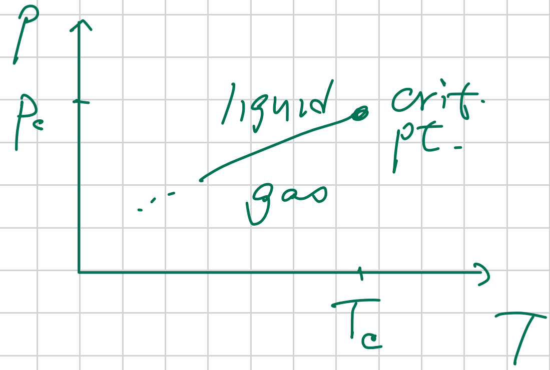
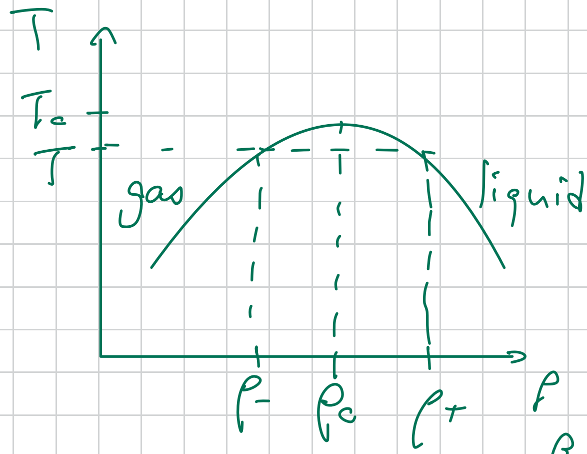
Statistical Field Theory

0. Motivation

- Universality: sometimes very different physical systems exhibit same behaviour.

Example liquid-gas system





$$|\rho_+ - \rho_-| \propto |T - T_c|^\beta$$

Example Ferromagnet

- T_c = Curie Temp.
- Magnetisation $M = 0$ for $T > T_c$
  $M \propto (T_c - T)^\beta$ for $T < T_c$

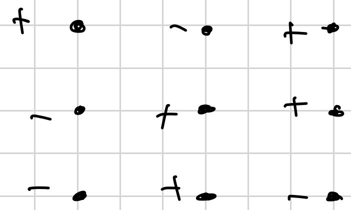
This course: Classical stat. phys. with fields.

1. From Spins to Fields

1.1 Ising Model

- Simple model for magnet

In d dimensional space, consider a lattice with N sites.
On i-th site, have "spin" $S_i \in \{\underset{(\downarrow)}{-1}, \underset{(\uparrow)}{+1}\}$



Configuration $\{S_i\}$ has energy

$$E = -B\sum_i S_i - J\sum_{\langle ij\rangle} S_i S_j .$$

sum over nearest neighbour pairs

How does physics depend on B, J, T ?

$\underline{J>0}$: spins prefer to align $\uparrow\uparrow$ or $\downarrow\downarrow$ (feromagnet)

$\underline{J<0}$: antialign $\uparrow\downarrow$ or $\downarrow\uparrow$ (antiferomagnet)

minimise E

Assume $J>0$, $\underline{B>0}$: spins prefer $\uparrow$
$\underline{B<0}$: spins prefer $\downarrow$

Intuition: low T $\Rightarrow$ minimise E $\Rightarrow$ ordered state
high T $\Rightarrow$ maximise S $\Rightarrow$ disordered state.

Def^n (Canonical ensemble)

$$p(S_i) = e^{-\beta E(S_i)} / Z \quad , \quad \beta = 1/T .$$

where $Z(T,B) = \sum_{\{s_i\}} e^{-\beta E(\{s_i\})}$.

Note: Set $k_B = 1$.

Def^n (Thermodynamic free energy)

$$F_{\text{thermo}}(T,B) = \langle E \rangle - TS = -T \log Z.$$

Def^n (Magnetisation)

$$m = \frac{1}{N} \left\langle \sum_{i=1}^{N} s_i \right\rangle \in [-1,1]$$

Distinguishes ordered ($m \neq 0$) and disordered ($m=0$).

$$m = \sum_{\{s_i\}} \frac{e^{-\beta E(\{s_i\})}}{Z} \cdot \frac{1}{N} \sum_i s_i = \frac{1}{N\beta} \frac{\partial \log Z}{\partial B}.$$

Want to compute Z.

- $d=1$: easy (Ex 1)
- $d=2$: square lattice, $B=0$ (Onsager)

Other cases has no exact sol^n.

Aim: to approximate in a way that correctly captures long-distance behaviour.

Define m for any $\{s_i\}$ by $m = \frac{1}{N} \sum s_i$.

$$Z = \sum_m \sum_{\{s_i\}|m} e^{-\beta E(\{s_i\})} =: \sum_m e^{-\beta F(m)}$$

$\{s_i\}$ s.t. $\frac{1}{N} \sum s_i = m$

Spacing between allowed values of m is $2/N$.

For large N, approximate m as continuous

$$\boxed{Z \simeq \frac{N}{2}\int_{-1}^{1} dm \; e^{-\beta F(m)}.}$$

where $F(m)$ is the effective free energy - depends on T, B, m.

$F(m)$ contains more information than F_{thermo}.

Let $f(m) = F(m)/N$ be effective free energy per lattice site, then

$$Z \propto \int dm \; e^{-\beta N f(m)},$$

with $\beta f(m) = \mathcal{O}(1)$ as $N \to \infty$. (intensive), then

Z dominated by the minimum of f

$$\left.\frac{\partial f}{\partial m}\right|_{m = m_{min}} = 0$$

m_{min} is the equilibrium value of magnetisation.

"saddle point approx":

$$Z \propto e^{-\beta N f(m_{min})}$$

$$\Rightarrow F_{thermo}(T, B) \approx F(m_{min}(T,B), T, B).$$

$$\langle m \rangle = -\frac{1}{N} \left.\frac{\partial F_{thermo}}{\partial B}\right|_{m_{min}}$$

$$= -\frac{1}{N}\left(\cancelto{0}{\frac{\partial F}{\partial m}} \frac{\partial m_{min}}{\partial B} + \frac{\partial F}{\partial B}\right)_{m_{min}}$$

Note $E = -BNm + (\text{indpt. of } B) \Rightarrow F = -BNm + (\qquad)$

$$\Rightarrow \langle m \rangle = m_{min}$$

Computing $F(m)$ is hard!

Attempt: use mean field approx.

Replace $S_i \to m$

$$\Rightarrow E = -B\sum_i m - J\sum_{\langle ij\rangle} m^2 = -BNm - \frac{1}{2}JNqm^2.$$

where q = # nearest neighbours.

$$\Rightarrow Z \approx \sum_m \Omega(m)\, e^{-\beta E(m)}$$

configurations with $\frac{1}{N}\sum S_i = m$.

let $N_\uparrow$ = # up spins, $N_\downarrow = N - N_\uparrow$ = # down spins

$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow - N}{N}$$

$$\Omega(m) = \frac{N!}{N_\uparrow!(N-N_\uparrow)!} = \frac{N!}{N_\uparrow!\,N_\downarrow!}$$

Stirling: $\log n! \approx n\log n - n$. for $n \gg 1$.

$$\Rightarrow \log\Omega \approx N(\log N - 1) - N_\uparrow(\log N_\uparrow - 1) - N_\downarrow(\log N_\downarrow - 1)$$

$$\Rightarrow \frac{1}{N}\log\Omega \approx \log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m)$$

$$\Rightarrow f(m) \approx -Bm - \frac{1}{2}Jqm^2 - T\left[\log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m)\right]$$

Minimise : $\frac{\partial f}{\partial m} = 0 \Rightarrow \beta(B + Jqm) = \frac{1}{2}\log\left(\frac{1+m}{1-m}\right)$

$$\Rightarrow m = \tanh[\beta(\underbrace{B + Jqm}_{B_{eff}})]$$

Each spin feels "effective field" $B_{eff}(m)$.

1.2 Landau Theory of Phase Transitions

At a phase transition, some quantity (an <u>order param.</u>) is not smooth. For us, this is m.

For small m, Taylor expand

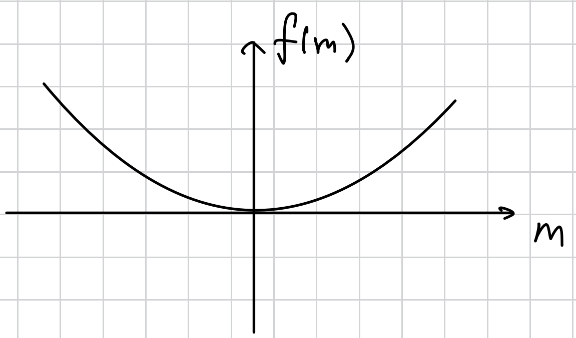
$$f(m) \approx -T\log 2 - Bm + \frac{1}{2}(T - Jq)m^2 + \frac{1}{12}Tm^4 + \dots$$

In eqm. $m = m_{min}$. How does this behave as we vary T and B ?

<u>$B = 0$</u> :

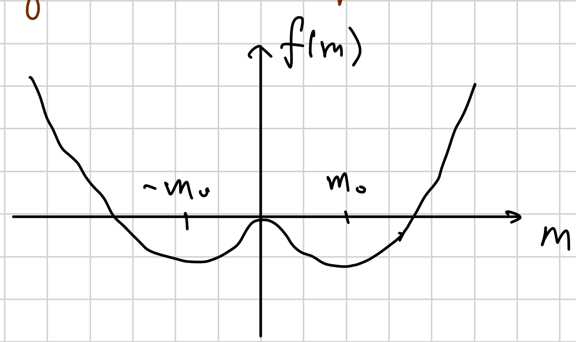
$$f(m) \approx \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Jm^4 + \dots$$

$T_c \equiv Jq$ critical temp.



$T > T_c$

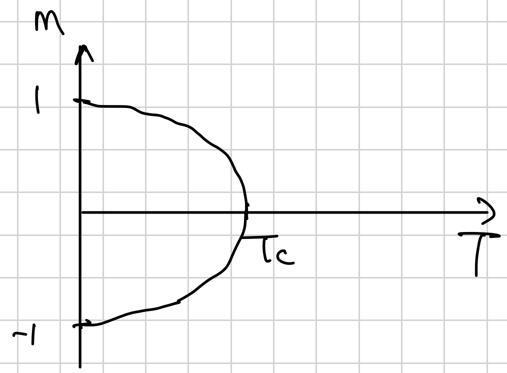
$m_{min} = 0$



$T < T_c$

$$m_{min} = \pm m_0 = \pm\sqrt{\frac{3(T_c - T)}{J}}$$

valid for small m_0 (T close to T_c)



Phase transition at $T = T_c$

$T > T_c$: $m = 0$ "disordered phase"

$T < T_c$: $m \neq 0$ "ordered phase".

m continuous at $T = T_c$, so this is a continuous phase transition or a second order phase transition.

- F invariant under Z_2 symmetry $\begin{matrix} m \mapsto -m \\ B \mapsto -B \end{matrix}$.
(inherited from $\begin{matrix} S_i \mapsto -S_i \\ B \mapsto -B \end{matrix}$ sym of Ising)

$\boxed{B=0}$ either $m = +m_0$ or $m = -m_0$. So the Z_2 sym doesn't preserve the ground state — "spontaneous symmetry breaking" (SSB).

- At finite N, Z is analytic. Phase transition only occurs when $N \to \infty$ (thermodynamic limit).
SSB occurs only for $N \to \infty$

$$\langle m \rangle = \lim_{B \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_i \langle S_i \rangle .$$

if we switch the limits, then $m = 0 \ \forall T$

(Finite N. Z_2 sym $\Rightarrow$ Z even in B $\Rightarrow$ F_{thermo} even in B
$\Rightarrow \left.\dfrac{\partial F_{\text{thermo}}}{\partial B}\right|_{B=0} = 0 \Rightarrow \langle m \rangle = 0$)

Def^n Heat Capacity $\quad C = \dfrac{\partial \langle E \rangle}{\partial T}$

$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta} \Rightarrow C = \beta^2 \frac{\partial^2 \log Z}{\partial \beta^2}$$

Have

$$\log Z = -\beta N f(m_{min}) = \begin{cases} \text{const.} & T > T_c \\ \frac{3N}{4}\frac{(T_c - T)^2}{T^2} + \text{const.} & T < T_c \end{cases}$$

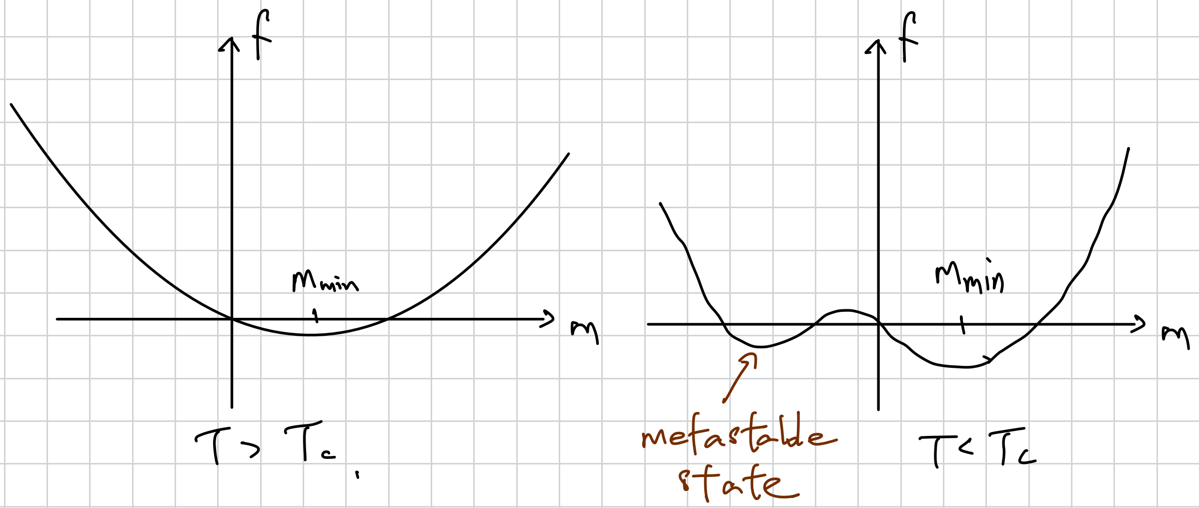
let $c := C/N$, then

$$c \to \begin{cases} 0 & \text{as } T \to T_c^+ \\ \frac{3}{2} & \text{as } T \to T_c^- \end{cases}$$

So c discts as $T \to T_c$.

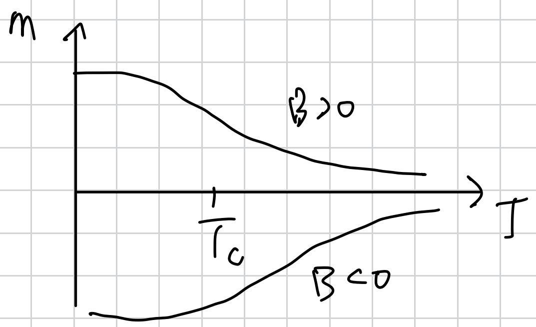
$\boxed{B > 0}$:

$$f(m) \approx -Bm + \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \dots$$



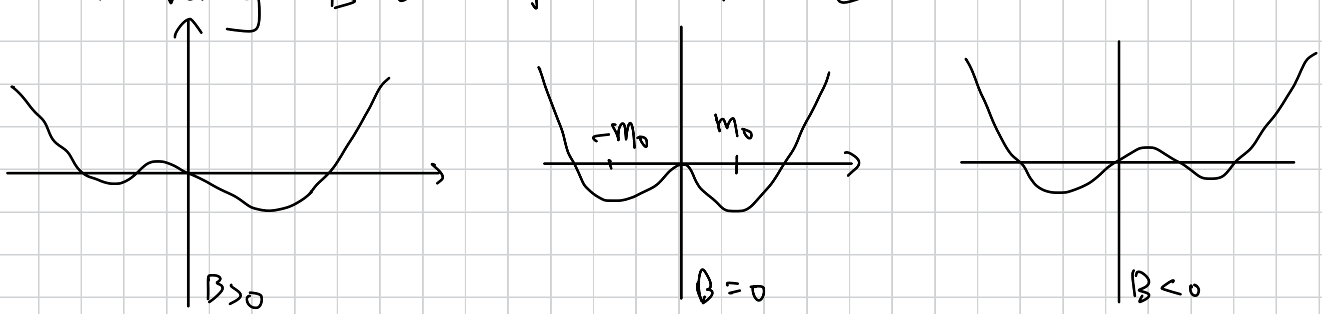
$T > T_c$ $T < T_c$

$m \simeq B/T$ for $T \to \infty$



m_{min} depends smoothly on T
$\Rightarrow$ No phase transition if T varied at fixed $B \neq 0$.

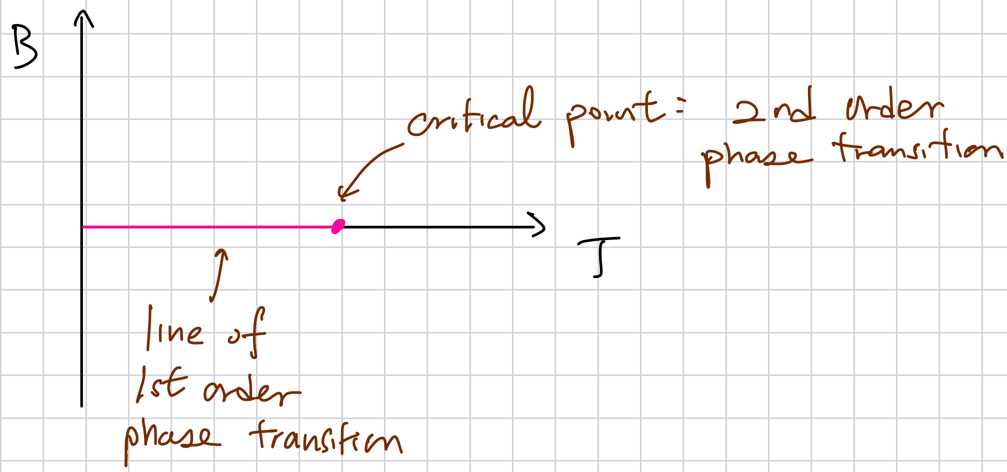
But: vary B at fixed $T < T_c$:



m jumps discretely from m_0 to $-m_0$ as B decreases from m +ve to -ve

This is a <u>first order phase transition</u> $\left(m_{eqm} = \frac{\partial f_{thermo}}{\partial B} \right)$

Phase diagram:



Behaviour near crit. pt. (CP)

- Fix $T = T_c \Rightarrow f(m) \approx -Bm + \frac{1}{12} T m^4$

minimise $\Rightarrow m^3 \sim B \Rightarrow m \sim B^{1/3}$ for small m.

- Magnetic susceptibility

$$\chi = \left(\frac{\partial m}{\partial B} \right)_T$$

$T > T_c$: $f(m) \approx -Bm + \frac{1}{2}(T - T_c) m^2 + \dots$

$$\Rightarrow m \simeq \frac{B}{T - T_c}$$

$$\Rightarrow \chi \simeq \frac{1}{T - T_c}$$

$T < T_c$: Write $m = m_0 + \delta m$. Solve for δm to leading order in B.

$$\Rightarrow m = m_0 + \frac{B}{2(T_c - T)}$$

$$\Rightarrow \chi \simeq \frac{1}{2(T_c - T)}.$$

So $\chi \sim \frac{1}{|T - T_c|}$

Validity of MFT

Does MFT approx. give correct result?

- $d=1$: No! There no phase transitions!
- $d=2,3$: Phase diagram is qualitatively correct. Quantitative predictions near CP incorrect.
- $d \geq 4$: Yes!

Similar for other systems. MFT gets phase structure wrong for $d \leq d_\ell$ (lower critical dimension) and MFT correct for $d \geq d_c$ (upper critical dim). Ising has $d_\ell = 1$, $d_c = 4$. Interesting : $d_\ell < d \leq d_c$.

Critical exponents

Near CP, MFT predicts

if $\underline{B=0}$, as $T \to T_c^-$, $m \sim (T_c - T)^\beta$, $\beta = \frac{1}{2}$.

as $T \to T_c$, $c \sim C_\pm |T - T_c|^{-\alpha}$, $\alpha = 0$ (discts $C_+ \neq C_-$)

$\chi \sim |T - T_c|^{-\gamma}$, $\gamma = 1$

If $\underline{T = T_c}$, as $B \to 0$, $m \sim B^{1/\delta}$ $\delta = 3$,

$\alpha, \beta, \gamma, \delta$ are <u>critical exponents</u>.

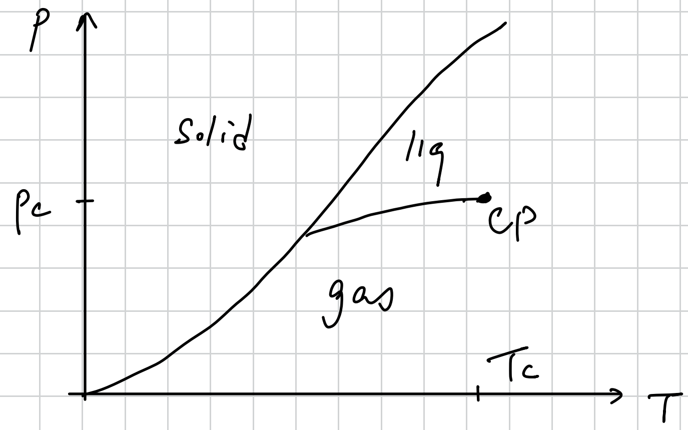
This behaviour is correct (e.g. use exact $d=2$ sol^n of Ising) but values of $\alpha, \beta, \gamma, \delta$ are <u>not</u>.

	MFT	$d=2$ (exact)	$d=3$ (numerical)
α	0 (discts)	0 (log)	0.110...
β	1/2	1/8	0.3164...
γ	1	7/4	1.2371...
δ	3	15	4.7898...

← reasonable agreement with experiments

Universality

Normal material



Liquid-gas phase transition similar to Ising (line of 1st order PT ending in CP).

$$B \longleftrightarrow p \qquad\qquad m \longleftrightarrow v := \frac{V}{N}. \ \left(\begin{matrix}\text{order}\\ \text{param.}\end{matrix}\right)$$

Use an EoS (e.g. van der Waals) to calculate the behaviour near CP.

$$v_{gas} - v_{liquid} \sim (T_c - T)^{\beta} \quad \text{as } T \to T_c^- \text{ along coexistence line} \ : \ \beta = \frac{1}{2}$$

$$\sim (p - p_c)^{1/\delta} \quad \text{as } p \to T_c \text{ at } T = T_c \ : \ \delta = 3$$

Def^n Isothermal compressibility

$$\kappa = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$$

$$\kappa \sim \frac{1}{|T - T_c|^{\gamma}} \quad \text{at } p = p_c \ : \ \gamma = 1$$

Heat capacity $\quad C_v \sim C_{\pm} |T - T_c|^{-\alpha} \ : \ \alpha = 0$ (discts)

Predictions for $\alpha, \beta, \gamma, \delta$ are same as MFT for Ising!

Expt: <u>actual</u> values for $\alpha, \beta, \gamma, \delta$ are same as correct values for $d = 3$ Ising!

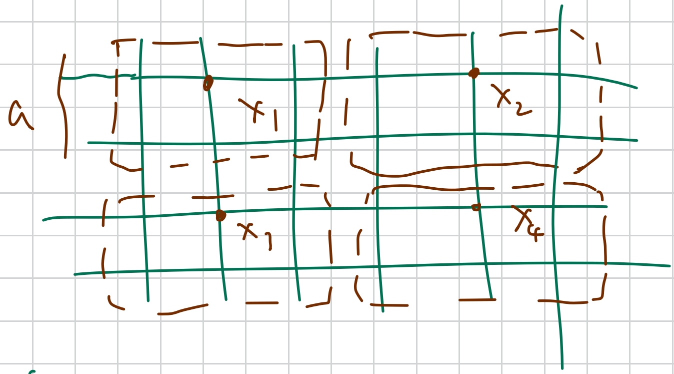
This is example of universality: different physical sys. can exhibit same behaviour at a CP. - suggests microscopic physics unimportant at CP. Systems governed by same CP belong to same universality class.

Landau - Ginzburg Theory

Aim: find out model that correctly describes long-distance physics near CP ⇒ can use it to calculate crit expts ∀ theories in same universality class.

LG theory generalises MFT (Landau theory) to allow for spatial variations in m: $\underline{m}(\underline{x})$ is now a field, produced from a microscopic model by "coarse graining". The microscopic degrees of freedom over regions of size $a \ll \xi$, where ξ is the length scale of physics we're interested in.

Example Ising: divide lattice into boxes, each with $N' \ll N$ sites, size $a \ll \xi$.



Define $\underline{m}(\underline{x})$ s.t.

- $\underline{m}(\underline{x}_i)$ = avg. of spins in box with centre $\underline{x}_i$

  $N \gg 1$ so discreteness of m can be ignored. $m \in [-1, 1]$.

- $\underline{m}(\underline{x})$ smooth

- $\tilde{\underline{m}}(\underline{k}) = 0$ for $|\underline{k}| > \pi/a$ ($\underline{m}$ doesn't vary on scales $\lesssim a$)

$\leftarrow$ Fourier transform

$$Z = \sum_{m(\underline{x})} \sum_{\{s_i\}|m(\underline{x})} e^{-\beta E[s_i]} =: \sum_{m(\underline{x})} e^{-\beta F[m(\underline{x})]}$$

Note $F[m(\underline{x})]$ is a <u>functional</u>: the LG free energy.

<u>Notation</u>: Write

$$Z = \int \mathcal{D}m(\underline{x})\, e^{-\beta F[m(\underline{x})]}$$

a functional integral — sum over all $m(\underline{x})$ that don't vary on scales $< a$.

Can interpret this as prob. (density) of field config $m(\underline{x})$:

$$p[m(\underline{x})] = \frac{e^{-\beta F[m(\underline{x})]}}{Z}$$

The form of $F[m(\underline{x})]$ is constrained by the following.

- Locality: spins only influence nearby spins

$$\Rightarrow F[m(\underline{x})] = \int d^d\underline{x}\, f(m(\underline{x}), \nabla_i m(\underline{x}), \nabla_i \nabla_j m(\underline{x}), \dots)$$

(not e.g. $\int d^dx\, d^dy\, f(m(x), m(y), \dots)$)

- Translational symmetry: inherited from discrete trans sym of lattice
- Rotational sym: inherited from discrete rot^n sym of lattice
- Z_2 sym: $B=0$ Ising has $S_i \leftrightarrow S_j$.

  $B \neq 0$ $\quad S_i \to -S_i$, $B \to -B$ sym

So assume F is invar. under $m(\underline{x}) \to -m(\underline{x})$, $B \to -B$.

• Analyticity : $F[m(\underline{x})]$ defined by coarse graining over finite number of spins, suggests it is analytic, i.e. can Taylor expand near $m=0$.

Dim. analysis : $a\nabla m \sim \frac{am}{\xi}$, $a^2\nabla\nabla \xi \sim \frac{a^2 m}{\xi^2}$, etc.

dim.-less

$\xi \gg a$ suggests ∇m more important than $\nabla\nabla m$ etc.

For $\underline{B=0}$, above

$$\Rightarrow F[m(\underline{x})] = \int d^d x \left(\tfrac{1}{2}\alpha_2(T)\, m^2 + \tfrac{1}{4}\alpha_4(T)\, m^4 + \tfrac{1}{2}\gamma(T)\,(\nabla m)^2 + \dots \right)$$

(Could include $F_0(T)$ c.f. $T\log 2$ in MFT. — ignore).

For $\underline{B \neq 0}$: Allow $B = B(\underline{x})$ doesn't vary on scales $\leq a$.
(arises from coarse graining Ising $B_i \to -B_i$. $E = -\sum_i B_i S_i + \dots$)
$\Rightarrow$ include $-\int d^d x\, B(\underline{x})\, m(\underline{x})$.

Coeff. $\alpha_2(T), \alpha_4(T), \dots$ hard to compute from 1st principles.
From MFT, expect $\alpha_2 \approx T - T_c$, $\alpha_4 \approx \frac{1}{3}T$, but we'll assume: coeff. analytic in T, $\alpha_2(T) > 0$ for $T > T_c$, α_2 has simple zero at $T = T_c$, $\alpha_4(T) > 0$, $\gamma(T) > 0$.

<u>Saddle point approx.</u>

Assume $\int \mathcal{D}m$ dominated by saddle point, i.e. $m(\underline{x})$ that minimises $F[m(\underline{x})]$. Vary $m(\underline{x}) \to m(\underline{x}) + \delta m$

$$\Rightarrow \delta F = \int d^d x \left(\alpha_2 m\, \delta m + \alpha_4 m^3 \delta m + \gamma \nabla m \cdot \nabla \delta m + \dots \right)$$

$$= \int d^d x \left(\alpha_2 m + \alpha_4 m^3 - \gamma \nabla^2 m + \dots \right) \delta m \quad \left(\text{IBP, } \delta m \to 0 \text{ at boundary} \right)$$

The $(\dots)$ is usually written as $\frac{\delta F}{\delta m(\underline{x})}$ (functional deriv.)

If $m(\underline{x})$ minimises F, then $\delta F = 0 \quad \forall \ \delta m(\underline{x})$.

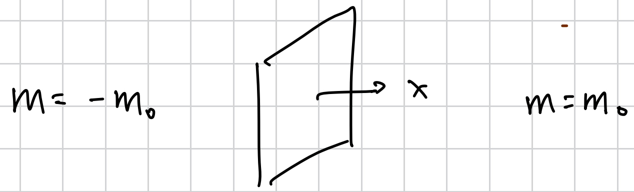
$$\Rightarrow \quad \gamma \nabla^2 m = \alpha_2 m + \alpha_4 m^3 + \dots$$

Simplest sol^n: $m = $ const. $\rightarrow$ MFT / Landau theory:

- $T > T_c \Rightarrow \alpha_2 > 0 \quad \Rightarrow m = 0$
- $T < T_c \Rightarrow \alpha_2 < 0 \Rightarrow m = \pm m_0 = \pm\sqrt{-\alpha_2/\alpha_4}$.

Domain walls

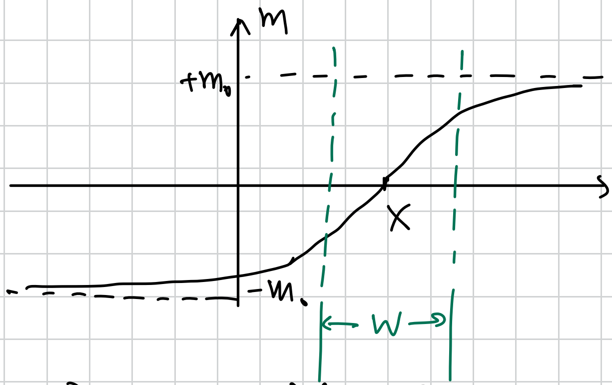
$\underline{T < T_c}$: 2 ground states $\pm m_0$. Could have $m \rightarrow \pm m_0$ as $x \rightarrow \pm\infty$.



Assume $m(\underline{x}) = m(x)$

$$\Rightarrow \gamma \frac{d^2 m}{dx^2} = \alpha_2 m + \alpha_4 m^3$$

Solved by $\quad m = m_0 \tanh\left(\frac{x - X}{W}\right)$, $\quad X = $ const., $W = \sqrt{-2\gamma/\alpha_2}$.



domain wall (kink) at $x = X$ of width W.

Free energy: If sys. has size L, then $F[m_0] \propto L^d$.

Cost of domain wall is

$$\Delta F = F[m(x)] - F[m_0] \sim L^{d-1}\sqrt{\frac{-\gamma\alpha_2^3}{\alpha_4^2}} \quad \text{(prop. to area of wall)}$$

Exercise Check ΔF : IBP w.r.t. x, use ODE

Near CP : $\alpha_2 \to 0 \Rightarrow W \to \infty$, $\Delta F \to 0$.

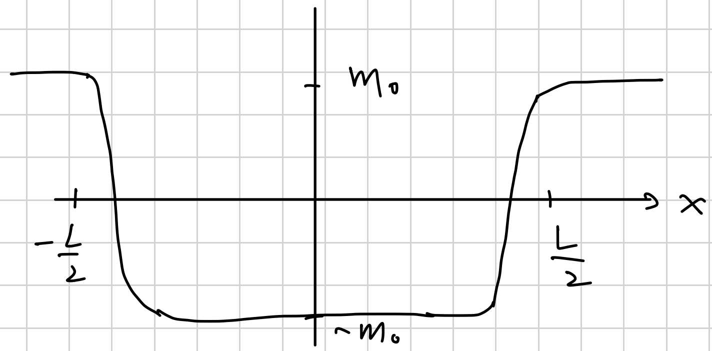
Domain walls explain why $d_\ell = 1$ for Ising :

Let $d = 1$, $-\frac{L}{2} \leq x \leq \frac{L}{2}$. Assume $\alpha_2(T) < 0$.

The b.c.'s are $m(\pm\frac{L}{2}) = \pm m_0$. Prob. of config. $m(x)$ is

$$\frac{e^{-\beta F[m(x)]}}{Z}.$$

2 Domain wall config. (approx. saddle pt.)



(assume $L \gg W$)

For well separated walls, $\Delta F_{2\text{walls}} \simeq 2\Delta F$.

$$\Rightarrow \frac{p(\text{2 domain walls at given } X, Y)}{p(m \equiv m_0)} = e^{-2\beta \Delta F}$$

$$\Rightarrow \frac{p(\text{2 dws})}{p(m \equiv m_0)} \sim \int_{-L/2}^{L/2} \frac{dX}{W} \int_{-L/2}^{L/2} \frac{dY}{W}\, e^{-2\beta\Delta F} \sim \left(\frac{L}{W}\right)^2 e^{-2\beta\Delta F}$$

For $d > 1$, RHS $\to 0$ as $L \to \infty$ ($\Delta F \to \infty$) "energy beats entropy".

$d = 1$: RHS $\to \infty$ as $L \to \infty$ "entropy beats energy".

much more probable to see pairs of domain walls than $m \equiv m_0$

since any region with $m \simeq \pm m_0$ and size $\gg W$ is

unstable to formation of domain wall — ordered phase

doesn't exist.

$\Rightarrow$ No spontaneous sym. breaking for $d = 1$.

2. My First Path Integral

Want to go beyond saddle point approx. to calculate

$$Z = \int \mathcal{D} m(\underline{x}) \, e^{-\beta F[m(\underline{x})]}$$

Change notation: $m(\underline{x}) \to \varphi(\underline{x})$. Set $B=0$.

$$F[\varphi(\underline{x})] = \int d^d x \left(\tfrac{1}{2}\alpha_2(T)\varphi^2 + \tfrac{1}{4}\alpha_4(T)\varphi^4 + \tfrac{1}{2}\gamma(T)(\nabla\varphi)^2 + \dots \right)$$

Evaluating path integral is

- easy if F quadratic in φ. ("free field theory").
- possible if non-quadratic terms are small.
- very hard o/w.

For $T > T_c$, let $\mu^2 = \alpha_2(T) > 0$, we'll consider quadratic approx. to F.

$$F[\varphi(\underline{x})] = \frac{1}{2}\int d^d x \left(\mu^2 \varphi^2 + \gamma(T)(\nabla\varphi)^2 + \dots \right) \qquad (*)$$

terms quadratic in φ, e.g. $(\nabla^2\varphi)^2$.

For $T < T_c$, $\alpha_2(T) < 0 \Rightarrow$ saddle point has $\langle\varphi\rangle = \pm m$.

Let $\tilde{\varphi}(\underline{x}) = \varphi(\underline{x}) - \langle\varphi\rangle$.

$$\Rightarrow \quad F = F[m_0] + \frac{1}{2}\int d^d x \left(\alpha_2'(T)\tilde{\varphi}^2 + \gamma(T)(\nabla\tilde{\varphi})^2 + \dots \right)$$

$\alpha_2'(T) = \alpha_2(T) + 3 m_0^2 \alpha_4(T) = -2\alpha_2(T) > 0$.

$\Rightarrow$ quadratic approx gives $(*)$ with $\varphi \to \varphi'$, and $\mu^2 = \alpha_2'(T) = 2|\alpha_2(T)|$.

2.1 Thermodynamic Free Energy

Aim is to calculate corrections to F_{thermo} from fluctuations in $\varphi(\underline{x})$. (Ignore $F_0(T)$, $F[m_0]$ here).

Defn (Fourier transform)

$$\varphi_{\underline{k}} := \int d^d x \; e^{-i\underline{k}\cdot\underline{x}} \varphi(\underline{x}).$$

For φ real, $\varphi^*_{\underline{k}} = \varphi_{-\underline{k}}$, where $\underline{k}$ wavevector, often called <u>momentum</u> (analogy with QFT).

φ doesn't vary on scales $< a$ $\Rightarrow$ $\varphi_{\underline{k}} = 0$ $\forall$ $|\underline{k}| > \Lambda \equiv \pi/a$. where Λ is UV cutoff.

The inverse FT is

$$\varphi(\underline{x}) = \int \frac{d^d k}{(2\pi)^d} \, e^{i\underline{k}\cdot\underline{x}} \varphi_{\underline{k}} \qquad (+)$$

In finite volume, consider a system occupying cubic region with $V = L^d$, then $\underline{k}$ take values

$$\underline{k} = \frac{2\pi}{L}\underline{n}, \qquad \underline{n} \in \mathbb{Z}^d$$

$$\Rightarrow \int \frac{d^d k}{(2\pi)^d} e^{i\underline{k}\cdot\underline{x}} \varphi_{\underline{k}} = \left(\frac{1}{L}\right)^d \sum_{\underline{n}} \cdots = \frac{1}{V}\sum_{\underline{k}} \cdots$$

$$\Rightarrow \varphi(\underline{x}) = \frac{1}{V}\sum_{\underline{k}} e^{i\underline{k}\cdot\underline{x}} \varphi_{\underline{k}}$$

Substitute (+) into quadratic approx. to F:

$$F[\varphi_{\underline{k}}] = \frac{1}{2}\int \frac{d^d k_1}{(2\pi)^d}\int\frac{d^d k_2}{(2\pi)^d}\int d^d x \left(\mu^2 - \gamma\, \underline{k}_1\cdot\underline{k}_2 + \ldots\right) \varphi_{\underline{k}_1}\varphi_{\underline{k}_2} e^{i(\underline{k}_1+\underline{k}_2)\cdot\underline{x}}.$$

Use $\int d^dx \; e^{i(\underline{k}_1+\underline{k}_2)\cdot\underline{x}} = (2\pi)^d \, \delta^{(d)}(\underline{k}_1+\underline{k}_2)$, then

$$F[\varphi_{\underline{k}}] = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \underbrace{(\mu^2 + \gamma \underline{k}^2 + \dots)}_{=K(\underline{k}^2)\,:\,\text{a series in } \underline{k}^2.} \varphi_{\underline{k}} \varphi_{-\underline{k}}$$

$$= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} K(\underline{k}^2) \, |\varphi_{\underline{k}}|^2$$

$$= \int' \frac{d^dk}{(2\pi)^d} K(\underline{k}^2) \, |\varphi_{\underline{k}}|^2 ,$$

where $\int'$ = integral over region $k_x \geq 0$, and using $\varphi_{-\underline{k}} = \varphi_{\underline{k}}^*$

$$= \frac{1}{V} \sum_{\underline{k}}' K(\underline{k}^2) \, |\varphi_{\underline{k}}|^2$$

Define measure as

normalisation const.

$$\int \mathcal{D}\varphi(\underline{x}) = N \prod_{\underline{k}}' \int d(\mathrm{Re}\,\varphi_{\underline{k}}) \, d(\mathrm{Im}\,\varphi_{\underline{k}}).$$

Since $|\underline{k}| < \Lambda$, we have finite product.

$\Rightarrow Z$

$$= N \left(\prod_{\underline{k}}' \int d(\mathrm{Re}\,\varphi_{\underline{k}}) \, d(\mathrm{Im}\,\varphi_{\underline{k}}) \right) \exp\left[-\frac{\beta}{V} \sum_{\underline{k}}' K(\underline{k}^2) |\varphi_{\underline{k}}|^2 \right]$$

$$= N \prod_{\underline{k}}' \int d(\mathrm{Re}\,\varphi_{\underline{k}}) \, d(\mathrm{Im}\,\varphi_{\underline{k}}) \exp\left[-\frac{\beta}{V} K(\underline{k}^2) \left((\mathrm{Re}\,\varphi_{\underline{k}})^2 + (\mathrm{Im}\,\varphi_{\underline{k}})^2 \right) \right]$$

Recall $\int_{-\infty}^{\infty} dx \; e^{-x^2/a} = \sqrt{\pi a}$.

$$\Rightarrow \quad e^{-\beta F_{\text{thermo}}} = Z = N \prod_{\underline{k}}' \left(\sqrt{\frac{\pi V T}{K(\underline{k}^2)}} \right)^2$$

$$\Rightarrow \quad \frac{F_{\text{thermo}}}{V} = -\frac{T}{V} \log Z = -\frac{T}{V} \sum_{\underline{k}}' \log\left(\frac{\pi V T}{K(\underline{k}^2)} \right) - \frac{T}{V} \log N.$$

Recall

$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta} = \frac{\partial(\beta F_{thermo})}{\partial \beta}$$

and

$$C = \frac{\partial \langle E \rangle}{\partial T} = -\beta^2 \frac{\partial \langle E \rangle}{\partial \beta} = -\beta^2 \frac{\partial^2}{\partial \beta^2}(\beta F_{thermo})$$

$$\Rightarrow \frac{C}{V} = -\beta^2 \frac{\partial^2}{\partial \beta^2}\left[-\frac{1}{V}\sum_{\underline{k}}{}' \log\left(\frac{\pi V T}{K(\underline{k}^2)}\right)\right]$$

For simplicity, take $\mu^2 = T - T_c$, $\gamma = \text{const.}$ and neglect higher order terms in K ($K = \mu^2 + \gamma \underline{k}^2$).

$$\Rightarrow \frac{C}{V} = \frac{1}{V}\sum_{\underline{k}}{}' \left[1 + \frac{2T}{\mu^2 + \gamma \underline{k}^2} + \frac{T^2}{(\mu^2+\gamma \underline{k}^2)^2}\right]$$

$$= \int' \frac{d^d k}{(2\pi)^d}\left[1 + \frac{2T}{\mu^2 + \gamma \underline{k}^2} + \frac{T^2}{(\mu^2+\gamma \underline{k}^2)^2}\right]$$

$$= \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\left[1 + \frac{2T}{\mu^2 + \gamma \underline{k}^2} + \frac{T^2}{(\mu^2+\gamma \underline{k}^2)^2}\right]$$

1st term: $\frac{1}{2}k_B$ per degree of freedom (equipartition)

Other terms may diverge as $T \to T_c$ ($\mu^2 \to 0$): integral may not converge when $\underline{k} = \underline{0}$, example of "IR divergence".

Final term $\propto \displaystyle\int_0^{\Lambda} \frac{dk\ k^{d-1}}{(\mu^2 + \gamma k^2)^2}$

$$= \frac{\mu^{d-4}}{\gamma^{d/2}} \int_0^{\Lambda\sqrt{\gamma/\mu^2}} \frac{dx\ x^{d-1}}{(1+x^2)^2} \qquad \left(\text{set } k = \sqrt{\frac{\mu^2}{\gamma}}\, x\right)$$

$$\sim \begin{cases} \Lambda^{d-4} & d > 4 \\ \mu^{d-4} & d < 4 \\ \log(\Lambda/\mu) & d = 4 \end{cases} \qquad \text{as } T \to T_c.$$

Similarly, second term $\propto \displaystyle\int_0^\Lambda \frac{dk\, k^{d-1}}{\mu^2 + \gamma k^2}$

$$\sim \begin{cases} \Lambda^{d-2} & d>2 \\ \mu^{-1} & d=1 \\ \log \Lambda/\mu & d=2 \end{cases} \qquad \text{as } T \to T_c$$

So for $d>4$, contribution of fluctuations is finite as $T \to T_c$. For $d \leq 4$, diverges as $T \to T_c$.

$$C \sim \mu^{d-4} \sim |T-T_c|^{-\alpha}$$

$$\Rightarrow \quad \alpha = 2 - \frac{d}{2}$$

So fluctuations explains why MFT $\alpha = 0$ wrong for $d \leq 4$, but this value is also wrong – can't neglect φ^4 for $d \leq 4$.

2.2 Correlation functions

We know

$$\langle \varphi(\underline{x}) \rangle = \begin{cases} 0 & T > T_c \\ \pm m_0 & T < T_c \end{cases}$$

Fluctuations around this are captured by correlation functions

$$\left\langle \left(\varphi(\underline{x}) - \langle \varphi(\underline{x}) \rangle\right)\left(\varphi(\underline{y}) - \langle \varphi(\underline{y}) \rangle\right) \right\rangle = \langle \varphi(\underline{x})\varphi(\underline{y}) \rangle - \langle \varphi(\underline{x}) \rangle \langle \varphi(\underline{y}) \rangle .$$

Compute this by including B-field in F (general, not just quad.)

$$F[\varphi, B] = \int d^dx \left(\tfrac{1}{2}\mu^2 \varphi^2 + \tfrac{1}{2}\gamma (\nabla\phi)^2 + \dots - B(\underline{x})\, \varphi(\underline{x}) \right).$$

allow spatial variation in B

and

$$Z[B(x)] = \int \mathcal{D}\varphi \, e^{-\beta F[\varphi, B]}$$

contains lots of info!

Example

$$\frac{1}{\beta} \frac{\delta \log Z}{\delta B(x)} = \frac{1}{\beta Z} \frac{\delta Z}{\delta B(x)}$$

$$= \frac{1}{Z} \int \mathcal{D}\varphi \; \varphi(x) \, e^{-\beta F} = \langle \varphi(x) \rangle_B$$

expectation value with B.

$$\frac{1}{\beta^2} \frac{\delta^2 \log Z}{\delta B(x) \delta B(y)} = \frac{1}{\beta^2 Z} \frac{\delta^2 Z}{\delta B(x) \delta B(y)} - \frac{1}{\beta^2 Z^2} \frac{\delta Z}{\delta B(x)} \cdot \frac{\delta Z}{\delta B(y)}.$$

$$= \langle \varphi(x) \varphi(y) \rangle_B - \langle \varphi(x) \rangle_B \langle \varphi(y) \rangle_B .$$

$$\Rightarrow \frac{1}{\beta^2} \frac{\delta^2 \log Z}{\delta B(x) \delta B(y)} \bigg|_{B=0} = \underbrace{\langle \varphi(x) \varphi(y) \rangle - \langle \varphi \rangle^2}_{\text{This is what we want to know!}} \qquad (*)$$

Now calculate $Z[B(x)]$ in Fourier space for quadratic F.

For $T > T_c$,

FT of B.

$$F[\varphi, B] = \int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{2} K(k^2) \varphi_k \varphi_{-k} - B_{-k} \varphi_k \right]$$

For $T < T_c$, set $\tilde{\varphi} = \varphi - \langle \varphi \rangle$, couple to external $\tilde{B}$ via $F[\tilde{\varphi}, \tilde{B}]$ of this form.

Complete square: $\hat{\varphi}_k = \varphi_k - \dfrac{B_k}{K(k^2)}$.

Shift $\varphi_k \to \hat{\varphi}_k$ in path, then $\int \cdots d[\varphi_k] = \int \cdots d[\hat{\varphi}_k]$.

$$\Rightarrow Z = N \int \prod_k{}' \, d(\mathrm{Re}\, \hat{\varphi}_k) \, d(\mathrm{Im}\, \hat{\varphi}_k) \, e^{-\beta F}$$

$$= e^{-\beta F_{\text{thermo}}} \exp\left(\frac{\beta}{2}\int \frac{d^dk}{(2\pi)^d} \frac{|B_{\underline{k}}|^2}{K(\underline{k}^2)}\right),$$

$$= e^{-\beta F_{\text{thermo}}} \exp\left(\frac{\beta}{2}\int d^dx\, d^dy\; B(\underline{x})\, G(\underline{x}-\underline{y})\, B(\underline{y})\right).$$

where $G(\underline{x}) \equiv \int \frac{d^dk}{(2\pi)^d} \frac{e^{-i\underline{k}\cdot\underline{x}}}{K(\underline{k}^2)}$

Then

$$\langle \varphi(\underline{x})\rangle = \left.\frac{\delta \log Z}{\delta B(\underline{x})}\right|_{B=0} = 0$$

and $(*)$ $\Rightarrow$ $\langle \varphi(\underline{x})\varphi(\underline{y})\rangle = \frac{1}{\beta} G(\underline{x}-\underline{y})$ for $T > T_c$.

For $T < T_c$,

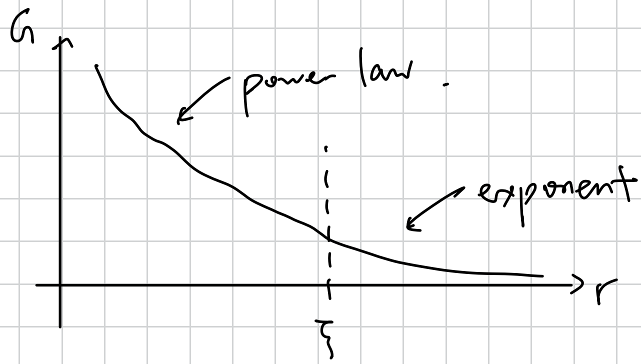
$$\langle \tilde{\varphi}(\underline{x})\rangle = 0$$

$$\langle \tilde{\varphi}(\underline{x})\tilde{\varphi}(\underline{y})\rangle = \langle \varphi(\underline{x})\varphi(\underline{y})\rangle - \langle\varphi\rangle^2 = \frac{1}{\beta} G(\underline{x}-\underline{y}).$$

Take $K = \mu^2 + \gamma \underline{k}^2$, and let $\xi^2 = \gamma/\mu^2$.

Claim

$$G(\underline{x}) \sim \begin{cases} 1/r^{d-2} & r \ll \xi, \\ \dfrac{e^{-r/\xi}}{\xi^{\frac{d-3}{2}}\, r^{\frac{d-1}{2}}} & r \gg \xi \end{cases} \qquad (r = |\underline{x}|)$$



Fluctuations occur are lengths $|\underline{x}| \lesssim \xi$, decay exponentially for $|\underline{x}| > \xi$. In general, this property defines the correlation length ξ.

$\xi \to \infty$ as $T \to T_c$ $\Rightarrow$ fluctuations on all length scales at CP.

Ex. 2 Q2 : $(-\gamma\nabla^2 + \mu^2)\, G(\underline{x}) = \delta^{(d)}(\underline{x})$, i.e. $G(\underline{x})$ is a Green f^n for $-\gamma\nabla^2+\mu^2$.

Define more critical exponents

$$\xi \sim \frac{1}{|T-T_c|^{\nu}} \quad \text{as} \quad T \to T_c$$

and, at CP ($\xi = \infty$), define

$$\langle \varphi(\underline{x})\, \varphi(\underline{0}) \rangle \sim \frac{1}{r^{d-2+\eta}} \qquad (r \gg a) \quad (\langle\varphi\rangle = 0 \text{ at CP})$$

	Prediction from MFT + quadratic fluctuation	correct values $d=2$	$d=3$	$d\geq 4$
η	0	$\frac{1}{4}$	0.0363	0
ν	$\frac{1}{2}$	1	0.6300	$\frac{1}{2}$.

Susceptibility

$$\chi = \left.\frac{\partial\langle\varphi\rangle}{\partial B}\right|_{B=0}$$

when no spatial variation. Generalise this,

$$\chi(\underline{x},\underline{y}) = \left.\frac{\delta\langle\varphi(\underline{x})\rangle_B}{\delta B(\underline{y})}\right|_{B=0}$$

$$= \frac{\delta}{\delta B(\underline{y})}\left[\frac{1}{\beta}\frac{\partial \log Z}{\delta B(\underline{x})}\right]\Bigg|_{B=0}$$

$$= \beta\left(\langle\varphi(\underline{x})\varphi(\underline{y})\rangle - \langle\varphi\rangle^2\right) = G(\underline{x}-\underline{y}).$$

If no spatial variation, then

$$\langle\varphi\rangle = \frac{1}{V}\int d^dx \,\langle\varphi(\underline{x})\rangle$$

$$\to \delta\langle\varphi\rangle = \frac{1}{V}\int d^dx\, d^dy\, \frac{\delta\langle\varphi(\underline{x})\rangle}{\delta B(\underline{y})}\,\delta B(\underline{y}).$$

$$= \frac{1}{V} \int d^dx \, d^dy \; \chi(\underline{x}, y) \; \delta B(y)$$

$$\Rightarrow \chi = \frac{1}{V} \int d^dx \, d^dy \; G(x-y)$$

$$= \int d^dx \; G(x)$$

$\int$ diverges at $|\underline{x}| \to \infty$ as $T \to T_c$.

$$\Rightarrow \chi = \int r^{d-1} \, dr \; \frac{e^{-r/\xi}}{\xi^{(d-3)/2} \, r^{(d-1)/2}} \underset{r=\xi y}{\sim} \xi^2 \sim \frac{1}{|T-T_c|}$$

$\Rightarrow$ predict $\gamma = 1$ as before.

Upper critical dimension

For $T < T_c$, $\quad \langle \varphi(\underline{x}) \rangle = \pm m_0$.

Ginzburg criterion: MFT can't be trusted when fluctuations in φ large compared to $\langle \varphi \rangle$.

$$\frac{\int_{\substack{|\underline{x}| \le \xi \\ |y| \le \xi}} d^dx \, d^dy \; \Big\langle (\varphi(\underline{x}) - \langle \varphi(\underline{x}) \rangle)(\varphi(y) - \langle \varphi(y) \rangle) \Big\rangle}{\int_{|\underline{x}| \le \xi} d^dx \; \langle \varphi(\underline{x}) \rangle \int_{|y| \le \xi} d^dy \; \langle \varphi(y) \rangle}.$$

$$\underset{\text{approx.}}{\overset{\text{quad.}}{\sim}} \quad \frac{\int_{|\underline{x}|, |y| \le \xi} d^dx \, d^dy \; \frac{1}{\beta} G(x-y)}{(\xi^d m_0)^2}$$

$$\underset{(\underline{x}, y) \mapsto (x' = \underline{x} - y, \; y' = y)}{\sim} \quad \frac{\xi^d}{\xi^{2d} m_0^2} \int_0^{\xi} r^{d-1} \, dr \cdot \frac{1}{r^{d-2}} \sim \frac{\xi^{2-d}}{m_0^2}$$

$$\sim \frac{\left(|T-T_c|^{-1/2}\right)^{2-d}}{\left(|T-T_c|^{1/2}\right)^2} \sim |T-T_c|^{\frac{d-4}{2}}.$$

$\Rightarrow$ MFT bad as $T \to T_c$ if $d < d_c = 4$. For $d \geq d_c$, MFT does work.

3. The Renormalisation Group.

Now we include non-quadratic terms in F, e.g. φ^4.

3.1 The big idea

rescale φ to set $\gamma = 1$

$$F[\varphi] = \int d^dx \left(\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 + g\varphi^4 + \dots \right).$$

contains important T-dependence $\mu^2 \sim T - T_c$

Allow <u>all</u> possible terms in F consistent with analyticity and symmeteries

<u>Example</u>
- φ^{18}, or $\varphi^5 (\nabla\varphi)^2 \nabla^2\varphi$ OK
- φ^{17} bad (break $\mathbb{Z}_2$)
- φ^{-2} bad (not analytic)

Each term has a coupling const. $(\mu^2, g, \dots) = \underline{g}$ - view as coords in ∞-dim <u>theory space</u>.

$$Z = \int \mathcal{D}\varphi \, e^{-F[\varphi]} \qquad \text{(absorb } \beta \text{ into } F \text{ now)}$$

Well-defined because of UV cutoff: $\varphi_{\underline{k}} = 0$ for $|\underline{k}| > \Lambda$, with $\Lambda \sim 1/a$, $a =$ size of coarse-graining boxes.

Suppose we are only care about physics on long-distance scale L — don't care about $\varphi_{\underline{k}}$ for $|\underline{k}| \gg L^{-1}$.

Start from theory with couplings g_0. Try to construct new theory with cutoff $\Lambda' = \Lambda/\zeta$, $\zeta > 1$, should work fine as long as $\Lambda' \gg L^{-1}$.

Write $\varphi_{\underline{k}} = \varphi_{\underline{k}}^- + \varphi_{\underline{k}}^+$.

$$\varphi_{\underline{k}}^- = \begin{cases} \varphi_{\underline{k}} & \text{if } |\underline{k}| < \Lambda' \\ 0 & \text{if } |\underline{k}| > \Lambda' \end{cases} \qquad (\text{IR modes})$$

$$\varphi_{\underline{k}}^+ = \begin{cases} \varphi_{\underline{k}} & \text{if } \Lambda' < |\underline{k}| < \Lambda \\ 0 & \text{o/w} \end{cases} \qquad (\text{UV modes})$$

Then

$$F[\varphi_{\underline{k}}] = \underbrace{F_0[\varphi_{\underline{k}}^-]}_{g = g_0} + F_0[\varphi_{\underline{k}}^+] + \underbrace{F_I[\varphi_{\underline{k}}^-, \varphi_{\underline{k}}^+]}_{\text{interaction}}$$

$$\Rightarrow Z = N \int \prod_{|\underline{k}|<\Lambda}' \underbrace{d\varphi_{\underline{k}}}_{d\varphi_{\underline{k}} = d(\mathrm{Re}\,\varphi_{\underline{k}})\, d(\mathrm{Im}\,\varphi_{\underline{k}})} e^{-F[\varphi_{\underline{k}}]}$$

$$= N \int \prod_{|\underline{k}|<\Lambda'}' d\varphi_{\underline{k}}^- \, e^{-F_0[\varphi_{\underline{k}}^-]} \cdot \int \prod_{\Lambda'<|\underline{k}|<\Lambda}' d\varphi_{\underline{k}}^+ \, e^{-F_0[\varphi_{\underline{k}}^+]} \, e^{-F_I[\varphi_{\underline{k}}^-, \varphi_{\underline{k}}^+]}$$

$$=: N' \int \prod_{|\underline{k}|<\Lambda'}' d\varphi_{\underline{k}}^- \, e^{-F'[\varphi_{\underline{k}}^-]}$$

$F'[\varphi_{\underline{k}}^{<}]$ is the Wilsonian effective free energy obtained by "integrating out" the UV modes to obtain a theory with a lower cutoff Λ'.

Must take same general form as F since we included all possible terms.

$$F'[\varphi] = \int d^d x \left(\tfrac{1}{2} \gamma' (\nabla\varphi)^2 + \tfrac{1}{2}\mu'^2 \varphi^2 + g' \varphi^4 + \dots \right).$$

BUT: values of coupling const will differ

(absorb φ-indpt part into $\mathcal{N}$).

Want to compare to original theory but they have different cutoffs, so rescale ("zoom out")

$$\underline{k}' = \zeta \underline{k} \Rightarrow |\underline{k}'| < \zeta \Lambda' = \Lambda$$

Similarly

$$\underline{x}' = x/\zeta \Rightarrow \underline{k} \cdot \underline{x} = \underline{k}' \cdot \underline{x}'$$

$$\Rightarrow \int d^d x \; \tfrac{1}{2} \gamma' (\nabla\varphi)^2 = \int \zeta^d \, d^d x' \; \tfrac{1}{2} \gamma' \zeta^{-2} (\nabla'\varphi)^2$$

Rescale the field

$$\varphi'(\underline{x}') = \zeta^{\frac{d-2}{2}} \sqrt{\gamma'} \, \varphi(\underline{x}).$$

$$\Rightarrow F'[\varphi'] = \int d^d x' \left(\tfrac{1}{2} (\nabla'\varphi')^2 + \tfrac{1}{2} \mu(\zeta)^2 \varphi'^2 + g(\zeta)\, \varphi'^4 + \dots \right)$$

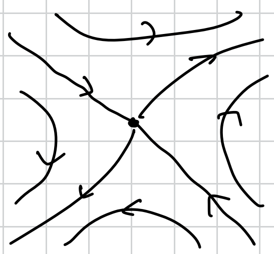
back to $\frac{1}{2}$

coupling depend on ζ
$g = g(\zeta)$ with $g(1) = g_0$

$$\Rightarrow F'[\varphi] = \int d^d x \left(\tfrac{1}{2} (\nabla\varphi)^2 + \tfrac{1}{2} \mu^2(\zeta)\, \varphi^2 + g(\zeta)\, \varphi^4 + \dots \right)$$

As ζ increases, we obtain a flow in theory space.

E.g. g :



Map $R(\zeta)$ from F to F' is renormalisation group transformation obeys $R(\zeta_1) R(\zeta_2) = R(\zeta_1 \zeta_2)$, but $R(\zeta)$ not invertible $\Rightarrow$ semigroup, not group

Summary: 3 steps in RG

① Integrate out high mom. modes $\frac{\Lambda}{\zeta} < |\underline{k}| < \Lambda$.

② Rescale $\underline{k}' = \zeta \underline{k}$, $\underline{x}' = \underline{x}/\zeta$

③ Rescale field to get canonically normalised gradient term

$\Rightarrow$ flow on couplings $g(\zeta)$. How does RG flow as $\zeta \to \infty$?

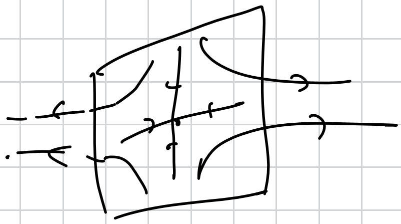
Possibilities:
- flow $\to \infty$ in theory space
- flow $\to$ fixed point (FP)
- flow $\to$ limit cycle (doesn't happen)

Focus on FP (possibly at ∞). After rescaling, $\xi' = \xi/\zeta$.
FP has $\xi' = \xi \Rightarrow \xi = 0$ or ∞.

- Disordered phase $(T > T_c)$: $\xi = 0$ is limit $T \to \infty$
- Ordered phase $(T < T_c)$: $\xi = 0$ is limit $T \to 0$.

A FP with $\xi = \infty$ can describe a CP of a continuous phase transition.

The critical surface of such a FP is the set of points that flow to FP under RG.



Let $\xi = \xi_0$ at some point on surface. Under RG $\xi = \xi_0/\zeta$ flows to FP, so $\xi \to \infty$ as $\zeta \to \infty$, so $\xi_0 = \infty$.

Hence, each point on crit. surface describes a different microscopic theory at a CP.

The long-distance physics of these CP's is the same – described by FP. This explains universality!

Our goal: calculate crit exp. $\forall$ theories on crit. surface by calculating for FP.

Beta functions

Start at an arbitrary point g_0 in theory space. Write $\zeta = e^s$. $s \geq 0$. write $g(s) = \begin{pmatrix} \mu^2(s) \\ g(s) \\ \vdots \end{pmatrix}$: ∞-dim vec. of all couplings after RG.

RG satisfies
$$R[e^{s+\delta s}] = R[e^{\delta s}]R[e^s],$$
so $g(s+\delta s)$ determined by applying RG with $\zeta = e^{\delta s} \approx 1+\delta s$

to theory with couplings $g(s)$.

Note: not $\beta(g(s), g_0)$

$$\Rightarrow \underline{g}(s+\delta s) = \underline{g}(s) + \delta s\, \underline{\beta}(\underline{g}(s))$$

for some

$$\underline{\beta} = \begin{pmatrix} \beta_{M^2}(\underline{g}) \\ \beta_g(\underline{g}) \\ \vdots \end{pmatrix}$$

cpts of $\underline{\beta}$ are called β-functions.

$$\Rightarrow \quad \frac{d\underline{g}}{ds} = \underline{\beta}(\underline{g}(s))$$

ODE determining RG flow.

$\underline{g}_*$ is a FP iff $\underline{\beta}(\underline{g}_*) = 0$.

Now consider flow <u>near</u> a FP: write $\underline{g}(s) = \underline{g}_* + \delta \underline{g}(s)$.

Linearise, then

$$\frac{d\, \delta g_a}{ds} = \sum_b \underbrace{\left.\frac{\partial \beta_a}{\partial g_b}\right|_{g=g_*}}_{=M_{ab}} \delta g_b \,,$$

i.e.

$$\frac{d\, \delta \underline{g}}{ds} = \underline{\underline{M}}\, \delta \underline{g}$$

Assuming $\underline{\underline{M}}$ diagonalisable, i.e. $\exists$ basis of evec. $\underline{v}_1, \underline{v}_2, \ldots$

with $M \underline{v}_i = \Delta_i \underline{v}_i$, $\Delta_1 \geq \Delta_2 \geq \ldots$.

Expand $\delta \underline{g} = \sum_i f_i(s)\, \underline{v}_i$

const.

$$\Rightarrow \quad f_i(s) = c_i e^{\Delta_i s}$$

So

$$\delta \underline{g} = \sum_i c_i e^{\Delta_i s}\, \underline{v}_i$$

describes RG flow near FP.

The cpts $\delta g_a = (\mu^2, g, \dots)$ don't transform simply under RG but linear combinations do.

let row vec. $\underline{w}_i$ be left evec. of $\underline{\underline{M}}$

$$\underline{w}_i \underline{\underline{M}} = \Delta_i \underline{w}_i$$

Can choose s.t. $\underline{w}_i \underline{v}_j = \delta_{ij}$

Define

$$\delta \hat{g}_i := \underline{w}_i \, \delta \underline{g}$$

$$\Rightarrow \delta \underline{g} = \sum_i \delta \hat{g}_i \, \underline{v}_i \, , \quad \delta g_i = c_i e^{\Delta_i s} .$$

So $\delta \hat{g}_i$ is a linear combination of couplings that transforms simply under RG near g_*.

We say $\delta \hat{g}_i$ has <u>scaling dimension</u> Δ_i

<u>Defn</u> $\delta \hat{g}_i$ is $\begin{Bmatrix} \text{relevant} \\ \text{irrelevant} \\ \text{marginal} \end{Bmatrix}$ if $\begin{Bmatrix} \Delta_i > 0 \\ \Delta_i < 0 \\ \Delta_i = 0 \end{Bmatrix}$.

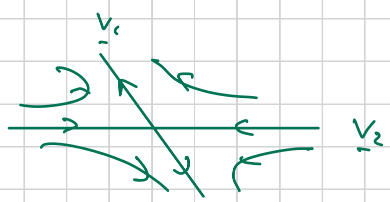
<u>Note</u> : Depends on g_* !

Consider theory near FP with $\delta \hat{g}_i \neq 0$ and $\delta \hat{g}_j = 0$ for $j \neq i$.

If $\delta \hat{g}_i$ is $\begin{Bmatrix} \text{relevant} \\ \text{irrelevant} \end{Bmatrix}$ then RG flow takes us $\begin{Bmatrix} \text{away from} \\ \text{back to} \end{Bmatrix}$ FP.

$\Rightarrow$ perturbed theory has $\begin{Bmatrix} \text{different} \\ \text{same} \end{Bmatrix}$ long-distance physics as FP.

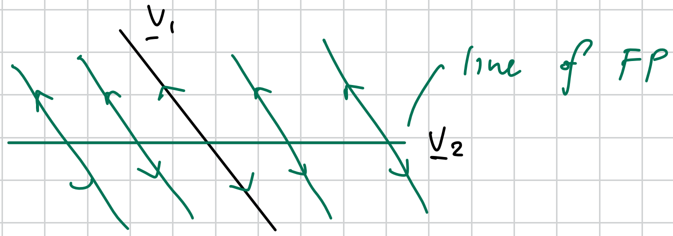
<u>Example</u> $\Delta_1 > 0$, $\Delta_2 < 0$



If $\exists$ marginal coupling, need to look at $O(\delta g^2)$ terms – usually cause $\left\{\begin{array}{l}\text{away from}\\ \text{towards}\end{array}\right\}$ FP, say coupling is $\left\{\begin{array}{l}\text{marginally relevant}\\ \quad \text{"} \quad \text{irrelevant}\end{array}\right\}$.

Could also have line (or surface) of FP.

$\Delta_1 > 0, \quad \Delta_2 = 0$
"exactly marginal" coupling



Near a FP, the crit surface is spanned by the set of (marginally) irrelevant couplings (i.e. $\mathrm{span}\{\underline{v}_i \mid \Delta_i < 0\}$).

Typically, $\exists$ just a few relevant (or marg-rel.) couplings. The crit surface corresponds to tuning these to zero.

codimension of crit surface = #(marginally) relevant couplings.

For Ising model ($d > d_c = 1$) with $B = 0$ ($\varphi \to -\varphi$ sym.), we reach CP by tuning 1 param: $T \Rightarrow$ crit. surface has co-dim 1.

$\Rightarrow$ FP must have exactly 1 relevant coupling $\delta\hat{g}_1$ with

$$\delta\hat{g}_1 \big|_{T=T_c} = 0$$

Couplings analytic in T

$$\Rightarrow \delta\hat{g}_1 \sim T - T_c$$

We'll identify $\delta\hat{g}_1 \longleftrightarrow t \equiv \dfrac{T-T_c}{T_c}$ (reduced temp.)

and write $\Delta_1 = \Delta_\varepsilon$. (More generally, B also relevant – see below).

3.2 Scaling

In RG step 3,

$$\varphi'(\underline{x}') = \zeta^{\frac{d-2}{2}} \sqrt{\gamma'}\, \varphi(\underline{x}) =: Z(g, \zeta)\, \varphi(\underline{x})$$

couplings before RG transf^n

Start at $g = g_0$, then $g = g(\zeta)$ along flow.

$$R(\zeta_1 \zeta_2) = R(\zeta_1) R(\zeta_2)$$

$$\Rightarrow Z(g_0, \zeta_1 \zeta_2) = \underbrace{Z(g(\zeta_2), \zeta_1)}_{R(\zeta_1)}\ \underbrace{Z(g_0, \zeta_2)}_{R(\zeta_2)}$$

If $g_0 = g_*$ (FP), then $g(\zeta) \equiv g_*$

$$\Rightarrow Z(g_*, \zeta_1 \zeta_2) = Z(g_*, \zeta_1)\, Z(g_*, \zeta_2)$$

Take $\partial/\partial\zeta_2$, set $\zeta_2 = 1$, and using $Z(1) = 1$,

$$\Rightarrow Z(g_0, \zeta) = \zeta^{\Delta_\varphi},$$

for some Δ_φ.

$$\zeta^{\Delta_\varphi} = e^{s \Delta_\varphi}.$$

So we say Δ_φ is the <u>scaling dimension</u> of the field φ.

$$\Rightarrow \langle \varphi'(\underline{x}')\, \varphi'(\underline{0}) \rangle = \zeta^{2\Delta_\varphi} \langle \varphi(\underline{x})\, \varphi(\underline{0}) \rangle \text{ at FP.}$$

but at FP, correlation f^n in primed variables is same as unprimed

$$\Rightarrow \langle \varphi'(\underline{x}')\, \varphi'(\underline{0}) \rangle = \langle \varphi(\underline{x}')\, \varphi(\underline{0}) \rangle = \zeta^{2\Delta_\varphi} \langle \varphi(\underline{x})\, \varphi(\underline{0}) \rangle$$

with $\zeta = r/r'$.

$$\Rightarrow \quad \underbrace{r'^{2\Delta_\varphi}\langle \varphi(\underline{x}')\,\varphi(\underline{0})\rangle}_{f^n \text{ of } r'} = \underbrace{r^{2\Delta_\varphi}\langle \varphi(\underline{x})\,\varphi(\underline{0})\rangle}_{f^n \text{ of } r}.$$

Since r, r' indpt,

$$\langle \varphi(\underline{x})\,\varphi(\underline{0})\rangle \propto \frac{1}{r^{2\Delta_\varphi}} \quad \text{at FP.}$$

but def^n of η is LHS $\sim \dfrac{1}{r^{d-2+\eta}}$ at FP, so

$$\Delta_\varphi = \frac{d-2}{2} + \underbrace{\frac{\eta}{2}}_{\text{anomalous dimension}}.$$

Puzzle: $\eta \neq 0$ appears to contradict dimensional analysis (DA).

In DA, measure dimensions in units of inverse length.

$[x] = -1, \quad [\partial/\partial x] = +1, \quad [F] = 0$ (so e^{-F} make sense).

$$F[\varphi] = \int d^dx \left(\tfrac{1}{2}(\nabla\varphi)^2 + \dots \right) \Rightarrow \underbrace{[\varphi]}_{\text{"engineering dimension" of } \varphi} = \frac{d-2}{2}.$$

If $\langle \varphi(\underline{x})\,\varphi(\underline{0})\rangle \propto \dfrac{1}{2\Delta_\varphi}$, so $\Delta_\varphi = \dfrac{d-2}{2}$. WRONG!

Resolution: $\exists$ another scale Λ, $[\Lambda] = +1$. actually

$$\langle \varphi(\underline{x})\,\varphi(\underline{0})\rangle \propto \frac{\Lambda^{-\eta}}{r^{d-2+\eta}}.$$

With B-field, F includes the term $\int d^dx\, B(\underline{x})\,\varphi(\underline{x})$.

At a FP, rewrite as

$$\int d^dx\, B(\underline{x})\,\varphi(\underline{x}) \overset{\text{at FP}}{=} \int d^dx\, \zeta^d B(\underline{x})\, \zeta^{-\Delta_\varphi}\varphi'(\underline{x}')$$

$$\Rightarrow B'(\underline{x}') = \zeta^{d-\Delta_\varphi} B(\underline{x}).$$

$$\Rightarrow \Delta_B = d - \Delta_\varphi = \frac{d+2-\eta}{2}.$$

For η small, $\Delta_B > 0 \Rightarrow B$ relevant.

Also $\xi' = \xi/\zeta \Rightarrow \Delta_\xi = -1$, but $\xi \sim t^{-\nu}$, $\xi' \sim (t')^{-\nu}$

$$\Rightarrow \zeta^{\Delta_\xi} \xi \sim \zeta^{-\nu\Delta_t} t^{-\nu}$$

$$\Rightarrow \Delta_t = -\Delta_\xi/\nu = 1/\nu$$

RG $\Rightarrow$ relations between crit expt:

① Specific heat (capacity)

$$c = \frac{C}{V} \sim \frac{\partial^2 f}{\partial t^2} + \text{smoother}, \quad f \equiv \frac{F_{\text{thermo}}}{V}.$$

Near Ising FP, $f = f(\delta g) = f(t, \delta\hat{g}_2, \delta\hat{g}_3, \dots)$ $\quad (B = 0)$.

Under RG, $t \to t'$, $\underline{x} \to \underline{x}' = \underline{x}/\zeta \Rightarrow V \to V' = V/\zeta^d$

Value of $F_{\text{thermo}} = -\log Z$ does not change under RG, so

$$f(t, \delta\hat{g}_2, \dots) V = f'(t', \delta\hat{g}_2', \dots) V'$$

$$\Rightarrow f(t, \delta\hat{g}_2, \dots) = f(\zeta^{\Delta_t} t, \zeta^{\Delta_2} \delta\hat{g}_2, \dots) \zeta^{-d}.$$

Set $\zeta = t^{-1/\Delta_t}$, then

$$f(t, \delta\hat{g}_2, \dots) = \underbrace{f(1, t^{-\Delta_2/\Delta_t} \delta\hat{g}_2, \dots)} t^{d/\Delta_t} \sim t^{d\nu}$$

$\rightarrow f(1, 0, 0, \dots)$ as $t \to 0$ $\quad (T \to T_c)$.

(since $\Delta_i < 0 \ \forall i \geq 2$).

Then $c \sim t^{d\nu - 2}$, but $c \sim t^{-\alpha}$, so

$$\boxed{\alpha = 2 - d\nu} \quad \text{(hyperscaling relation).}$$

② $T < T_c$ (ordered phase), at $T \approx T_c$, $\langle \varphi \rangle \sim t^\beta$.

By scaling, $\Delta_\varphi = \beta \Delta_t = \beta/\nu$.

$$\Rightarrow \quad \beta = \nu \Delta_\varphi = \frac{1}{2}(d-2+\eta)\nu$$

③ $\chi = \left(\frac{\partial\langle\varphi\rangle}{\partial B}\right)_T \sim t^{-\gamma} \Rightarrow \Delta_\varphi - \Delta_B = -\gamma/\nu.$

$$\Rightarrow \gamma = \nu(2-\eta)$$

④ Near CP, $\langle\varphi\rangle \sim B^{1/\delta} \Rightarrow \delta = \Delta_B / \Delta_\varphi = \frac{d+2-\eta}{d-2+\eta}.$

So can compute $\alpha, \beta, \gamma, \delta$ from η, ν.

This works well for $d=2,3$ Ising, ok for $d=4$ MFT (with quadratic fluctuations), disagrees with MFT for $d>4$ for β, δ.

Operators

Expressions built from φ and its derivatives are called operators

e.g. $\varphi^n, \varphi^m(\nabla\varphi)^2, \ldots$

Near FP with

$$F[g, \varphi] = F[g_*, \varphi] + \int d^dx \sum_a \delta g_a \mathcal{O}_a(\underline{x}).$$

e.g. $\delta g_1 = \delta(\mu^2), \quad \delta(g_2) = \delta g_2 \ldots$

$\mathcal{O}_1 = \frac{1}{2}\varphi^2 \qquad \mathcal{O}_2 = \varphi^4, \ldots$

Consider $\delta g = \sum \delta\hat{g}_i \underline{v}_i$

$$\Rightarrow \sum_a \delta g_a \mathcal{O}_a = \sum_i \delta\hat{g}_i \underbrace{\sum_a (\underline{v}_i)_a \mathcal{O}_a}_{\mathcal{O}_i}$$

Under RG,

$$\int d^dx \sum_i \delta\hat{g}_i \mathcal{O}_i(x) = \int d^dx' \, \zeta^d \sum_i \zeta^{-\Delta_i} \delta\hat{g}_i \mathcal{O}_i(x).$$

So

$$\mathcal{O}_i'(x') = \zeta^{d-\Delta_i}\,\mathcal{O}_i(x).$$

and say $\mathcal{O}_i$ has scaling dim $\Delta_{\mathcal{O}_i} = d - \Delta_i$.

Say $\mathcal{O}_i$ is $\left\{\begin{matrix}\text{relevant}\\ \text{irrelevant}\end{matrix}\right\}$ if $\left\{\begin{matrix}\Delta_{\mathcal{O}_i} < d\\ \Delta_{\mathcal{O}_i} > d\end{matrix}\right\}$.

If $\mathcal{O}_i$ has m factors of φ and n derivatives (e.g. $(\nabla\varphi)^2\varphi^2$ has $m=4$, $n=2$), then engineering dimension is $m\frac{(d-2)}{2}+n$.
($\mathbb{Z}_2$ sym $\Rightarrow$ m even, and $\mathcal{O}_i$ scaler $\Rightarrow$ n even).

3.3 Gaussian Fixed Point

Apply RG starting from most general quadratic theory

$$F_0[\varphi] = \int d^dx \left(\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu_0\varphi^2 + \frac{1}{2}\sum_{n=2}^{\infty} h_0^{(n)}\varphi(-\nabla^2)^n\varphi\right)$$

(use IBP to bring to this form).

$$= \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{2}\left(\underline{k}^2 + \mu_0^2 + \sum_{n=2}^{\infty} h_0^{(n)}(\underline{k}^2)^n\right)|\varphi_{\underline{k}}|^2.$$

Write $\varphi_{\underline{k}} = \varphi_{\underline{k}}^- + \varphi_{\underline{k}}^+$

$|\underline{k}| < \Lambda/\zeta$ $\qquad$ $\frac{\Lambda}{\zeta} < |\underline{k}| < \Lambda$.

$$\Rightarrow F_0[\varphi] = F_0[\varphi^-] + F_0[\varphi^+]$$

$$e^{-F'[\varphi']} = \left(\int \mathcal{D}\varphi^+ e^{-F_0[\varphi^+]}\right) e^{-F_0[\varphi^-]}$$

$$=: N e^{-F_0[\varphi^-]}$$

some const.

Now scale $\underline{k}' = \zeta\underline{k}$, $\varphi'_{\underline{k}'} = \zeta^{-w}\varphi^-_{\underline{k}}$.

$$F_0[\varphi^-] = \int^{\Lambda/s} \frac{d^dk}{(2\pi)^d}\,\frac{1}{2}\left(\underline{k}^2 + \mu^2 + \sum_{n=2}^{\infty} h_0^{(n)}(\underline{k}^2)^n\right)|\varphi_{\underline{k}}^-|^2.$$

$$= \int^\Lambda \frac{d^dk'}{(2\pi)^d} \cdot \frac{1}{2\zeta^d} \left(\frac{k'^2}{\zeta^2} + \mu_0^2 + \sum_{n=2}^{\infty} \frac{h_0^{(n)} (k'^2)^n}{\zeta^{2n}} \right) \zeta^{2w} |\varphi'_{k'}|^2$$

For canonical $(\nabla\varphi)^2$ term, need $w = \frac{d+2}{2}$.

Relabel $\underline{k}' \to \underline{k}$, $\varphi' \to \varphi$:

$$F_0'[\varphi] = \int^\Lambda \frac{d^dk}{(2\pi)^d} \frac{1}{2} \left(\underline{k}^2 + \mu^2(\zeta) + \sum_{n=2}^{\infty} h^{(n)}(\zeta) (\underline{k}^2)^n \right) |\varphi_{\underline{k}}|^2$$

where $\mu^2(\zeta) = \zeta^2 \mu_0^2$, $h^{(n)}(\zeta) = h_0^{(n)} / \zeta^{2n-2}$.

So quadratic form of F is preserved by RG.

As $\zeta \to \infty$, $h^{(n)} \to 0$, and $\mu^2 \to \begin{cases} \infty & \mu_0^2 \neq 0 \\ 0 & \mu_0^2 = 0 \end{cases}$

$\mu^2 = \infty$, $h^{(n)} = 0$ corresponds to $T = 0$ or ∞ $(\xi \to 0)$

$\mu^2 = 0$, $h^{(n)} = 0$ is the Gaussian FP (GFP)

$$F_{GFP}[\varphi] = \int d^dx \, \frac{1}{2} (\nabla\varphi)^2 .$$

Calculate Δ_φ at GFP

$$\varphi'(\underline{x}') = \int \frac{d^dk'}{(2\pi)^d} \varphi'_{\underline{k}'} e^{i\underline{k}'\cdot\underline{x}'}$$

$$= \int \frac{d^dk}{(2\pi)^d} \zeta^d \zeta^{-w} \varphi^-_{\underline{k}} e^{i\underline{k}\cdot\underline{x}} = \zeta^{d-w} \varphi^-(x).$$

$$\Rightarrow \Delta_\varphi = d - w = \frac{d-2}{2}$$ (same as engineering dim.).

Exercise (a) Show M_{ab} has a subset of evals $\Delta_1 = 2$, $\Delta_n = -(2n-2)$, $n = 2, 3, \ldots$

(b) The corresponding operators / couplings with well-defined scaling dims are

$$\mathcal{O}_1 = \frac{1}{2}\varphi^2, \qquad \mathcal{O}_n = \varphi(-\nabla^2)^n \varphi.$$

$\Delta_{\mathcal{O}}$: $d - \Delta_1 = d - 2$ $\qquad$ $d - \Delta_n = d + 2n - 2$. (same as eng dim.)

$\delta \hat{g}_1 = \delta(\mu^2) +$ parts from non-quadratic couplings

$\delta \hat{g}_n = \delta h^{(n)} +$ "

(c) $\exists \infty$ many other evals Δ_i, these have $\delta \hat{g}_i =$ lin. combⁿ only of non-quad couplings and $\mathcal{O}_i =$ combⁿ of non-quad. and quad

Claim At GFP, the full set of operator scaling dims agrees with the engineering dims. $\frac{m(d-2)}{2} + n$, (m, n even) (excluding $m = n = 2 \Leftrightarrow \frac{1}{2}(\nabla\varphi)^2$ fixed in RG).

$m = 4, n = 0 \longleftrightarrow$ eng. dim. of $\varphi^4 \Rightarrow 2(d-2)$, but operator is lin. combⁿ of φ^4 and $\Lambda^{d-2}\varphi^2$ (see later). This is relevant iff $2(d-2) < d \Leftrightarrow d < 4$.

So for $d < 4$, $\exists$ (at least) 2 relevant operators/couplings $\Rightarrow$ GFP doesn't describe Ising CP! We'll see $\exists$ another FP.

$d > 4$: GFP has only 1 relevant operator : $\frac{1}{2}\varphi^2$ (Ex.).

and GFP does describe Ising CP.

$d = 4$: $m = 4, n = 0$ is marginally irrel., behave like $d > 4$.

Dangerously Irrelevant

$\boxed{d > 4}$: GFP describes Ising CP, but scaling gave incorrect results for β, δ - why?

Scaling ($t < 0$) : $\langle \varphi \rangle \sim |t|^\beta \Rightarrow \beta = \frac{\Delta_\varphi}{\Delta_t}$ wrong for $d > 4$,

but actually MFT told us $\langle \varphi \rangle \sim (-\mu^2/g)^{1/2}$. g important!

Near GFP, $\mu^2 = \delta\mu^2 = \underset{\Delta=2}{\delta\hat{g}_1} + \underset{\Delta>2}{\ldots}$ $\Rightarrow \mu^2 \sim \delta\hat{g}_1$ as $\tau\to\infty$.

$g = \delta g \underset{\text{(c) from ex.}}{=}$ lin. comb^n of those $\delta\hat{g}_i$ with $m \geq 4$ ($\therefore$ irrelevant)

The least irrel. has $m=4, n=0$

$\Rightarrow \Delta_0 = 2(d-2) \Rightarrow \Delta = d - d_0 = 4-d$ (= eng. dim. of g).

Call this $\delta\hat{g}_{1'}$, so $g \sim \delta\hat{g}_{1'}$ as $\tau\to\infty$

$$\Rightarrow \left(-\mu^2/g\right)^{1/2} \sim \left(-\frac{\delta\hat{g}_1}{\delta\hat{g}_{1'}}\right)^{1/2} \sim \left(\frac{-t}{\delta\hat{g}_{1'}}\right)^{1/2}$$

So $\beta = \frac{1}{2}$, and RHS has dim. $\frac{1}{2}[2-(4-d)] = \frac{d-2}{2} = \Delta_\varphi$.

Usually irrel. couplings/operators can be ignored at long distances as couplings $\to 0$ under RG, but if such a coupling multiplies a quantity of interest and changes scaling, it is <u>dangerously irrelevant</u>.

($d<4$: we'll see $g \neq 0$ at Ising FP, so denominator $\to$ const.)

<u>Interactions that break Z_2.</u>

We assumed Z_2 sym $F[\varphi] = F[-\varphi]$ excludes φ^m in F for odd m. Sym is preserved by RG.

Now relax this assumption: odd m possible.

$m=1$: $\int d^dx\, g_1\varphi$, $g_1 \equiv B$, $\Delta_B = \frac{d+2}{2}$ at GFP $\Rightarrow$ relevant

$m=3$: $\int d^dx\, g_3\varphi^3$ looks relevant for $d<6$ but can always set $g_3 = 0$ by redefining $\varphi \to \varphi + c$ (not allowed with Z_2 sym.)

$(g\varphi^4 \to g\varphi^4 + 4cg\varphi^3 + \dots)$

Emergence of Rotational Sym.

We assumed F invar under all rot^n in $O(d)$, but original lattice is invar. only under finite $G \subset O(d)$, e.g. $d=2$ square lattice has $G = D_8$. So should only demand F is invar. only under G.

Cubic lattice: lowest engineering dimension operator invar. w.r.t. G but not $O(d)$ is

$$\mathcal{O} = \varphi \sum_{i=1}^{d} \partial_i^4 \varphi.$$

Expect such terms to be present in F. But at GFP, can show $\Delta_{\mathcal{O}} = 4 + 2\Delta_\varphi = d+2 \Rightarrow$ irrel.

$\Rightarrow$ doesn't affect long-distance physics

$O(d)$ is emergent: a sym. of long-distance physics that isn't present microscopically

3.4 RG with interactions

Non-quad with lowest eng. dim. is φ^4.

Start from

$$F[\varphi] = \int d^dx \left(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\mu_0^2\varphi^2 + \underbrace{g_0}_{\geq 0.}\varphi^4 \right).$$

$$=: \underbrace{F_0[\varphi^-] + F_0[\varphi^+]}_{\text{quad.}} + \underbrace{F_I[\varphi^-, \varphi^+]}_{\text{from } \varphi^4.}$$

$$e^{-F'[\varphi^-]} := e^{-F_0[\varphi^-]} \int \mathcal{D}\varphi^+ \, e^{-F_0[\varphi^+]} \, e^{-F_I[\varphi^-,\varphi^+]}$$

Define

$$\langle (\ldots) \rangle_+ \cong \frac{\int \mathcal{D}\varphi^+ (\ldots) \, e^{-F_0[\varphi^+]}}{Z_+}$$

where $Z_+ = \int \mathcal{D}\varphi^+ \, e^{-F_0[\varphi^+]}$. This is expectation value treating φ^+ as a r.v. with Gaussian prob. density $e^{-F_0[\varphi^+]}$.

$$\therefore \quad F'[\varphi^-] = F_0[\varphi^-] - \underbrace{\log \langle e^{-F_I[\varphi^-,\varphi^+]} \rangle_+}_{\text{compute this}} - \underbrace{\log Z_+}_{\text{indpt. of } \varphi^- \Rightarrow \text{ ignore.}}$$

To compute, assume g_0 small, and use pert^n thy. ($F_I = \mathcal{O}(g_0)$).

$$\log \langle e^{-F_I} \rangle_+ = \log \left(1 - \langle F_I \rangle_+ + \frac{1}{2} \langle F_I^2 \rangle_+ + \ldots \right)$$

$$= -\langle F_I \rangle + \frac{1}{2} \left(\langle F_I^2 \rangle_+ - \langle F_I \rangle_+^2 \right) + \ldots$$

(In general, n-th term is $(-1)^n \times n^{\text{th}}$ "cumulant" of F_I.)

To $\mathcal{O}(g_0)$:

$$F_I = g_0 \int d^dx \prod_{i=1}^{4} \underbrace{\int \frac{d^dk_i}{(2\pi)^d} \varphi_{\underline{k}_i} e^{i \underline{k}_i \cdot \underline{x}}}_{\varphi(\underline{x})}$$

$$= g_0 \int \left[\prod_{i=1}^{4} \frac{d^dk_i}{(2\pi)^d} \left(\varphi^+_{\underline{k}_i} + \varphi^-_{\underline{k}_i} \right) \right] (2\pi)^d \delta^{(d)}\left(\sum \underline{k}_i \right).$$

5 types of term:

- $(\varphi^-)^4$: $\langle \ldots \rangle_+$ trivial – just gives $g_0 \int d^dx \, (\varphi^-)^4$ in F'.
- $4 (\varphi^-)^3 \varphi^+$: odd in $\varphi^+ \Rightarrow \langle \ldots \rangle_+ = 0$
- $6 (\varphi^-)^2 (\varphi^+)^2$: interesting.

- $4\varphi^-(\varphi^+)^3$: odd in $\varphi^+ \Rightarrow \langle\ldots\rangle_+ = 0$.
- $(\varphi^+)^4$: gives const. term in F'.

<u>Note</u> : $6 = \binom{4}{2}$ = # ways of choosing where 2 k_i's label φ^- and which label φ^+.

Need to compute

same kind of correlation f^n computed previously.

$$\langle F_I \rangle_+ \supset 6 g_0 \int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} \varphi^-_{\underline{k}_1} \varphi^-_{\underline{k}_2} \langle \varphi^+_{\underline{k}_3} \varphi^+_{\underline{k}_4} \rangle_+ \, \delta^{(d)}(\Sigma \underline{k}_i).$$

Recall $\langle \varphi^+_{\underline{k}} \varphi^+_{\underline{k}'} \rangle = (2\pi)^d \delta^{(d)}(\underline{k}+\underline{k}') G_0(\underline{k})$. , where $G_0(\underline{k}) = \dfrac{1}{k^2+\mu_0^2}$.

$$\Rightarrow \langle F_I \rangle_+ \supset 6 g_0 \int^{\Lambda/\zeta} \frac{d^d k}{(2\pi)^d} \varphi^-_{\underline{k}} \varphi^-_{-\underline{k}} \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+\mu_0^2}$$

This is a correction to the $(\varphi^-)^2$ term in F'

$$\therefore \quad \mu_0^2 \longrightarrow \mu'^2 = \mu_0^2 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+\mu_0^2}.$$

from $F_0[\varphi^-]$

Coeff. of $(\nabla\varphi^-)^2$ and $(\varphi^-)^4$ unchanged to $\mathcal{O}(g_0)$, so under $\underline{k}' = \zeta \underline{k}$, we have $\varphi'_{\underline{k}'} = \zeta^{-\frac{(d+2)}{2}} \varphi^-_{\underline{k}}$ as before.

$$\Rightarrow \mu^2(\zeta) = \zeta^2 \left(\mu_0^2 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+\mu_0^2} \right) + \mathcal{O}(g_0^2).$$

$$g(\zeta) = \zeta^{4-d} g_0 + \mathcal{O}(g_0^2).$$

β-functions : for $\zeta = e^s$,

$$\Rightarrow \beta_g \equiv \frac{dg}{ds} = (4-d) g + \mathcal{O}(g_0^2) = (4-d) g + \mathcal{O}(g^2).$$

Consider

$$\int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d}\, g(q) = \int_{\Lambda/\zeta}^{\Lambda} \frac{dq}{(2\pi)^d}\, q^{d-1}\Omega_{d-1}\, g(q),$$

where Ω_n = area of n-sphere, $(\Omega_1 = 2\pi,\ \Omega_2 = 4\pi,\ \Omega_3 = 2\pi^2)$

and

$$\frac{d}{ds}\int_{\Lambda e^{-s}}^{\Lambda} dq\, f(q) = -f(\Lambda e^{-s})\frac{d}{ds}(\Lambda e^{-s}) = \Lambda e^{-s} f(\Lambda e^{-s}).$$

$$\Rightarrow \beta_{\mu^2} \equiv \frac{d\mu^2}{ds} = 2\mu^2 + \frac{12 g_0}{(2\pi)^d}\frac{\Lambda^d \Omega_{d-1} e^{(4-d)s}}{\Lambda^2 + \underbrace{\mu_0^2 e^{2s}}_{\mu^2 + \mathcal{O}(g_0)}} + \mathcal{O}(g_0^2).$$

$$= 2\mu^2 + \frac{12\Lambda^d \Omega_{d-1}}{(2\pi)^d}\frac{g}{\Lambda^2+\mu^2} + \mathcal{O}(g^2).$$

Let's ignore all except these 2 couplings at GFP

$$\underline{\underline{M}} = \begin{pmatrix} \partial\beta_{\mu^2}/\partial\mu^2 & \partial\beta_{\mu^2}/\partial g \\ \partial\beta_g/\partial\mu^2 & \partial\beta_g/\partial g \end{pmatrix}\Bigg|_{\mu^2 = g = 0} = \begin{pmatrix} 2 & \overbrace{\frac{12\Lambda^{d-2}\Omega_{d-1}}{(2\pi)^d}}^{=:(d-2)c\Lambda^{d-2}} \\ 0 & 4-d \end{pmatrix}$$

$$\Rightarrow \Delta_1 = 2,\quad \underline{v}_1 = \begin{pmatrix}1\\0\end{pmatrix};\qquad \Delta_2 = 4-d,\quad \underline{v}_2 = \begin{pmatrix}-c\\1\end{pmatrix}$$

$$\underline{w}_1 = (1 \quad c\Lambda^{d-2})\qquad\qquad \underline{w}_2 = (0 \quad 1)$$

$$\underline{v}_i \Rightarrow \mathcal{O}_1 = \tfrac{1}{2}\varphi^2 \qquad\qquad \mathcal{O}_2 = \varphi^4 - \tfrac{1}{2}c\Lambda^{d-2}\varphi^2.$$

$$\underline{w}_i \Rightarrow \delta\hat{g}_1 = \overbrace{\mu^2 + c\Lambda^{d-2} g}^{\text{has dim } 2} \qquad\qquad \delta\hat{g}_2 = \overbrace{g}^{\text{has dim } 4-d.}$$

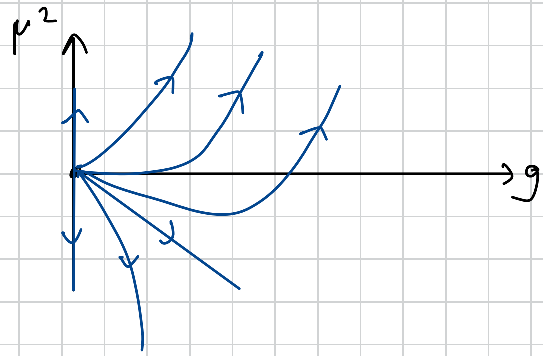
$\delta\mu = \mu,\ \delta g = g$ at GFP

$\Rightarrow$ supports our claim about GFP.

For $d>4$: g irrel., $\delta\hat{g} \sim t$.

$\mu^2 = -c\Lambda^{d-2} g$: crit surface.

For $d<4$: g relevant $\Rightarrow$ g rows under RG. $\Rightarrow$ pert^n thy breaks down



For $d=4$: marginal $\Rightarrow$ need to go to $O(g_0^2)$.

For $O(g_0^2)$, $\quad F' \supset -\frac{1}{2}\left(\langle F_I^2\rangle_+ - \langle F_I\rangle_+^2\right)$.

This gives 2 types of term

- $\left(\int d^dx\, f(x)\right)^2$: cancelled by $\langle F_I\rangle_+^2$.
- $\int d^dx\, f(x)$: not cancelled.

Focus on terms with 2 φ^- from each φ^4.

$$\langle F_I^2\rangle_+ \supset \binom{4}{2}^2 g_0^2 \int_0^{\Lambda/\zeta}\left[\prod_{i=1}^{4}\frac{d^dk_i}{(2\pi)^d}\varphi^-_{k_i}\right]\int_{\Lambda/\zeta}^{\Lambda}\left[\prod_{j=1}^{4}\frac{d^dq_j}{(2\pi)^d}\right]\langle\varphi^+_{q_1}\varphi^+_{q_2}\varphi^+_{q_3}\varphi^+_{q_4}\rangle_+$$

$$\cdot (2\pi)^{2d}\,\delta^{(d)}(\underline{k}_1+\underline{k}_2+\underline{q}_1+\underline{q}_2)\,\delta^{(d)}(\underline{k}_3+\underline{k}_4+\underline{q}_3+\underline{q}_4).$$

\# ways of choosing 2 $\underline{k}$'s and 2 $\underline{q}$'s from each φ^4.

Claim $\langle \varphi^{+}_{q_1} \varphi^{+}_{q_2} \varphi^{+}_{q_3} \varphi^{+}_{q_4} \rangle_+ = \langle \varphi_{q_1} \varphi_{q_2} \rangle_+ \langle \varphi_{q_3} \varphi_{q_4} \rangle_+$

$+ \langle \varphi_1 \varphi_3 \rangle_+ \langle \varphi_2 \varphi_4 \rangle_+$

$+ \langle \varphi_1 \varphi_4 \rangle_+ \langle \varphi_2 \varphi_3 \rangle_+ .$

Use the fact that

$$\langle \varphi^{+}_{q} \varphi^{+}_{q'} \rangle_+ = (2\pi)^d \delta^{(d)}(q + q') \, G_0(q).$$

1st term : something of form $\left(\int d^d x \, f(x) \right)^2 \Rightarrow$ cancelled by $\langle F_I \rangle_+^2$.

2nd, 3rd term:

$$2 \binom{4}{2}^2 g_0^2 \int_0^{\Lambda/\zeta} \left[\prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} \varphi^-_{k_i} \right] f(k_1 + k_2) \, (2\pi)^d \delta^{(d)}(\Sigma k_i) .$$

where $f(k_1 + k_2) = \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} \frac{1}{(k_1 + k_2 + q)^2 + \mu_0^2}$

$$= \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + \mu_0^2)^2} \left(1 + O((k_1 + k_2)^2) \right).$$

generate terms $\sim k^2 (\varphi^-)^4$

$\sim (\varphi^-)^2 (\nabla \varphi^-)^2$

irrelevant

$$\therefore \; g_0 \to g_0' = g_0 - 36 g_0^2 \int_{\Lambda/\zeta} \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + \mu_0^2)^2}$$

Coeff of $(\nabla \varphi^-)^2$ changes at $O(g_0^2)$ (see later)

$\therefore \; \varphi'_{k'} = \zeta^{-\frac{d+2}{2}} \left(1 + O(g_0^2) \right) \varphi^-_k .$

only affects $O(g_0^3)$ in $g(\zeta)$

$$\Rightarrow \; g(\zeta) = \zeta^{4-d} \left(g_0 - 36 \, g_0^2 \int \ldots \right)$$

For $\underline{d=4}$, at $\mathcal{O}(g_0)$, g marginal. But at $\mathcal{O}(g_0^2)$, see that g decreases under RG, so g is marginally irrel. at GFP. (so $d=4$ is like $d>4$: GFP describe Ising CP).

We have
$$\mu^2(\zeta) = \zeta^2(\mu_0^2 + a g_0 + g_0^2)$$
$$g(\zeta) = \zeta^{4-d}(g_0 - b g_0^2 + \mathcal{O}(g_0^3))$$

where
$$a = \frac{12\,\Omega_{d-1}}{(2\pi)^d}\int_{\Lambda/\zeta}^{\Lambda} \frac{dq\, q^{d-1}}{q^2+\mu_0^2},$$

$$b = \frac{36\,\Omega_{d-1}}{(2\pi)^d}\int_{\Lambda/\zeta}^{\Lambda} \frac{dq\, q^{d-1}}{(q^2+\mu_0^2)^2}$$

$$\Rightarrow \beta_{\mu^2} = 2\mu^2 + \frac{12\,\Omega_{d-1}\Lambda^d g}{(2\pi)^d(\Lambda^2+\mu^2)} + \mathcal{O}(g^2)$$
$$\beta_g = (4-d)g - \frac{36\,\Omega_{d-1}}{(2\pi)^d}\frac{\Lambda^d g^2}{(\Lambda^2+\mu^2)^2} + \mathcal{O}(g^3)$$

Look for fixed points: $d \geq 4$.: $\beta_g = 0 \Rightarrow g=0$ (as $g \geq 0$). then $\beta_{\mu^2}=0 \Rightarrow \mu^2=0$ GFP.

For $d<4$: $4-d>0$. Then
$$\beta_g = 0 \Rightarrow g=0, \quad \text{or} \quad g = (4-d)\,\mathcal{O}(\Lambda^{4-d})$$
$$\Downarrow$$
$$\mu^2 = 0 \Rightarrow \text{GFP}.$$

Suggests maybe $\exists$ another FP, but in pert^n thy, $(4-d)\,\mathcal{O}(\Lambda^{4-d})$ not small, so lies outside our pert^n approx.

Recall $\underline{g} = (\mu^2, g, \underline{h})$. We assumed $\underline{h}(0) = 0$ and didn't

all other couplings

calculate $\underline{h}(\zeta)$. Can repeat analysis allowing small $\underline{h}(0)$

— treat perturbatively. Conclusion unaffected.

Feynman Diagrams

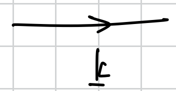
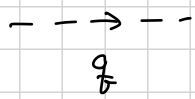
$\mathcal{O}(g_0^p)$: $F' \supset \dfrac{(-1)^{p+1}}{p!} \langle F_I^p \rangle_+$. Each term in $\langle F_I^p \rangle_+$ can be represented by a diagram. Terms with integrand of form $g_0^p (\varphi^-)^n (\varphi^+)^l$ are represented by diagrams with

- n external solid lines labelled by $\underline{k}_i$
- l internal dotted lines labelled by $\underline{q}_i$.

Joined up in all possible ways $\Leftrightarrow$ diff. Wick contractions

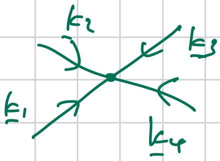
- p verticies where 4 lines meet.

Each diagram corresponds to an $\int$.

-  $\displaystyle\int_0^{\Lambda/\zeta} \frac{d^d k}{(2\pi)^d} \varphi_{\underline{k}}^-$
-  $G_0(\underline{q})$ propagator
- Each loop $\displaystyle\int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d}$
- Vertex $g_0 (2\pi)^d \delta^{(d)}(\Sigma$ momenta into vertex$)$

Symmetry factor : # topologically equiv. diagrams

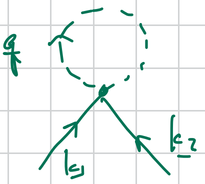
Example At $\mathcal{O}(g_0)$, we have



$$= \int_0^{\Lambda/\zeta} \left(\prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} \varphi^-_{k_i} \right) g_0 (2\pi)^d \delta^{(d)}\left(\Sigma k_i\right)$$

$$= g_0 \int d^d x \; \varphi(x)^4 .$$

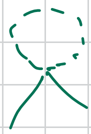
(At "tree level", i.e. no loop.)

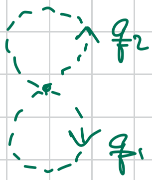


$$= \int \left(\prod_{i=1}^{2} \frac{d^d k_i}{(2\pi)^d} \varphi^-_{k_i} \right) \int \frac{d^d q}{(2\pi)^d} G_0(q)\, g_0 (2\pi)^d \, \delta^{(d)}(k_1 + k_2 + \cancel{q} - \cancel{q})$$

(1-loop correction to μ^2)

Sym. factor $= \binom{4}{2} = 6$ ways of choosing which 2 are the k lines emerging from vertex will be dotted.

So only need to compute  and multiply by sym. factor 6.



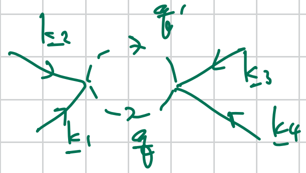
$$= \int_{\Lambda/\zeta}^{\Lambda} \prod_{i=1}^{2} \frac{d^d q_i}{(2\pi)^d} \; \frac{(2\pi)^d \delta^{(d)}(q_1 + q_2 - q_1 - q_2)}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)} .$$

Interpret $(2\pi)^d \delta^{(d)}(0) = \int d^d x \, e^{i x \cdot 0} = V$

$\Rightarrow$ term in $F' \supset \int d^d x$ (const.) (ignored above).

Sym factors $= 3$. (# ways joining dotted line).

At $\mathcal{O}(g_0^2)$, our $\mathcal{O}(g_0^2)$ correction to $g\varphi^4$ comes from



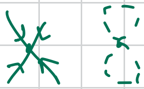
(1 loop correction to g)

Sym factor $= \binom{4}{2} \cdot 2 \cdot \binom{4}{2} = 72$.

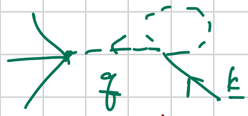
paring of dotted lines

Disconnected diagrams always cancel out in F'.

e.g. can ignore



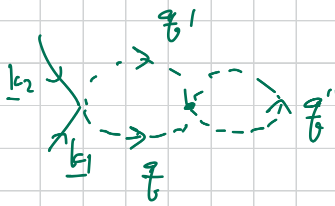
,



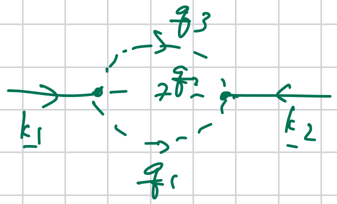
δ forces $\underline{k} = \underline{q}$, but $|\underline{q}| > \Lambda/\zeta$, $|\underline{k}| < \Lambda/\zeta$

$\Rightarrow$ diagram $= 0$

2-loop diagrams:



$$= g_0^2 \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{2} C(\zeta, \Lambda) \, \varphi^-_{\underline{k}} \varphi^-_{-\underline{k}}$$



$$= g_0^2 \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{2} \underbrace{A(\underline{k}, \zeta, \Lambda)}_{= A(0,\zeta,\Lambda) + \frac{1}{2} \underline{k}^2 A''(0,\zeta,\Lambda) + \dots} \varphi^-_{\underline{k}} \varphi^-_{-\underline{k}}.$$

These generate $\mathcal{O}(g_0^2)$ in μ'^2:

$$\mu'^2 = \mu^2 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} + g_0^2 \left[C(\zeta,\Lambda) + A(0,\zeta,\Lambda) \right]$$

but also shift coeff. of $(\nabla\varphi)^2 \sim \underline{k}^2 \varphi^2$.

$$F' \supset \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{2} \left(1 + g_0^2 A''(0,\zeta,\Lambda) \right) \underline{k}^2 \varphi^-_{\underline{k}} \varphi^-_{-\underline{k}}$$

In final step of RG, $\underline{k}' = \zeta \underline{k}$.

$$\varphi^-_{\underline{k}'} = \zeta^{-\frac{(d+2)}{2}} \left(1 + \tfrac{1}{2} g_0^2 A''\right)^{\frac{1}{2}} \varphi^-_{\underline{k}}$$

$$= \zeta^{-\frac{d+2}{2}} \left(1 + \tfrac{1}{4} g_0^2 A''\right) \varphi^-_{\underline{k}} \qquad (*)$$

This is origin of anomalous dimension. Recall

$$\Delta_\varphi = \frac{d-2}{2} + \frac{\eta}{2} \Leftrightarrow \varphi'(\underline{x}') = \zeta^{\frac{d-2+\eta}{2}} \varphi^-(\underline{x})$$

$$\Leftrightarrow \varphi'_{\underline{k}'} = \zeta^{-\frac{d+2}{2} + \frac{\eta}{2}} \varphi^-_{\underline{k}}$$

$$= \zeta^{-\frac{d+2}{2}} e^{\frac{\eta}{2} \log \zeta} \varphi^-_{\underline{k}}$$

$$= \zeta^{-\frac{d+2}{2}} \left(1 + \frac{\eta}{2} \log \zeta + \dots\right) \varphi^-_{\underline{k}}$$

So $(*)$ gives $\eta = O(g_*^2)$ at a FP (μ_*^2, g_*) ($\Rightarrow \eta = 0$ at GFP).

3.5 The epsilon expansion

Some non-trivial FP $g = (4-d)\, O(\Lambda^{4-d})$.

<u>Idea</u>: let $d = 4 - \varepsilon$, $\varepsilon > 0$ <u>small</u>. Hope that setting $\varepsilon = 1$ is valid and tell us something.

let $\tilde{g} = \Lambda^{-\varepsilon} g$ (dimless), then

$$\frac{d\mu^2}{ds} = 2\mu^2 + \frac{12\, \Omega_{d-1} \Lambda^4}{(2\pi)^d (\Lambda^2 + \mu^2)} \tilde{g} + O(\tilde{g}^2).$$

$$\frac{d\tilde{g}}{ds} = \varepsilon \tilde{g} - \frac{36\, \Omega_{d-1} \Lambda^4}{(2\pi)^d (\Lambda^2 + \mu^2)} \tilde{g}^2 + O(\tilde{g}^3).$$

and $\Omega_{d-1} = \dfrac{2\pi^{d/2}}{\Gamma(d/2)}$, $\Omega_3 = 2\pi^2 \Rightarrow \dfrac{\Omega_{d-1}}{(2\pi)^d} = \dfrac{1}{8\pi^2} + O(\varepsilon)$.

Two FP: (i) $\mu^2 = \tilde{g} = 0$ (GFP)

(ii) $\tilde{g}_* = \frac{2\pi^2}{9} \frac{(\Lambda^2+\mu_*^2)^2}{\Lambda^4} \varepsilon$ $\quad\Rightarrow\quad$ $\tilde{g}_* = \frac{2\pi^2}{9}\varepsilon + \mathcal{O}(\varepsilon^2)$

$\mu_*^2 = -\frac{3}{4\pi^2} \frac{\Lambda^4}{\Lambda^2+\mu_*^2} \tilde{g}_*$ $\qquad\qquad$ $\mu_*^2 = -\frac{1}{6}\Lambda^2\varepsilon + \mathcal{O}(\varepsilon^2)$.

This is the Wilson-Fisher FP: valid for $\varepsilon \ll 1$. (Can show $\underline{h} = \mathcal{O}(\varepsilon^2)$ or smaller).

Linearise $\quad \mu^2 = \mu_*^2 + \delta\mu^2, \quad \tilde{g} = \tilde{g}_* + \delta\tilde{g}$

$$\Rightarrow \frac{d}{ds}\begin{pmatrix}\delta\mu^2 \\ \delta\tilde{g}\end{pmatrix} = \begin{pmatrix} 2-\varepsilon/3 & M_{12} \\ 0 & -\varepsilon\end{pmatrix}\begin{pmatrix}\delta\mu^2 \\ \delta\tilde{g}\end{pmatrix}. \qquad M_{12} = \frac{3}{2\pi^2}\Lambda^2\left(1+\frac{\varepsilon}{6}\right)$$

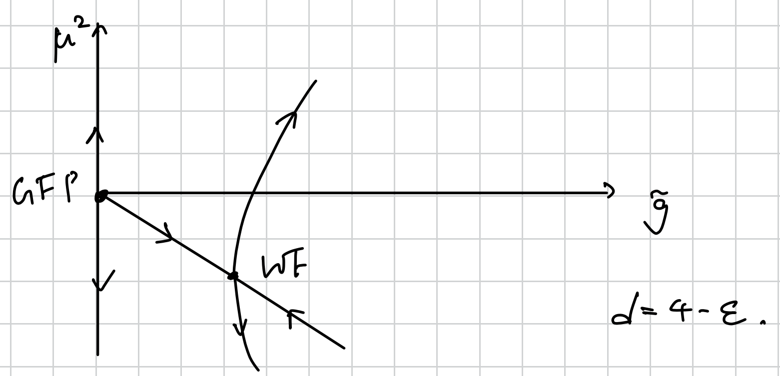
$\therefore \Delta_1 = 2 - \frac{\varepsilon}{3} + \mathcal{O}(\varepsilon^2), \quad \underline{v}_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad \underline{w}_1 = (1 \;\; a)$

$\Delta_2 = -\varepsilon + \mathcal{O}(\varepsilon^2), \quad \underline{v}_2 = \begin{pmatrix}-a\\1\end{pmatrix}, \quad \underline{w}_2 = (0 \;\; 1), \quad a = \frac{M_{12}}{\Delta_1 - \Delta_2}.$

$$\Rightarrow \left| \begin{array}{llll} \delta\hat{g}_1 = \underline{w}_1\, \delta g = \delta\mu^2 + a\,\delta\tilde{g} & & \text{relevant} \\ \delta\hat{g}_2 = \underline{w}_2\, \delta g & & \text{irrelevant.} \end{array}\right.$$

Single relevant coupling, as req'd for Ising CP!

$\therefore$ Identify $\delta\hat{g}_1 \sim t \Rightarrow \Delta_t = 2 - \frac{\varepsilon}{3} + \mathcal{O}(\varepsilon^2)$.



(expect qualitatively same for $d=3$).

Crit-expt at WF (recall §3.2)

$$\nu = \frac{1}{\Delta_\varepsilon} = \frac{1}{2} + \frac{\varepsilon}{12} + \mathcal{O}(\varepsilon^2)$$

$$\eta = \mathcal{O}(\varepsilon^2) \qquad \left(\text{Actual value: } \frac{\varepsilon^2}{6} + \dots\right).$$

Scaling $\Rightarrow$

$$\alpha = 2 - d\nu = \frac{\varepsilon}{6}$$

$$\beta = \frac{1}{2}(d-2+\eta)\nu = \frac{1}{2} - \frac{\varepsilon}{6}$$

$$\gamma = \nu(2-\eta) = 1 + \frac{\varepsilon}{6}$$

$$\delta = \frac{d+2-\eta}{d-2+\eta} = 3 + \varepsilon$$

	α	β	γ	δ	η	ν
MFT	0	1/2	1	3	0	1/2
$\varepsilon = 1$	0.17	0.33	1.17	4	0	0.58
$d = 3$	0.11	0.33	1.24	4.79	0.04	0.63

<u>$d=2$ Case</u>: $\quad F = \int d^2x \, \frac{1}{2}(\nabla\varphi)^2 + \dots$

$\Delta_\varphi = 0$ at GFP $\Rightarrow$ ∞-many relevant operator at GFP

$$\varphi^{2n} + c_n \Lambda^2 \varphi^{2(n-1)} + \dots$$

turns out $\exists$ ∞-many FP

Rough idea: flow to n-th FP from GFP by turning on $\varphi^{2(n+1)}$.

$$F_0 = \int d^2x \left(\frac{1}{2}(\nabla\varphi)^2 + g_{2(n+1)} \varphi^{2(n+1)} \right).$$

$\varphi^2, \varphi^4, \dots$ generated by RG: tune coeff to 0 to reach FP

which has a relevant interactions ($\sim$ lin. comb. of $\varphi^2, \dots, \varphi^{2n}$).

Each FP has diff. crit. expt.

4. Continuous Symmetry

4.1 Symmetry

Characterise phases of matter by 2 sym. groups.

G: Sym. of effective free energy (sym. of theory)

H: " ground state

e.g. $B=0$ Ising has $\varphi \to -\varphi$ sym, so $G = \mathbb{Z}_2$. Disordered phase $\langle \varphi \rangle = 0$ has $H = \mathbb{Z}_2$. Ordered phase $\langle \varphi \rangle = \pm m_0$ has $H = \{id\}$ (SSB).

($B \neq 0$ Ising has $G = \{id\}$ — view B as param of theory so $B \to -B$, $\varphi \to -\varphi$ not allowed)

Useful because

- for fixed G, diff. $H \Leftrightarrow$ diff. phases
- nature of CP (i.e. universality class) determined by G and no. of param.

4.2 O(N) models

N order param.: combine into vector $\varphi(\underline{x}) = (\varphi_1(\underline{x}), \dots, \varphi_N(\underline{x}))$

Assume F universal under $G = O(N)$ (unrelated to $O(d)$ spatial rot^n) with $\varphi_a(\underline{x}) \to R_{ab}\varphi_b(\underline{x})$, $a,b = 1,\dots,N$.

$R^T R = I \Leftrightarrow R \in O(N)$.

$$F[\varphi] = \int d^d x \left(\frac{\gamma}{2} \underbrace{(\nabla\varphi)^2}_{\nabla\varphi_a \cdot \nabla\varphi_a} + \frac{1}{2}\mu^2 \underbrace{\varphi^2}_{\varphi_a\varphi_a} + g(\varphi^2)^2 + \dots \right).$$

(To calculate correlation f^n, couple to $\underline{B}(\underline{x})$ via
$$-\int d^dx \ \underline{B}(\underline{x}) \cdot \varphi(\underline{x}).$$
)

$\underline{N=1}$: $O(1) = \mathbb{Z}_2 \Rightarrow$ Ising

$\underline{N=2}$: $\psi(\underline{x}) := \varphi_1(\underline{x}) + i\varphi_2(\underline{x})$

$$\Rightarrow F[\psi] = \int d^dx \left(\frac{\gamma}{2} \nabla\psi^* \cdot \nabla\psi + \frac{1}{2}\mu^2 |\psi|^2 + g|\psi|^4 + \dots \right)$$

$O(2) = U(1)$ act as $\psi \to e^{i\alpha}\psi$.

This is XY-model : arises by coarse graining a microscopic theory of spins free to rotate in a plane:

$$E = -J \sum_{\langle ij \rangle} \underline{S}_i \cdot \underline{S}_j \quad , \quad \underline{S}_i : 2d \text{ vec}, \ |\underline{S}_i| = 1.$$

(describes certain types of magnet, superfluid, BE condensates)

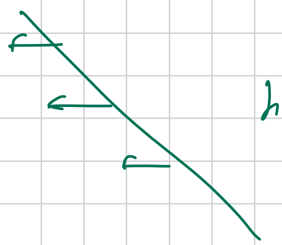
$\underline{N=3}$: Coarse grain Heisenberg model. S_i : 3d vec, $|\underline{S}_i| = 1$.

Consider ordered phase for $N \geq 2$: $\mu^2 < 0$, F minimised by $|\varphi| = M_0 \equiv \sqrt{-\mu^2/4g}$.

Direction of φ undetermined: φ can lie anywhere on S^{N-1}.

Example $(N=2, d=1)$

and



has same energy.

There are now configurations that minimises the "Potential energy" $\frac{1}{2}\mu^2|\varphi|^2 + g(\varphi^2)^2 + \dots$, i.e. $|\varphi| = M_0$, but where direction of φ varies in space, e.g.

These get energy only from deriv. terms, e.g. $(\nabla\varphi)^2$ in F. Can lower these energy by increasing length scale over which variations occur

These excitations, arising from SSB of a cts sym, are called Goldstone bosons. They are said to be gapless because energy $\to 0$ as wavelength $\to \infty$.

Choice of (const.) φ breaks $O(N) = G$ to $H = O(N-1)$

e.g. $\varphi = (M_0, \underbrace{0, \dots, 0}_{O(N-1) \text{ rotates these}})$.

The vacuum manifold is the space of ground states $- S^{N-1}$.

View $(\varphi_1, \dots, \varphi_N)$ as coords in $\mathbb{R}^N$. Change of coords in $\mathbb{R}^N \longleftrightarrow$ field redef^n $\varphi \to \varphi'(\varphi)$.

Choose e.g. polar coords $(\underset{|\varphi|}{\overset{\shortparallel}{M}}, G_1, \dots, G_{N-1})$ s.t.

$S^{N-1} = \{M = M_0\}$ is parameterised by $G_1, \dots, G_{N-1}$.

Then $(G_1(x), \dots, G_{N-1}(x))$ are the GS bosons. They describe fluctuations tangential to S^{N-1}.

$$\left(\begin{array}{l} \text{In general, vacuum manifold} = G/H \\ \# \text{ GBs} = \dim G - \dim H \end{array}\right).$$

A scaler on S^{N-1} invar. under G must be const.

$\Rightarrow \nexists$ term in F built from $(G_1, \dots, G_N)$ w/o deriv.

Example (XY-model)

$$f(\psi) := \frac{1}{2}\mu^2 |\psi|^2 + g|\psi|^4 ,$$

$\mu^2 < 0 \Rightarrow$ "Mexican hat" potential

Ground state: $M(x) = M_0 = \sqrt{-\mu^2/4g}$, $\theta = \theta_0$ const.

Let $\psi(x) = M(x)\, e^{i\theta(x)}$

Set of ground states is S^1, param. by θ_0)

Fluctuations: write $M(x) = M_0 + \tilde{M}(x)$, $\theta = \theta(x)$.

$$\Rightarrow F = \int d^dx \left(\frac{\gamma}{2}(\nabla\tilde{M})^2 + |\mu^2|\,\tilde{M}^2 + g\,\tilde{M}^4 + \frac{\gamma}{2}M_0^2(\nabla\theta)^2 + \gamma M_0 \tilde{M}(\nabla\theta)^2 + \dots \right)$$

θ is GB – has only deriv. interactions ($\because$ F invar. w.r.t. $G = U(1)$, sym $\theta \to \theta +$ const.)

$\underline{N=3}$: $\underline{\psi}(x) = M(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$, $M = M(x)$, $\theta = \theta(x)$, $\varphi = \varphi(x)$.

ground state: $M = M_0$, $\theta = \theta_0$, $\varphi = \varphi_0$.

fluctuations: $M = M_0 + \tilde{M}(x)$, $\theta = \theta(x)$, $\varphi = \varphi(x)$.

$$F = \int d^dx \left(\frac{\gamma}{2}(\nabla\tilde{M})^2 + |\mu^2|\,\tilde{M}^2 + g\,\tilde{M}^4 + \frac{\gamma}{2}M_0^2\left[(\nabla\theta)^2 + \sin^2\theta(\nabla\varphi)^2\right] + \dots \right)$$

c.f. metric on S^2
– invar under $O(3)$

$\theta(x)$, $\varphi(x)$ are the GBs. (interacting).

Crit. expts

Include $-\int d^dx\, B_a(x)\, \varphi_a(x)$ in F. For $B_a(x) = B\, n_a$, $|n_a|=1$,

let $\chi = \dfrac{\partial (n_a \varphi_a)}{\partial B}$,

$$\left.\begin{aligned} C &\sim |T-T_c|^{-\alpha} \\ |\varphi| &\sim |T-T_c|^{\beta} \\ \chi &\sim |T-T_c|^{-\gamma} \end{aligned}\right\} \text{ as } T \to T_c \text{ with } B_a = 0$$

$|\varphi| \sim |B|^{1/\delta}$ as $B_a \to 0$, $T = T_c$.

In MFT, $\alpha, \beta, \gamma, \delta$ has same value as for $N=1$.

Correlation functions : use quadratic approx.

$T > T_c$: Same as for $N=1$

$$Z = e^{-F_{thermo}} \exp\left[\frac{1}{2} \int d^dx\, d^dy\; B_a(x)\, G_0(x-y)\, B_a(y) \right]$$

where

$$G_0(x-y) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot(x-y)}}{\gamma k^2 + \mu^2}.$$

$$\Rightarrow \langle \varphi_a(x)\, \varphi_b(y) \rangle = \left.\frac{\delta^2 \log Z}{\delta B_a(x)\, \delta B_b(y)}\right|_{B=0} = \delta_{ab}\, G_0(x-y).$$

$T < T_c$: assume $\langle \varphi \rangle = (M_0, 0, \dots, 0)$ (WLOG). Define $\tilde{\varphi} = \varphi - \langle \varphi \rangle$. For small $\tilde{\varphi}$, $\tilde{\varphi}_a$, $a = 2, 3, \dots, N$ are the GBs.

$$\frac{1}{2}\mu^2 \varphi^2 + g(\varphi^2)^2 = \text{const.} + \frac{1}{2}\mu_1^2 \tilde{\varphi}_1^2$$

to quadratic order in $\tilde{\varphi}$. $\mu_1^2 = -2\mu^2$ as for $N=1$, but $\tilde{\varphi}_a^2$, $a = 2, \dots, N$ is absent.

To compute $\langle \tilde\varphi \tilde\varphi \rangle$ add $-\int d^dx\, \tilde B_a(x) \tilde\varphi_a(x)$.

$$Z = e^{-F_{\text{thermo}}} \exp\left[\frac{1}{2} \int d^dx\, d^dy \left(\tilde B_1(x) G_L(x-y) \tilde B_1(y) + \sum_{a=2}^{N} \tilde B_a(x) G_T(x-y) \tilde B_a(y) \right)\right]$$

where
$$G_L(x-y) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot(x-y)}}{\gamma k^2 + \mu_1^2}$$
$$G_T(x-y) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot(x-y)}}{\gamma k^2}$$

$$\Rightarrow \langle \tilde\varphi_a(x) \tilde\varphi_b(y) \rangle = \begin{cases} G_L(x-y) & a=b=1 \quad \text{"longitudinal fluc."} \\ G_T(x-y) & a=b>1 \quad \text{"transverse fluc."} \\ 0 & \text{o/w} \end{cases}$$

Correlation f^n for $T > T_c$ and corr. f^n of <u>longitudinal mode</u> for $T < T_c$ behave just like $N=1$:

$$G_{0/L}(x-y) \sim \begin{cases} e^{-r/\xi} & r \gg \xi \\ r^{-(d-2)} & r \ll \xi \end{cases}$$

where $\xi^2 \sim 1/\mu^2 \sim 1/\mu_1^2 \sim 1/|T-T_c|$

Define ν by $\xi \sim 1/|T-T_c|^\nu \Rightarrow \nu = 1/2$ is MFT prediction.

Define η by $\langle \varphi_a \varphi_b \rangle \underset{\text{large } r}{\sim} \delta_{ab}\, r^{-(d-2+\eta)}$ at $T=T_c \Rightarrow \eta = 0$ is MFT prediction.

For $T<T_c$, G_T has the same form as $G_{0/L}|_{T=T_c}$. So <u>Goldstone modes have ∞ correlation length</u>. i.e. in SSB of a cts sym, there are long range correlations in the broken phase (not just at CP).

$$G_T \sim r^{-(d-2)}$$

Can show this is exact $\forall\, T<T_c$. $\Rightarrow$ no anomalous dimensions for Goldstone modes.

The CP with diff. values of N describes diff. universality class. Will see $d_c = 4$ as for $N=1$. (ie. CP described by GFP for $d \geq 4$, non-trivial FP for $d < 4$).

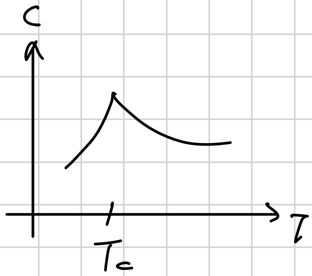
Numerical data for $d=3$:

	η	ν
MFT	0	1/2
$N=1$ (Ising)	0.0363	0.6300
$N=2$	0.0385	0.6719
$N=3$	0.0386	0.702

($\alpha, \beta, \gamma, \delta$ follow from scaling as for $N=1$).

$\alpha = 2 - 3\nu \Rightarrow \alpha < 0$ for $N \geq 2$.

$\Rightarrow c$ exhibit cusp (seen in He: $N=2$)



Mermin - Wagner Theorem

Note $\displaystyle\int_0^\Lambda \frac{d^dk}{k^2} \sim \int_0^\Lambda \frac{k^{d-1}dk}{k^2} = \infty$ if $d \leq 2$

$\Rightarrow G_T = \infty$ for $d \leq 2$.

For $d \leq 2$, MFT predicted phase transition, but quadratic fluctuations diverge in ordered phase ($N \geq 2$) $\Rightarrow$ MFT not reliable.

Thm (MW Thm) There is no ordered phase for $d \leq 2$. $\therefore$ SSB of a cts sym can't occur for $d \leq 2 \Rightarrow$ there are no GBs for $d \leq 2$.

∴ lower crit dim for O(N) model is

$$d_\ell = \begin{cases} 2 & N \geq 2 \quad \text{(cts sym)} \\ 1 & N = 1 \quad \text{(Ising. discrete sym.)} \end{cases}$$

RG for O(N):

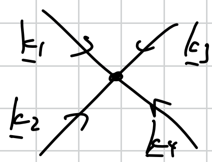
$$F[\varphi] = \int d^d x \left(\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 + g(\varphi^2)^2 + \dots \right) \quad (\text{scale at } \gamma = 1)$$

<u>GFP</u>: $\mu^2 = g = \dots = 0$. At GFP, $\Delta_{\varphi_a} = \frac{d-2}{2}$, $\Delta_1 = 2$, $\Delta_2 = 4 - d$ (same as $N=1$)

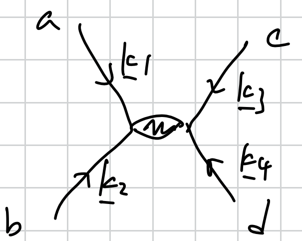
g $\qquad\qquad$ $\mu^2 + cg\Lambda^{d-2}$

Repeat what we did for $N=1$: Start at $g_0 = (\mu_0^2, g_0, 0, \dots)$ and use pert^n thy in g_0.

φ^4 vertex has more structure $(\varphi^2)^2 = \varphi_a \varphi_a \varphi_b \varphi_b$.



$N = 1$: $g_0 (2\pi)^d \delta^{(d)}(\Sigma \underline{k}_i)$

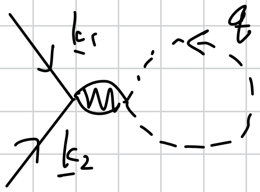


$N > 1$: $g_0 \delta_{ab} \delta_{cd} (2\pi)^d \delta^{(d)}(\Sigma \underline{k}_i)$.

$$\langle \varphi^+_{a\underline{k}} \varphi^+_{b\underline{k}'} \rangle_+ = (2\pi)^d \delta^{(d)}(\underline{k} - \underline{k}') G_0(\underline{k}) \delta_{ab}$$

⇒ Propagator $a \xrightarrow[\underline{q}]{} b$: $\delta_{ab} G_0(\underline{q})$.

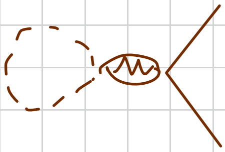
So at $O(g_0)$ we have



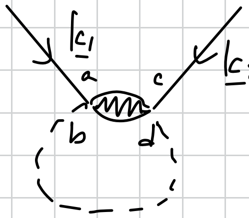
$$: 2\int_0^{\Lambda/\zeta}\left(\prod_{i=1}^{2}\frac{d^dk_i}{(2\pi)^d}\right)\varphi^-_{a\underline{k}_1}\varphi^-_{b\underline{k}_2}\int_{\Lambda/\zeta}^{\Lambda}\frac{d^dq}{(2\pi)^d}\,\delta_{cd}\,G_0(\underline{q})$$

$$\cdot\, g_0(2\pi)^d\,\delta_{ab}\,\delta_{cd}\,\delta^{(d)}(\underline{k}_1+\underline{k}_2+\cancel{\underline{q}-\underline{q}})$$

sym factor, also



$$= 2Ng_0\underbrace{\int_0^{\Lambda/\zeta}\frac{d^dk}{(2\pi)^d}\,\varphi^-_{a,\underline{k}}\,\varphi^-_{b,-\underline{k}}\int_{\Lambda/\zeta}^{\Lambda}\frac{d^dq}{(2\pi)^d}\,G_0(\underline{q})}_{I}.$$



$$: 4\int_0^{\Lambda/\zeta}\left(\prod_{i=1}^{2}\frac{d^dk_i}{(2\pi)^d}\right)\varphi^-_{a\underline{k}_1}\varphi^-_{b\underline{k}_2}\int_{\Lambda/\zeta}^{\Lambda}\frac{d^dq}{(2\pi)^d}\,\delta_{bd}\,G_0(\underline{q})$$

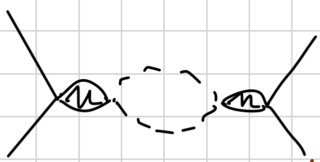
$$\cdot\, g_0(2\pi)^d\,\delta_{ab}\,\delta_{cd}\,\delta^{(d)}(\underline{k}_1+\underline{k}_2+\underline{q}-\underline{q})$$

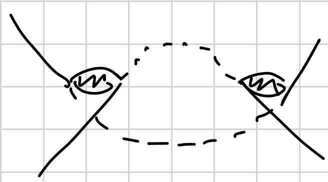
$$= 4g_0 I$$

After RG step 1, $\mu'^2 = \mu_0^2 + 4(N+2)g_0\int_{\Lambda/\zeta}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+\mu^2}$.

<u>Note</u> factor of N arises on 1st diagram because label a on φ_a^+ isn't determined by external lines.

Sim. at $O(g_0^2)$





factor of N ← contracting $\delta\cdot\delta$

dotted lines from L/R on 1st vertex

connecting dotted lines

Sym. factor $2\times2\times2 = 8$ $\quad$ $2\times4\times2\times2 = 32$ $\quad$ $4\times4\times2 = 32$

1 dotted into L, 1 into R on 2nd vertex

swap vertices

$$\Rightarrow \quad g_0' = g_0 - 4(N+8)\, g_0^2 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+\mu^2)^2}$$

note: $(-\frac{1}{2})$.

β- f^ns can be read off from $N=1$ results by adjusting coeffs

$$d-4=\varepsilon, \quad \tilde{g} = \Lambda^{-\varepsilon} g \Rightarrow \begin{cases} \dfrac{d\mu^2}{ds} = 2\mu^2 + \dfrac{N+2}{2\pi^2} \dfrac{\Lambda^4}{\Lambda^2+\mu^2} \tilde{g} + O(\tilde{g}^2) \\ \dfrac{d\tilde{g}}{ds} = \varepsilon \tilde{g} - \dfrac{N+8}{2\pi^2} \dfrac{\Lambda^4}{(\Lambda^2+\mu^2)^2} \tilde{g}^2 + O(\tilde{g}^3) \end{cases}$$

linearise around GFP $\Rightarrow \Delta_1 = 2, \ \Delta_2 = \varepsilon$

$\Rightarrow$ 1 relevant direction if $\varepsilon < 0$ $(d>4)$

WF FP:
$$\mu_*^2 = -\frac{1}{2} \frac{N+2}{N+8} \Lambda^2 \varepsilon + O(\varepsilon^2)$$
$$\tilde{g}_* = \frac{2\pi^2}{N+8} \varepsilon + O(\varepsilon^2)$$

Linearise around FP $\Rightarrow$
$$\Delta_t = \Delta_1 = 2 - \frac{N+2}{N+8}\varepsilon + O(\varepsilon^2)$$
$$\Delta_g = \Delta_2 = -\varepsilon + O(\varepsilon^2)$$

$\Rightarrow$ 1 relevant direction for $\varepsilon > 0$ $(d<4) \Rightarrow d_c = 4$.

and $\eta = O(\varepsilon^2)$ as before. (2 loops: $\eta = \dfrac{N+2}{2(N+8)} \varepsilon^2$)

Scaling:
$$\nu = \frac{1}{\Delta_t} = \frac{1}{2} + \frac{N+2}{4(N+8)}\varepsilon + O(\varepsilon^2)$$
$$\alpha = \frac{4-N}{2(N+8)}\varepsilon$$

(predicts incorrect sign for α for $N=2,3$)

$$\beta = \frac{1}{2} - \frac{3}{2(N+8)}\varepsilon$$
$$\gamma = 1 + \frac{N+2}{2(N+8)}\varepsilon$$
$$\delta = 3 + \varepsilon \quad \text{(indpt. of } N\text{)}$$