}$, as a real form of $\mathfrak{g}_\mathbb{C}$, so usually $\kappa(X,Y) \in \mathbb{R}$.

Let $\{T_a\}$ be a basis for $\mathfrak{g}$, recall

$$\mathrm{ad}_{T_a} T_b = [T_a, T_b] = f^c{}_{ab} T_c.$$

Finite dim $\mathfrak{g}$ allows a matrix interpretation

$$(\mathrm{ad}_{T_a})^c{}_b = f^c{}_{ab}.$$

<u>Example</u> For $su(2)$, $[T_a, T_b] = \varepsilon_{abc} T_c$.

$$\mathrm{ad}_{T_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{etc.}$$

Then

$$\kappa(T_a, T_b) = \mathrm{Tr}\left[(\mathrm{ad}_{T_a})^e{}_d \; (\mathrm{ad}_{T_b})^d{}_e\right]$$

$$= f^c{}_{ad}\, f^d{}_{bc} =: \kappa_{ab}.$$

For general $X, Y \in \mathfrak{g}$,

$$\kappa(X, Y) = X^a Y^b \kappa(T_a, T_b) = X^a X^b \kappa_{ab}.$$

Claim: $\kappa(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = \kappa(X, Y) \quad \forall g \in G,\; X, Y \in L(G)$.

where $\mathrm{Ad}_g X = g X g^{-1}$.

Let $g = e + tZ + O(t^2)$

$$\kappa(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = \kappa(X + t\,\mathrm{ad}_Z X,\; Y + t\,\mathrm{ad}_Z Y)$$

$$= \kappa(X, Y) + t\underbrace{\left(\kappa(\mathrm{ad}_Z X, Y) + \kappa(X, \mathrm{ad}_Z Y)\right)}_{\text{By invariance} = 0}$$

$$\Rightarrow \kappa([Z, X], Y) + \kappa(X, [Z, Y]) = 0$$

$$\Rightarrow \kappa(Y, [Z, X]) = \kappa([Y, Z], X).$$

This is invariance of Killing form.

Put the Killing form to use

Thm (Cartan) The Killing form of a Lie algebra $\mathfrak{g}$ is nondegenerate iff $\mathfrak{g}$ is semisimple

Pf ($\Rightarrow$): Suppose $\mathfrak{g}$ is not semisimple, then $\exists$ non-trivial abelian ideal $\mathfrak{a} \subset \mathfrak{g}$, i.e. $[X, A] \in \mathfrak{a} \;\forall A \in \mathfrak{a},\; X \in \mathfrak{g}$.

Basis for ideal $\mathfrak{a}$: $\{T_i \mid i = 1, \ldots, \dim \mathfrak{a}\}$.

Basis for $\mathfrak{g}$: $\{T_B\} = \{T_i\} \cup \{T_\alpha \mid \alpha = 1, \dots, \dim\mathfrak{g} - \dim\mathfrak{a}\}$.

$\mathfrak{a}$ Abelian $\Rightarrow [T_i, T_j] = 0 \Rightarrow f_{ij}{}^{B} = 0$. (1)

$\mathfrak{a}$ ideal $\Rightarrow [T_i, T_\alpha] \in \mathfrak{a} \Rightarrow f_{i\alpha}{}^{\beta} = 0$ (2)

(1), (2) $\Rightarrow f_{iB}{}^{\beta} = 0 = f_{Bi}{}^{\beta}$ (3).

Consider $\kappa(X, A) = \kappa_{Bi} X^B A^i$

$$\kappa_{Bi} = f^C{}_{BD} f^D{}_{iC}.$$

Have $\{C\} = \{j\} \cup \{\beta\}$, then

$$\kappa_{Bi} = f^\alpha{}_{BD} f^D{}_{i\alpha} + f^j{}_{BD} \underbrace{\cancel{f^C{}_{ij}}}_{\to 0 \text{ by } (1)}$$

$$= \underbrace{\cancel{f^\alpha{}_{Bj}}}_{0 \text{ by } (2)} f^j{}_{i\alpha} + f^\alpha{}_{B\beta} \underbrace{\cancel{f^\beta{}_{i\alpha}}}_{\to 0 \text{ by } (2)} = 0$$

So our assumption $\Rightarrow$ for $A \in \mathfrak{a}$, $\nexists X \in \mathfrak{g}$ s.t. $\kappa(X, A) \neq 0$

$\Rightarrow \kappa$ is degenerate $\square$.

<u>Prop</u> If κ non-degenerate, then it is invertible, i.e. for $[\kappa_{ab}]$, we can find $[\kappa]^{-1}$ s.t.

$$(\kappa_{ab})(\kappa^{-1})^{bc} = \delta_a{}^c.$$

<u>Def</u> A Lie group is <u>semisimple</u> if its Lie alg. is semisimple

<u>Prop</u> If the Killing form of a real Lie alg. $\mathfrak{g} = L(G)$ is -ve def, then G is compact, and $\mathfrak{g}$ is said to be of <u>compact type</u>.

[Aside :

Prop. A compact group which is not semisimple has an alg with neg. semidef. Killing form (not neg. def.) .

Note : There are also non-compact groups whose alg. have -ve. semidef Killing form.]

Prop Every semisimple, complex Lie alg $L(G)_{\mathbb{C}}$. has a real form with $\kappa_{ab} = -\kappa \delta_{ab}$, $\kappa \in \mathbb{R}^+$.

By above, G compact, and the real form is of compact type.

Def^n Any basis for which $\kappa_{ab} \propto \delta_{ab}$, if it exists, is called an adaptive basis.

In an adaptive basis $\{T_a\}$,

$$\kappa\left([T_c, T_a], T_b\right) = f^d{}_{ca}\, \underbrace{\kappa\left(T_d, T_b\right)}_{\delta_{db}} = -\kappa f^b{}_{ca}$$

By invariance,

$$= \kappa\left(T_c, [T_a, T_b]\right) = f^d{}_{ab}\, \kappa(T_c, T_d) = -\kappa f^c{}_{ab}.$$

Hence,

$$f^c{}_{ab} = f^b{}_{ca} = -f^b{}_{ac}.$$

anti-sym when swapping upper/lower indices.

In adaptive basis, we can write

$$f_{abc} := f^c{}_{ab} = f^a{}_{bc} = f^b{}_{ca} \quad \text{etc.}$$

6.3 Casimir Elements

A Casimir element is a polynomial of elements of a Lie algebra which commutes with all elts of the Lie algebra

Casimir are elts of the universal enveloping algebra (UEA) of the Lie alg. The UEA of $\mathfrak{g}$ is a formal span of $\{I, \mathfrak{g}, \mathfrak{g}\otimes\mathfrak{g}, \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}, \ldots\}$ subject to the rule

$$XY - YX = [X, Y],$$

where $[\cdot,\cdot]$ Lie bracket

XY may not be in $\mathfrak{g}$, $XY \in \mathfrak{g}\otimes\mathfrak{g}$

New identities such as

$$[X, YZ] = [X, Y]Z + Y[X, Z]$$

The universal, quadratic Casimir elt (of the UEA) of a Lie alg.

$$C := T_b T_b \quad (\text{sum over } b)$$

Check that

$$\begin{aligned}[T_a, T_b T_b] &= T_b [T_a, T_b] + [T_a, T_b] T_b \\ &= T_b f_{abc} T_c + f_{abc} T_c T_b \\ &= f_{abc} T_b T_c - f_{acb} T_c T_b \\ &= 0\end{aligned}$$

So C commutes with all $X = X^a T_a \in \mathfrak{g}$.

Some Lie alg. have higher order (polynomial) Casimirs, but these are not universal.

Consider the universal quad. Casimir in rep. d of a real Lie alg. $\mathfrak{g} = L(G)$

$$C_d := \sum_a d(T_a)\, d(T_a)$$

By above,

$$[d(X), C_d] = 0 \quad \forall X \in \mathfrak{g}$$

Let $D(g) = \exp(d(X))$ be corresponding group. rep. of G. If d irred., then D is irred, and by Schur's lemma,

$$C_d D(g) = D(g) C_d \quad \forall g \in G.$$

$$\Rightarrow C_d = c_d I \quad , \text{ i.e. } C_d \propto \text{identity}.$$

Where $c_d \in \mathbb{R}$.

Example ($L(SU(2))$) Adapted basis: $T_a = -\frac{i}{2\sqrt{2}} \sigma_a$.

Normalised s.t. $\kappa_{ab} = -\delta_{ab}$.

Look at $SU(2)$ irrep. $\Lambda = 1$ ($j = 1/2$)

$$C_{1/2} = -\frac{1}{8} \sigma_a \sigma_a = -\frac{3}{8} I. \Rightarrow c_{1/2} = -\frac{3}{8}$$

the quadratic Casimir of the "fundamental rep".

More generally,

$$C_j = -\frac{1}{2} |\underline{J}|^2 = -\frac{1}{2} j(j+1) I.$$

6.4 Cartan-Weyl Basis

Def^n Let $\mathfrak{g}$ Lie alg. Elt $X \in \mathfrak{g}$ is ad-diagonalisable if $ad_X : \mathfrak{g} \to \mathfrak{g}$ is diagonalisable

Def^n A maximal Abelian subgroup $\mathfrak{h}$ is not contained in any larger Abelian subalgebra.

Def^n If $\mathfrak{g}$ is complex, semisimple Lie algebra, then a Cartan subalgebra (CSA) $\mathfrak{h}$ of $\mathfrak{g}$ is a complex space s.t.

(1) For all $H_1, H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$ (Abelian)

(2) For all $X \in \mathfrak{g}$, if $[H, X] = 0$ $\forall H \in \mathfrak{h}$, then $X \in \mathfrak{h}$,
(Maximal)

(3) $\forall H \in \mathfrak{h}$, ad_H is diagonalisable.

Rmk: • If $\mathfrak{g}$ is not semisimple, then $\mathfrak{g}$ may or may not have a CSA.

• It can be proved that semisimple Lie alg. have CSA.

• The choice of a group's CSA may not be unique but all possible CSA of $\mathfrak{g}$ will have the same dimension

Def^n The rank of a Lie alg $\mathfrak{g}$ is the dim. of its CSA.

Example $L(SU(2))_{\mathbb{C}}$: $H = \sigma_3$ is a 1-dim CSA of $\mathfrak{su}(2)_{\mathbb{C}}$.

$$\mathfrak{h} = \text{span}_{\mathbb{C}}\{\sigma_3\}.$$

Example $L(SU(n))_{\mathbb{C}} = \{ X \in Mat_n(\mathbb{C}) \mid Tr X = 0 \}$.

Can choose the diag. elts as the CSA.

$$(H_i)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i} - \delta_{(\alpha+1),i}\,\delta_{(\beta+1),i} \quad (\text{no sum over } i)$$

Rank $= n-1$.

Evecs of ad_H : $\forall H, H' \in \mathfrak{h}$, $[H, H'] = 0$

$$\Rightarrow [ad_H, ad_{H'}] = 0 \quad (ad \text{ a rep})$$

All the maps of H commute $\Rightarrow$ simultaneously diag.

By the spectral decomposition thm, $\mathfrak{g}$ is spanned by the simultaneous evecs of the $\{ad_H\}$

Choose one $H \in \mathfrak{h}$ WLOG, then every other elt. of $\mathfrak{h}$ is a zero evec. of ad_H:

$$\mathrm{ad}_H H' = [H, H'] = 0, \quad , \; H' \in \mathfrak{h}.$$

Consider a basis for $\mathfrak{h}$: $\{H_i \mid i = 1, \dots, r\}$, where

$$r = \dim \underset{\underset{\text{CSA}}{\uparrow}}{\mathfrak{h}} = \operatorname{rank} \mathfrak{g}.$$

Since CSA is maximal, there is no other nonzero evec.
Label the rest by their evals by case: α.

$$\mathrm{ad}_{H_i} E_\alpha = [H_i, E_\alpha] = \alpha_i E_\alpha, \quad \alpha_i \in \mathbb{C}, \; i = 1, \dots, r$$

Evals α are called <u>roots</u>.

<u>Defⁿ</u> The set of all nonzero roots of a Lie alg. is its <u>root set</u> $\Phi = \{\alpha, \beta, \gamma, \dots\}$.

<u>Defⁿ</u> The evecs $E_\alpha, E_\beta, \dots$ are <u>adder</u> / <u>step</u> operators.

<u>Prop</u> The nonzero simultaneous evecs of H are non-degenerate, i.e. the E_α are unique, up to a scaling.

A general elt. of the CSA $H \in \mathfrak{h}$, is a linear combⁿ of $\{H_i\}$, say $H = \rho^i H_i$, $\rho^i \in \mathbb{C}$.

$$\mathrm{ad}_H E_\alpha = [H, E_\alpha] = [\rho^i H_i, E_\alpha] = \rho^i \alpha_i E_\alpha =: \alpha(H) E_\alpha,$$

where $\alpha(H) := \rho^i \alpha_i$.

<u>Claim</u> The roots $\alpha, \beta, \gamma, \dots$ are vectors in the dual vec. space $\mathfrak{h}^*$.

Def^n Given a vec. space V over field $\mathbb{F}$, the <u>dual vector space</u> V^* is the vec. space of lin. f^n $f: V \to \mathbb{F}$.

Note • $\dim V^* = \dim V$.

• Given a basis $\{v_i\}$ for V, we can find a basis for V^*, $\{v_i^*\}$, s.t. $v_i^*(v_j) = \delta_{ij}$.

Claim The roots are vectors in the dual vec. space $\mathfrak{h}^*$.

$\alpha : \mathfrak{h} \to \mathbb{C}$, $\alpha(H) \in \mathbb{C}$.

Pf: linearity:

$$\alpha(H+H')E_\alpha = [H+H', E_\alpha] = [H, E_\alpha] + [H', E_\alpha]$$
$$= (\alpha(H) + \alpha(H'))E_\alpha$$

Def^n The <u>Cartan-Weyl basis</u> for $\mathfrak{g}$ is given by

$$\{H_i \mid i = 1, \dots, r\} \cup \{E_\alpha \mid \alpha \in \Phi\}$$

Note $\dim \mathfrak{h} = r = \operatorname{rank} \mathfrak{g}$, $\dim \mathfrak{g} - \dim \mathfrak{h} = |\Phi|$, where Φ is the order of the root set. We will see that $|\Phi| > r$, so not all roots α are LI in $\mathfrak{h}^*$.

Next, use killing form to define an inner product in $\mathfrak{h}^*$.

Claim $\kappa(H, E_\alpha) = 0$ $\forall H \in \mathfrak{h}$, $\alpha \in \Phi$.

Pf: Given some root $\alpha \in \Phi$, $\exists H' \in \mathfrak{h}$ s.t. $\alpha(H') \neq 0$.

$$\alpha(H')\kappa(H, E_\alpha) = \kappa(H, \alpha(H')E_\alpha)$$
$$= \kappa(H, [H', E_\alpha])$$
$$= \kappa(\underbrace{[H, H']}_{=0}, E_\alpha) = 0.$$

$\Rightarrow \kappa(H, E_\alpha) = 0$ $\square$

Claim $\kappa(E_\alpha, E_\beta) = 0 \quad \forall \alpha, \beta \in \Phi, \ \alpha+\beta \neq 0$.

Pf: For $H \in \mathfrak{h}$,

$$\begin{aligned}
&(\alpha(H) + \beta(H))\kappa(E_\alpha, E_\beta) \\
&= \kappa([H, E_\alpha], E_\beta) + \kappa(E_\alpha, [H, E_\beta]) \\
&= 0. \quad \text{by invariance.}
\end{aligned}$$

□

Claim If $\alpha \in \Phi$, then $-\alpha \in \Phi$ and $\kappa(E_\alpha, E_{-\alpha}) \neq 0$.

Pf: From previous claim,

$$\kappa(E_\alpha, H) = 0 \quad \forall H \in \mathfrak{h}$$

$$\kappa(E_\alpha, E_\beta) = 0 \quad \forall \beta \neq -\alpha. \ \alpha, \beta \in \Phi.$$

However, $\mathfrak{g}$ semisimple $\Rightarrow$ κ nondegenerate, so $\exists X \in \mathfrak{g}$ s.t. $\kappa(E_\alpha, X) \neq 0$.

Process of elimination $\Rightarrow$ $-\alpha \in \Phi$. and $\kappa(E_\alpha, E_{-\alpha}) \neq 0$. □

Claim $\forall H \in \mathfrak{h}$, $\exists H' \in \mathfrak{h}$ s.t. $\kappa(H, H') \neq 0$

Pf: Suppose $\exists$ $H \in \mathfrak{h}$ s.t. $\kappa(H, H') = 0 \ \forall H' \in \mathfrak{h}$. Also, $\kappa(H, E_\alpha) = 0 \ \forall \alpha \in \Phi$, then $\kappa(H, X) = 0 \ \forall X \in \mathfrak{g} \Rightarrow \kappa$ degenerate ※ □.

Consequence of nondegeneracy: κ can be inverted within $\mathfrak{h}$.

Basis for $\mathfrak{h}$: $\{H_i\}$. Any 2 general elt of $\mathfrak{h}$: $H = \rho^i H_i$, $H' = \rho'^j H_j$. Then

$$\kappa(H, H') = \kappa_{ij} \rho^i \rho'^j.$$

and claim $\Rightarrow$ $\det[\kappa_{ij}] \neq 0$. So can invert $[\kappa_{ij}]$, ie. $\exists \kappa^{-1}$ s.t.

$$(\kappa^{-1})^{ik} \kappa_{kj} = \delta_{ij}$$

Given any $\alpha, \beta \in \Phi$, define an inner product on h^* as

$$(\alpha, \beta) := (\kappa^{-1})^{ij} \alpha_i \beta_j .$$

Recall $[H_i, H_j] = 0$, $[H_i, E_\alpha] = \alpha_i E_\alpha$, $i = 1, \dots, r$, $\alpha \in \Phi$.

Need to find $[E_\alpha, E_\beta]$.

(Recall for $SU(2)$ $[E_+, E_-] = H$).

For any $H \in h$, $\alpha, \beta \in \Phi$, we have

$$\begin{aligned} \mathrm{ad}_H [E_\alpha, E_\beta] &= [H, [E_\alpha, E_\beta]] \\ &= -[E_\beta, \underbrace{[H, E_\alpha]}_{=\alpha(H) E_\alpha}] - [E_\alpha, \underbrace{[E_\beta, H]}_{=-\beta(H) E_\beta}] . \\ &= (\alpha(H) + \beta(H)) [E_\alpha, E_\beta]. \end{aligned}$$

If $\alpha(H) + \beta(H) \neq 0$, then

$$[E_\alpha, E_\beta] = 0 .$$

or

Normalisation factor

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} .$$

an evec. of ad_H.
$\Rightarrow \alpha + \beta \in \Phi$.

If $\alpha(H) + \beta(H) = 0$, then $\mathrm{ad}_H [E_\alpha, E_{-\alpha}] = 0$

$$\Rightarrow [E_\alpha, E_{-\alpha}] \in h.$$

How to write this $[E_\alpha, E_{-\alpha}] \in h$?

Using Killing form.

$$\begin{aligned} \kappa([E_\alpha, E_{-\alpha}], H) &= \kappa(E_\alpha, [E_{-\alpha}, H]) \\ &= \alpha(H) \underbrace{\kappa(E_\alpha, E_{-\alpha})}_{\neq 0} . \end{aligned}$$

Define a normalised elt H_α by

$$H_\alpha = \frac{[E_\alpha, E_{-\alpha}]}{\kappa(E_\alpha, E_{-\alpha})}.$$

s.t. $\kappa(H_\alpha, H) = \alpha(H) \; \forall H \in h$.

In components, $H_\alpha = \rho_\alpha^i H_i$, $H = \rho^i H_i$.

$$\kappa_{ij} \rho_\alpha^i \rho^j = \alpha_j \rho^j$$

But ρ^i arbitrary, so

$$\rho_\alpha^i = (\kappa^{-1})^{ij} \alpha_j.$$

and

$$H_\alpha = (\kappa^{-1})^{ij} \alpha_i H_j.$$

Summarising

$$[H_i, H_j] = 0,$$
$$[H_i, E_\alpha] = \alpha_i E_\alpha,$$
$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \alpha+\beta \in \Phi \\ \kappa(E_\alpha, E_{-\alpha}) H_\alpha & \alpha+\beta = 0 \\ 0 & \text{o/w}. \end{cases}$$

Special elts:

$$\begin{aligned} [H_\alpha, E_\beta] &= (\kappa^{-1})^{ij} \alpha_j [H_i, E_\beta] \\ &= (\kappa^{-1})^{ij} \alpha_j \beta_i E_\beta. \\ &= (\alpha, \beta) E_\beta. \end{aligned}$$

$sl(2, \mathbb{C})$ subalgebras

New normalisation

$$e_\alpha := \sqrt{\frac{2}{(\alpha,\alpha)\kappa(E_\alpha, E_{-\alpha})}} \; E_\alpha.$$

$$h_\alpha := \frac{2}{(\alpha,\alpha)} H_\alpha.$$

Then
$$[h_\alpha, h_\beta] = 0,$$
$$[h_\alpha, e_\beta] = \frac{2(\alpha,\beta)}{(\alpha.\alpha)} e_\beta$$
$$[e_\alpha, e_\beta] = \begin{cases} n_{\alpha\beta}\, e_{\alpha+\beta} & \alpha+\beta\in\Phi \\ h_\alpha & \alpha+\beta=0 \\ 0 & o/w \end{cases}$$

For each $\alpha\in\Phi$, there is an $sl(2,\mathbb{C})$ $(\cong su(2)_{\mathbb{C}})$ subalgebra with basis $\{h_\alpha, e_\alpha, e_{-\alpha}\}$. with
$$[h_\alpha, e_{\pm\alpha}] = \pm 2 e_{\pm\alpha}$$
$$[e_\alpha, e_{-\alpha}] = h_\alpha.$$

Write these subalgebra $sl(2)_\alpha$

6.5 The real geometry of roots.

Plan: (i) $(\alpha,\beta)\in\mathbb{R}$

(ii) h^* is spanned by Φ.

(iii) $\exists$ real vec. space $h^*_{\mathbb{R}} \subset h^*$ which is spanned by Φ and $h^*_{\mathbb{R}}$ contain all the roots. in Φ.

(iv) $(\alpha,\alpha)\geq 0$, so length can be defined as $|\alpha| = \sqrt{(\alpha,\alpha)}$.

(v) We can determine lengths and angles between roots.

Let $\alpha,\beta\in\Phi$.

Def^n The α-root string passing through β is the set of roots
$$S_{\alpha,\beta} = \{\beta+p\alpha\in\Phi \mid p\in\mathbb{Z}\}.$$

Claim All such roots $\beta+p\alpha\in\Phi$ have $p = n_-, n_-+1, \dots, n_+-1, n_+$, where n_- and n_+ satisfy
$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -(n_+ + n_-)\in\mathbb{Z}.$$

Pf: Consider the vec. space (spanned by)

$$V_{\alpha,\beta} = \text{span}_{\mathbb{C}}\{ e_{\beta+p\alpha} \mid \beta+p\alpha \in S_{\alpha,\beta} \}.$$

Action of $sl(2)_\alpha$ on $V_{\alpha,\beta}$ is given by the adjoint rep.

$$\text{ad}_{h_\alpha} e_{\beta+p\alpha} = [h_\alpha, e_{\beta+p\alpha}] = \frac{2(\alpha, \beta+p\alpha)}{(\alpha,\alpha)} e_{\beta+p\alpha}$$
$$= \left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2p \right) e_{\beta+p\alpha}.$$

Also have

$$\text{ad}_{e_{\pm\alpha}} e_{\beta+p\alpha} = [e_{\pm\alpha}, e_{\beta+p\alpha}] \propto \begin{cases} e_{\beta+(p\pm1)\alpha} & \beta+(p\pm1)\alpha \in \Phi \\ 0 & \text{o/w.} \end{cases} \in V_{\alpha,\beta}$$

$V_{\alpha,\beta}$ is closed under $sl(2)_\alpha$ and thus is a rep^n space for $sl(2)_\alpha$.

The weight set for this irrep can be identified as follows.

$$S = \left\{ \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2p \;\middle|\; \beta+p\alpha \in \Phi,\ p \in \mathbb{Z} \right\}$$
$$= \{ -\Lambda, -\Lambda+2, \dots, \Lambda-2, \Lambda \}.$$

For some n_- and n_+

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_- = -\Lambda$$

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_+ = \Lambda.$$

$$\Rightarrow \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -n_+ - n_- \in \mathbb{Z}.$$

So root strings $\Leftrightarrow$ irreps of $sl(2,\mathbb{C})$ "quantisation condition". □

Defn A root string $S_{\alpha,\beta}$ has length $l_{\alpha,\beta} := n_+ - n_- + 1$
$(= \dim$ of $sl(2)_\alpha$ irrep.$)$.

Claim: For $\alpha, \beta \in \Phi$, we can write

$$(\alpha, \beta) = \sum_{\gamma \in \Phi} (\alpha, \gamma)(\gamma, \beta)$$

Pf: Cartan-Weyl basis $[H_i, H_j] = 0$, $[H_i, E_\gamma] = \gamma_i E_\alpha$.
$\{ad_{H_i}\}$ diag, with diag. entries either 0 or nonzero γ_i

$$K_{ij} = \kappa(H_i, H_j) = \mathrm{Tr}(ad_{H_i} \circ ad_{H_j})$$
$$= \sum_{\gamma \in \Phi} \gamma_i \gamma_j . \qquad (\mathcal{N} = 1).$$

Recall $(\alpha, \beta) = \alpha_i (\kappa^{-1})^{ij} \beta_j$. Also, define $\alpha^i := (\kappa^{-1})^{ij} \alpha_j$, then

$$(\alpha, \beta) = \alpha_i \beta^i = \alpha^i \beta^j \kappa_{ij} = \sum_{\gamma \in \Phi} \alpha^i \gamma_i \gamma_j \beta^j = \sum_{\gamma \in \Phi} (\alpha, \gamma)(\gamma, \beta). \ (*) \ \square.$$

Claim $(\alpha, \beta) \in \mathbb{R}$.

Pf: Divide $(*)$ by $\frac{(\alpha,\alpha)(\beta,\beta)}{4}$ $(\neq 0)$, then

$$\frac{2}{(\beta,\beta)} \cdot \underbrace{\frac{2(\alpha,\beta)}{(\alpha,\alpha)}}_{\in \mathbb{Z}} = \sum_{\gamma \in \Phi} \underbrace{\frac{2(\alpha,\gamma)}{(\alpha,\alpha)}}_{\in \mathbb{Z}} \cdot \underbrace{\frac{2(\beta,\gamma)}{(\beta,\beta)}}_{\in \mathbb{Z}}$$

So either $(\alpha, \beta) = 0$, or $(\beta, \beta) \in \mathbb{R} \setminus \{0\}$, $\Rightarrow (\alpha, \alpha) \in \mathbb{R}$ and $(\alpha, \beta) \in \mathbb{R}$. $\square$.

We know roots $\alpha \in \Phi$ are elts of h^*. Generally more roots than $\dim h^*$, i.e. $|\Phi| \geq \dim h^*$.

Claim: The roots span h^*.

Pf: Suppose not, then $\exists$ some $\lambda \in h^*$ s.t. $(\lambda, \alpha) = 0 \ \forall \alpha \in h^*$.
Corresponding vec in h: $H_\lambda := \lambda^i H_i \in h$.

$\mathfrak{h}$ is CSA $\Rightarrow$ $[H_\lambda, H] = 0 \quad \forall H \in \mathfrak{h}$.

Also have $[H_\lambda, E_\alpha] = \lambda^i [H_i, E_\alpha] = \lambda^i \alpha_i E_\alpha = \underbrace{(\lambda, \alpha)}_{=0} E_\alpha = 0$

$\therefore$ H_λ has zero bracket w/ every elt of C-W basis,

contradicts the algebra $\mathfrak{g}$ being semisimple. $\#$

$\Rightarrow$ Roots span $\mathfrak{h}^*$ $\square$.

Choose a basis : $\{ \alpha_{(i)} \in \Phi \mid i = 1, \dots, r \} \subset \Phi$. These will be called "simple roots"

Let $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$ be the subspace

$$\mathfrak{h}^*_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}} \{ \alpha_{(i)} \} \qquad , i = 1, \dots, r$$

Claim All roots lie in $\mathfrak{h}^*_{\mathbb{R}}$.

Pf: Let $\beta = c^i \alpha_{(i)} \in \mathfrak{h}^*$, sum over i. Inner product w/ any $\alpha_{(j)}$.

$$(\beta, \alpha_{(j)}) = c^i (\alpha_{(i)}, \alpha_{(j)})$$

By (i), both inner product are real $\Rightarrow$ $c^i \in \mathbb{R} \Rightarrow \beta \in \mathfrak{h}^*_{\mathbb{R}}$. $\square$.

Claim $\forall \lambda \in \mathfrak{h}^*_{\mathbb{R}}$, $(\lambda, \lambda) \geq 0$, with $(\lambda, \lambda) = 0 \Leftrightarrow \lambda = 0$.

Pf:
$$(\lambda, \lambda) = \sum_{\gamma \in \Phi} (\lambda, \gamma)^2 \geq 0.$$

$$(\lambda, \lambda) = 0 \Rightarrow (\lambda, \gamma) = 0 \quad \forall \gamma \in \Phi.$$

The roots span $\mathfrak{h}^*_{\mathbb{R}}$, so $\lambda = 0$ $\square$.

Def^n $|\alpha| = \sqrt{(\alpha, \alpha)}$ as the length (or norm) of a root in Φ.
(or extend to any $|\lambda| = \sqrt{(\lambda, \lambda)}$ for any $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$) For any $\alpha, \beta \in \Phi$ define angle between them through

$$(\alpha, \beta) = |\alpha| |\beta| \cos\theta$$

Recall quantisation condition: $2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$.

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2\frac{|\beta|}{|\alpha|}\cos\theta \quad \text{and} \quad \frac{2(\alpha,\beta)}{(\beta,\beta)} = 2\frac{|\alpha|}{|\beta|}\cos\theta$$

$$\Rightarrow \quad 4\cos^2\theta \in \mathbb{Z}.$$

$$\Rightarrow \quad |\theta| = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi\right\}.$$

6.6 Simple roots

$\mathfrak{h}^*_{\mathbb{R}}$: pick a hyperplane of dim. $r-1$ s.t. no root is contained in the hyperplane. The hyperplane bisects $\mathfrak{h}^*_{\mathbb{R}}$ into 2 halves, one we call positive, one negative

If α is positive, $-\alpha$ is negative. Root set is divided into $\Phi = \Phi_+ \cup \Phi_-$.

(i) $\alpha \in \Phi_+ \Leftrightarrow -\alpha \in \Phi_-$

(ii) $\alpha, \beta \in \Phi_+ \Rightarrow \alpha+\beta \in \Phi_+$ if a root

Def^n A simple root is a +ve root which cannot be written as a sum of +ve roots. The set of all simple roots is Φ_s.

Claim If $\alpha, \beta \in \Phi_s$, then $\alpha-\beta \notin \Phi$ not a root.

Pf: Suppose $\alpha-\beta$ is a root.

Case 1: $\alpha-\beta \in \Phi_+$, then $\alpha = (\alpha-\beta)+\beta \Rightarrow \alpha$ not simple #.

Case 2: $\alpha-\beta \in \Phi_-$, then $\beta-\alpha \in \Phi_+$ and $\beta = (\beta-\alpha)+\alpha$

$\Rightarrow \beta$ not simple #

$\therefore \alpha-\beta \notin \Phi$. $\square$.

Claim For $\alpha, \beta \in \Phi_s$, the α-string through β has length

$$l_{\alpha,\beta} = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \{1,2,3,\dots\}.$$

Pf: $S_{\alpha,\beta} = \{\beta + p\alpha \mid p\in\mathbb{Z},\ n_- \le p \le n_+\}$, $n_- \le 0$, $n_+ \ge 0$. and found
$n_+ + n_- = -\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

(i) $\Rightarrow$ $n_- = 0$ $\Rightarrow$ $n_+ = -\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}$

$$\Rightarrow l_{\alpha,\beta} = n_+ + 1 = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}. \qquad \square$$

Claim For $\alpha, \beta \in \Phi_S$, $(\alpha,\beta) \le 0$. and $(\alpha,\alpha) > 0$.

Claim Any $\beta \in \Phi_+$ can be written as lin. comb. of simple roots with +ve, integer coeff.

Pf: If $\beta \in \Phi_S$, done. If $\beta \notin \Phi_S$, $\exists \beta_1, \beta_2 \in \Phi_+$ s.t. $\beta = \beta_1 + \beta_2$.
If β_1, β_2 simple, done. Else, repeat until done. $\square$.

Claim All roots $\alpha \in \Phi$ can be written as $\alpha = \sum_i p_i \alpha_{(i)}$, with $p_i \in \mathbb{Z}$, $\alpha_{(i)} \in \Phi_S$. ie. Simple roots span $h^*_{\mathbb{R}}$.

Pf: From above, $\alpha \in \Phi_+ \Rightarrow$ applies. For $\alpha \in \Phi_-$, apply to $-\alpha$.
Then all non-zero p_i are either +ve or -ve integers. $\square$

Claim The simple roots form a L.I. set.

Pf: All $\lambda \in h^*_{\mathbb{R}}$ can be written as

$$\lambda = \sum_i c_i \alpha_{(i)}, \quad c_i \in \mathbb{R}, \alpha_{(i)} \in \Phi_S.$$

L.I.: $\lambda = 0 \Leftrightarrow c_i = 0 \ \forall i$.

Let $J_\pm = \{ i \mid c_i \gtrless 0 \}$. Define $\lambda_+ = \sum_{i\in J_+} c_i \alpha_{(i)}$ and
$\lambda_- = -\sum_{i\in J_-} c_i \alpha_{(i)} = \sum_{i\in J_-} b_i \alpha_{(i)}$, where $b_i = -c_i$, $b_i > 0$.

If $\lambda \ne 0$, $\lambda = \lambda_+ - \lambda_- = \sum_{i\in J_+} c_i \alpha_{(i)} - \sum_{i\in J_-} b_i \alpha_{(i)}$

$$\Rightarrow (\lambda,\lambda) = \underbrace{(\lambda_+,\lambda_+)}_{\ge 0} + \underbrace{(\lambda_-,\lambda_-)}_{\ge 0} - 2(\lambda_+,\lambda_-) \ge -2(\lambda_+,\lambda_-).$$

and $(\lambda_+, \lambda_-) = \sum_{i\in J_+} \sum_{j \in J_-} \underbrace{c_i b_j}_{>0} \underbrace{(\alpha_{(i)}, \alpha_{(j)})}_{\leq 0} \leq 0$. $\Rightarrow (\lambda, \lambda) > 0$

So simple roots are L.I.. □

Claim There are $r = \dim \mathfrak{h}^*_{\mathbb{R}}$ simple roots, i.e., $|\Phi_S| = r$.

Pf: $|\Phi_S| = r$, and Φ_S a basis. □

6.7 Classification

Simple roots form a basis for $\mathfrak{h}^*_{\mathbb{R}}$. Lie algebra from this basis: Chevalley basis.

Define the $r \times r$ Cartan matrix A, $r = \dim \mathfrak{h}^*_{\mathbb{R}}$, with elt.

$$A_{ji} = \frac{2(\alpha_{(j)}, \alpha_{(i)})}{(\alpha_{(i)}, \alpha_{(i)})}.$$

not sym. generally.

Recall that $A_{ji} \in \mathbb{Z}$. (in fact, for $j \neq i$, $A_{ji} \in \mathbb{Z}_{\leq 0}$).

From §6.4,

$$[h_\alpha, h_\beta] = 0$$

$$[h_\alpha, e_\beta] = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} e_\beta$$

$$[e_\alpha, e_\beta] = \begin{cases} n_{\alpha\beta}\, e_{\alpha+\beta} & \alpha+\beta \in \Phi \\ h_\alpha & \alpha+\beta = 0 \\ 0 & \text{o/w} \end{cases}$$

Chevalley:

$$[h_{\alpha_{(i)}}, h_{\alpha_{(j)}}] = 0$$

$$[h_{\alpha_{(i)}}, e_{\pm\alpha_{(j)}}] = \pm A_{ji}\, e_{\pm\alpha_{(j)}}$$

$$[e_{\alpha_{(i)}}, e_{-\alpha_{(j)}}] = \delta_{ij}\, h_j$$

Recall $\alpha_{(i)} - \alpha_{(j)} \notin \Phi$ as diff. of simple roots is not a root.

$$[e_{\alpha(i)}, e_{\alpha(j)}] = \mathrm{ad}_{e_{\alpha(i)}}(e_{\alpha(j)}) \propto e_{\alpha(i)+\alpha(j)} \quad \text{if } \alpha(i)+\alpha(j) \in \bar{\Phi}.$$

If $\alpha(i)+\alpha(j) \in \bar{\Phi}$, then it's part of a root string. Write this as $n\alpha(i) + \alpha(j)$ with $n = 0, \dots, -A_{ji}$, ie. length of string is

$$l = 1 - A_{ji}$$

Root string ends, so

$$(\mathrm{ad}_{e_{\alpha(i)}})^{1-A_{ji}}(e_{\alpha(j)}) = 0 \qquad (\text{Serre relation}).$$

The bracket and Serre relation completely characterise the Lie algebra. Any finite-dim, simple, complex Lie algebra is uniquely determined by its Cartan matrix.

Constraints on Cartan matrices:

(i) $\forall i$, $A_{ii} = 2$ (by def^n)

(ii) $A_{ji} = 0 \Rightarrow A_{ij} = 0$

(iii) $A_{ji} \in \mathbb{Z}_{\le 0}$ follows from $(\alpha(j), \alpha(i)) \le 0$ for $j \ne i$.

(iv) $\det A > 0$. Write $A = \kappa D$ where

$$\kappa(\alpha(i), \alpha(j)) = (\alpha(i), \alpha(j)).$$

where

$$D_{jk} = \frac{2}{(\alpha(j), \alpha(k))}\delta_{jk}.$$

κ sym. D diag w/ +ve elt. $\det A = \det\kappa \det D > 0$.

[$\{H_i\}$ basis for h. $\kappa(H_i, H_j) = \kappa_{ij}$

$H_\alpha = (\kappa^{-1})^{ij}\alpha_j H_i$

$$\kappa(H_\alpha, H_\beta) = (\kappa^{-1})^{ij}\alpha_j (\kappa^{-1})^{kl}\beta_l \underbrace{\kappa(H_i, H_k)}_{=\kappa_{ik}} = \alpha_k(\kappa^{-1})^{kl}\beta_l = (\alpha, \beta)$$]

(v) For simple Lie alg., A is irred., i.e. A cannot be made block triangular by a permutation transform P. $A \mapsto PAP^{-1}$.

If A reducible $\Leftrightarrow$ $\mathfrak{g}$ is semi simple

(vi) Claim: $A_{ij} A_{ji} \in \{0, 1, 2, 3\}$ for $i \neq j$ (no sum)

Pf: $(\alpha_{(i)}, \alpha_{(j)}) = |\alpha_{(i)}| \, |\alpha_{(j)}| \cos \phi_{ij}$

$$\Rightarrow A_{ij} A_{ji} = 4 \cos^2 \phi_{ij} \in \{0, 1, 2, 3, 4\}.$$

But for $\phi = 0, \pi, \ldots$, $\alpha_{(j)} = \pm \alpha_{(i)} \Rightarrow$ not LI.

So $0 \leq \cos^2 \phi_{ij} \leq 1$, and $A_{ij} A_{ji} \in \{0, 1, 2, 3\}$ for $i \neq j$ $\square$

Consider case of rank 2 algebra $(r = 2)$

$$A = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}, \text{ with } m, n \in \mathbb{Z}_{\geq 0}$$

$\det A = 4 - mn > 0 \Rightarrow (m,n) \in \{(1,1), (1,2), (1,3), (3,1), (2,1)\}$.

$m = 0 \Leftrightarrow n = 0$ gives reducible A.

6.8 Dynkin diagrams

1-to-1 correspondence between Cartan matrices and diagrams.

Constraints on acceptable A's $\Rightarrow$ rules for admissible diagrams

Def^n: Given a Cartan matrix A, its corresponding Dynkin diag. is defined via

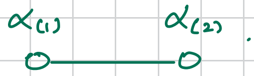
1. Draw a node for each simple root $\alpha_{(i)}$
2. Joins pair of nodes, say $\alpha_{(i)}, \alpha_{(j)}$, with

$$\max(|A_{ji}|, |A_{ij}|) \in \{0, 1, 2, 3\}$$

lines.

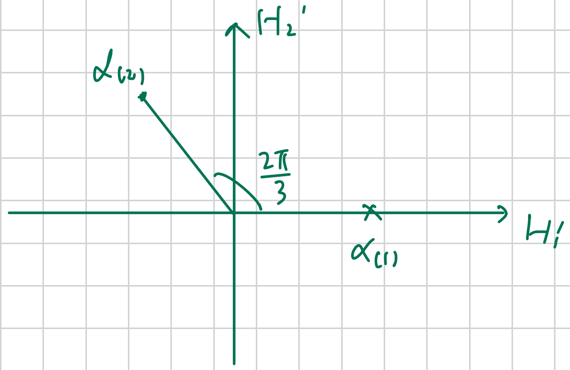
3. If more than one line connects 2 nodes, draw an arrow head pointing from longer root $|\alpha_{(l)}|$ to shorter one $|\alpha_{(s)}| < |\alpha_{(l)}|$.

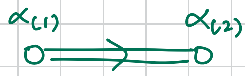
<u>Example</u> $r=2$

- $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\quad A_{12}$ 

$|\alpha_{(1)}| = |\alpha_{(2)}|$
$\cos\phi_{12} = -\frac{1}{2}$
$\Rightarrow \Phi_{12} = 2\pi/3$

Choose axes for $h^*_{\mathbb{R}}$ s.t. $\alpha_{(1)} = (1,0)$, $\alpha_{(2)} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



- $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ 

$|\alpha_{(2)}| = \frac{1}{\sqrt{2}} |\alpha_{(1)}|$.

- $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$  (ex. sheet)

- $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  (A reducible).

<u>Def^n</u> A Dynkin diagram is <u>admissible</u> if the corresponding Cartan matrix satisfies its constraints, equivalently, if the diagram is connected and correspond to a system of indpt. unit vectors $\hat{\alpha}_{(i)} = \frac{\alpha_{(i)}}{|\alpha_{(i)}|}$ s.t. the angles ϕ_{ij} between 2 roots of $\hat{\alpha}_{(i)}, \hat{\alpha}_{(j)}$ $(i \neq j)$ are $\left\{\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}\right\}$ from $(\hat{\alpha}_{(i)}, \hat{\alpha}_{(j)}) = \cos\phi_{ij}$.

Cartan matrix constraints $\Rightarrow$ Dynkin diagram constraints

(i) The diagram must be connected, o/w A reducible, as it can be block-diag by perm^{n}.

(ii) Given an admissible diagram, any subdiagram obtained by removing some nodes and all lines connected to them will be admissible, as long as subdiagram is connected.

Pf: Given a set of r LI vecs, subset is still LI. $\square$

(iii) There are at most $r-1$ pairs of nodes connected by lines

Pf: If $\hat{\alpha}_{(i)}$, $\hat{\alpha}_{(j)}$ connected, $(\hat{\alpha}_{(i)}, \hat{\alpha}_{(j)}) \leq \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

$$0 < \left(\sum_{i=1}^{r} \hat{\alpha}_{(i)} , \sum_{i=1}^{r} \hat{\alpha}_{(i)} \right) = r + 2 \sum_{i<j} (\hat{\alpha}_{(i)}, \hat{\alpha}_{(j)}) .$$

$$\leq r - p,$$

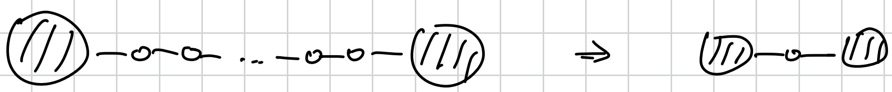
where $p = \#$ connected pairs $\Rightarrow p < r \Rightarrow p \leq r-1$. $\square$

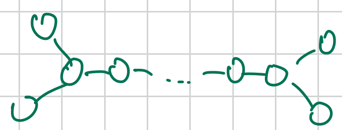
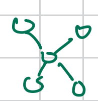
(iv) No admissible diagram contain cycles / loop as a subdiagram.

(v) No node can have more than 3 lines attached to it.

Pf: Notes.

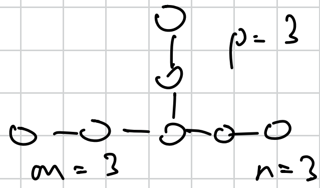
(vi) Replacing a linear chain of roots in an admissible diagram by a single node leaves an admissible diagram.

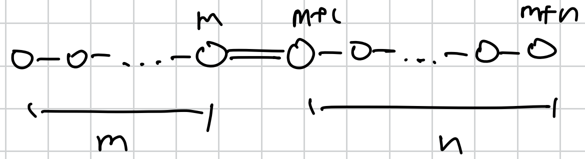


Example  ruled out, since  contradicts (v).

(vii) A diagram with a node having 3 lines attached to 3 linear chains of length m, n, p is admissible only if

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{p}>1.$$

e.g.  is inadmissible.

(viii)  is admissible only if

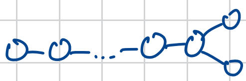
$(m-1)(n-1)<2$. $\Rightarrow$ only $\circ-\circ=\circ-\circ$, $\circ=\circ-\circ-\cdots-\circ$, $\circ-\cdots-\circ=\circ$

Cartan's classification

Thm Any admissible diagram is one of the following

Infinite families

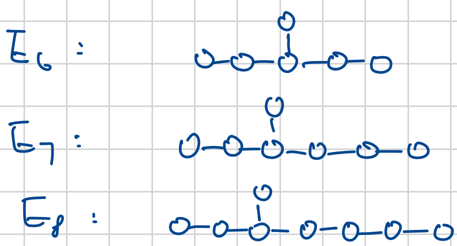
r rank
nodes

		Correspond to Lie alg.
A_r : $\circ-\circ-\cdots-\circ-\circ$	$(r\geq 1)$	$L(SU(r+1))_{\mathbb{C}}$
B_r : $\circ-\circ-\cdots-\circ-\circ\Rightarrow\circ$	$(r\geq 2)$	$L(SO(2r+1))_{\mathbb{C}}$
C_r : $\circ-\circ-\cdots-\circ-\circ\Leftarrow\circ$	$(r\geq 3)$. $(C_2=B_2)$	$L(Sp(2r))_{\mathbb{C}}$
D_r : 	$(r\geq 4)$. $(D_3=A_3)$.	$L(SO(2r))_{\mathbb{C}}$.

Rmk $C_2=B_2 \Rightarrow L(Sp(4))_{\mathbb{C}} \cong L(SO(5))_{\mathbb{C}}$

$D_3=A_3 \Rightarrow L(SO(6))_{\mathbb{C}} \cong L(SU(4))_{\mathbb{C}}$.

Exceptional diagrams



F_4 : $\circ-\circ\Rightarrow\circ-\circ$

G_2 : $\circ\Rrightarrow\circ$

6.9 Reconstruction

1. Dynkin diagram $\Rightarrow$ Cartan matrix
2. Cartan matrix $\Rightarrow$ relative lengths and angles between simple roots
3. Use root strings to find other +ve roots (-ve = -(+ve)).
4. Roots $\Rightarrow$ Cartan-Weyl basis

$$\{H_i, E_\alpha \mid i=1,\dots,r \;;\; \alpha \in \Phi\}.$$

5. Lie Brackets of whole algebra can be deduced from Chevalley and Serre and Jacobi identity.

7. Classification of Representations

7.1 Basics

Let d be an N-dim rep. of $\mathfrak{g}$ and let the C-W basis for $\mathfrak{g}$ be $\{H_i, E_\alpha\}$. Since d is a rep.,

$$[d(H_i), d(H_j)] = d([H_i, H_j]) = 0$$

and $d(H_i)$ are simul. diag., and the rep^n space $V \cong \mathbb{C}^N$ is spanned by the simul. evec. of $\{d(H_i)\}$.

Let V_λ be the espace

$$V_\lambda := \{v \in V \mid d(H_i)\, v = \lambda_i v \;;\; \lambda_i \in \mathbb{C}\}.$$

Def^n $\lambda = (\lambda_1, \dots, \lambda_r)$, $r = \operatorname{rank} \mathfrak{g}$, is a <u>weight</u> of rep. d.

Def^n The set of weights of rep d, S_d, is the <u>weight set</u> of d.

The rep. space $V = \bigoplus_{\lambda \in S_d} V_\lambda$.

Note: Weights in general can have nontrivial multiplicity. Any V_λ could appear more than once in direct sum.

Let $v \in V_\lambda$.

Claim: $d(E_\alpha)v \in V_{\lambda+\alpha}$ if $\lambda+\alpha \in S_d$, else $d(E_\alpha)v = 0$

Pf: $d(H_i)\, d(E_\alpha) v = d(E_\alpha)\, d(H_i) v + \underbrace{[d(H_i), d(E_\alpha)]}_{\substack{= d([H_i, E_\alpha]) \\ = d(\alpha_i E_\alpha)}} v$

$= \lambda_i\, d(E_\alpha) v + \alpha_i\, d(E_\alpha) v$

$= (\lambda_i + \alpha_i)\, d(E_\alpha) v$ $\square$

Consider the action of $sl(2)_\alpha$ subalg. generators $\{d(h_\alpha), d(e_\alpha), d(e_{-\alpha})\}$ on V. Each defines a lin. map $V \to V$, so V is a valid rep. space of $sl(2)_\alpha$.

$$h_\alpha = \frac{2}{(\alpha,\alpha)} \underbrace{(\kappa^{-1})^{ij} \alpha_i H_j}_{= H_\alpha}$$

For $v \in V_\lambda$

$$\Rightarrow d(h_\alpha) v = \frac{2}{(\alpha,\alpha)} (\kappa^{-1})^{ij} \alpha_i\, d(H_j) v$$

$$= \frac{2}{(\alpha,\alpha)} (\kappa^{-1})^{ij} \alpha_i \lambda_j v.$$

$$= \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} v.$$

This is a weight of $sl(2)_\alpha$, so must be an integer, so

$$\frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

since all weights of $sl(2,\mathbb{C})$ are integers. This is <u>quantisation condition II</u>.

7.2 Roots and weight lattices

Def^n The <u>root lattices</u> $\mathcal{L}[\mathfrak{g}]$ of Lie alg. $\mathfrak{g}$ with simple roots $\alpha_{(i)}$ is

$$\mathcal{L}[\mathfrak{g}] := \mathrm{span}_{\mathbb{Z}}\{\alpha_{(i)}\}.$$

Def^n Simple <u>co-roots</u> are the simple roots normalised as

$$\check{\alpha}_{(i)} := \frac{2}{(\alpha_{(i)}, \alpha_{(i)})}\alpha_{(i)}$$

Def^n <u>Co-root lattice</u>

$$\check{\mathcal{L}}[\mathfrak{g}] := \mathrm{span}_{\mathbb{Z}}\{\check{\alpha}_{(i)}\}.$$

Def^n <u>weight lattice</u> is dual (in the following sense) to the co-root lattice

$$\mathcal{L}_W[\mathfrak{g}] = \check{\mathcal{L}}^*[\mathfrak{g}] = \left\{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid (\lambda, \check{\alpha}) \in \mathbb{Z},\ \check{\alpha} \in \check{\mathcal{L}}[\mathfrak{g}]\right\}.$$

Writing $\check{\alpha} = n^i \check{\alpha}_{(i)}$ with $n^i \in \mathbb{Z}$.

$$(\lambda, \check{\alpha}_{(i)}) = \frac{2(\lambda, \alpha_{(i)})}{(\alpha_{(i)}, \alpha_{(i)})} \in \mathbb{Z}.$$

For any rep. d of $\mathfrak{g}$, the weight set S_d is a subset of the weight lattice. $S_d \subset \mathcal{L}_W[\mathfrak{g}]$.

The co-roots $\{\check{\alpha}_{(i)}\}$ form a basis $\check{\mathcal{L}}[\mathfrak{g}]$.

Form a basis for $\mathcal{L}_W[\mathfrak{g}]$ from $\{w_{(i)}\}$ s.t. $(\check{\alpha}_{(i)}, w_{(j)}) = \delta_{ij}$.

Def^n The $\{w_{(j)}\}$ are called the <u>fundamental weights</u>

The simple roots span $\mathfrak{h}^*_{\mathbb{R}}$, so we can write

$$w_{(j)} = \sum_{k=1}^{r} B_{jk}\,\alpha_{(k)}, \quad B_{jk} \in \mathbb{R}.$$

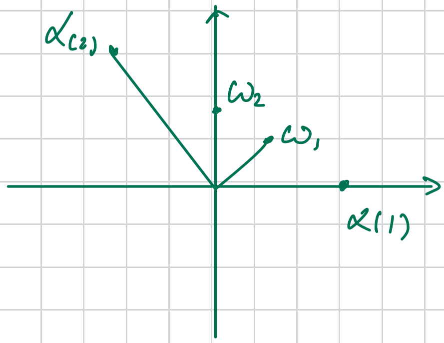
$$\delta_i^{\,j} = (\check{\alpha}_{(i)}, w_{(j)})$$

$$= \sum_{k=1}^{r} B_{jk} \frac{2(\alpha_{(i)}, \alpha_{(k)})}{(\alpha_{(i)}, \alpha_{(i)})} = \sum_k B_{jk} A_{ki}$$

$$\Rightarrow \quad B = A^{-1}, \quad \text{and} \quad \alpha_{(i)} = \sum_j A_{ij} w_{(j)}.$$

Example $\mathfrak{g} = A_2 = L(SU(3))_{\mathbb{C}}$. $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $B = A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

$|\alpha_{(1)}| = |\alpha_{(2)}|$ and $\phi_{12} = 2\pi/3$.



Say $\alpha_{(1)} = (1, 0)$, $\alpha_{(2)} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$$w_{(1)} = \frac{1}{3}(2\alpha_{(1)} + \alpha_{(2)}) = \frac{1}{2}\left(1, \frac{1}{\sqrt{3}}\right)$$

$$w_{(2)} = \frac{1}{3}(\alpha_{(1)} + 2\alpha_{(2)}) = \left(0, \frac{1}{\sqrt{3}}\right).$$

Def^n Any weight $\lambda \in S_d \subset \mathcal{L}_W[\mathfrak{g}]$ can be written as a lin. comb.,

$$\lambda = \sum_{i=1}^{r} \lambda^i w_{(i)},$$

with the set of $\lambda^i \in \mathbb{Z}$ called the Dynkin labels of λ. Write $\lambda = [\lambda^1, \ldots, \lambda^r]$. A vector in V_λ can be written $|\lambda^1, \ldots, \lambda^r\rangle$.

Thinking about a specific rep^n d (finite-dim)

Def^n Every finite-dim rep^n d of $\mathfrak{g}$ has at least one highest weight $\Lambda \in S_d$ s.t.

$$d(E^\alpha)v = 0 \quad \forall \alpha \in \Phi_+, \quad \forall v \in V_\Lambda.$$

Def^n Dynkin labels of a rep.^{n} are those of its highest weight(s). $\{\Lambda^i\} = [\Lambda^1, \ldots, \Lambda^r]$.

Prop. If Λ is a highest weight, then $\Lambda^i \geq 0$.

Prop. If d irred., then Λ is unique and V_Λ is 1-dimensional.
(Λ is non-degenerate)

Thinking about all possible rep.

Def'n A weight λ on $L_W[\mathfrak{g}]$ is dominant if all $\lambda^i \geq 0$.

Prop. (Highest weight thm). For any dominant rep., $\exists$ a unique, irred., finite-dim rep. d_λ with highest weight λ.

Given an irrep. d_Λ, with highest weight Λ, we can find the vecs in the espace, corresponding to other weights, by applying lowering operators, say $v_\Lambda \in V_\Lambda$,

$$v_\lambda = d_\Lambda(E_{-\alpha_\ell}) \cdots d_\Lambda(E_{-\alpha_j})\, d_\Lambda(E_{-\alpha_i})\, v_\Lambda .$$

for $\alpha_i, \alpha_j, \ldots, \alpha_\ell \in \Phi_+$, ie every weight of d_Λ can be written as a $\lambda = \Lambda - \mu$ with $\mu = \sum_{i=1}^r \mu^i \underbrace{\alpha_{(i)}}_{\text{simple roots}}$, with $\mu^i \in \mathbb{Z}_{\geq 0}$

Lem (No holes lemma) For any finite-dim rep d of $\mathfrak{g}$, if

$$\lambda = \sum_{i=1}^r \lambda^i \underbrace{\omega_{(i)}}_{\text{fund weights}} \in S_d \text{ — weight set}$$

then $\lambda - m_{(i)}\alpha_{(i)} \in S_d$, where $\alpha_{(i)} \in \Phi_S$ and $m_{(i)} = 0, 1, \ldots, \lambda^i$

This gives algo. for finding all weights of d_Λ, given highest weight Λ.

7.3 Representation of $su(3)_{\mathbb{C}}$.

$\alpha_{(i)} = A_{ij} w_{(j)}$, $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\alpha_{(1)} = [2, -1]$, $\alpha_{(2)} = [-1, 2]$

Investigate a few dominant weights.

$$\Lambda = [\Lambda^1, \Lambda^2] = \Lambda^1 \omega_{(1)} + \Lambda^2 \omega_{(2)}$$

- $d_{[0,0]}$: $\Lambda = 0$. Trivial rep. Scalers, invariants
- $d_{[1,0]}$: $\Lambda = \omega_{(1)}$. <u>Fundamental</u> irrep, f. $\Lambda = \omega_{(1)} \in S_f$

other weights $\lambda_1 = \Lambda - \alpha_{(1)} = \omega_{(1)} - \alpha_{(1)}$

$$= \omega_{(1)} - (2\omega_{(1)} - \omega_{(2)})$$

$$= \omega_{(2)} - \omega_{(1)} = [-1, 1] \in S_f.$$

$$\lambda_2 = \lambda_1 - \alpha_{(2)} = -\omega_{(2)} = [0, -1] \in S_f.$$

Done. So $\dim d_{[0,0]} = 3$.

- $d_{[0,1]}$: $\Lambda = \omega_{(2)}$ <u>Antifundamental</u> $\bar{f}$

[Mathematician refer to any irrep. with a found weight as its highest weight as a fundamental rep.]

- $d_{[1,1]}$: $\Lambda = \omega_{(1)} + \omega_{(2)} = [1, 1]$ $(= \alpha_{(1)} + \alpha_{(2)})$

$$\begin{tikzcd}
 & \Lambda = [1,1] \arrow[dl, "-\alpha_{(1)}"'] \arrow[dr, "-\alpha_{(2)}"] & \\
{[-1,2]} \arrow[dr, "-\alpha_{(2)}"] \arrow[dd, "-2\alpha_{(2)}"'] & & {[2,-1]} \arrow[dl, "-\alpha_{(1)}"'] \arrow[dd, "-2\alpha_{(1)}"] \\
 & {[0,0]} \arrow[dl, "-\alpha_{(2)}"] \arrow[dr, "-\alpha_{(1)}"] & \\
{[1,-2]} \arrow[dr, "-\alpha_{(1)}"] & & {[-2,1]} \arrow[dl, "-\alpha_{(2)}"] \\
 & {[-1,-1]} &
\end{tikzcd}$$

Weights : $\{\alpha_{(1)}, \alpha_{(2)}, \alpha_{(1)} + \alpha_{(2)}, -\alpha_{(1)}, -\alpha_{(2)}, -\alpha_{(1)} - \alpha_{(2)}, 0, 0\}$.

Caution: algo is not guaranteed to get multiplicity correct.

Know $SU(3)$ has dim 8. Observe that $d_{[1,1]}$ is the adjoint rep. which must have dim 8 as well.

7.4 Decomposition of tensor products

Generalise what we did for $su(2)$.

Let d_Λ and $d_{\Lambda'}$ be irreps of $\mathfrak{g}$ with spaces $V^{(\Lambda)}, V^{(\Lambda')}$

$$V^{(\Lambda)} = \bigoplus_{\lambda \in S_\lambda} V_\lambda \quad . \quad V^{(\Lambda')} = \dots$$

If $v_\lambda \in V_\lambda$, $v_{\lambda'} \in V_{\lambda'}$, then for any $H \in \mathfrak{h}$,

$$(d_\Lambda \otimes d_{\Lambda'})(H)(v_\lambda \otimes v_{\lambda'}) = d_\Lambda(H) v_\lambda \otimes v_{\lambda'} + v_\lambda \otimes d_{\Lambda'}(H) v_{\lambda'} .$$
$$= \underbrace{(\lambda + \lambda')}_{\text{add weights}} (v_\lambda \otimes v_{\lambda'})$$

Vectors of the form $v_\lambda \otimes v_{\lambda'}$ span $V^{(\Lambda)} \otimes V^{(\Lambda')}$, so

$$S_{\Lambda \otimes \Lambda'} \subset \{ \lambda + \lambda' \mid \lambda \in S_\Lambda, \lambda' \in S_{\Lambda'} \}$$

noting multiplicities. Decomposition

1. Find a highest weight of the tensor product rep.
   — corresponds to the highest weight of some irrep.
2. Subtract weight set of that irrep.
3. Repeat with leftover weights

Example $SU(3)_{\mathbb{C}}$:

$d_{[1,0]} \otimes d_{[1,0]}$:

$$S_{[1,0]} = \{ [1,0], [-1,1], [0,-1] \}.$$

$$S_{[1,0]\otimes[1,0]} = \{ \underbrace{[2,0], [-2,2], [0,-2]}_{\text{multiplicity } 1}, \underbrace{[-1,0], [1,-1], [0,1]}_{\text{multiplicity } 2} \}$$

Find $\quad d_{[1,0]} \otimes d_{[1,0]} = d_{[2,0]} \oplus d_{[0,1]}$

$$\underset{\text{fund.}}{\underline{3}} \otimes \underline{3} = \underline{6} \oplus \underset{\text{anti-fund.}}{\underline{\bar{3}}}$$

7.5 SU(3) Flavour

- Approx sym. in QCD (3-flavours)

$$m_u < m_d \overset{1/27}{\ll} m_s \overset{1/9}{\ll} m_{proton}$$

Can think of $m_u = m_d = m_s$ and the 3 quarks as elts of vec.

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$$

Transform $\quad \underline{3}$ fund $\qquad \underline{\bar{3}}$ antifund.

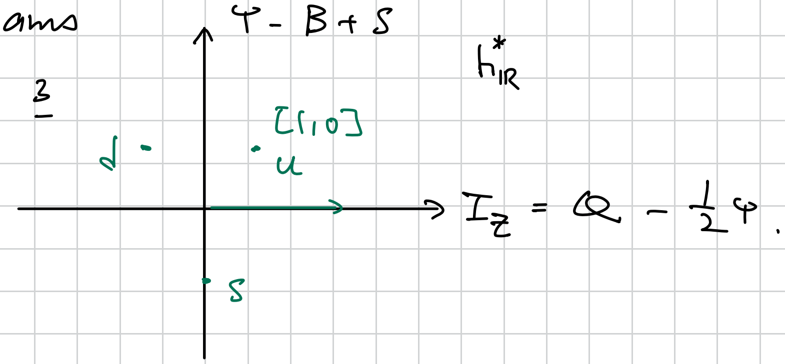
- Interactions $\Rightarrow$ quarks into hadrons.

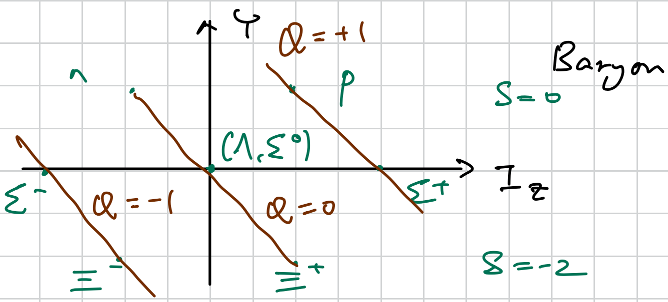
How do the products of q and $\bar{q}$ transform ?

Mesons : $(q\bar{q})$: $\underline{3} \otimes \underline{\bar{3}} = \underline{1} \oplus \underline{8}$

Baryons : (qqq) : $\underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{1} \oplus \underline{8} \oplus \underline{8} \oplus \underline{10}$

Weight diagrams





8. Gauge Theory

8.2 Non-abelian Gauge Theory

Consider an N-component scaler field φ.

$$\varphi : \mathbb{R}^{1,3} \to V, \quad V = \mathbb{R}^N \text{ or } \mathbb{C}^N$$

Inner product for V, $(\, , \,)$

Lagrangian

$$\mathcal{L}_\varphi = \underbrace{(\partial_\mu \varphi, \partial^\mu \varphi)}_{\text{kinetic}} - \underbrace{W((\varphi, \varphi))}_{\text{potential}}$$

Say $\mathcal{L}_\varphi$ is invar. under some global transf^n of some group G. for $g \in G$,

$$\varphi(x) \mapsto \Delta(g)\, \varphi(x),$$

Δ is an N-dim rep. of G., i.e.

$$(\Delta(g)\varphi, \Delta(g)\varphi) = (\varphi, \varphi) \quad \forall g \in G.$$

G compact, N finite $\Rightarrow$ Δ a unitary rep.

$$\Delta(g)^\dagger = \Delta(g)^{-1}.$$

Say we wish to "gauge" the sym/thy s.t. we have $\mathcal{L}$ invar. under local transf^n $g(x) \in G$.

$$\varphi(x) \mapsto [\Delta(g)(x)]\, \varphi(x),$$

Small group transf^n

$$\Delta(g) = \exp(\varepsilon d(X)) = I + \varepsilon d(X) + \dots$$

where $X \in L(G)$ and d a N-dim rep. of $L(G)$

Now

$$\varphi(x) \mapsto \varphi(x) + \varepsilon\, d(X) \varphi =: \varphi(x) + \delta_X \varphi(x),$$

where $\delta_X \varphi := \varepsilon\, d(X)\, \varphi$.

Introduce a gauge-covar. derivative involving a gauge field $A_\mu : \mathbb{R}^{1,3} \to L(G)$.

$$D_\mu := \partial_\mu + d(A_\mu)$$

s.t. $D_\mu \varphi$ transform as φ. Require

$$D_\mu \varphi \longrightarrow \Delta(g)\, D_\mu(\varphi),$$

or want

$$\delta_X (D_\mu \varphi) = \varepsilon\, d(X)\, D_\mu \varphi$$

Not^n. usually drop the "representation" notation

$$\Delta(g) \to g$$

$$d(X) \to X$$

$$d(A_\mu) \to A_\mu$$

How does A_μ transform? Let $A_\mu \mapsto A'_\mu$.

$$D_\mu \varphi \mapsto (\partial_\mu + A'_\mu)\, g \varphi \overset{!}{=} g\, D_\mu \varphi.$$

$$\Rightarrow g^{-1}(\partial_\mu + A'_\mu)\, g \varphi = D_\mu \varphi$$

$$\Rightarrow [g^{-1}\partial_\mu g + \partial_\mu + g^{-1} A'_\mu g]\varphi = (\partial_\mu + A_\mu)\varphi$$

$$\Rightarrow \quad A'_\mu = g A_\mu g^{-1} - (\partial_\mu g) g^{-1}.$$

Write $g = I + \varepsilon X \Rightarrow A'_\mu = A_\mu + \underbrace{\varepsilon [X, A_\mu] - \varepsilon \partial_\mu X}_{= \delta_X A_\mu}$.

Field strength Tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in L(G)$$

Note that $[D_\mu, D_\nu]\varphi = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]\varphi$

$$= F_{\mu\nu}\varphi.$$

$$\Rightarrow F_{\mu\nu}\varphi \longmapsto g F_{\mu\nu}\varphi = (g F_{\mu\nu} g^{-1})(g\varphi)$$

$$\Rightarrow F_{\mu\nu} \longmapsto g F_{\mu\nu} g^{-1}.$$

With $g = I + \varepsilon X \Rightarrow \delta_X F_{\mu\nu} = \varepsilon [X, F_{\mu\nu}]$

$\mathcal{L}$: Lorentz scaler. Look at $F_{\mu\nu}F^{\mu\nu}$ not gauge invar.

Use killing form Yang-Mills

$$\mathcal{L}_{YM} = -\frac{1}{g_s^2}\sum_{\mu,\nu} \kappa(F_{\mu\nu}, F^{\mu\nu}),$$

where g_s = gauge coupling

$$= -\frac{1}{2g_s^2} \operatorname{Tr} F_{\mu\nu}F^{\mu\nu}$$

Choice of adapted basis and normalisation.

Can rescale fields:

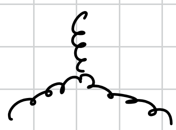
$$A'_\mu = \frac{1}{g_s} A_\mu \Rightarrow \mathcal{L}_{YM} = -\frac{1}{2}\operatorname{Tr} F'_{\mu\nu}F'^{\mu\nu}.$$

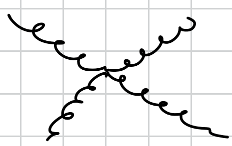
Then $D_\mu = \partial_\mu + g_s A'_\mu$.

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu + g_s [A'_\mu, A'_\nu]$$

Feynman diagrams

$(\partial A)(\partial A)$ terms 

$g_s[A, A]\partial A$ 

$g_s^2 [A, A]^2$ 

Example (Standard Model)

Gauge $\quad G = \underset{\text{colour}}{SU(3)} \times \underbrace{SU(2)_L \times U(1)_Y}_{\text{electroweak}}$ ← hypercharge

left

electroweak $\xrightarrow{SSB}$ $U(1)_{EM}$

$$\mathfrak{g} = su(3) \oplus su(2) \oplus u(1)$$

Gauge bosons

A_μ: gluons g $\qquad A_1, A_2, A_3 \qquad B$

$\downarrow$ SSB $\qquad \downarrow$ SSB

$W^\pm, Z^0 \qquad$ photon

Weak hypercharge : $\quad Y = 2(Q - T_3)$

Weak isospin T_3 3rd cpt : $T_3 = \begin{cases} \pm\frac{1}{2} & SU(2)_L \text{ doublets} \\ 0 & \text{" singlets.} \end{cases}$

EM charge $\quad Q = \frac{Y}{2} + T_3$.

SM content — not^n : $(SU(3)\text{ rep}, SU(2)_L \text{ rep})_Y$

- Scaler Higgs $(\underline{1}, \underline{2})_1$ ← $Y = 2(0 + \frac{1}{2}) = 1$.

- Lepton (e⁻, muon, neutrino)

$\underset{\text{LH spinor}}{(\underline{1}, \underline{2})_{-1}} \oplus \underset{\text{RH spinor}}{(\underline{1}, \underline{1})_{-2}}$ $\qquad \begin{array}{c} T_3 \\ 1/2 \\ -1/2 \end{array}$ $\quad$ doublet $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$, $\quad$ singlet e_R

- Quarks $\quad \underset{\begin{pmatrix} u \\ d \end{pmatrix}_L}{(\underline{3}, \underline{2})_{1/3}} \oplus \underset{u_R}{(\underline{3}, \underline{1})_{4/3}} \oplus (\underline{3}, \underline{-1})_{-2/3}$