Quantum Field Theory

0. Introduction

Why QFT? Construct laws of nature that are compatible with locality

Goal: combine SR in QM.

lessons:
- Fields are central
- # particles not preserved.

Principles: QFT governed by locality, symmetries, and renormalisation.

Conventions:
- $\hbar = c = 1$ $\quad [L = T = M^{-1}]$
- measure things in energy (or mass) → eV.
- $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$

1. Classical Field Theory

Classical mech: action

$$S(t_1, t_2) = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m \dot{\underline{x}}^2 - V(\underline{x}) \right).$$

3 facts:

1) Classical e.o.m. obtained by extremise S
2) B.C. are supplied extremally
3) S is built based on symmetries.

In field theory.

- place space and time on the same footing
- interested in a local object, i.e. "field".

Example Scaler field

$$\phi(x^i, t) : \mathbb{R}^{3,1} \to \mathbb{R}.$$

Notation $\phi(x^i, t) = \phi(x^\mu) = \phi(x).$

In general,

$$\underbrace{\phi_a}_{\text{spaces of field}}(x^\mu) : \mathbb{R}^{3,1} \to \mathbb{C}, \mathbb{R}^{3,1}, \dots$$

Example EM potential

$$A^\mu = (\phi(x), \underline{A}(x)) \quad : \text{gauge field.}$$

$$\underline{E} = -\nabla\phi - \frac{\partial \underline{A}}{\partial t} \quad , \quad \underline{B} = \nabla \times \underline{A}.$$

Goal: describe action principles for fields

Lagrangian

Lagrangian density

$$L = T - V \qquad\qquad L = \int d^3x \, \mathcal{L}(x).$$

$$S = \int dt \, L \quad \longrightarrow \quad S = \int dt \int d^3x \, \mathcal{L} = \int d^4x \, \mathcal{L}(x).$$

$\mathcal{L}(x)$ is a f^n of the fields and its derivatives

Consider

$$\mathcal{L} = \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x), \partial^2\phi, \dots)$$

For simplicity, consider

$$\mathcal{L} = \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)).$$

To get e.o.m. :

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right)$$

$$= \int d^4x \left[\delta\phi_a \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right) + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right)}_{\text{Tot. deriv.}} \right]$$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0 .}$$

Example Simplest action, massive scaler field

$$\mathcal{L} = \overbrace{\frac{1}{2}\eta^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi}^{\text{Kinetic}} - \overbrace{\frac{1}{2} m^2 \phi^2}^{\text{mass-term}} \qquad (m^2 > 0 \ \text{const.})$$

$$= \underbrace{\frac{1}{2}(\partial_t \phi)^2}_{\text{Kinetic}} - \underbrace{\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2} m^2 \phi^2}_{\text{Potential}}$$

For eom, use E-L

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

$$\Rightarrow \boxed{m^2 \phi + \partial_\mu \partial^\mu \phi = 0} \qquad (\text{Klein - Gordan eqn})$$

$$\partial_\mu \partial^\mu = \Box \Rightarrow \Box\phi + m^2\phi = 0$$

Example Maxwell theory

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

e.o.m. :
$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = 0}$$

$$\nabla\cdot\underline{E} = 0 \qquad E^i = F^{0i}$$

$$\Rightarrow \quad \nabla\times\underline{B} = \frac{\partial E}{\partial t} \qquad \varepsilon^{ijk}B^k = F^{ij}.$$

Hamiltonian

link Lagrangian and Hamiltonian formalism.

In the Hamiltonian formalism, we introduce the concept of canonical momenta

$$\Pi^a(x) = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi_a)}$$

Then, Hamiltonian density is

$$\mathcal{H} = \Pi^a\partial_t\phi_a - \mathcal{L}$$

We favour Π^a over $\partial_t\phi_a$. The Hamiltonian is

$$H = \int d^3x\ \mathcal{H}.$$

Example Massive scaler field

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi \qquad (\dot\phi = \partial_t\phi)$$

$$H = \int d^3x\ (\Pi\dot\phi - \mathcal{L}) = \int d^3x\left(\overbrace{\underbrace{\frac{1}{2}\Pi^2}_{\text{KE}} + \frac{1}{2}(\nabla\phi)^2}^{\text{kanetic}}\underbrace{+\overbrace{m^2\phi^2}^{\text{mass term}}}_{\text{PE.}}\right)$$

Symmetries

Symmetries are important

- dictate actions
- dictate fields (operators)
- dictate constrain observables.

Examples
- $U(1)$ symmetries + $SU(2)$
- Spacetime translations
- Lorentz (boost)
- Rotations
- Discrete symmetries
- Conformal
- Supersymmetry

$$\left.\begin{array}{l}\text{Spacetime translations}\\ \text{Lorentz (boost)}\\ \text{Rotations}\end{array}\right\}\text{Poincaré}$$

Lorentz Invariance

Most important one in QFT.

Defⁿ The group of linear and homogeneous coord transⁿ

$$x^\mu \longmapsto \Lambda^\mu{}_\nu x^\nu$$

which preserves the interval

$$s^2 = x^\mu x^\nu \eta_{\mu\nu}$$

$$s^2 \mapsto s'^2 = s^2$$

This implies

$$\underbrace{\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}}_{}\qquad (\eta = \Lambda^T\eta\Lambda).$$

↑ Defining properties of Λ

Examples

1. Rotation: $t = t'$, $\Lambda^i{}_j = R^i{}_j$ (orth. matrix)

Rot on (x,y) plane

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Boost : mix time and space . e.g. boost on x-dir

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where η is rapidity,

$$\cosh\eta = \frac{1}{\sqrt{1-v^2}} \, , \quad \sinh\eta = \frac{v}{\sqrt{1-v^2}} \, .$$

More generally,

$$\eta = \Lambda^T \eta \Lambda \qquad (*)$$

Infer prop. about Λ.

First take det of (*)

$$\Rightarrow \boxed{(\det\Lambda)^2 = 1}$$

e.g. boost + rot^n

$$\Rightarrow \det\Lambda = \begin{cases} 1 & \text{proper Lor. Trans}^n \\ -1 & \text{improper} \quad " \end{cases}$$

e.g. parity, time reversal.

For proper transf^n, we can write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\eta}{}_{\nu} + \epsilon^{\mu}{}_{\nu} + O(\epsilon^2) \qquad (**)$$

Replace (**) in (*)

$$\begin{aligned} \Rightarrow \eta_{\rho\sigma} &= \eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} \\ &= \eta_{\mu\nu} \left(\delta^{\mu}{}_{\rho} + \epsilon^{\mu}{}_{\rho} + \dots \right) \left(\delta^{\nu}{}_{\sigma} + \epsilon^{\nu}{}_{\sigma} + \dots \right) \\ &= \eta_{\rho\sigma} + \epsilon_{\rho\sigma} + \epsilon_{\sigma\rho} + O(\epsilon^2) \end{aligned}$$

$$\Rightarrow \boxed{\epsilon_{\rho\sigma} = -\epsilon_{\sigma\rho}}$$

infinitetestimal param is anti-sym ⇒ 6 indep choices of $\epsilon_{\sigma\rho}$

3 boost. / 3 rot^n

Fields revisited

$\phi_a(x)$ has a definite trans^n rule under Lorentz.

Under Lorentz,

$$\phi_a(x) \longmapsto \phi'_a(x) = D[\Lambda]^b{}_a \, \phi_b(\Lambda^{-1}x)$$

where $D[\Lambda]^b{}_a$ forms a representation of the Lorentz group.

$$D[\Lambda_1]\, D[\Lambda_2] = D[\Lambda_1 \Lambda_2]$$

$$D[\Lambda^{-1}] = D[\Lambda]^{-1}$$

$$D[\underset{\substack{\uparrow \\ \text{identity}}}{\mathbb{1}}] = 1$$

Example Trivial rep $D[\Lambda] = 1$

⇒ Scaler field: $\phi(x) \longmapsto \phi'(x) = \phi(\Lambda^{-1}x)$

Example Vector representation

$$D[\Lambda]^\mu{}_\nu = \Lambda^\mu{}_\nu$$

$$A^\mu(x) \longmapsto A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

Covector :

$$\partial_\mu \phi \longmapsto \partial_\mu \phi' = (\Lambda^{-1})^\nu{}_\mu \, \partial_\nu \phi .$$

Actions revisited

Take mass scaler

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial_\nu \phi \, \eta^{\mu\nu} - \frac{1}{2} m^2 \phi^2$$

$$S = \int d^4x \, \mathcal{L}(x).$$

What's the effect of a Lorentz trans^n?

$y = \Lambda^{-1} x$

$$\phi(x) \longmapsto \phi'(x) = \phi(\Lambda^{-1} x) = \phi(y)$$

$$\partial_\mu \phi \mapsto (\Lambda^{-1})^\nu{}_\mu \, \partial_\nu \phi(\Lambda^{-1} x) = (\Lambda^{-1})^\nu{}_\mu \, \partial_\nu \phi(y)$$

$$\eta^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \longmapsto (\Lambda^{-1})^\rho{}_\mu \, \partial_\rho \phi(y) \, (\Lambda^{-1})^\sigma{}_\nu \, \partial_\sigma \phi(y) \, \eta^{\mu\nu}$$

$$= \partial_\rho \phi(y) \, \partial_\sigma \phi(y) \, \eta^{\rho\sigma}$$

$$\Rightarrow \quad \mathcal{L}(x) \longmapsto \mathcal{L}'(x) = \mathcal{L}(y).,$$

$$S = \int d^4x \, \mathcal{L}(x) \longmapsto \int d^4x \, \mathcal{L}(y)$$

$$= \int d^4y \, \mathcal{L}(y) = S$$

Jacobian $\det(\Lambda) = 1$

Guidance to write possible $\mathcal{L}$ come from symmetries.
In particular, Lorentz invar., and local.

Noether's theorem.

Thm (Noether's) 1) Every cts sym. of the action gives rise to a conserved current j^μ, i.e. $\partial_\mu j^\mu = 0$, or

$$\frac{d}{dt} j^0 + \nabla \cdot \underline{J} = 0$$

when e.o.m. are used.

2) A conserved current implies that $\exists$ a conserved charge Q.$^{(*)}$ where

$$Q = \int d^3x \, j^0 .$$

(*) : provided suitable boundary conditions.

Def^n A transf^n is cts if $\exists$ a param. that can be taken to be small. A continuous transformation is

$$\delta\phi_a(x) := \phi_a'(x) - \phi_a(x)$$

Note: coords are fixed.

When this transf^n is a sym of action $S[\phi]$,

$$\delta S = S[\phi'] - S[\phi] = 0$$

Implication for $\mathcal{L}$ is

$$\delta\mathcal{L} = \mathcal{L}'(x) - \mathcal{L}(x) = \partial_\mu F^\mu .$$

can transform up to a total derivative.

Pf: 1) cts sym $\begin{cases} \text{internal sym : act on fields, not coords} \\ \text{local sym.: act both on fields and coords.} \end{cases}$

Take $\delta\phi_a(x) = \phi_a'(x) - \phi_a(x)$. Now,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)$$

$$= \left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right)\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)$$

$$\overset{\text{sym.}}{=} \partial_\mu F^\mu .$$

$$\Rightarrow \quad \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right)}_{\text{e.o.m.} = 0}\delta\phi_a - \partial_\mu\left(\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - F^\mu}_{j^\mu}\right) = 0 .$$

Impose e.o.m., we have $\partial_\mu j^\mu = 0$, with

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a - F^\mu .$$

2) Given $Q = \int d^3x \; j^0$ with $\partial_\mu j^\mu = 0$,

Conservation ⇒ $dQ/dt = 0$.

$$\frac{dQ}{dt} = \int_V d^3x \, \frac{\partial j^0}{\partial t} = \int_V d^3x \; -\nabla \cdot \underline{J}$$

$$= -\int_{\partial V} \underline{J} \cdot d\underline{S} = 0 .$$

Usually $V = \mathbb{R}^3$. Natural assumption $j \xrightarrow{|x|\to\infty} 0$. (*). □

Energy-momentum Tensor

Implement Noether's thm for translations

$$x^\mu \to x^\mu - \epsilon^\mu .$$

with ϵ^μ const. vec.

This is a local sym. Under translations, field transform

$$\phi_a \to \phi_a'(x) = \phi_a(x+\epsilon)$$

$$= \phi_a x + \epsilon^\mu \partial_\mu \phi_a(x) + \dots$$

So

$$\boxed{\delta\phi_a = \phi_a'(x) - \phi_a(x) = \epsilon^\mu \partial_\mu \phi_a}$$

The Lagrangian then transform as

$$\delta\mathcal{L} = \mathcal{L}'(x) - \mathcal{L}(x) = \epsilon^\mu \partial_\mu \mathcal{L} = \partial_\mu \Big(\underbrace{\epsilon^\mu \mathcal{L}}_{F^\mu .} \Big) .$$

The conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a - F^\mu .$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \epsilon^\nu \partial_\nu \phi_a - \epsilon^\mu \mathcal{L}$$

$$= \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu{}_\nu \mathcal{L} \right) \epsilon^\nu \equiv T^\mu{}_\nu \epsilon^\nu$$

where $T^\mu{}_\nu$ is the energy-momentum tensor.

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu{}_\nu \mathcal{L}$$

where

$$\partial_\mu T^\mu{}_\nu = 0$$

since $\partial_\mu j^\mu = 0$.

From here, we can construct 4 conserved charges.

- Time translation : $\epsilon^\mu = (1, 0, 0, 0)$

$$\Rightarrow \text{Energy} \quad E = \int d^3x \; T^{00}$$

- Spatial translation

$$\Rightarrow \text{Momentum} \quad p^i = \int d^3x \; T^{0i} .$$

Example Massive scaler field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L} = \eta^{\alpha\mu} \partial_\alpha \phi \, \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L}$$

$$\Rightarrow T^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

Energy : $E = \int d^3x \; T^{00} = \int d^3x \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right)$

$\uparrow$ Hamiltonian

Momentum : $P^i = \int d^3x \; T^{0i} = \int d^3x \; \dot{\phi} \, \partial^i \phi .$

$$= \int d^3x \; \Pi \, \partial^i \phi .$$

Note : The energy-mom tensor with this def^n is not always symmetric $T^{\mu\nu} \Leftrightarrow T^{\nu\mu}$. For this example, we do have $T^{\mu\nu} = T^{\nu\mu}$. One way to make $T^{\mu\nu}$ symmetric is, for example, to add improvement term

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho J^{\rho\mu\nu} \Rightarrow \partial_\mu \Theta^{\mu\nu} = 0$$

s.t. $J^{\rho\mu\nu} = - J^{\mu\rho\nu}$, $\partial_\mu \partial_\rho J^{\rho\mu\nu} = 0.$

Why is it nice to have $T^{\mu\nu} = T^{\nu\mu}$? It connects to GR

$$\Theta^{\mu\nu} = \left(- \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g^{\mu\nu}} \left(\sqrt{-g} \, \mathcal{L} \right) \right) \Bigg|_{g_{\mu\nu} = \eta_{\mu\nu}} .$$

Internal Symmetry

Example is a complex scaler field.

$$\psi(x) = \phi_1(x) + i \phi_2(x)$$

with ϕ_1, ϕ_2 real scaler field. A simple Lagrangian (local, Lorentz invar)

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - V(|\psi|^2) \qquad (|\psi|^2 = \psi^* \psi).$$

E.O.M.

$$\partial_\mu \partial^\mu \psi + \frac{\partial V}{\partial \psi^*} = 0$$

$$\partial_\mu \partial^\mu \psi^* + \frac{\partial V}{\partial \psi} = 0$$

Internal symmetry in this example is

$$\psi(x) \to \psi'(x) = e^{i\alpha} \psi(x)$$

with $\alpha \in \mathbb{R}$. const.

$$\psi'(x) = e^{i\alpha}\psi(x)$$
$$= (1 + i\alpha + \mathcal{O}(\alpha^2))\,\psi(x)$$
$$\Rightarrow \delta\psi = \psi'(x) - \psi(x) = i\alpha\,\psi(x)$$
$$\delta\psi^* = -i\alpha\,\psi^*$$

Lagrangian is clearly invar. under this transf^n.

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} \qquad (F^\mu = 0).$$
$$\delta\mathcal{L} = 0.$$

Current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\delta\psi^*.$$
$$= \partial^\mu\psi^* \cdot i\alpha\psi + \partial^\mu\psi(-i\alpha\psi^*)$$
$$= i\alpha\left((\partial^\mu\psi^*)\psi - (\partial^\mu\psi)\psi^*\right).$$
$$\overset{\alpha=1}{=} i\left(\psi\,\partial^\mu\psi^* - \psi^*\,\partial^\mu\psi\right).$$

2. Quantum Fields - Free theories

In QM, the basic recipe is

$$[x^i, p^j] = i\hbar\,\delta^{ij}.$$

In QFT, $\phi_a(x)$, $\Pi^a(x)$. are the variables, and the rule

$$[\underbrace{\phi_a(\underline{x},t),\ \Pi^b(\underline{x}',t)}_{\text{equal time}}] = i\,\delta^{(3)}(\underline{x}-\underline{x}')\,\delta_a{}^b.$$

2.1 Canonical Quantisation

Note : In DT notes, Schrödinger picture $(t=0)$

Here, Heisenburg picture $(t\neq0)$

The theory is

$$\mathcal{L} = \frac{1}{2}\eta_{\mu\nu}\,\partial^{\mu}\phi\,\partial^{\nu}\phi - \frac{1}{2}m^2\phi^2 \qquad \{ \text{ Free theory}$$

E.o.m. :

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

One sol^n to the e.o.m. :

$$\phi(x) \sim \exp(i\underline{k}\cdot\underline{x} + i\omega t).$$

where

$$-\omega^2 + \underline{k}^2 + m^2 = 0$$

$$\Rightarrow \omega_k = \pm\sqrt{\underline{k}^2 + m^2}.$$

Notation : $\omega_k \equiv +\sqrt{\underline{k}^2 + m^2} > 0$.

Then the sol^n is

$$\phi(x) = \frac{1}{(2\pi)^3}\int d^3k \left[a(\underline{k})\, e^{i\underline{k}\cdot\underline{x} - i\omega_k t} + b(\underline{k})\, e^{i\underline{k}\cdot\underline{x} + i\omega_k t} \right]$$

arbitrary functions

Note : • $\phi(x) = \phi^*(x)$

$$\phi^*(x) = \frac{1}{(2\pi)^3}\int d^3k \left[a^*(\underline{k})\, e^{-i\underline{k}\cdot\underline{x} + i\omega_k t} + b^*(\underline{k})\, e^{i\underline{k}\cdot\underline{x} - i\omega_k t} \right]$$

Flip $\underline{k} \to -\underline{k}$

$$\Rightarrow \quad a^*(-\underline{k}) = b(\underline{k}), \qquad b^*(-\underline{k}) = a(\underline{k})$$

Then

$$\phi(\underline{x}, t) = \int \frac{d^3k}{(2\pi)^3} \left[a(\underline{k})\, e^{i\underline{k}\cdot\underline{x} - i\omega t} + a^*(\underline{k})\, e^{-i\underline{k}\cdot\underline{x} + i\omega t} \right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \left[a(\underline{k})\, e^{-ik\cdot x} + a^*(\underline{k})\, e^{ik\cdot x} \right]$$

where $k\cdot x = k^\mu x_\mu = \omega t - \underline{k}\cdot\underline{x}$.

- Normalisation

$$\phi(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3} \underbrace{\frac{1}{\sqrt{2\omega}}}_{\text{just a choice of normalisation}} \left[a(\underline{k})\, e^{-ik\cdot x} + a^*(\underline{k})\, e^{ik\cdot x} \right]$$

Next, quantise field. We declare

$$\left.\begin{aligned} &[\phi(\underline{x},t), \phi(\underline{x}',t)] = 0 \\ &[\Pi(\underline{x},t), \Pi(\underline{x}',t)] = 0 \\ &[\phi(\underline{x},t), \Pi(\underline{x}',t)] = i\,\delta^{(3)}(\underline{x}-\underline{x}') . \end{aligned}\right\} (*)$$

Where

$$\Pi(x) = \dot{\phi}(x), = \int \frac{d^3k}{(2\pi)^3} \frac{1}{i} \sqrt{\frac{\omega}{2}} \left(a(\underline{k})\, e^{-ik\cdot x} - a^*(\underline{k})\, e^{ik\cdot x} \right) .$$

Claim: A consequence of $(*)$ is $a(\underline{k})$ promoted to be an operator. $a^*(\underline{k}) \longrightarrow a^{\dagger}(\underline{k})$, and

$$\left.\begin{aligned} &[a(\underline{k}), a(\underline{k}')] = 0 \\ &[a^{\dagger}(\underline{k}), a^{\dagger}(\underline{k}')] = 0 \\ &[a(\underline{k}), a^{\dagger}(\underline{k}')] = (2\pi)^3\, \delta(\underline{k}-\underline{k}') . \end{aligned}\right\} (\star)$$

Pf: Two ways

- Write $a, a^{\dagger}$ in terms of Π, ϕ : $(*) \Rightarrow (\star)$
- Use $(\star)$ and check $(*)$.

$$[\phi(\underline{x},t), \Pi(\underline{x}',t)]$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2i} \sqrt{\frac{\omega_p}{\omega_q}} \Big[a(p)\, e^{i\underline{p}\cdot\underline{x} - i\omega_p t} + a^{\dagger}(p)\, e^{-i\underline{p}\cdot\underline{x} + i\omega_p t},$$
$$a(q)\, e^{i\underline{q}\cdot\underline{x} - i\omega_q t} + a^{\dagger}(q)\, e^{-i\underline{q}\cdot\underline{x} + i\omega_q t} \Big]$$

$$\overset{(\star)}{=} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2i} \sqrt{\frac{\omega_p}{\omega_q}} \Big(-[a(p), a^\dagger(q)]\, e^{i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}}\, e^{it(\omega_q - \omega_p)}$$

$$+ [a^\dagger(p), a(q)]\, e^{-i\underline{p}\cdot\underline{x} + i\underline{q}\cdot\underline{y}}\, e^{it(\omega_p - \omega_q)} \Big).$$

Note $[a(p), a^\dagger(q)] = (2\pi)^3 \delta^{(3)}(\underline{p} - \underline{q})$, so $\omega_p = \omega_q$.

$$\circeq i \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x - x')} = i\, \delta^{(3)}(\underline{x} - \underline{x}')$$

Check also $[\phi(\underline{x}, t), \phi(\underline{x}', t)] = 0$

$[\Pi(\underline{x}, t), \Pi(\underline{x}', t)] = 0$. $\square$

2.2 Hamiltonian

What are the consequences of quantisation of it?

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right).$$

Step 1: Write in terms of a, $a^\dagger$.

write $a_p = a(\underline{p})$.

$$\Rightarrow H = \frac{1}{2} \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6} \Big(-\frac{\sqrt{\omega_p\omega_q}}{2} (a_p e^{-ipx} - a_p^\dagger e^{ipx})(a_q e^{-iqx} - a_q^\dagger e^{iqx})$$

$$- \frac{1}{2}\frac{1}{\sqrt{\omega_p\omega_q}}\, \underline{p}\cdot\underline{q}\, (\ldots) + \frac{m^2}{2\sqrt{\omega_p\omega_q}} (\ldots) \Big).$$

$$\int d^3x\, e^{\pm i(\underline{x}\cdot(\underline{p} \pm \underline{q})}$$

$$\shortparallel$$

$$(2\pi)^2\, \delta^{(3)}(\underline{p} - \underline{q}).$$

$$\Rightarrow H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega} \Big[\underbrace{(-\omega^2 + \underline{p}^2 + m^2)}_{=0} (a_p a_{-p} e^{-2i\omega t} + a_p^\dagger a_{-p}^\dagger e^{2i\omega t})$$

$$+ (\omega^2 + \underline{p}^2 + m^2)(a_p^\dagger a_p + a_p a_p^\dagger) \Big]$$

↑ did not move left/right a's, a†'s (*)!

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3}\, \omega\, (a_p^\dagger a_p + a_p a_p^\dagger)$$

$$= \int \frac{d^3p}{(2\pi)^3} \Big(\underbrace{\omega a_p^\dagger a_p}_{H \text{ in QM for SHO.}} + \underbrace{\frac{\omega}{2} (2\pi)^3 \delta^{(3)}(\underline{0})}_{!!} \Big)$$

use $[a(\underline{p}), a^\dagger(\underline{p}')] = (2\pi)^3 \delta^{(3)}(\underline{p} - \underline{p}')$

To understand the second term, let's introduce the vacuum state $|0\rangle$. We define $|0\rangle$ as

$$a(\underline{p})\,|0\rangle = 0 \qquad \forall\ \underline{p}$$

$$a(\underline{p}) : \text{destruction} \quad , \quad a^{\dagger}(\underline{p}) : \text{creation}$$

Then

$$H|0\rangle = E_0\,|0\rangle = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\omega(2\pi)^3\,\delta(0)\,|0\rangle = \infty\,|0\rangle$$

Infinite vacuum energy!

Nature of divergence: two divergences

1) Infrared divergence due to $\delta^{(3)}(0)$.

$$(2\pi)^3\,\delta^{(3)}(\underline{0}) = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x\; e^{i\underline{x}\cdot\underline{p}}\Big|_{\underline{p}=\underline{0}}.$$

$$= \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x$$

$$= \lim_{L\to\infty} \mathrm{Vol}(L).$$

A sol^n:

$$\mathcal{E}_0 = \frac{E_0}{V} \quad \text{energy density}.$$

2) Ultraviolet divergence

$$\mathcal{E}_0 = \frac{E_0}{V} = \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\,\omega_{\underline{p}}.$$

cutoff

$$= \frac{1}{2}\int^{p_{max}}\frac{d^3p}{(2\pi)^3}\,\sqrt{\underline{p}^2+m^2}.$$

$$\xrightarrow[p_{max}\to\infty]{} \infty$$

We are assuming here that the theory is valid for all ranges of frequencies / energies. This is absurd, hence set a maximum (cutoff).

Punchline: practical approach to it. choose

$$H|0\rangle = 0$$,

selecting zero-point energy.

The approach of ∞'s is due to ambiguities in ordering of operators (e.g. see (*))

To fix the ambiguity of multiplying fields, introduce a rule: normal ordering

Def^n (normal ordering) If we have a string of operators $\phi_1(x_1)\dots\phi_n(x_n)$, the quantity

$$:\phi_1(x_1)\dots\phi_n(x_n):$$

is the usual product with all $a(p)$ operators placed to the right of $a^\dagger(p')$

Example $H = :\int d^3x\, \mathcal{H}: = \int \frac{d^3p}{(2\pi)^3} \omega a_p^\dagger a_p$

$\Rightarrow$ First destroy, then create.

$$\Rightarrow \quad H|0\rangle = 0|0\rangle.$$

2.3 Fock space

All states (vacuum + excited)

$$[H, a_p^\dagger] = \omega_p a_p$$

$$[H, a_p] = -\omega_p a_p^\dagger .$$

This tells us that we can construct energy eigenstate by acting with $a^\dagger$ on $|0\rangle$.

Consider

$$|\underline{p}\rangle = a^\dagger(\underline{p})\,|0\rangle .$$

single particle state of mass m, relativistic dispersion $E^2 = \omega_p^2 = \underline{p}^2 + m^2$.

satisfies

$$H|\underline{p}\rangle = \underbrace{\omega_p}_{\text{energy}}\,|\underline{p}\rangle$$

For $\underline{P} = -:\int d^3x\ \Pi\,\nabla\phi: = \int \frac{d^3p}{(2\pi)^3}\,\underline{p}\,a_p^\dagger a_p$., then

$$\underline{P}\,| \ = \underline{p}\,|\underline{p}\rangle \quad \text{momentum eigenstate}$$

Multiparticle state :

$$|p_1, \dots, p_n\rangle = a^\dagger_{p_1} \dots a^\dagger_{p_n}|0\rangle .$$

Note Because $a^\dagger$'s commute, the state is sym under exchanges of p_i's , e.g. $|p_1, p_2\rangle = |p_2, p_1\rangle$

Fock space (Hilbert space) is all possible combinations of $a^\dagger$s acting on vacuum.

$$|0\rangle, \quad a^\dagger_p|0\rangle, \quad a^\dagger_p a^\dagger_q |0\rangle, \dots$$

Useful to introduce the number operator

$$N \equiv \int \frac{d^3p}{(2\pi)^3}\, a_p^\dagger a_p .$$

satisfies

$$N|p_1, \dots, p_n\rangle = n\,|p_1, \dots, p_n\rangle$$

Then Fock space $= \oplus\, \mathcal{H}_n$: sum of the n-particle Hilbert space

Notice that $[H, N] = 0 \Rightarrow$ # particles conserved — this is a special feature of free theory. (will not be true in interacting theories).

Relativistic Normalisation

We can choose $\langle 0|0\rangle = 1$. With this choice, then

$$\langle \mathbf{p}, \mathbf{q}\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$$

$$\big\downarrow$$

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$$

But this is not Lorentz invar.

We would like the states in a Lorentz invar. theory transforms

$$|\mathbf{p}\rangle \longrightarrow |\mathbf{p}'\rangle = \underset{\uparrow}{U(\Lambda)}\, |\mathbf{p}\rangle$$

Definite transformation in reps of unitary transf^n.

To fix this, a shortcut would be consider

$$|\mathbf{q}\rangle = \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}\rangle \langle \mathbf{p}|\mathbf{q}\rangle.$$

$$\Rightarrow \quad 1 = \underbrace{\int \frac{d^3p}{(2\pi)^3}}_{(*)} \underbrace{|\mathbf{p}\rangle\langle \mathbf{p}|}$$

not Lorentz invar

Strategy is to make $(*)$ Lorentz invar. This will tell us how to normalise states.

Consider

$$\underbrace{\int d^4p}_{LI} \; \underbrace{\delta(p^2 - m^2)}_{LI} \; \underbrace{\overset{\text{step } f^n}{\Theta(p^0)}}_{LI}$$

$$= \int d^4p \; \delta\big((p^0)^2 - \mathbf{p}^2 - m^2\big) \, \Theta(p^0).$$

$$\downarrow \quad p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$$

$$= \int d^3p \, \frac{1}{2\sqrt{\mathbf{p}^2 + m^2}} = \int \frac{d^3p}{2\omega}.$$

Sol^n :

$$||p\rangle = \sqrt{2\omega}\, a_p^\dagger\ |0\rangle$$

Notation : write $||p\rangle = |p\rangle$

Note : In other ref.

$$\phi = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left(\tilde{a}(k) e^{-ikx} + \dots \right)$$

2.4 Causality

Canonical quantisation seems to not respect Lorentz invar.

One reason is

$$[\phi(\underline{x},t), \Pi(\underline{x},t)] = i\,\delta^{(3)}(\underline{x}-\underline{y})$$

Q: What happens at arbitrary spacetime separations?

In a relativistic, we would like for spacelike separated events to <u>not</u> influence each other $\Rightarrow$ causality.

Consider

$$\begin{aligned} \Delta(x-y) &:= [\phi(x), \phi(y)] \\ &= \phi(x)\phi(y) - \phi(y)\phi(x) \,. \end{aligned}$$

like a "field measurement"

Properties:

1) Evaluate for free theory

$$\Delta(X-y) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{\omega_k\omega_p}} \left([a_k, a^\dagger_p]\, e^{-ikx} e^{ipy} + [a^\dagger_k, a_p]\, e^{ikx} e^{-ipy} \right)$$

$$= \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p}}_{LI} \underbrace{\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right)}_{LI}$$

classical number

$\Rightarrow \Delta(X-y)$ is Lor. inv. for free theory a c-number

2) Timelike separated points

Take $(x-y)_T = (t,0,0,0)$. then

$$\Delta(x-y)_T = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{-i\omega_p t} - e^{i\omega_p t}\right) \sim e^{-imt} - e^{imt} \neq 0.$$

3) Spacelike separated points

Take $(x-y) = (0, \underline{x}-y)$

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{i\underline{p}\cdot(\underline{x}-\underline{y})} - \underbrace{e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}}_{\underline{p}\mapsto-\underline{p}}\right) = 0.$$

⇒ Spacelike separated don't influence each other.

Lesson: $\Delta(x-y)$ vanishes outside lightcone.

2.5 Propagators

Q: prepare a particle at a point y. What is prob. of measuring it at x ?

Look at

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip\cdot(x-y)} =: D(x-y)$$

with $D(x-y)$ the propagator.

For spacelike separation $(x-y)^2<0$,

$$D(x-y)\sim e^{-m|\underline{x}-\underline{y}|}\neq 0.$$

still

$$[\phi(x),\phi(y)] = D(x-y) - D(y-x) = 0$$

if $(x-y)^2<0$.

Feynman Propagator

Most important quantities in (pert.) QFT.

$$\Delta_F(x-y) := \langle 0 | \overset{\text{time ordering}}{T} \phi(x) \phi(y) | 0 \rangle .$$

$$= \begin{cases} D(x-y) & \text{if} \quad x^0 > y^0 \\ D(y-x) & \text{if} \quad y^0 > x^0 . \end{cases}$$

and

$$T \phi(x) \phi(y) = \begin{cases} \phi(x) \phi(y) & \text{if } x^0 > y^0 \\ \phi(y) \phi(x) & \text{if } y^0 > x^0 . \end{cases}$$

Why time ordering? amplitude $\langle \underset{t \to +\infty}{f} | \underset{t \to -\infty}{i} \rangle$

Claim: $\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}$

Pf: $\langle 0 | T \phi(x) \phi(y) | 0 \rangle$

$$= \langle 0 | \phi(x) \phi(y) | 0 \rangle \Theta(x^0 - y^0) + \langle 0 | \phi(y) \phi(x) | 0 \rangle \Theta(y^0 - x^0)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\omega_k (x^0 - y^0)} e^{i\underline{k}\cdot(\underline{x}-\underline{y})} \Theta(x^0 - y^0)$$

$$+ \int \frac{d^3k}{(2\pi)^2} \frac{1}{2\omega_k} e^{-i\omega_k (y^0 - x^0)} e^{i\underline{k}\cdot(\underline{y}-\underline{x})} \Theta(x^0 - y^0)$$

$\underline{k} \longmapsto -\underline{k}$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{i\underline{k}\cdot(\underline{x}-\underline{y})} \left(e^{-i\omega_k \tau} \Theta(\tau) + e^{i\omega_k \tau} \Theta(-\tau) \right) \quad (*)$$

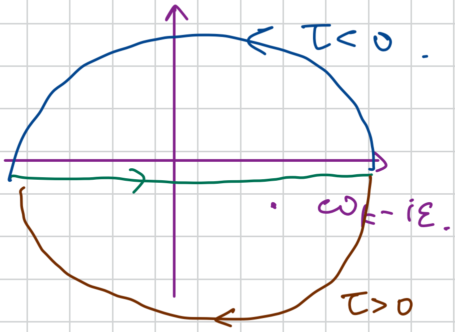
with $\tau = x^0 - y^0$. , and WTS:

$$e^{-i\omega_k \tau} \Theta(\tau) + e^{i\omega_k \tau} \Theta(-\tau) \overset{!}{=} \lim_{\varepsilon \to 0} \frac{-2\omega_k}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega^2 - \omega_k^2 + i\varepsilon} \quad (**)$$

$$\text{RHS} \overset{\circ}{=} \lim_{\varepsilon \to 0} \frac{-2\omega_k}{2\pi i} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega\tau} \left(\frac{1}{\omega - (\omega_k - i\varepsilon)} - \frac{1}{\omega - (-\omega_k + i\varepsilon)} \right) + \mathcal{O}(\varepsilon^2)$$

with $\varepsilon = 2\omega_k \tilde{\varepsilon}$, rewrite $\tilde{\varepsilon} \to \varepsilon$

$\textcircled{=} \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\left(-I_1 + I_2\right)$



$I_1 = \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\tau}}{\omega-(\omega_k - i\varepsilon)}$

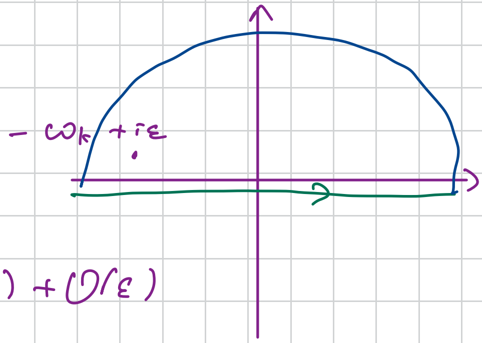
If $\tau<0$, then

$$e^{-i\omega\tau} = e^{\mathrm{Im}\,\omega\cdot\tau}\, e^{-i\,\mathrm{Re}\,\omega\cdot\tau}$$

closing above $\Rightarrow I_1 = 0$

If $\tau>0$, then closing below

$$\Rightarrow I_1 = -2\pi i\, e^{-i\omega_k\tau}\,\Theta(\tau) + \mathcal{O}(\varepsilon).$$



$I_2 = \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\tau}}{\omega-(-\omega_k + i\varepsilon)}$

If $\tau<0$, closing above,

$$\Rightarrow I_2 = 2\pi i\, e^{i\omega_k\tau}\,\Theta(-\tau) + \mathcal{O}(\varepsilon)$$

If $\tau<0$, closing below, $I_2 = 0$.

Then $(**)$ is true.

Replacing $(**)$ in $(*)$,

$$\Delta_F(x-y) = \langle 0|T\phi(x)\,\phi(y)|0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3}\cdot\frac{i}{2\pi}\, e^{i\underline{k}\cdot(\underline{x}-\underline{y})} \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\tau}}{\omega^2-\omega_k^2+i\varepsilon}$$

$\varepsilon\to 0$ (implicit).

$\omega_k^2 = \underline{k}^2 + m^2$

$$= \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2-m^2+i\varepsilon}\, e^{-ik\cdot(x-y)}.$$

$k^2 = -\omega^2 + \underline{k}^2$.

$\square$.

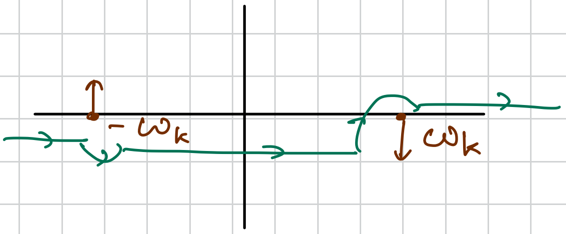
Properties / comments:

1. $\Delta_F(x-y)$ is Lor. inv.

2. $\omega \neq \sqrt{\underline{k}^2+m^2}$, $\Delta_F(x-y)$ is off-shell. (doesn't satisfy eom)

3. i coming from contour integral - don't forget it.

4. ε is a way to keep track of time ordering.



5. $\Delta_F(x-y)$ is a Green's f^n

$$(\partial_\mu \partial^\mu + m^2)\, \Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} (-k^2+m^2)\, e^{-ik\cdot(x-y)}$$

$$= -i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}$$

$$= -i\, \delta^{(4)}(x-y).$$

$$\Rightarrow \boxed{(\partial_\mu \partial^\mu + m^2)\, \Delta_F(x-y) = -i\, \delta^{(4)}(x-y).}$$

6. Claim: $T\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta_F(x-y).$

Pf: Choose $x^0 > y^0$, then

$$T\phi(x)\phi(y) = :\phi(x)\phi(y) + D(x-y)$$

and choose $y^0 > x^0$, then

$$T\phi(x)\phi(y) = :\phi(x)\phi(y): + D(y-x).$$

3. QFT: Interactions

3.1 Couplings in QFT

Free theory: where we can fully characterise the Fock space.

As we will see, scattering in this theory is boring.

Want to do more complicated things, we need a strategy.

- Perturbative QFT : $\mathcal{L} = \underbrace{\mathcal{L}_{free}}_{\mathcal{L}_0} + \underbrace{\mathcal{L}_{int}}_{\text{controlled (small pert)}}$

- How to organise/classify $\mathcal{L}_{int}$? (EFTs).

Example $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

$$\mathcal{L}_{int} = -\sum_{n\geq 3} \lambda_n \phi^n .$$

What are the conditions to treat $\mathcal{L}_{int}$ as something small?

Naive answer: $\lambda_n \ll 1$. Wrong! Why? λ_n has units.

Recall $c = \hbar = 1 \Rightarrow [M] = [L^{-1}] = [T^{-1}] = 1$, then

↑ dimension

$$[c] = [\hbar] = 0$$

Apply this to the action

$$S = \int d^4x\, \mathcal{L} \quad , \quad [S] = [\hbar] = 0 , \quad [d^4x] = -4 .$$

$$\Rightarrow [\mathcal{L}] = 4$$

Then $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$.

$$[\mathcal{L}_0] = 4 \quad , \quad [\partial_\mu] = 1 , \quad [m] = 1 , \Rightarrow [\phi] = 1 .$$

field has units of inverse length.

Next, $\mathcal{L}_{int} \sim -\lambda_n \phi^n \Rightarrow [\lambda_n] = 4-n$.

So λ_n dimensionful. Generally it makes no sense to write $\lambda_n \ll 1$.

To construct a dimensionless param. (that we can say it's small), consider E, which is a typical energy scale in the problem

$$\Rightarrow \underbrace{\lambda_n (E)^{n-4}}_{\ll 1 \ (\text{this ok})} \rightarrow [\lambda_n E^{n-4}] = 0 .$$

This also indicates that there are different classes of λ_n's.

1. $[\lambda_n] = 4-n > 0$; i.e. $n=3$ ⇒ $[\lambda_3]=1$. , $\frac{\lambda_3}{E}$ dimensionless

If high energy, small ($\lambda_3 \ll E$). If low energy, big. ($\lambda_3 \gg E$)

relevant — big, important

2. $[\lambda_n] = 4-n = 0$, $n=4$.

$\lambda_4 \ll 1$ makes this pert. ⇒ marginal coupling

3. $[\lambda_n] = 4-n < 0$, $n > 4$. The combination $\lambda_n E^{n-4}$ can be made small at low energies ⇒ ignore ⇒ irrelevant

Study 2 classes of theory

- ϕ^4 theory : $\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$.

- Scaler Yukawa theory :

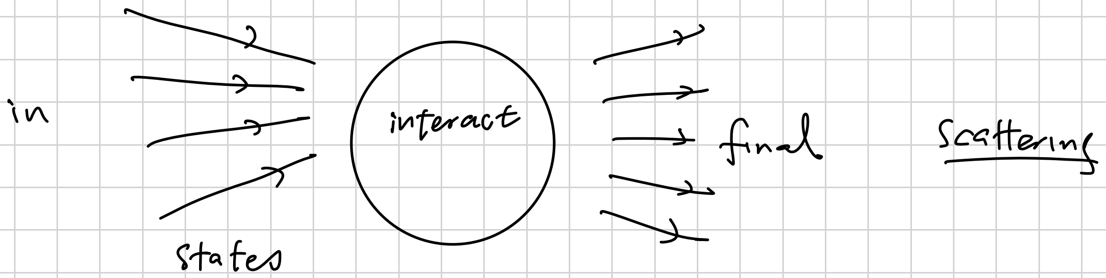
$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \partial_\mu\psi\,\partial^\mu\psi^\dagger - M^2\psi\psi^\dagger$$

$$\mathcal{L}_{int} = -g\,\psi^\dagger\psi\,\phi.$$

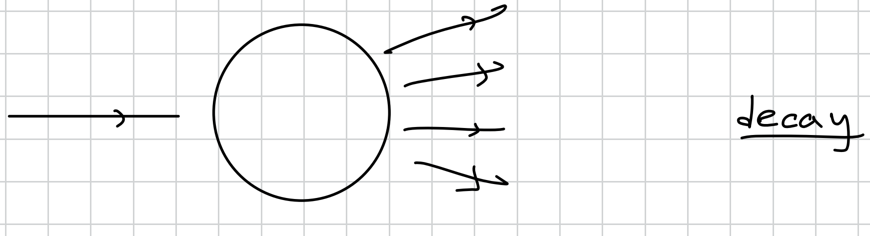
S-matrix (cross sections, decay rates).

Goal: evaluate a <u>scattering process</u> in QFT with $\mathcal{L}_{int}$.

3.2 LSZ reduction formula



or



We need:

1. Define initial and final states
2. Relate |initial⟩ with |final⟩ (prob. to transition)
3. Method to evaluate S-matrix.

Step 1: Define initial and final ⇒ Asymptotic states

$$\underbrace{\langle \text{final}\,;\, t_f \mid \text{initial}\ t_i \rangle}_{\text{Schr pic}} = \underbrace{\langle \text{final} \mid S \mid \text{initial} \rangle}_{\substack{\text{Heisenberg pic:} \\ \text{operator evolve : } S = S\text{ matrix}}}$$

Assumption: interactions only occur in a finite interval of time. If

$$\left.\begin{array}{ll} |\text{initial}\rangle \text{ occurs at} & t_i \to -\infty \\ |\text{final}\rangle & t_f \to +\infty \end{array}\right\} \text{Free states} \Rightarrow \text{asy. states}$$

This means

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p(t)\, e^{-ipx} + a_p^\dagger(t)\, e^{ipx} \right)$$

⇒ interactions (modification of e.o.m.) add time dependence to a and $a^\dagger$.

The assumption is then

$$a_p(t) = e^{iH(t-t_0)} \underbrace{a_p^0(t_0)}_{\text{at } t_0 \text{ free}} e^{-iH(t-t_0)}.$$

where $H = \overset{\text{free}}{H_0} + H_{int}$. This is because

$$i\partial_t \phi = [\phi, H]$$

The implementation of asymp. states

$$\lim_{t\to\pm\infty} a_p(t) = a^0_p \quad : \underline{\text{free operators}}$$

As we prepare states, we will write, e.g.

$$|\text{initial}, -\infty\rangle = \sqrt{2\omega_1}\,\sqrt{2\omega_2}\; a^\dagger_{p_1}(-\infty)\, a^\dagger_{p_2}(-\infty)\,|\Omega\rangle$$

$$|\text{final}, +\infty\rangle = \sqrt{2\omega_3}\,\sqrt{2\omega_4}\; a^\dagger_{p_3}(+\infty)\, a^\dagger_{p_4}(+\infty)\,|\Omega\rangle .$$

Here $|\Omega\rangle$ is the vacuum of the whole (int.) theory ($|\Omega\rangle \neq |0\rangle$), with $\omega_i = \sqrt{p_i^2 + m^2}$ and $p_i \neq p_j$ for $i \neq j$.

Goal: evaluate S-matrix for this choice of initial and final state

$$\langle f|S|i\rangle = \langle \text{final}, +\infty \,|\, \text{initial}, -\infty\rangle$$

$$= \prod_{j=1}^{4}\sqrt{2\omega_j}\,\langle\Omega|T\,\underbrace{a_{p_3}(+\infty)\,a_{p_4}(+\infty)\,a^\dagger_{p_1}(-\infty)\,a^\dagger_{p_2}(-\infty)}_{(1)}|\Omega\rangle \qquad (3)$$

Claim: $\sqrt{2\omega_p}\left(a^\dagger_p(\infty) - a^\dagger_p(-\infty)\right) = -i\int d^4x\; e^{-ipx}\,(\Box + m^2)\,\phi(x).$

for any interacting theory that complies with assumption of asymp. states.
(2)

Pf: Free theory: $a_p(\infty) = a_p(-\infty) \equiv a^0_p$. , and $(1) = 0$ by e.o.m.

Interacting: Recall from free theory

$$\sqrt{2\omega_p}\; a^0_p = i\int d^3x\; e^{ipx}\,\overleftrightarrow{\partial_t}\,\phi_0$$

$$\sqrt{2\omega_p}\; a^{0\dagger}_p = -i\int d^3x\; e^{-ipx}\,\overleftrightarrow{\partial_t}\,\phi_0$$

where $f(t)\,\overleftrightarrow{\partial_t}\,g(t) = f\,\partial_t g - (\partial_t f)\,g$

To prove claim, start from RHS (1).

$$-i\int d^4x\; e^{-ipx}(\Box + m^2)\,\phi(x).$$

$$= -i\int d^4x\; e^{-ipx}(\partial_t^2 - \nabla^2 + m^2)\,\phi(x).$$

$$\overset{\text{IBP}\times 2}{=} -i\int d^4x\; e^{ipx}(\partial_t^2 + \underbrace{p^2 + m^2}_{=\omega_p^2,})\,\phi(x)$$

$$= -i\int d^4x\; \partial_t\left(e^{-ipx}\,\partial_t\phi - \partial_t(e^{-ipx})\,\phi\right).$$

$$= -i\int d^4x\; \partial_t\left(e^{-ipx}\,\overleftrightarrow{\partial_t}\,\phi\right)$$

int. over t,
use (2)

$$= \sqrt{2\omega_p}\left(a_p^\dagger(\infty) - a_p^\dagger(-\infty)\right). \qquad \Box$$

Analogously,

$$\boxed{\sqrt{2\omega_p}\left(a_p(\infty) - a_p(-\infty)\right) = i\int d^4x\; e^{ipx}(\Box + m^2)\,\phi(x).} \qquad (4)$$

Next, we can rewrite (3) as follows

$$\langle f|S|i\rangle = \prod_{j=1}^{4}\sqrt{2\omega_p}\,\langle\Omega|\,T\left(a_{p_3}(\infty) - a_{p_3}(-\infty)\right)\left(a_{p_4}(\infty) - a_{p_4}(-\infty)\right)$$
$$\left(a_{p_1}^\dagger(-\infty) - a_{p_1}^\dagger(\infty)\right)\left(a_{p_2}^\dagger(-\infty) - a_{p_2}^\dagger(\infty)\right)|\Omega\rangle.$$

$$\overset{(1),(4)}{=} \prod_{j=1}^{4} i\int d^4x_j\; \overbrace{e^{-ip_1x_1}(\Box_1 + m^2)\, e^{-ip_2x_2}(\Box_2 + m^2)}^{\text{initial}}\; \underbrace{e^{ip_3x_3}(\Box_3 + m^2)}$$
$$\underbrace{e^{ip_4x_4}(\Box_4 + m^2)}_{\text{final.}}\,\langle\Omega|\,T\,\phi(x_1)\phi(x_2)\,\phi(x_3)\,\phi(x_4)|\Omega\rangle$$

This is <u>LSZ formula</u> for 2→2 scattering.

Simple generalisation if initial states has n particles,
final n' ,

and in the middle,

$$\langle \Omega | T \phi(x_1) \dots \phi(x_n)\ \phi(x_{1'}) \dots \phi(x_{n'}) | \Omega \rangle .$$: n+n'-point <u>correlation function</u>

Shorthand notation : $\langle \phi_1 \dots \phi_n\ \phi_{1'} \dots \phi_{n'} \rangle$, but

- always time-ordered
- always against $|\Omega\rangle$.

Apply LSZ to Scaler Yukawa theory.

$$\mathcal{L}_0 = \underbrace{\partial_\mu \psi \partial^\mu \psi^\dagger - M^2 \psi^\dagger \psi}\ + \underbrace{\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2}_{a_p,\ a_p^\dagger} .$$

⇒ Fock space for "ϕ" type particle

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_p e^{-ipx} + c_p^\dagger e^{ipx} \right).$$

$$[\psi(\underline{x}, t),\ \Pi_\psi(\underline{y}, t)] = i\delta^{(3)}(\underline{x} - \underline{y})$$

$$\Downarrow$$

$$[b_p, b_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\underline{p} - \underline{p}')$$

$$[c_p, c_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\underline{p} - \underline{p}') .$$

Hamiltonian : $$H_0 = \int \frac{d^3p}{(2\pi)^3} \left(\omega_p a_p^\dagger a_p + \tilde{\omega}_p b_p^\dagger b_p + \tilde{\omega}_p c_p^\dagger c_p \right)$$

where $\omega_p = \sqrt{\underline{p}^2 + m^2}$, $\tilde{\omega}_p = \sqrt{\underline{p}^2 + M^2}$.

Charge : $$Q = i \int d^3x :(\dot{\psi}^* \psi - \psi^* \dot{\psi}):$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(c_p^\dagger c_p - b_p^\dagger b_p \right).$$

Fock space:

1-particle : $a_p^\dagger|0\rangle \to E=\omega_p\,,\ Q=0$

$b_p^\dagger|0\rangle \to E=\tilde{\omega}_p\,,\ Q=-1$

$c_p^\dagger|0\rangle \to E=\tilde{\omega}_p\,,\ Q=+1.$

Turn on interactions on Scaler Yukawa theory.

Gen of (1)+(4), for example

$$\sqrt{2\tilde{\omega}_p}\,\left(b_p^\dagger(\infty) - b_p^\dagger(-\infty)\right) = -i\int d^4x\; e^{-ipx}\left(\Box_x + M^2\right)\psi^\dagger(x).$$

Example initial : $b_p^\dagger(-\infty)\,|\Omega\rangle$

final : $c_{q_1}^\dagger(+\infty)\; b_{q_2}^\dagger(+\infty)\; b_{q_3}^\dagger(+\infty)\;|\Omega\rangle.$

Then $\langle f|S|i\rangle$

$$= i\int d^4x \prod_{i=1}^{3} i\int d^4y_i\; e^{-ipx}\left(\Box_x + M^2\right) e^{iq_i y_i}\left(\Box_x + M^2\right)$$
$$\langle\Omega|T\,\psi^\dagger(x)\,\psi^\dagger(y_1)\,\psi(y_2)\,\psi(y_3)|\Omega\rangle$$

Goal: How to evaluate $\langle\Omega|T\ldots|\Omega\rangle$?

3.3 Schwinger - Dyson Equation

- Prescription that favours the Lagrangian form ⇒ SRT.

Assumption 1: At any given moment of time, the Hilbert space of the interacting theory is the same as the free theory Fock Space

causality

$$\left.\begin{aligned}&[\phi(\underline{x},t),\ \phi(\underline{x}',t)] = 0\\ &[\phi(\underline{x},t),\ \Pi(\underline{x}',t)] = i\,\delta^{(3)}(\underline{x}-\underline{x}')\end{aligned}\right|\ (1)$$

true also for interacting theory

← Q.M.

Assumption 2: Our fields will satisfy e.o.m., i.e. if

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}[\phi],$$

then

assume for simplicity no $\partial\phi$.

$$(\Box + m^2)\,\phi(x) - \overbrace{\mathcal{L}'_{int}[\phi]} = 0\,, \qquad (2)$$

where $\mathcal{L}'_{int} = \partial \mathcal{L}_{int}/\partial\phi$.

Equivalent statement in Hamiltonian approach

$$i\partial_t\phi = [\phi, H]$$

(this is same as e.o.m.)

Claim: $(\Box_x + m^2)\langle\phi_x\phi_y\rangle = \langle(\Box_x + m^2)\phi_x\,\phi_y\rangle - i\delta^{(4)}(x-y).$

(Recall $\langle\phi_x\phi_y\rangle = \langle\Omega|\,T\,\phi(x)\,\phi(y)\,|\Omega\rangle$.

Pf: In free theory, (2) $\Rightarrow$ $(\Box_x + m^2)\phi^0_x = 0$, then

$$(\Box_x + m^2)\underbrace{\langle\phi_x\phi_y\rangle}_{\Delta_F(x-y).} = -i\delta^4(x-y)$$

So true for free theory.

Next, for an interacting theory, first look at

$$\begin{aligned}
\partial_{x^0}\langle\phi_x\phi_y\rangle &= \partial_{x^0}\Big(\langle\Omega|\phi_x\phi_y|\Omega\rangle\,\Theta(x^0-y^0) \\
&\qquad + \langle\Omega|\phi_y\phi_x|\Omega\rangle\,\Theta(y^0-x^0)\Big) \\
&= \langle\partial_{x^0}\phi_x\phi_y\rangle + \langle\Omega|\phi_x\phi_y|\Omega\rangle\,\underbrace{\partial_{x^0}\Theta(x^0-y^0)}_{=\,\delta(x^0-y^0)} \\
&\qquad + \langle\Omega|\phi_y\phi_x|\Omega\rangle\,\underbrace{\partial_{x^0}\Theta(y^0-x^0)}_{=\,-\delta(y^0-x^0)}. \\
&= \langle\partial_{x^0}\phi_x\phi_y\rangle + \underbrace{\delta(x^0-y^0)\langle\Omega|[\phi_x,\phi_y]|\Omega\rangle}_{(1)\,\Rightarrow\,0} \\
&= \langle\partial_{x^0}\phi_x\phi_y\rangle
\end{aligned}$$

Take a second derivative,

$$\begin{aligned}
\partial^2_{x^0}\langle\phi_x\phi_y\rangle &= \partial_{x^0}\langle\partial_{x^0}\phi_x\phi_y\rangle \\
&= \langle\partial^2_{x^0}\phi_x\phi_y\rangle + \delta(x^0-y^0)\langle\Omega|\,[\overbrace{\partial_{x^0}\phi_x}^{=\,\Pi}, \phi_y]\,|\Omega\rangle \\
&= \langle\partial^2_{x^0}\phi_x\phi_y\rangle - i\delta^{(4)}(x-y).
\end{aligned}$$

Finally, ∇_x^2 and m^2 do not interfere with time ordering, so

$$(\Box_x + m^2)\langle \phi_x \phi_y \rangle = \langle (\Box_x + m^2)\phi_x \phi_y \rangle - i\,\delta^{(4)}(x-y). \qquad \Box$$

Use (2), we can rewrite

$$(\Box_x + m^2)\langle \phi_x \phi_y \rangle = \langle \mathcal{L}_{int}'[\phi_x]\,\phi_y \rangle - i\,\delta^{(4)}(x-y)$$

Natural generalisation

$$(\Box_1 + m^2)\langle \phi_1 \dots \phi_n \rangle = \langle (\Box_1 + m^2)\phi_1 \dots \phi_n \rangle - i\sum_{j=2}^{n} \delta^{(4)}(x_1 - x_j)\langle \phi_2 \dots \phi_{j-1}\phi_{j+1} \dots \phi_n \rangle$$

$$= \langle \mathcal{L}_{int}'[\phi_1]\,\phi_2 \dots \phi_n \rangle - i\sum_{j=2}^{n} \delta^{(4)}(x_1 - x_j)\langle \phi_2 \dots \phi_{j-1}\phi_{j+1} \dots \phi_n \rangle.$$

This is the Schw-Dyson eqn.

Example Free theory, 2 point, $m^2 = 0$.

Write $\Delta_F(x_1 - x_2) \equiv \Delta_{12}$. We know

$$\Box_1 \Delta_{12} = -i\,\delta(x_1 - x_2). \qquad (*)$$

$$\langle \phi_1 \phi_2 \rangle = \int d^4x\, \delta^{(4)}(x - x_1)\langle \phi_x \phi_2 \rangle$$

$$\overset{(*)}{=} i\int d^4x\, (\Box_x \Delta_{1x})\langle \phi_x \phi_2 \rangle$$

$$\overset{\text{IBP} \times 2}{=} i\int d^4x\, \Delta_{1x}\, \Box_x \langle \phi_x \phi_2 \rangle$$

$$\overset{\text{SD eqn}}{=} i\int d^4x\, \Delta_{1x}\,(-i\,\delta(x - x_2)) = \Delta_{12}.$$

Example Free theory, 4 point, $m^2 = 0$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$$

$$= \int d^4x\, \delta^{(4)}(x - x_1)\langle \phi_x \phi_2 \phi_3 \phi_4 \rangle$$

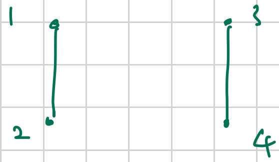
$$= i \int d^4x \left(\Box_x \Delta_{1x}\right) \langle \phi_x \, \phi_2 \, \phi_3 \, \phi_4 \rangle$$

$$= i \int d^4x \; \Delta_{1x} \left(\Box_x \langle \phi_x \, \phi_2 \, \phi_3 \, \phi_4 \rangle\right)$$

$$= i \int d^4x \; \Delta_{1x} \Big(-i\, \delta^{(4)}(x-x_2) \langle \phi_3 \phi_4 \rangle - i\, \delta^{(4)}(x-x_3) \langle \phi_2 \phi_4 \rangle$$
$$- i\, \delta^{(4)}(x-x_4) \langle \phi_2 \phi_3 \rangle \Big)$$

$$= \Delta_{12} \Delta_{34} + \Delta_{13} \Delta_{24} + \Delta_{14} \Delta_{23}$$

=



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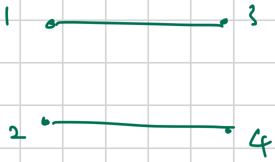

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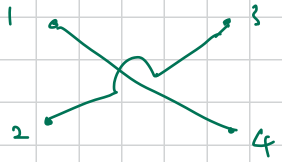


Aside : Wick Thm

$$T\, \phi_1 \phi_2 = :\phi_1 \phi_2: + \Delta_{12}$$

Generalisation

$$T \phi_1 \cdots \phi_n = :\phi_1 \ldots \phi_n: + \text{"all possible contractions"}$$

replace two pairs of field with

$$\Delta_{ij} \equiv \overbrace{\phi_i \phi_j}$$

Example $T \phi_1 \ldots \phi_4 = :\phi_1 \ldots \phi_4: + :\phi_1 \phi_2: \overbrace{\phi_3 \phi_4} +$ permutations

$+ \overbrace{\phi_1 \phi_2} \, \overbrace{\phi_3 \phi_4} +$ perm.

$$\Rightarrow \langle 0 | T \phi_1 \ldots \phi_4 | 0 \rangle = \overbrace{\phi_1 \phi_2} \, \overbrace{\phi_3 \phi_4} + \text{perm.}$$

Example $\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi$. $\mathcal{L}_{int} = \frac{g}{3!} \phi^3$

Schw - Dyson:

$$\Box_x \langle \phi_x \phi_1 \ldots \phi_n \rangle = \frac{g}{2} \langle \phi_x^2 \phi_1 \ldots \phi_n \rangle - i \sum_j \delta(x-x_j) \langle \phi_1 \ldots \phi_{j-1} \phi_{j+1} \ldots \phi_n \rangle \quad (*)$$

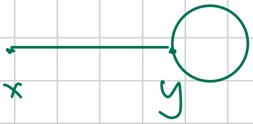
(a)
$$\begin{aligned}\langle\phi_x\rangle &= \int d^4y\ \delta(x-y)\,\langle\phi_y\rangle && \big\downarrow \text{ def}^n \text{ of } \Delta_F\\
&= i\int d^4y\ (\Box_y\Delta_{xy})\,\langle\phi_y\rangle && \big\downarrow \text{ IBP}\\
&= i\int d^4y\ \Delta_{xy}\,\Box_y\langle\phi_y\rangle && \big\downarrow \ (*).\\
&= i\int d^4y\ \Delta_{xy}\cdot\frac{g}{2}\,\langle\phi_y^2\rangle\end{aligned}$$

(b)
$$\begin{aligned}\langle\phi_1\phi_2\rangle &= i\int d^4y\ \Delta_{y1}\,\Box_y\langle\phi_y\phi_2\rangle\\
&\qquad\big\downarrow (*)\\
&= i\int d^4y\ \Delta_{y1}\left(\frac{g}{2}\langle\phi_y^2\phi_2\rangle - i\,\delta(y-x_2)\right).\\
&= \underbrace{\Delta_{12}}_{\text{free}} + \underbrace{\frac{ig}{2}\int d^4y\ \Delta_{1y}\,\langle\phi_y^2\phi_2\rangle}_{\mathcal{O}(g).}.\end{aligned}$$

Now, back to (a):

$$\langle\phi_x\rangle \overset{(b)}{=} \frac{ig}{2}\int d^4y\ \Delta_{xy}\Big(\overset{\mathcal{O}(g)}{\Delta_{yy}} + \frac{ig}{2}\int d^4z\ \Delta_{yz}\,\overset{\mathcal{O}(g^2)}{\langle\phi_z^2\phi_y\rangle}\Big).$$

$$= \frac{ig}{2}$$



$$+\ \mathcal{O}(g^2).$$

(c)
$$\begin{aligned}\langle\phi_1\phi_2\phi_3\rangle &= i\int d^4x\ \Delta_{x1}\,\Box_x\langle\phi_x\phi_2\phi_3\rangle\\
&= i\int d^4x\ \Delta_{1x}\Big(\frac{g}{2}\langle\phi_x^2\phi_2\phi_3\rangle - i\,\delta(x-x_2)\langle\phi_3\rangle\\
&\qquad - i\,\delta(x-x_3)\langle\phi_2\rangle\Big)\\
&\sim \mathcal{O}(g)\end{aligned}$$

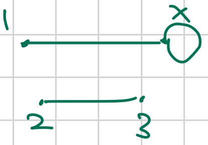
(d)
$$\begin{aligned}\langle\phi_1\phi_2\phi_3\phi_4\rangle &= i\int d^4x\ \Delta_{1x}\,\Box_x\langle\phi_x\phi_2\phi_3\phi_4\rangle\\
&= i\int d^4x\ \Delta_{1x}\Big(\frac{g}{2}\langle\phi_x^2\phi_2\phi_3\phi_4\rangle - i\,\delta(x-x_2)\langle\phi_3\phi_4\rangle\\
&\qquad - i\,\delta(x-x_3)\langle\phi_2\phi_4\rangle - i\,\delta(x-x_4)\langle\phi_2\phi_3\rangle\Big).\\
&= \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} + \mathcal{O}(g).\end{aligned}$$

Then in (c),

$$\langle \phi_1 \phi_2 \phi_3 \rangle \overset{(d),(a)}{=} \frac{ig}{2} \int d^4x \; \Delta_{1x} \left(\Delta_{xx} \Delta_{23} + 2 \Delta_{x2} \Delta_{x3} \right)$$

$$+ \frac{ig}{2} \Delta_{12} \int d^4y \; \Delta_{y3} \Delta_{yy} + \frac{ig}{2} \Delta_{13} \int d^4y \; \Delta_{y2} \Delta_{yy} + O(g^2).$$

$= ig\Big(\frac{1}{2}$



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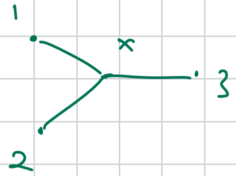
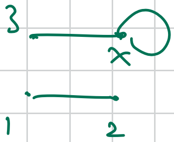
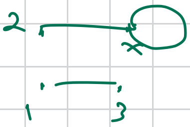


$+ \frac{1}{2}$



$+ \frac{1}{2}$



$\Big) + O(g^2).$

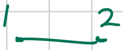
Then in (b), using (c)

$$\langle \phi_1 \phi_2 \rangle = \Delta_{12} + \frac{ig}{2} \int d^4y \; \Delta_{1y} \langle \phi_y^2 \phi_2 \rangle$$

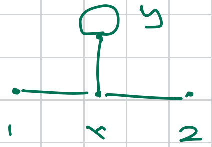
$$= \Delta_{12} + \frac{(ig)^2}{2} \int d^4x \int d^4y \; \Delta_{1x} \Delta_{2x} \left(\Delta_{yx} \Delta_{yy} \right)$$

$$+ \left(\frac{ig}{2} \right)^2 \int d^4x \int d^4y \; \Delta_{1x} \Delta_{2y} \left(\Delta_{xx} \Delta_{yy} + 2 \Delta_{xy}^2 \right) + O(g^3)$$

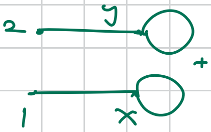
$=$



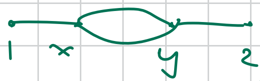
$+ (ig)^2 \Big(\frac{1}{2}$



$+ \frac{1}{4}$

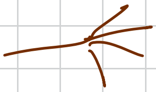


$+ \frac{1}{2}$



$\Big) + O(g^3).$

3.4 Feynman Diagrams



$$\langle \phi_1 \cdots \phi_n \rangle = i \int d^4x \; \Delta_{1x} \Big(\langle \overset{\text{proliferate}}{\mathcal{L}'_{int}[\phi_1]} \phi_2 \cdots \phi_n \rangle - i \sum_{j=2}^{n} \delta(x - x_j) \underbrace{\langle \phi_2 \cdots \phi_{j-1} \phi_{j+1} \cdots \phi_n \rangle}_{\text{contractions}} \Big)$$

⇓

loops

Via example of ϕ^3, we saw that there was an iterative process of proliferate + contract and at each order in g, we have all possible combinations of it.

$\langle \phi_1 \dots \phi_n \rangle$ via Feynmann Diagrams:

1) Start with n-extremal points $i = 1, \dots, n$.

Draw a line from each point

2) A line can either

- contract ; join another existing line. This gives $\Delta_F(x_i - x_j)$
- split , where this gives rise to vertex.

The coeff. will be come with $i\lambda_n$ $\left(\mathcal{L}_{int} = \frac{\lambda_n}{n!} \phi_n \right)$.

\# of lines coming out is $(n-1)$.

3) At any given order λ_n, the result is the sum of all diagrams with all lines contracted integrated over positions of vertex.

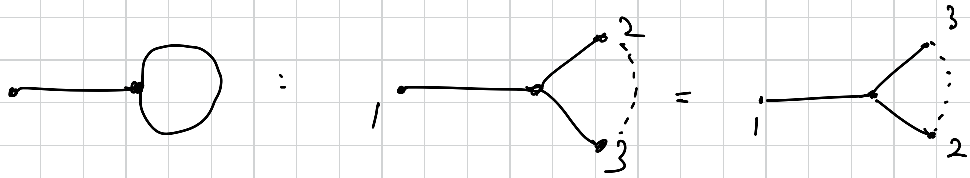
4) Symmetry factors : We write

$$\mathcal{L}_{int} = \frac{i\lambda_n}{n!} \phi^n \rightarrow \text{the vertex } i\lambda_n.$$

Warning : this might be an over count.

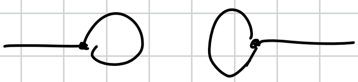
In S-D eqn, $\mathcal{L}'_{int}$, hence $\frac{1}{n!} \rightarrow \frac{1}{(n-1)!}$. then additional contractions remove $(n-1)!$ But there are cases where there are not enough contractions. For example,

(i)



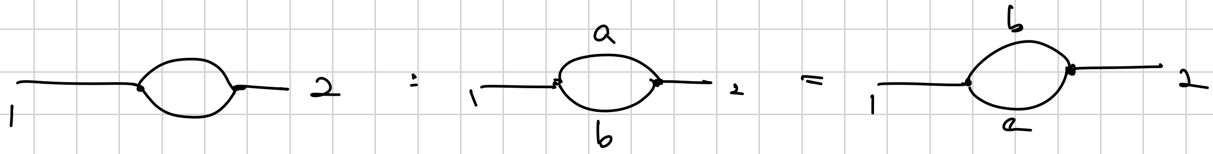
⇒ symmetry factor of 2

(ii)



→ sym. fac. of 4

(iii)



$\Rightarrow$ sym. fac. 2.

<u>Example</u> (Scaler Yukawa theory) $\mathcal{L}_{int} = -g\psi^{+}\psi\phi$.

Assumption :

(1) e.o.m. :

$$(\Box + m^2)\phi - (-g\psi^{+}\psi) = 0$$

$$(\Box + M^2)\psi - (-g\psi\phi) = 0$$

(2)

$$[\phi(x,t), \partial_t\phi(y,t)] = i\delta^{(3)}(x-y)$$

$$[\psi(x,t), \partial_t\psi^{+}(y,t)] = i\delta^{(3)}(x-y).$$

Then S-D eqn :

$$(\Box_x + m^2)\langle\phi_x\phi_1\cdots\phi_n\,\psi_{1'}\cdots\psi_{n'}\,\psi^{+}_{1''}\cdots\psi^{+}_{n''}\rangle$$

$$= -g\langle\psi_x\psi_x^{+}\cdots\rangle - i\sum_{j=1}^{n}\delta(x-x_j)\langle\phi_1\cdots\phi_{j-1}\phi_{j+1}\cdots\phi_n\,\psi_{\cdots}\psi\,\psi^{+}_{\cdots}\psi^{+}\rangle.$$

$$(\Box_x + M^2)\langle\psi_x\,\phi_1\cdots\phi_n\,\psi_{1'}\cdots\psi_{n'}\,\psi^{+}_{1''}\cdots\psi^{+}_{n''}\rangle$$

$$= -g\langle\psi_x\phi_x\,\phi_1\cdots\,\psi\cdots\,\psi^{+}\cdots\rangle - i\sum_{j=1}^{n}\delta(x-x_j)\langle\phi\cdots\phi\,\psi\cdots\psi\,\psi^{+}_{1''}\cdots\psi^{+}_{j-1}\psi^{+}_{j+1}\cdots\psi^{+}_{n''}\rangle.$$

We also have two ways to introduce $\delta(x-y)$ in the correlator.

$$(\Box_x + m^2)\,\Delta_F(x-y) = -i\delta(x-y).$$

where $\Delta_F(x-y) = \langle 0|\,T\phi(x)\,\phi(y)\,|0\rangle$. , or

$$(\Box_x + M^2)\,\hat{\Delta}_F(x-y) = -i\delta(x-y)$$

where

$$\hat{\Delta}_F(x-y) = \langle 0|\,T\,\psi^{+}(x)\,\psi(y)\,|0\rangle.$$

$$= \int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-M^2+i\varepsilon}\,e^{-ip\cdot(x-y)}.$$

<u>Note</u> : $\langle 0|\,T\psi(x)\,\psi(y)\,|0\rangle = 0$.

Say, we want to evaluate

$$\langle \phi_1 \psi_2^\dagger \psi_3 \rangle$$

$$= \int d^4x \; \delta(x-x_1) \; \langle \phi_x \psi_2^\dagger \psi_3 \rangle$$

$$= i \int d^4x \quad \Delta_{1x} \, (\Box_x + m^2) \, \langle \phi_x \psi_2^\dagger \psi_3 \rangle$$

$$= -ig \int d^4x \; \Delta_{1x} \, \langle \psi_x \psi_x^\dagger \psi_2^\dagger \psi_3 \rangle$$

$$= -i^2 g \int d^4x \int d^4y \; \Delta_{1x} \, \hat{\Delta}_{xy} \, (\Box_y + M^2) \, \langle \psi_y \psi_x^\dagger \psi_2^\dagger \psi_3 \rangle$$

$$= -i^2 g \int d^4x \int d^4y \; \Delta_{1x} \, \hat{\Delta}_{xy} \Big(-g \langle \psi_y \phi_y \psi_x^\dagger \psi_2^\dagger \psi_3 \rangle$$

$$- i \delta^{(4)}(x-y) \langle \psi_2^\dagger \psi_3 \rangle - i \delta^{(4)}(y-x_2) \langle \psi_x^\dagger \psi_3 \rangle \Big).$$

$$= (-ig) \int d^4x \int d^4y \; \Delta_{1x} \, \hat{\Delta}_{xy} \Big(\delta(x-y) \underbrace{\langle \psi_2^\dagger \psi_3 \rangle}_{\substack{\text{free thy} \\ + \mathcal{O}(g)}} + \delta(y-x_2) \underbrace{\langle \psi_x^\dagger \psi_3 \rangle}_{\text{free} + \mathcal{O}(g)} \Big) + \mathcal{O}(g^2)$$

$$= (-ig) \int d^4x \int d^4y \; \Delta_{1x} \, \hat{\Delta}_{xx} \, \hat{\Delta}_{23} + (-ig) \int d^4x \; \Delta_{1x} \, \hat{\Delta}_{x2} \, \hat{\Delta}_{x3} + \mathcal{O}(g^2).$$

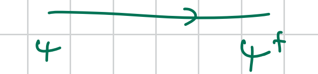
Writing

$$\Delta_{ij} = \langle \phi_i \phi_j \rangle = \int \frac{d^4p}{(2\pi)^4} \; \frac{e^{-ip(x_i - x_j)}}{p^2 - m^2 + i\varepsilon}$$

$$\Rightarrow (\Box + m^2) \Delta_{ij} = -i \, \delta^{(4)}(x_i - x_j) \quad : \quad - - - - - - -$$

$$\hat{\Delta}_{ij} = \langle \psi_i \psi_j^\dagger \rangle = \int \frac{d^4p}{(2\pi)^4} \; \frac{e^{-ip(x_i - x_j)}}{p^2 - M^2 + i\varepsilon}$$

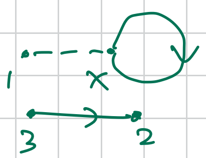
$$\Rightarrow (\Box + M^2) \hat{\Delta}_{ij} = -i \, \delta^{(4)}(x_i - x_j) \; :$$



Then

$$\langle \phi_1 \psi_2^\dagger \psi_3 \rangle = -ig \int d^4x \; \Delta_{1x} \left(\hat{\Delta}_{xx} \hat{\Delta}_{23} + \hat{\Delta}_{x2} \hat{\Delta}_{x3} \right) + \mathcal{O}(g^2)$$

$= -ig \Big($



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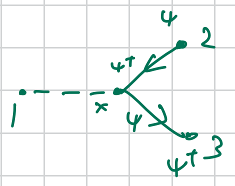
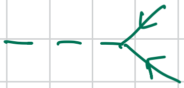


$\Big) + \mathcal{O}(g^2)$

Note: 1) Arrows keep a "flow" (due to L_{int}). It means that  is wrong!

$\langle \phi_1 \psi_2 \psi_3 \rangle = 0$

2) No symmetry factors!

3.5 Scattering Process

$$\langle \text{final}, +\infty \mid \text{initial}, -\infty \rangle = \langle f | S | i \rangle$$

Work with an example, in S-Yukawa theory, to see mechanics

Example (Nuclear Scattering).

$a^\dagger |\Omega\rangle \Rightarrow$ "meson"

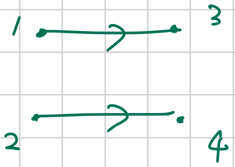
$b^\dagger |\Omega\rangle \Rightarrow$ "anti-nucleon"

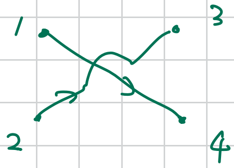
$c^\dagger |\Omega\rangle \Rightarrow$ "nucleon"

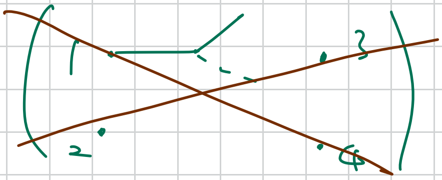
The process is $\underset{p_1\ \ p_2}{\underset{\text{initial}}{|c^\dagger c^\dagger\rangle}} \rightarrow \underset{p_3\ \ p_4}{\underset{\text{final}}{|c^\dagger c^\dagger\rangle}}$.

$$\langle f | S | i \rangle \overset{LSZ}{=} i^4 \prod_{j=1}^{4} \int d^4 x_j \overbrace{e^{-ip_1x_1} (\Box_1 + M^2)\, e^{-ip_2x_2} (\Box_2 + M^2)}^{\text{incoming}}$$
$$\underbrace{e^{ip_3x_3} (\Box_3 + M^2)\, e^{ip_4x_4} (\Box_4 + M^2)}_{\text{outgoing}} \langle \psi_1(x_1)\, \psi_2(x_2)\, \psi_3^\dagger(x_3)\, \psi_4^\dagger(x_4) \rangle.$$

Use Feynman diagram to evaluate $\langle \psi_1 \psi_2 \psi_3^\dagger \psi_4^\dagger \rangle$.

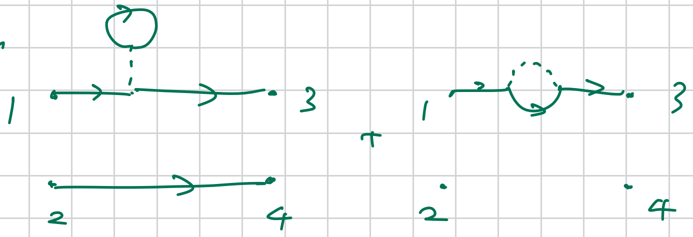
$\langle \psi_1 \psi_2 \psi_3^\dagger \psi_4^\dagger \rangle =$ 

$+$ 

$+ ig$ 

proliferate one,
contract everything else. but not possible

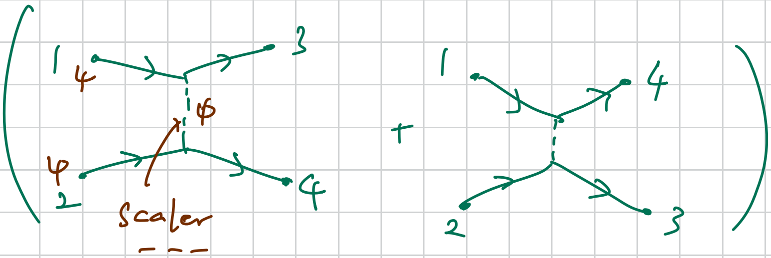
$+ (ig)^2 \Bigg($



+ perm. (3→4) + perm. vertex on 2→4 $\Bigg)$

Disconnected contributions

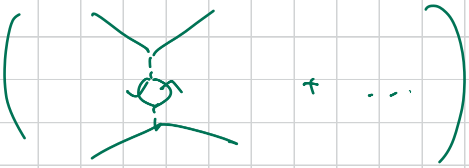
$+ (ig)^2 \Bigg($



$\Bigg)$

connected diagrams
+ free

$+ (ig)^4 \Bigg($

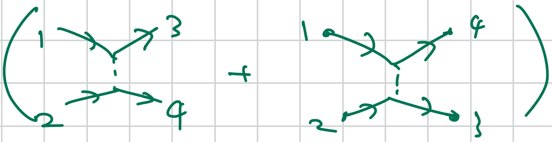


$\Bigg)$

connected diagrams.
+loop.

and

$$\langle \psi_1 \psi_2 \psi_3^\dagger \psi_4^\dagger \rangle_c = (ig)^2 \Bigg($$



$$\Bigg) + \mathcal{O}(g^3)$$

$$= (ig)^2 \int d^4x \int d^4y \Big(\hat{\Delta}_{1x} \, \hat{\Delta}_{x3} \, \Delta_{xy} \, \hat{\Delta}_{2y} \, \hat{\Delta}_{y4}$$

$$+ \hat{\Delta}_{1x} \, \hat{\Delta}_{x4} \, \Delta_{xy} \, \hat{\Delta}_{2y} \, \hat{\Delta}_{y3} \Big) + \dots$$

Use in $\langle f | S | i \rangle_c = \langle f | S - \mathbb{1} | i \rangle$

$$\langle f | S | i \rangle_c = i^4 \prod_{j=1}^{4} \int d^4x_j \, e^{ip_1x_1} e^{-ip_2x_2} e^{ip_1x_3} e^{ip_4x_4}$$

$$(ig)^2 \int d^4x \int d^4y \Big((-i)^4 \, \delta(x-x_1) \, \delta(x-x_3) \, \Delta_{xy} \, \delta(y-x_2) \, \delta(y-x_4)$$

$$+ (3 \leftrightarrow 4) \Big)$$

$$= (-ig)^2 \int d^4x \, d^4y \Big(e^{-ip_1x} e^{-ip_2y} e^{ip_3x} e^{ip_4y} \, \Delta_{xy} + (3 \leftrightarrow 4) \Big)$$

$$= (-ig)^2 \int d^4x\, d^4y \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \Big(e^{-ip_1x} e^{-ip_2y} e^{ip_3x} e^{ip_4y} e^{-ik(x-y)} + (3 \leftrightarrow 4) \Big)$$

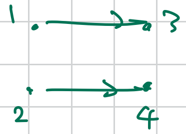
$$= (-ig)^2 \int \frac{d4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \Big((2\pi)^4 \delta(-p_1+p_3-k) (2\pi)^4 \delta(-p_2+p_4+k) + (3 \leftrightarrow 4) \Big).$$

$$= (-ig)^2 (2\pi)^4 \delta(-p_1-p_2+p_3+p_4) \left(\frac{i}{(-p_1+p_3)^2-m^2} + \frac{i}{(-p_1+p_4)-m^2} \right) + \mathcal{O}(g^3).$$

<u>Notes</u>: 1) Conservation of momentum & connected v.s. disconnected.

$$\langle f|S|i\rangle_c = \delta^{(4)}\Big(\sum_{\text{initial}} p_i - \sum_{\text{final}} p_f\Big)(\ldots)$$

$$\langle f|S|i\rangle_{\text{disconnected}} =$$



$$+ \ldots = \delta^{(4)}(p_1-p_3)\,\delta^{(4)}(p_2-p_4)(\ldots).$$

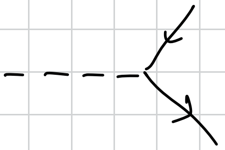
2) $$\underbrace{\langle f|S|i\rangle_c}_{\text{S-matrix}} = i(2\pi)^4\, \delta\Big(\sum_i p_i - \sum_f p_f\Big) \underbrace{\mathcal{A}(p_i, p_f, g)}_{\text{Amplitude}}$$

Feynman Rules

For the S-matrix in Scaler-Yukawa theory.

1) Draw Feynman diagrams relevant for the process

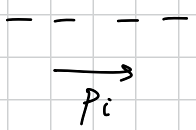
- use - - - - for ϕ (mesons) $(\Delta_F, a^\dagger)$
  $\longrightarrow$ for ψ (nucleons) $(c_{in}^\dagger, \hat{\Delta})$
  $\longleftarrow$ for ψ^+ (antinucleons) $(b_{in}^\dagger, \hat{\Delta})$.
- lines split via

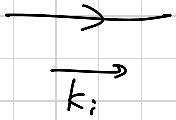


- only include connected diagram.

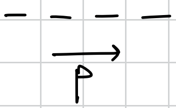
2). For each diagram, their contribution to S-matrix is

(i) Assign momenta to each line

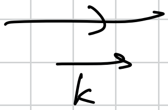




(ii) In the internal lines

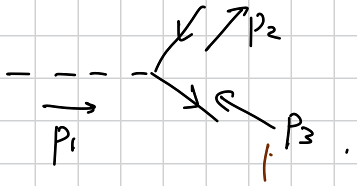


$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon}$$



$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-M^2+i\varepsilon}.$$

(iii) To each vertex, write



direction is arbitrary

$$= (-ig)\,(2\pi)^4\ \delta(p_1+p_3-p_2),$$

$$= (-ig)\,(2\pi)^4\ \delta(p_2-p_1-p_3)$$

(iv) External lines - do nothing.

3) Sum over all diagrams(*). Integrate over under momenta

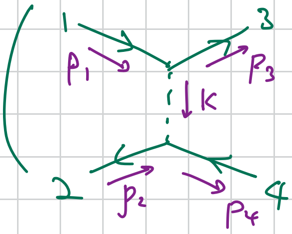
(*): sym factors if exist.

Example $\psi^\dagger\psi \to \psi^\dagger\psi$.

$b^\dagger_{p_2}\ c^\dagger_{p_1}\ \ b^\dagger_{p_4}\ c^\dagger_{p_3}$

Goal: $\langle f|S|i\rangle_c$ in leading order to g.

$g^2\Big($



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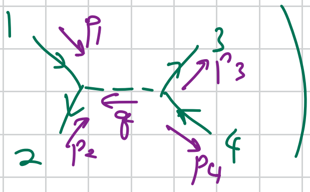


$\Big) + \mathcal{O}(g^3).$

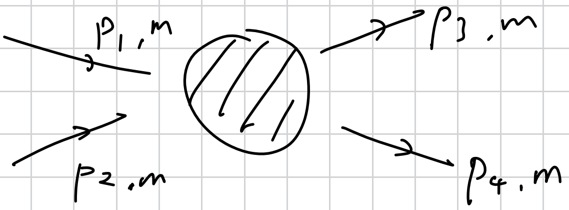
$$\langle f|S|i\rangle_c = \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}\,(-ig)^2(2\pi)^8\,\delta^{(4)}(p_1-k-p_3)\,\delta^{(4)}(p_2+k-p_4)$$

$$+\int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2+i\varepsilon}\,(-ig)^2(2\pi)^8\,\delta^{(4)}(p_1+p_2+q)\,\delta^{(4)}(-p_3-p_4-q)$$

$$= i(-ig)^2(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4)\left[\frac{1}{(p_4-p_2)^2-m^2+i\varepsilon}-\frac{1}{(p_3+p_4)^2-m^2+i\varepsilon}\right]$$
$$+\dots$$

Mandlestem Variables

In 2-2 scattering, it is useful to introduce some notation

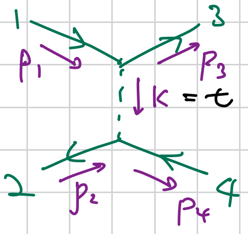


Conservation of 4-mom:

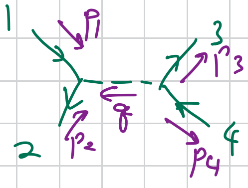
$$P_1 + P_2 = P_3 + P_4$$

Define

$$s \equiv (P_1+P_2)^2 = (P_3+P_4)^2$$
$$t \equiv (P_1-P_3)^2 = (P_2-P_4)^2$$
$$u \equiv (P_1-P_4)^2 = (P_2-P_3)^2$$



t-channel



s-channel

Identity: $s+t+u = \sum_i m_i^2 = 4m^2$

$s = 4E^2$, with E = total centre of mass energy

4. Dirac Equations

4.1 Representations Lorentz group.

$$\phi^a(x) \to D[\Lambda]^a{}_b \, \phi^b(x)$$

where $D[\Lambda]$ is a rep. of Lorentz.

$$D[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}R^{\rho\sigma}\right).$$

and $\Omega_{\rho\sigma} = -\Omega_{\sigma\rho}$ 6 numbers (3 rot + 3 boost). $R^{\rho\sigma}$ is a generator of the Lorentz algebra. hence it satisfies

$$[R^{\rho\sigma}, R^{\tau\nu}] = \eta^{\sigma\tau}R^{\rho\nu} - \eta^{\rho\tau}R^{\sigma\nu} + \eta^{\rho\nu}R^{\sigma\tau} - \eta^{\sigma\nu}R^{\rho\tau} \qquad (*)$$

What are examples of R's ?

Example Infinite-dim rep.

$$R^{\rho\sigma} \to L^{\rho\sigma} = (x^\rho \partial^\sigma - x^\sigma \partial^\rho)$$

Check that the L satisfy $(*)$.

Example A 4x4 rep.

$$R^{\rho\sigma} \to (M^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$$

M satisfy $(*)$.

$$\Lambda \equiv D[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$$

$$V^\mu \to D[\Lambda]^\mu{}_\nu V^\nu = \Lambda^\mu{}_\nu V^\nu.$$

Rot^n about z-axis

$$(M^{xy})^\mu{}_\nu = \begin{pmatrix} & & & \\ & & -1 & \\ & 1 & & \\ & & & \end{pmatrix} \xrightarrow[\Omega_{xy}=\theta]{} \Lambda = \exp\left(\tfrac{1}{2}\Omega M\right) = \begin{pmatrix} 1 & & & \\ & c & -s & \\ & s & c & \\ & & & 1 \end{pmatrix}$$

Boost along x

$$(M^{0x})^\mu{}_\nu = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix} \xrightarrow[\Omega_{0x}=\eta]{} \Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta & -\sinh\eta & & \\ -\sinh\eta & \cosh\eta & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Spinor Representations

We want to build a different R's. One way to construct is to consider first Clifford algebra.

Def^n $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_{4\times4}$, where γ^μ 4x4.

Equivalently, $(\gamma^0)^2 = \mathbb{1}$, $(\gamma^i)^2 = -\mathbb{1}$, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu\neq\nu$.

A simple rep. of γ's (chiral rep.)

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

where σ^i Pauli matrices. $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$

Claim Define $S^{\rho\sigma} := \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\gamma^\sigma\gamma^\rho$. then S's form a rep. of Lorentz.

Pf: Check that $S^{\rho\sigma}$ satisfy (*)

What transforms under it?

Def^n (Spinor)

$$\psi^a \longrightarrow D[\Lambda]^a{}_b\,\psi^b(\Lambda^{-1}x) = S[\Lambda]^a{}_b\,\psi^b(\Lambda^{-1}x)$$

where $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$.

Note: S's are 4x4 rep. of Lorentz. Is S's $\overset{!}{\simeq}$ M's, the other 4x4 rep? Answer: No!

Rotations of spinors:

$$S^{ij} = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = -\frac{i}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \qquad (i\neq j)$$

$$S^{12} = S^{xy} = -\frac{i}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \Rightarrow S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{12}\cdot 2S^{12}\right) = \exp\left(-\frac{i\Omega_{12}}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\right)$$

What happens $\Omega_{12} = \theta = 2\pi$?

$$S[\Lambda]_{2\pi} = \begin{pmatrix} e^{-i\pi\sigma_3} & 0 \\ 0 & e^{i\pi\sigma_3} \end{pmatrix} = -\mathbb{1} \Rightarrow \psi \to -\psi$$

$$\Lambda = M[\Lambda]_{2\pi} = \mathbb{1} \Rightarrow V \to V$$

Boost of spinor :

$$S^{0i} = \frac{1}{2} \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$\Rightarrow S[\Lambda] = \begin{pmatrix} e^{\eta\cdot\sigma/2} & 0 \\ 0 & e^{-\eta\cdot\sigma/2} \end{pmatrix}, \quad \text{where } \Omega_{i0} = -\Omega_{0i} = \eta_i .$$

Very different from $\Lambda = \exp(\frac{1}{2}\Omega M)$. $\Rightarrow$ spinors are not vectors.

5.2 Action

We have introduced

$$\psi(x) \to S[\Lambda]\, \psi(\Lambda^{-1}x) .$$

The next task is to construct comb^n of ψ's s.t. they are Lorentz inv. $\Rightarrow$ actions / Lagrangians.

We need some object $\mathcal{O}(x) \to \mathcal{O}(\Lambda^{-1}x)\, S[\Lambda]^{-1}$. s.t.

$$\mathcal{O}(x)\,\psi(x) \to \mathcal{O}\, S[\Lambda]^{-1} S[\Lambda]\psi = \mathcal{O}\psi \in \mathbb{C}, \mathbb{R}.$$

A first go to determine $\mathcal{O}$ is to consider

$$\psi^\dagger(x) = (\psi^*(x))^T .$$

From the transf^n of ψ,

$$\psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)\, S[\Lambda]^\dagger .$$

where

$$S[\Lambda]^\dagger = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger\right) .$$

In the chiral rep.,

$$\gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger \Rightarrow \gamma^{0\dagger} = \gamma^0 \; , \; (\gamma^i)^\dagger = -\gamma^i$$

This means,

$$(S^{\rho\sigma})^\dagger = \frac{1}{4}\left[(\gamma^\sigma)^\dagger, (\gamma^\rho)^\dagger\right] = -\gamma^0 S^{\rho\sigma} \gamma^0$$

then

$$S[\Lambda]^\dagger = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger\right)$$

$$= \exp\left(\frac{1}{2}\Omega_{\rho\sigma}(-\gamma^0 S^{\rho\sigma}\gamma^0)\right)$$

$(\gamma^0)^2 = \mathbb{1}$ ↘

$$= \gamma^0 \exp\left(-\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^0$$

What is $S[\Lambda]^{-1}$?

$$S[\Lambda]^{-1} = \exp\left(-\frac{1}{2}\Omega_{\rho\sigma}\Omega^{\rho\sigma}\right)$$

Hence $\psi^\dagger$ does not transform with $S[\Lambda]^{-1}$ (it is not $\mathcal{O}$)

<u>Claim</u> Define $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$ Adjoint spinor, then $\bar{\psi}(x)\psi(x)$ is Lorentz invar.

<u>Pf</u> :

$$\bar{\psi}(x) \longrightarrow \psi^\dagger(\Lambda^{-1}x)\, S[\Lambda]^\dagger\, \gamma^0$$

$$= \psi^\dagger(\Lambda^{-1}x)\, \gamma^0 S[\Lambda]^{-1} \gamma^0 \gamma^0$$

$$= \bar{\psi}(\Lambda^{-1}x)\, S[\Lambda]^{-1}$$

$\Rightarrow \bar{\psi}(x)\psi(x)$ Lorentz scaler. □

( )( )( ) for each $\mu = 0, \dots, 3$

<u>Claim</u> $\bar{\psi}\gamma^\mu\psi$ transforms as a 4-vector

<u>Pf</u> : Need to prove

$$\underbrace{S[\Lambda]^{-1}\gamma^\mu S[\Lambda]}_{\text{from def}^n} = \underbrace{\Lambda^\mu{}_\nu \gamma^\nu}_{\text{expect from a 4-vec.}} .$$

Will show this infinitesimally.

$$S[\Lambda] = \mathbb{1} + \tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \mathcal{O}(\Omega^2)$$

$$\Lambda = \mathbb{1} + \tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma} + \mathcal{O}(\Omega^2).$$

$$\text{LHS} = \left(\mathbb{1} - \tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \dots\right)\gamma^\mu\left(\mathbb{1} + \tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \dots\right)$$

$$= \gamma^\mu - \tfrac{1}{2}\Omega_{\rho\sigma}[S^{\rho\sigma},\gamma^\mu] + \dots$$

$$\text{RHS} = \left(\mathbb{1} + \tfrac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma} + \dots\right)^\mu{}_\nu\gamma^\nu$$

$$= \gamma^\mu + \tfrac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu + \dots$$

For LHS = RHS, to leading order of Ω,

$$-[S^{\rho\sigma},\gamma^\mu) \overset{!}{=} (M^{\rho\sigma})^\mu{}_\nu\gamma^\nu. \qquad (*)$$

$$(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu = (\eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu)\gamma^\nu = \eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho.$$

$$[S^{\rho\sigma},\gamma^\mu] = \gamma^\mu\eta^{\sigma\mu} - \gamma^\sigma\eta^{\rho\mu}$$

So $(*)$ true. So LHS = RHS $\Rightarrow$ $\bar\psi\gamma^\mu\psi$ transf. as 4-vec. □

__Claim__ Can also show,

- $\bar\psi\gamma^\mu\gamma^\nu\psi \to$ 2-tensor
- $\bar\psi\partial_\mu\psi \longrightarrow$ covector

With this we can build a __minimal__ action

least # of deriv. and powers of field.

$$S = \int d^4x \left(\bar\psi\, i\gamma^\mu\partial_\mu\psi - m\bar\psi(x)\psi(x)\right)$$

This is __Dirac action__.

reality of the action:

$$(\bar\psi i\gamma^\mu\partial_\mu\psi)^\dagger$$

$$= \underbrace{-i\partial_\mu(\bar\psi\gamma^\mu\psi)}_{\text{Total deriv}} + i\bar\psi\gamma^\mu\partial_\mu\psi.$$

E.o.m. :
$$\frac{\partial\mathcal{L}}{\partial\bar\psi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)}\right), \quad \frac{\partial\mathcal{L}}{\partial\psi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right).$$

$$\Rightarrow \quad (i\gamma^\mu\partial_\mu - m)\psi = 0, \qquad \bar\psi(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0.$$

Note : Here, the simplest action just has one deriv due to ∃ γ^μ. Hence, 1st order ODE.

There is also 2nd order e.o.m.

$$(i\gamma^\mu\partial_\mu + m)(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\Rightarrow (-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi = 0$$

$$\Rightarrow \Big(-\underbrace{\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}}_{\eta^{\mu\nu}}\partial_\mu\partial_\nu - m^2\Big)\psi = 0.$$

$$\Rightarrow (\partial^2 + m^2)\psi = 0$$

$\Rightarrow$ ψ also satisfy KG.

5.3 Symmetries

(i) Space-time translations

$$x^\mu \to x^\mu + \epsilon^\mu, \quad \epsilon^\mu \text{ const. vec.}$$

$$\delta\psi = \epsilon^\mu\partial_\mu\psi$$

Energy-momentum tensor

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi + \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)}}_{=0}\partial_\nu\bar\psi - \underbrace{\delta^\mu{}_\nu\mathcal{L}}_{=0\ (\text{e.o.m.})} = \bar\psi i\gamma^\mu\partial_\nu\psi. \quad \text{on-shell}$$

$$\Rightarrow E = H = \int d^3x\, T^{00} = \int d^3x\; i\bar\psi\gamma^0\dot\psi.$$

(ii) Lorentz transf^n

$$\psi \to S[\Lambda]\,\psi(\Lambda^{-1}x)$$

infinitesimally,

$$\delta\psi = \tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\psi - \Omega^{\rho\sigma}x_\sigma\partial_\rho\psi.$$

Current

$$(J^\mu)^{\lambda\nu} = \underbrace{-x^\nu T^{\mu\lambda} + x^\lambda T^{\mu\nu}}_{\substack{\text{usual ang. mom.} \\ \text{(same exp. as, e.g. } \phi)}} + \underbrace{i\bar{\psi}\gamma^\mu S^{\lambda\nu}\psi}_{\substack{\text{new, intrinsic} \\ \Rightarrow \text{ spin}}}.$$

(iii) Internal vector sym.

$$\psi \to e^{-i\alpha}\psi, \qquad \alpha \in \mathbb{R} \text{ const.}$$

$$\bar{\psi} \to e^{i\alpha}\bar{\psi}.$$

Current

$$j^\mu = \bar{\psi}\gamma^\mu\psi.$$

Charge

$$Q = \int d^3x \ \bar{\psi}\gamma^0\psi = \int d^3x \ \psi^\dagger\psi.$$

5.4 Plane wave solutions

Find solutions to

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

We also know

$$(\partial^2 + m^2)\psi = 0$$

From KG, we can see

$$\psi(x) \sim u(p)e^{-ip\cdot x} + v(p)e^{ip\cdot x}$$

with $p^2 = m^2$ and $u(p)$, $v(p)$ are spinor.

Now, if we impose Dirac. eqn on plane waves,

$$(-p^\mu\gamma_\mu + m)\,u(p) = 0 \quad \Rightarrow \quad (-\not{p} + m)\,u = 0 \qquad (1)$$

$$(p^\mu\gamma_\mu + m)\,v(p) = 0 \quad \Rightarrow \quad (\not{p} + m)\,v = 0 \qquad (2)$$

Notation : $p^\mu\gamma_\mu = \not{p}$, $\gamma^\mu\partial_\mu = \not{\partial}$

Solve for (1), (2) using explicit rep. for γ.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (\mathbb{1}, \sigma^i)$, $\bar{\sigma}^\mu = (\mathbb{1}. -\sigma^i)$.

Then (1) :

$$\begin{pmatrix} m & p\cdot\sigma \\ -p\cdot\bar{\sigma} & m \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0,$$

where u_1, u_2 are 2-cpt objects.

$$\Rightarrow \begin{cases} mu_1 = (p\cdot\sigma)\, u_2 \\ mu_2 = (p\cdot\bar{\sigma})\, u_1 \end{cases}$$

But these are the same, since

$$m^2 = (p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2$$

Then

$$\overset{s}{u}(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^s \end{pmatrix}. \quad , s = 1, 2,$$

and $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Similarly, solving (2),

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\ \xi^s \end{pmatrix}$$

Then the most general sol^n

$$\psi(x) = \sum_{s=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[b_p^s\, u^s(p) e^{-ip\cdot x} + c_p^{s*}\, v^s(p) e^{ip\cdot x} \right]$$

5.5 Weyl (Chiral) Spinor

In the chiral rep.,

$$S[\Lambda_{rot}] = \begin{pmatrix} e^{i\varphi\cdot\sigma/2} & 0 \\ 0 & e^{i\varphi\cdot\sigma/2} \end{pmatrix}, \quad \Omega_{ij} = -\varepsilon_{ijk}\varphi^k.$$

$$S[\Lambda_{boost}] = \begin{pmatrix} e^{\chi\cdot\sigma/2} & 0 \\ 0 & e^{-\chi\cdot\sigma/2} \end{pmatrix}, \quad \Omega_{i0} = \chi_i.$$

$\Rightarrow$ Reducible rep. due to 2x2 block structure

Then in chiral rep.

$$\psi = \begin{pmatrix} u_+ \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_- \end{pmatrix}$$

$u_\pm$ two cpt. objects where under rot^n

$$u_+ \longrightarrow e^{i\underline{\varphi}\cdot\underline{\sigma}/2} u_+ , \quad u_- \longrightarrow e^{i\underline{\varphi}\cdot\underline{\sigma}/2} u_-$$

same sign

and under boost

$$u_+ \longrightarrow e^{\underline{\chi}\cdot\underline{\sigma}/2} u_+ , \quad u_- \longrightarrow e^{-\underline{\chi}\cdot\underline{\sigma}/2} u_- .$$

opp. sign

$u_\pm$ are weyl (chiral) spinors.

This is obvious in the chiral rep for γ's, but it is general property. We can make this manifest by introducing

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 .$$

It satisfies

$$\{\gamma^5, \gamma^\mu\} = 0 , \qquad (\gamma^5)^2 = \mathbb{1} ,$$

$$[S^{\mu\nu}, \gamma^5] = 0 \qquad \text{invar. under Lor.}$$

In chiral rep,

$$\gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} .$$

Then we can define propagation operator.

$$P_+ \equiv \frac{1}{2}(\mathbb{1} + \gamma^5) \qquad P_- = \frac{1}{2}(\mathbb{1} - \gamma^5)$$

with

$$P_+^2 = P_+ , \quad P_-^2 = P_- , \quad P_+P_- = 0 .$$

Then

$$\psi_+ = P_+\psi = \begin{pmatrix} u_+ \\ 0 \end{pmatrix}$$

chiral rep.

$$\psi_- = P_-\psi = \begin{pmatrix} 0 \\ u_- \end{pmatrix} .$$

Final remark: Dirac. Lagrangian in terms of $u_\pm$

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi.$$

chiral rep.

$$= \underbrace{i u_-^\dagger \sigma^\mu \partial_\mu u_-}_{\text{Lor. inv.}} + \underbrace{i u_+^\dagger \bar{\sigma}^\mu \partial_\mu u_+}_{\text{Lor. inv.}} - m(\underbrace{u_+^\dagger u_- + u_-^\dagger u_+}_{\text{mix term.}}).$$

no mass term for just u_+ or u_-.

6. Quantising Dirac Field: Free Theory

6.1 Canonical Quantisation

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

Canonical mom:

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)} = i\psi\gamma^0 = i\psi^\dagger.$$

We want to declare some rule/commutator between ψ, Π.

Route 1:

$$[\psi_a(\underline{x},t), \Pi_b(\underline{y},t)] = i\delta_{ab}\delta^{(3)}(\underline{x}-\underline{y}).$$

$$[\psi_a(\underline{x},t), \psi_b(\underline{y},t)] = 0.$$

$$i[\psi_a(\underline{x},t), \psi_b^\dagger(\underline{y},t)] = i\delta_{ab}\delta^{(3)}(\underline{x}-\underline{y})$$

Route 2:

$$\{\psi_a(\underline{x},t), \psi_b^\dagger(\underline{y},t)\} = i\delta_{ab}\delta^{(3)}(\underline{x}-\underline{y})$$

$$\{\psi_a(\underline{x},t), \psi_b(\underline{y},t)\} = 0.$$

For both routes,

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_{\underline{p}}^s u^s(\underline{p}) e^{-ipx} + c_{\underline{p}}^{s\dagger} v^s(\underline{p}) e^{ipx}\right).$$

Inverting,

$$b_{\mathbf{p}}^{s} = \frac{1}{\sqrt{2\omega_p}} \int d^3x \; e^{ipx} \; \bar{u}^{s}(\mathbf{p}) \, \gamma^0 \, \psi(x)$$

$$c_{\mathbf{p}}^{s} = \frac{1}{\sqrt{2\omega_p}} \int d^3x \; e^{-ipx} \; \bar{\psi}(x) \, \gamma^0 \, v^{s}(\mathbf{p})$$

and

$$H = \int d^3x \; \Pi \dot{\psi}$$

$$= i \int d^3x \; \psi^\dagger \dot{\psi}$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} \; \omega_p \left(b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^{s} - c_{\mathbf{p}}^{s} c_{\mathbf{p}}^{\dagger} \right)$$

Goal: construct a Fock space, ie.

(a) states have +ve norm

(b) spectrum of H to be bounded.

For route 1:

$$[b_{\mathbf{p}}^{s}, b_{\mathbf{p}'}^{s'\dagger}] = (2\pi)^3 \, \delta^{(3)}(\mathbf{p}-\mathbf{p}') \, \delta_{ss'}$$

$$[c_{\mathbf{p}}^{s}, c_{\mathbf{p}'}^{s'\dagger}] = -(2\pi)^3 \, \delta^{(3)}(\mathbf{p}-\mathbf{p}') \, \delta_{ss'} \qquad (*)$$

all other zero, and

$$[H, b_{\mathbf{p}}^{s\dagger}] = \omega_p \, b_{\mathbf{p}}^{s\dagger}$$

$$[H, c_{\mathbf{p}}^{s\dagger}] = \omega_p \, c_{\mathbf{p}}^{s\dagger} .$$

Create with $b^\dagger$, $c^\dagger$, then (b) is ok, but (a) fails, due to (*).

Create with $b^\dagger$, c, then (a) is ok, but (b) fails.

For route 2:

$$\{ b_{\mathbf{p}}^{s}, b_{\mathbf{p}'}^{s'\dagger} \} = (2\pi)^3 \, \delta(\mathbf{p}-\mathbf{p}') \, \delta_{ss'}$$

$$\{ c_{\mathbf{p}}^{s}, c_{\mathbf{p}'}^{s'\dagger} \} = (2\pi)^3 \, \delta(\mathbf{p}-\mathbf{p}') \, \delta_{ss'}$$

and

$$[H, b_p^{s\dagger}] = \omega_p b_p^{s\dagger}$$

$$[H, c_p^{s\dagger}] = \omega_p c_p^{\dagger}$$

With this we can create a correct Fock space.

Following route 2, the Fock space is then

- ground state: $b_p^s|0\rangle = c_p^s|0\rangle = 0$.

- one particle space: $|b\rangle = \sqrt{2\omega_p}\; b_p^{s\dagger}|0\rangle$

  $|c\rangle = \sqrt{2\omega_p}\; c_p^{s\dagger}|0\rangle$.

- Hamiltonian (and normal ordering).

$$H = \sum_s \int \frac{d^3p}{(2\pi)^3}\, \omega_p \, :\left(b_p^{s\dagger} b_p^s - c_p^s c_p^{\dagger} \right):$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(b_p^{s\dagger} b_p^s + c_p^{s\dagger} c_p^s \right)$$

$$\Rightarrow \quad H|0\rangle = 0, \qquad H|b\rangle = \omega_p|b\rangle, \quad H|c\rangle = \omega_p|c\rangle.$$

- U(1) internal sym:

$$Q = \sum_s \int \frac{d^3p}{(2\pi)^3} \left(b_p^{s\dagger} b_p^s - c_p^{s\dagger} c_p^s \right)$$

$$Q|b\rangle = |b\rangle, \qquad Q|c\rangle = -|c\rangle$$

- Ang mom: J_z in the rest frame $(p = 0)$.

$$J_z|b, s\rangle = \pm \tfrac{1}{2}|b^s\rangle, \qquad s = \underset{+}{1}, \underset{-}{2}.$$

$$J_z|c, s\rangle = \pm \tfrac{1}{2}|c^s\rangle.$$

Our Fock space has charged particles (±) and with intrinsic spin $\pm\frac{1}{2}$.

Multiple particle states, e.g.

$$\sqrt{2\omega_{p_1}}\sqrt{2\omega_{p_2}}\, b_{p_1}^{s_1\dagger}\, b_{p_2}^{s_2\dagger}\,|0\rangle = |p_1, s_1 ; p_2, s_2\rangle = -|p_2, s_2 ; p_1, s_1\rangle$$

Note: Hamiltonian and vacuum energy before normalising

$$H = \sum_s \int \frac{d^3p}{(2\pi)^3}\,\omega_p \left(b_p^{s\dagger} b_p^s - c_p^{s\dagger} c_p^s \right)$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3}\,\omega_p \left(b_p^{s\dagger} b_p^s + c_p^{s\dagger} c_p^s \right) - \int \frac{d^3p}{(2\pi)^3}\,\omega_p \cdot 2(2\pi)^3 \delta^{(3)}(\underline{0}) \ldots$$

Feynman Propagator

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_p^s u^s(p)\, e^{-ipx} + c_p^{s\dagger} v^s(p)\, e^{ipx} \right)$$

$$\bar{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_p^{\dagger}\, \bar{u}^s(p)\, e^{ipx} + c_p^s\, \bar{v}^s(p)\, e^{-ipx} \right).$$

First evaluate

$$\langle 0|\, \psi_a(x)\, \bar{\psi}_b(y)\, |0\rangle$$

$$= \sum_{s,s'} \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2\omega_p}} \frac{1}{\sqrt{2\omega_{p'}}}\, e^{-ipx} e^{ip'y}\, u_a^s(p)\, \bar{u}_b^{s'}(p')\, \langle 0|\, b_p^s\, b_{p'}^{s'\dagger}\, |0\rangle$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p}\, u_a^s(p)\, \bar{u}_b^s(p)\, e^{-ip(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} (\not{p} + m)_{ab}\, e^{-ip(x-y)}.$$

$$= (i\not{\partial}_x + m) \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p}\, e^{-ip(x-y)}}_{D(x-y)\,:\ \text{propagator of scaler}} = (i\not{\partial}_x + m)\, D(x-y).$$

Next.

$$\langle 0|\, \bar{\psi}_b(y)\, \psi_a(x)\, |0\rangle$$

key! dictates time ordering rules

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} (\not{p} - m)\, e^{ip(x-y)} = -(i\not{\partial}_x + m)\, D(y-x).$$

The rules for normal ordering for spinors are

$$T(\psi_a(x)\,\bar\psi_b(y)) = \begin{cases} \psi_a(x)\,\bar\psi_b(y) & x^0 > y^0 \\ -\bar\psi_b(y)\,\psi_a(x) & y^0 > x^0 \end{cases}.$$

Next, as we construct Feynman propagator

$$\langle 0|T\psi_a(x)\,\bar\psi_b(y)|0\rangle = (i\not\partial_x + m)_{ab}\,\underset{\substack{\uparrow \\ \text{F.P. for massive} \\ \text{scaler field}}}{\Delta_F(x-y)} \equiv \underset{\substack{\uparrow \\ \text{F.P. for} \\ \text{Dirac spinor}}}{S_F(x-y)}.$$

$\rightarrow$ $S_F(x-y)$ is Lorentz covariant.

It follows that

$$(i\not\partial_x - m)\,S_F(x-y) = i\,\delta^{(4)}(x-y).$$

Since $(\Box_x + m^2)\,\Delta_F(x-y) = -i\,\delta^{(4)}(x-y)$.

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{i(\not p + m)}{p^2 - m^2 + i\epsilon}\,e^{-ip(x-y)}$$

Wick Thm for Spinors:

$$T(\psi(x)\,\bar\psi(y)) = :\psi(x)\,\bar\psi(y): + S_F(x-y)$$

Note that

$$T(\psi_1\psi_2\psi_3\psi_4) \underset{\substack{\uparrow \\ x_3^0 > x_1^0 > x_4^0 > x_2^0}}{=} (-1)^{\overset{3 = \#\text{ swaps}}{3}}\,\psi_3\,\psi_1\,\psi_4\,\psi_2$$

Spin and Statistics theorem

We have encountered

scaler field (spin 0) $\longrightarrow$ comm relations (Bose statistics)

spinor field (spin 1/2) $\rightarrow$ anti-comm relations (Fermi stat.).

There are 3 ways that we can discover this.

(a) Stability : Hamiltonian of free theory is bounded. Good Fock space

(b) Lorentz inv. of S-matrix : implications in $\langle T\, \mathcal{O} \ldots \mathcal{O} \rangle$

(c) Casuality : $[\underset{\uparrow}{\mathcal{O}(x)}, \underset{\uparrow}{\mathcal{O}(y)}] = 0$ if spacelike $(x-y)^2 < 0$.

observables

Note : $\{\psi(x), \psi(y)\} = 0$ if $(x-y)^2 < 0$.

6.2 QFT for spinors : Interactions

Example to illustrate how to evaluate S-matrix will be Yukawa theory.

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} \mu^2 \phi^2 + \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi .$$

$$\mathcal{L}_{int} = -\lambda \phi \bar{\psi} \psi .$$

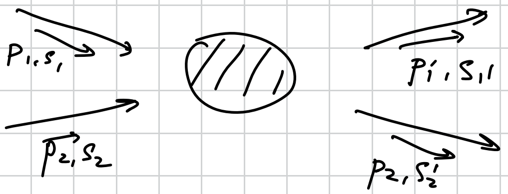
was an EFT for nuclear force $\begin{cases} \phi : \text{meson} \\ \psi, \bar{\psi} : \text{(anti)-nucleons (spin)} \end{cases}$

Dimensions : $[\mathcal{L}] = 4$, $[\partial_\mu] = 1$, $[\phi] = 1$, $[\mu] = 1$, $[\psi] = 3/2$

$[m] = 1$, $[\lambda] = 0$

So λ is marginal

LSZ formula (spinors)



$$|i, -\infty\rangle = \sqrt{2\omega_{p_1}} \sqrt{2\omega_{p_2}} \quad b_{p_1}^{s_1 \dagger}(-\infty) \; c_{p_2}^{s_2 \dagger}(-\infty) \; |\Omega\rangle$$

$$|f, \infty\rangle = \sqrt{2\omega_{p_1'}} \sqrt{2\omega_{p_2'}} \quad b_{p_1'}^{s_1' \dagger}(\infty) \; c_{p_2'}^{s_2 \dagger}(\infty) \; |\Omega\rangle .$$

S-matrix for this process is

$\langle f|S|i\rangle$

$$= \sqrt{2\omega_{p_1}}\sqrt{2\omega_{p_2}}\sqrt{2\omega_{p_3}}\sqrt{2\omega_{p_4}}\ \langle\Omega|T(c^{s_2'}_{p_2'}(\infty)\, b^{s_1'}_{p_1'}(\infty)\, b^{s_1\dagger}_{p_1}(-\infty)\, c^{s_2\dagger}_{p_2}(-\infty)|\Omega\rangle \quad (*).$$

We want to relate $\begin{matrix} b_p^{s\dagger}(\infty) \\ c_p^{s\dagger}(\infty)\end{matrix}$ with $\begin{matrix} b_p^{s\dagger}(-\infty) \\ c_p^{s\dagger}(-\infty)\end{matrix}$ (and for destruction too).

Consider

$$i\int d^4x\ \bar{\psi}(i\overleftarrow{\partial\!\!\!/}+m)\,u_s e^{-ip\cdot x} \qquad (\dagger)$$

with $(p\!\!\!/-m)\,u_s(p)=0$ and $\omega_p^2=\vec{p}^2+m^2$.

Note: In free thy this is 0.

$$(\dagger) \overset{IBP}{=} i\int d^4x\ \bar{\psi}(i\gamma^0\overleftarrow{\partial_0} - i\gamma^i\overrightarrow{\partial_i} + m)\,u_s(p)\,e^{-ip\cdot x}$$

$$= i\int d^4x\ \bar{\psi}(x)\left(i\gamma^0\overleftarrow{\partial_0} - i\gamma^i(ip^i) + m\right)u_s(p)\,e^{-ip\cdot x}.$$

$$= i\int d^4x\ \bar{\psi}(x)\left(i\gamma^0\overleftarrow{\partial_0} + p^0\gamma^0\right)u_s(p)e^{-ip\cdot x} \qquad \overset{\text{using}}{(p^0\gamma^0 - p^i\gamma^i - m)u_s(p)=0}$$

$$= i\int d^4x\ \bar{\psi}(x)\left(i\gamma^0\overleftarrow{\partial_0} + i\gamma^0\overrightarrow{\partial_0}\right)u_s(p)\ e^{-ipx}$$

$$= i\int d^4x\ \partial_0\left(i\bar{\psi}\gamma^0 u_s(p)\,e^{-ip\cdot x}\right).$$

Total time deriv. Use that at $t\to\pm\infty$, Hilbert space equals free thy Fock space

$$= -\sqrt{2\omega_p}\left(b_p^{s\dagger}(+\infty) - b_p^{s\dagger}(-\infty)\right) \qquad (A)$$

Similarly,

$$\sqrt{2\omega_p}\left(c_p^{s\dagger}(-\infty) - c_p^{s\dagger}(+\infty)\right) = -i\int d^4x\ e^{-ipx}\,\bar{v}_s(p)\,(-i\partial\!\!\!/+m)\psi(x). \qquad (B).$$

Replacing (A), (B) (and destruction) in (*),

$$\langle f|S|i\rangle = \prod_{j=1}^{4} i\int d^4x_j \, e^{-ip_2x_2}\, \bar{v}_{s_2}(p_2)\left(-i\not{\partial}_2+m\right) e^{ip_1'x_3}\,\bar{u}_{s_1}(p_1')\left(-i\not{\partial}_3+m\right)$$

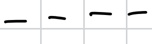
$$\langle\Omega|T\bar{\psi}(x_4)\,\psi(x_3)\,\bar{\psi}(x_1)\,\psi(x_2)|\Omega\rangle$$

$$\left(\left(i\overleftarrow{\not{\partial}}_1+m\right)u_s(p_1)e^{-ip_1\cdot x_1}\right)\left(\left(i\overleftarrow{\partial}_4+m\right)v_{s_2'}(p_2')e^{ip_2'\cdot x_4}\right)$$

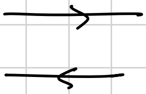
Feynman rules

Rules for $\langle f|S|i\rangle$ in Yukawa thy. Emphasis on spinors.

Not^n:



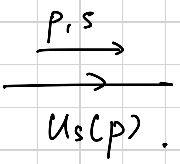
: ϕ



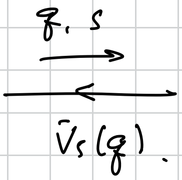
} ψ, $\bar{\psi}$.

1) Ingoing fermions

$b_p^{s\dagger}$:

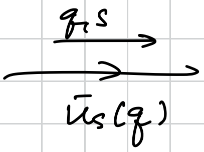


$c_q^{s\dagger}$:

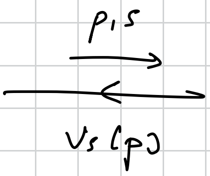


2) Outgoing fermions

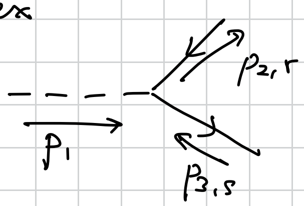
b_q^s :



c_p^s :

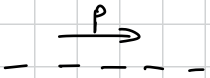


3) Vertex

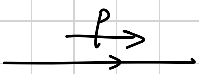


$= (-i\lambda)\,(2\pi)^4\,\delta^{(4)}(p_1+p_3-p_2)$.

4) Internal lines



$$= \int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-\mu^2+i\varepsilon}$$



$$= \int\frac{d^4p}{(2\pi)^4}\,\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}.$$

note: this is 4x4 matrix

5) Add extra minus sign for statistics of fermion.

What is behind rule (5)?

Consider

$$\langle \Omega | T(\psi(x_1)\,\psi(x_2)\,\bar\psi(x_3)\,\bar\psi(x_4))|\Omega\rangle = \langle \psi_1 \psi_2 \bar\psi_3 \bar\psi_4\rangle .$$

In free thy,

- Wick's thm
- Schw - Dyson eqn. for fermions }

$$(i\not\partial_x - m)\,\langle \psi(x)\,\bar\psi_1 \dots \psi_n\rangle = i\sum_{\substack{j=1 \\ \psi \text{ with } \bar\psi, \\ \text{and signed}}}^{n} \delta(x-x_j).$$

$$\langle \psi_1 \psi_2 \bar\psi_3 \bar\psi_4\rangle = \int d^4x \;\; \delta(x-x_1)\,\langle \psi_x \psi_2 \bar\psi_3 \bar\psi_4\rangle$$

Recall $(i\not\partial_x - m)\, S_F(x-y) = i\delta(x-y)$, and hence

$$S_F(x-y)\,(i\overleftarrow{\not\partial}_y + m) = -i\delta(x-y).$$

Hence

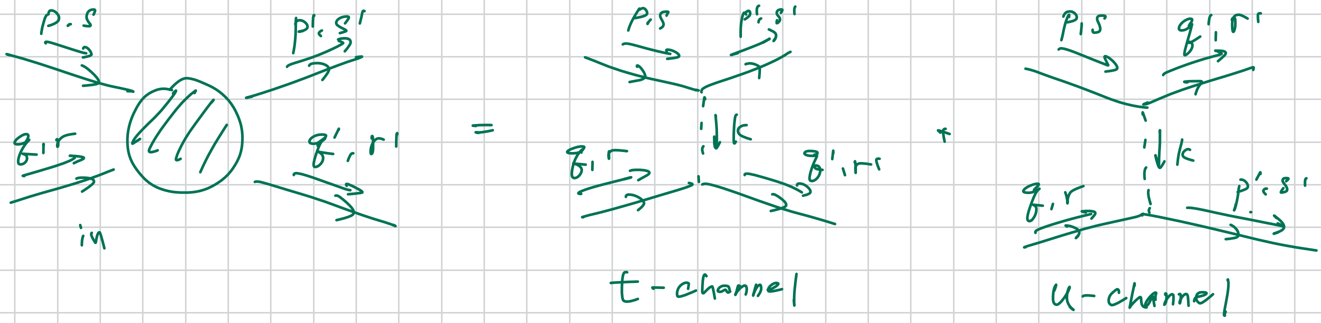
$$\langle \psi_1 \psi_2 \bar\psi_3 \bar\psi_4\rangle = i\int d^4x \;\; S_F(x-x_1)\,(i\overleftarrow{\not\partial}_x + m)\,\langle \psi_x \psi_2 \bar\psi_3 \bar\psi_4\rangle$$

$$\overset{\text{IBP}}{=} i\int d^4x \; S_F(x_1-x)\,(-i\overrightarrow{\not\partial}_x + m)\,\langle \psi_x \psi_2 \bar\psi_3 \bar\psi_4\rangle .$$

odd perm to contract x_1 3.

$$\overset{\text{S-D}}{=} -i^2\int d^4x \; S_F(x_1-x)\,\Big[-\delta(x_3-x)\,S_F(x_2-x_4) + \delta(x_4-x)\,S_F(x_2-x_3)\Big]$$

Example Nucleon scattering: $\psi\psi \to \psi\psi$ at tree level.
$(b^\dagger b^\dagger \to b^\dagger b^\dagger)$.



$$\langle f|S|i\rangle_{c,\,\substack{\text{tree-}\\\text{level}}} = \bar{u}_{s'}(p')\cdot u_s(p)\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}\,(-i\lambda)(2\pi)^4\,\delta^{(4)}(p-p'-k).$$

$$\bar{u}_{r'}(q')\,u_r(q)\cdot(-i\lambda)\,(2\pi)^4\,\delta(q+k-q').$$

$$-\,\bar{u}_{r'}(q')\,u_s(p)\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-\mu^2+i\varepsilon}\,(-i\lambda)\,(2\pi)^4\,\delta(p-q'-k)$$

$$\bar{u}_{s'}(p')\,u_r(q)\,(-i\lambda)\,(2\pi)^4\,\delta(q+k-p')$$

$$= i(2\pi)^4\,\delta(p+q-p'-q')\,(-i\lambda)^2\cdot$$

$$\left(\frac{(\bar{u}_{s'}(p')\cdot u_s(p))(\bar{u}_{r'}(q')\cdot u_r(q))}{(p-p')^2-\mu^2} - \frac{(\bar{u}_{r'}(q')\cdot u_s(p))(\bar{u}_{s'}(p')\cdot u_r(q))}{(p-q')^2-\mu^2}\right)$$

Spin sums

A common quantity to evaluate in particle physics is the cross section. In short, master formula

$$\underset{\substack{\nearrow\\\text{\# events}\\\text{per unit}\\\text{time}}}{N} = \underset{\substack{\uparrow\\\text{Flux}\\\text{(\# incoming particle}\\\text{per unit area, unit time)}}}{F}\;\sigma \;\nwarrow\;\text{cross section}$$

Diff cross section

$$d\sigma = \frac{\text{Prob. per unit time}}{\text{unit flux}}$$

Prob.
$$P = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}$$

Recall $\langle f|S|i\rangle = i(2\pi)^4\,\delta\left(\sum_i p_i - \sum_f p_f\right)\mathcal{A}$

$$\Rightarrow |\langle f|S|i\rangle|^2 = (2\pi)^4\,\delta\left(\sum_i p_i - \sum_f p_f\right)\underbrace{(2\pi)\delta(0)}_{=\int d^4x\, e^{ix\cdot 0}\quad \text{"V.T"}}|\mathcal{A}|^2.$$

Focus on $|A|^2$. and further just consider

$$B \equiv \left(\bar{u}_{s'}(p') \cdot u_s(p)\right)\left(\bar{u}_{r'}(q') \cdot u_r(q)\right),$$

Take

$$|B|^2 = (\bar{u}_{s'} \cdot u_s)(\bar{u}_{r'} \cdot u_r)\left(\bar{u}_{s'} \cdot u_s\right)^\dagger (\bar{u}_{r'} \cdot u_r)^\dagger$$

$$= (\bar{u}_{s'} \cdot u_s)(\bar{u}_s \cdot u_{s'})\ (\bar{u}_{r'} \cdot u_r)(\bar{u}_r \cdot u_{r'})$$

Further input from particle physics.

$$P \underset{\substack{\downarrow \\ \psi\psi\to\psi\psi}}{=} \underset{\uparrow}{\frac{1}{4}} \sum_{\substack{r,s \\ r',s'}} |A|^2$$

is the spin-averaged prob. (amplitude) incoming particles

Then

$$\sum_{\substack{s,s' \\ r,r'}} |B|^2 \underset{\uparrow}{\textcircled{=}} \mathrm{Tr}\left((\not{p}'+m)(\not{p}+m)\right)\mathrm{Tr}\left((\not{q}'+m)(\not{q}+m)\right).$$

$$\sum_s u^s_a(p)\,\bar{u}^s_b(p) = (\not{p} + m\mathbb{1})_{ab}$$

$$\textcircled{=}\ \mathrm{Tr}\left(p'_\mu p_\nu \gamma^\mu\gamma^\nu + \gamma^\mu(p_\mu + p'_\mu)m + m^2\right)\mathrm{Tr}(\ldots).$$

7. Quantum Electrodynamics

7.1 Classical Field Theory

- A_μ = gauge field → vector rep. of Lorentz.
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ → field strength

Action Principle and e.o.m.

Lagrangian for Maxwell theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

EoM:

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0 \Rightarrow \partial_\mu F^{\mu\nu} = 0 . \qquad (1)$$

Bianchi identity:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

(consequence of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F = dA$).

Gauge symmetry

How many d.o.f. do we have?

Naive: A_μ has 4 comp., maybe 4 dof? No!

First inspect e.o.m.

$$(1): \quad \partial_\mu \partial^\mu A_\nu - \partial_\nu \partial_\mu A^\mu = 0$$

$$\nu = i : \quad \boxed{\Box A_i - \partial_i(\partial_0 A^0 + \partial_j A^j) = 0 .} \qquad (2)$$

$$\nu = 0 : \quad \Box A_0 - \partial_0(\partial_0 A^0 + \partial_j A^j) = 0$$

$$\Downarrow$$

$$\boxed{\nabla^2 A_0 + \partial_0(\nabla \cdot \underline{A}) = 0} \qquad (3)$$

No time deriv. of A_0.

Solve for A_0 in terms of $\underline{A}$

$$\Rightarrow A_0(\underline{x}) = \int d^3x' \frac{1}{4\pi|\underline{x}-\underline{x}'|}\left(\nabla \cdot \frac{\partial \underline{A}}{\partial t}\right)(\underline{x}') .$$

Out of 4 cpt, down to 3 e.o.m.

Still, we have redundancy in our system.

Note, if

$$A_\mu \longrightarrow A_\mu + \partial_\mu \lambda(x) ,$$

gauge transf^n.

where $\lambda(x)$ smooth arb. f^n, then

$$F_{\mu\nu} \longrightarrow F_{\mu\nu} + (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\lambda = F_{\mu\nu}$$

invariant. So for any A_μ, A_μ' that differ by this transf^n solves e.o.m. $\Rightarrow$ <u>gauge symmetry</u>.

Why is this redundancy?

1) Noether's thm.

Place it on the same footing as any other cts transf^n.

$$J^\mu = \frac{\partial L}{\partial(\partial_\mu A_\nu)} \delta A_\nu = -F^{\mu\nu}\partial_\nu\lambda \underset{\text{e.o.m.}}{=} -\partial_\nu(F^{\mu\nu}\lambda).$$

The so-called charge is

$$Q = \int d^3x \; J^0 = \int d^3x \; \partial_i(F^{0i}\lambda) \underset{\substack{\text{no sources}\\ \text{field decay at } \infty}}{=} 0$$

$\Rightarrow$ "gauge transf^n do not give rise to charges" (***)

2) Re-write Maxwell eqn

$$\underbrace{(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)}_{\text{not invertible}} A_\nu = 0$$

$$\Rightarrow (\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\partial_\nu\lambda = 0$$

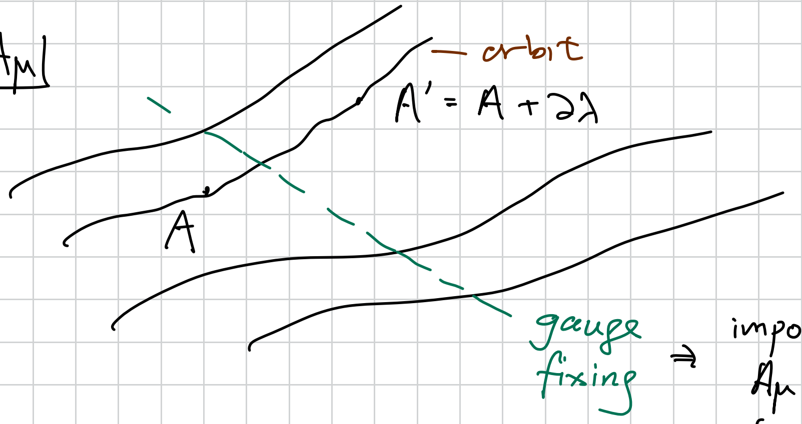
$$\Rightarrow (\Box\partial_\mu - \partial_\mu\Box)\lambda = 0$$

Given some initial data, we cannot distinguish between A_μ and $A_\mu' = A_\mu + \partial_\mu\lambda$.

$\Rightarrow$ out of 3 cpt, down to 2 cpts. $\Rightarrow$ 2 dof.

When studying theories that have a gauge sym. we need to be aware that physics (obs) do not depend on λ. (***)

space of A_μ



$\Rightarrow$ impose conditions on A_μ s.t. gauge transf^n disappears

Common choices of gauge fixing

1) Coulomb gauge

$$\boxed{\nabla \cdot \underline{A} = 0.}$$

If $\underline{A}'$ has $\nabla \cdot \underline{A} = \varphi(x)$, can find λ s.t. $\nabla^2 \lambda = \varphi$.

In this gauge, from (3),

$$\Rightarrow \boxed{A_0 = 0}$$

Pro: Full g.f. ; Con: breaks Lor. inv.

2) Lorentz gauge

$$\boxed{\partial_\mu A^\mu = 0}$$

If A' has $\partial_\mu A'^\mu = f(x)$, then find $\Box \lambda = f$. But this is only partial gauge fixing since can still do transf^n with

$$A_\mu \to A_\mu + \partial_\mu \lambda'$$

s.t. $\Box \lambda' = 0$.

Pro: Lor. inv. ; Con: partial

7.2 QFT: Free theory

7.2.1 Canonical Quantisation in Coulomb Gauge

CG: $\nabla \cdot \underline{A} = 0$, $A_0 = 0$, then e.o.m. (2)+(3) is

$$\Box A_i = 0$$

This tells us $A_\mu \sim \varepsilon_\mu(p)\, e^{-ip\cdot x} + \varepsilon^*_\mu(p)\, e^{ip\cdot x}$ with $p^2=0$

and choice of gauge

$$\varepsilon_0(p) = 0$$

$$\underline{p}\cdot\underline{\varepsilon}(p) = 0$$

Choosing a frame, e.g. $P_\mu = (E, 0, 0, E)$ $(p^2=0)$. Two sol^n to ε_μ

$$\varepsilon^1_\mu = (0,1,0,0) \quad , \quad \varepsilon^2_\mu = (0,0,1,0)$$

In general, polarisation vectors ε_μ satisfy

$$\underline{\varepsilon}_r(p)\cdot\underline{p} = 0,$$

$$\underline{\varepsilon}_r \cdot \underline{\varepsilon}_s = \delta_{rs}$$

and completeness relation

$$\sum_{r=1}^{2} \varepsilon^i_r(p)\, \varepsilon^j_r(p) = \delta^{ij} - \frac{p^i p^j}{|\underline{p}|^2}.$$

Therefore, general sol^n in CG for A_μ.

$$A_\mu(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\underline{p}|}} \sum_{r=1}^{2} \varepsilon^r_\mu \left(a^r_p e^{-ip\cdot x} + a^{r\dagger}_p e^{ip\cdot x}\right)$$

Next, we also need canonical mom

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} \equiv E^i$$

Note since $\nabla\cdot\underline{A} = 0$, $\nabla\cdot\underline{E} = 0$.

Commutation relation:

Naive guess: $[A_i(\underline{x}, t), \Pi_j(\underline{y}, t)] = i\delta_{ij}\, \delta^{(3)}(\underline{x}-\underline{y})$ $(*)$

This cannot be right. Why?

In the quantum theory, we still want to respect our gauge choice (fixing). The commutator $(*)$ does not respect it.

$$[\underbrace{\nabla\cdot \underline{A}}_{=0},\ \underbrace{\nabla\cdot \underline{\Pi}}_{=0}] = i\,\delta_{ij}\,\nabla^2 \delta^{(3)}(\underline{x}-\underline{y}).$$

So (*) is wrong!

The correct guess is

$$[A_i(\underline{x},t),\ \Pi_j(\underline{y},t)] = i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta^{(3)}(\underline{x}-\underline{y})$$

$$= i\int\frac{d^3p}{(2\pi)^3}\left(\delta_{ij} - \frac{p_ip_j}{|\underline{p}|^2}\right)e^{i\underline{p}\cdot(\underline{x}-\underline{y})}.$$

We can take a derivative

$$[\partial^i A_i\ ,\ \Pi_j] = i\int\frac{d^3p}{(2\pi)^3}\left(p_i - \frac{p_i\,\underline{p}^2}{\underline{p}^2}\right)e^{i\underline{p}\cdot(\underline{x}-\underline{y})} = 0.$$

From here, can show

$$\left.\begin{aligned}&[a^r_{\underline{p}},\ a^s_{\underline{q}}] = [a^{r\dagger}_{\underline{p}},\ a^{s\dagger}_{\underline{q}}] = 0\\ &[a^r_{\underline{p}},\ a^{s\dagger}_{\underline{q}}] = (2\pi)^3\,\delta^{rs}\,\delta^{(3)}(\underline{p}-\underline{q}).\end{aligned}\right\}\Rightarrow \text{Healthy Fock space.}$$

Hamiltonian

$$H = \int d^3x\,(\Pi_i\dot{A}_i - \mathcal{L})$$

$$= \frac{1}{2}\int d^3x\,(\underline{E}\cdot\underline{E} + \underline{B}\cdot\underline{B})$$

where $E^i = F^{0i}$, $F^{ij} = B_k\,\varepsilon^{ijk}$. Then in normal ordering,

$$H = \int\frac{d^3p}{(2\pi)^3}\,|\underline{p}|^2\sum_{r=1}^{2}a^{r\dagger}_{\underline{p}}a^r_{\underline{p}}.$$

7.2.2 Propagator

In CG, the Feynman propagator is

$$\langle 0|\,T A_i(x) A_j(y)\,|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\varepsilon}\left(\delta_{ij} - \frac{p_ip_j}{|\underline{p}|^2}\right)e^{-ip\cdot(x-y)}$$

Using A_i in terms of a^+, a's.

Goal is to construct $\langle 0|T A_\mu A_\nu|0\rangle$ in Lorentz gauge.

Claim $\langle 0|T A_\mu(x) A_\nu(y)|0\rangle$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2+i\varepsilon}\left(-\eta_{\mu\nu} - (\alpha-1)\frac{p_i p_j}{p^2}\right) e^{-ip(x-y)}$$

Note: This is Lor. inv.

Pf: Key: FP are Green's f^n, so

$$\text{"eom"} \langle T A_\mu(x) A_\nu(y)\rangle = \text{"} -i\,\delta^{(4)}(x-y) \text{"}.$$

Illustrate with massive scaler field $\Delta_F \to G$.

$$(\Box + m^2)\, G(x) = J(x)$$

$\nwarrow$ source

Transform to Fourier space

$$\Rightarrow (\Box+m^2)\int \frac{d^4p}{(2\pi)^4} G(p)\, e^{ip\cdot x} = \int \frac{d^4p}{(2\pi)^4} J(p)\, e^{ip\cdot x}$$

$$\Rightarrow (-p^2+m^2)\, G(p) = J(p)$$

$$\Rightarrow G(p) = \frac{J(p)}{-p^2+m^2}$$

If $J(x) = -i\,\delta(x)$,

$$G(p) = \frac{i}{-p^2+m^2}$$

$$\Rightarrow G(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{-p^2+m^2} e^{ip\cdot x}$$

For the gauge field, do sth similar.

$$\partial^\mu F_{\mu\nu} = J_\nu$$

In terms of A_μ,

$$(\eta_{\mu\nu}\Box - \partial_\nu \partial_\mu) A^\mu = J_\nu$$

$$\overset{FT}{\Longrightarrow} \quad \underbrace{\left(-p^2\eta_{\mu\nu} + p_\mu p_\nu\right)}_{\hat{\Pi}_{\mu\nu}} A^\mu(p) = J_\nu(p).$$

$\hat{\Pi}_{\mu\nu} \rightsquigarrow$ need to find inverse $\hat{\Pi}_{\mu\nu}^{-1}$

$$\Rightarrow A^\mu(p) = \left(\hat{\Pi}^{-1}\right)^{\mu\nu} J_\nu(p)$$

Issue: $\hat{\Pi}$ not invertible due to gauge sym

$$\hat{\Pi}_{\mu\nu}\, p^\nu = 0 \quad \Rightarrow \quad \text{eval} = 0 .$$

To fix this, consider

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 ,$$

Lagrange multiplier which enforces $\partial_\mu A^\mu = 0$

E.o.m.:

$$\left(\eta^{\lambda\nu}\Box + \left(\frac{1}{\alpha} - 1\right)\partial^\lambda \partial^\nu\right) A_\lambda = 0$$

Construct Green's f^n. In mom. space,

$$\underbrace{\left(-\eta_{\lambda\nu}p^2 + \left(\frac{1}{\alpha} - 1\right) p_\nu p_\lambda\right)}_{\Pi_{\lambda\nu}} A_\lambda(p) = J_\nu(p).$$

$\Pi_{\lambda\nu}$: now have inverse

We have

$$\Pi_{\mu\nu} = -\eta_{\mu\nu}p^2 + \left(\frac{1}{\alpha} - 1\right) p_\mu p_\nu$$

$$\left(\Pi^{-1}\right)_{\mu\nu} = -\frac{\eta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2}}{p^2} \equiv \tilde{\Pi}_{\mu\nu} .$$

So $\Pi^{\mu\nu}\tilde{\Pi}_{\mu\lambda} = \delta^\nu{}_\lambda$.

Solve for Green's f^n at $J \sim \delta(x-y)$

$$\Rightarrow \langle 0| T\, A_\mu(x)\, A_\nu(y) |0\rangle = \int \frac{d^4p}{(2\pi)^4}\, i\, \tilde{\Pi}_{\mu\nu}\, e^{-ip(x-y)} . \qquad \square$$

Notes:

i) Minus sign is correct: $\Pi_{\mu\nu} = -\eta_{\mu\nu}p^2 + \cdots$. Recall in CG,

$\Delta_i(x)$ propagate and their sign should be true.

2) α is not physical – observables do not depend on it.

- Useful to keep α to check S-matrix indpt. of it.
- convenient to select value of α to make prop. easier to manipulate, e.g. $\alpha = 1$: Feymann gauge,

$\alpha = 0$: Landau gauge

$\alpha = \infty$: Unitary gauge

§. QED: Interactions

Couple light (A_μ) to matter (ψ, ϕ).

EoM point of view: in Maxwell eqn. we have

$$\partial_\mu F^{\mu\nu} = j^\nu$$

current contains, e.g. charge density

$$\underbrace{\partial_\nu \partial_\mu}_{\text{sym}} \underbrace{F^{\mu\nu}}_{\text{antisym}} = \partial_\nu j^\nu$$

$$\Rightarrow \partial_\nu j^\nu = 0 \quad \text{conserved}$$

Action point of view: $\partial_\nu j^\nu = 0$ can also to tied to gauge inv.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu . \qquad (*)$$

Do a gauge transf^n $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$.

$$S = \int d^4x \, \mathcal{L} \rightarrow \int d^4x \left(-\frac{1}{4} F^2 - j^\mu (A_\mu + \partial_\mu \lambda) \right).$$

$$= S - \int d^4x \, [\partial_\mu (j^\mu \lambda) - (\partial_\mu j^\mu) \lambda]$$

If $j \to 0$ as $x \to \infty$, then

$$S \to S + \int d^4x \, (\partial_\mu j^\mu) \lambda .$$

Action is gauge inv. if $\partial_\mu j^\mu = 0$.

For Ψ, this gives guidance on how to write an action that is gauge inv. For Ψ (Dirac spinor), we had global (internal) $U(1)$ sym.

$$j^\mu = \bar{\Psi} \gamma^\mu \Psi .$$

Couple A_μ to Ψ, we would write

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\Psi} (i \not{\partial} - m) \Psi . - e j^\mu A_\mu$$

where e is a const. (coupling).

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} (i \not{\partial} - m) \Psi - e \bar{\Psi} \gamma^\mu \Psi A_\mu$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} (i \not{D} - m) \Psi .$$

with $\not{D} = D_\mu \gamma^\mu$, and

$$D_\mu \Psi = \partial_\mu \Psi + i e A_\mu \Psi .$$

is the covariant derivative.

Now, Lagrangian is inv.

$$A_\mu \to A_\mu + \partial_\mu \lambda(x)$$
$$\Psi \to e^{-ie\lambda(x)} \Psi .$$
$$\bar{\Psi} \to e^{ie\lambda(x)} \bar{\Psi} .$$

Rule: couple light to matter is to make global sym. of matter into a local sym and combines with gauge inv.

If $\lambda(x)$ const. $\Rightarrow$ $U(1)$ sym.

Check : $D_\mu \psi \to e^{-ie\lambda(x)} D_\mu \psi$. ,

$\bar{\psi} D_\mu \psi \to \bar{\psi} D_\mu \psi$.

Noether's charge :

Consider (*) , the charge would be

$$Q = -\int d^3x \, F^{0i} \partial_i \lambda .$$

$$= \int d^3x \, (-\partial_i F^{0i}) \lambda + \overset{0}{\cancel{\text{total deriv}}}$$

$$= -\int d^3x \, j^0 \lambda(x) .$$

$$= \int d^3x \, j^0 \qquad \text{(Pick } \lambda = -1 \text{)}.$$

$$= e \int d^3x \, \bar{\psi} \gamma^0 \psi .$$

Lesson : for theory with a gauge sym. we can construct a change from the global part of the transf^n

$\lambda(x) \to$ const. (large gauge transf^n).

$\Rightarrow$ only the matter is giving rise to conserved quantity.

Example e , electric charge also controls the coupling $\alpha = e^2/4\pi\hbar c \sim 1/137$.

Next, we can also couple A_μ to scalers

real scaler field $\to$ no internal (j^μ), does not interact to A_μ .

complex scaler field $\to$ $\varphi = \phi_1 + i\phi_2$.

$\varphi \to e^{i\lambda} \varphi$, λ const.

Construct an action that is gauge inv.

$$D_\mu \varphi = \partial_\mu \varphi + ie A_\mu \varphi .$$

then

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) - m|\varphi|^2 .$$

This is scaler QED (SQED).

Action / Lagrangian is now invar. under

$$A_\mu \to A_\mu + \partial_\mu \lambda ,$$

$$\varphi \to e^{i\lambda} \varphi .$$

for $\lambda(x)$ arb.

Interaction terms:

$$\mathcal{L}_{int} = ie \Big(\underbrace{(\partial_\mu \varphi)^* \varphi - \varphi^* \partial_\mu \varphi}_{j^\mu} \Big) A^\mu + \underbrace{e^2 A_\mu A^\mu \varphi^* \varphi}_{\text{extra term}} .$$

Noether's current for global

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} \delta\varphi^*$$

$$= (-ie\lambda)\Big((\partial_\mu \varphi^*) \varphi - \varphi^* \partial_\mu \varphi - 2ie A_\mu \varphi^* \varphi \Big)$$

Feynman Rules for QED.

New ingredient is A_μ (photon).

From Q.C. in Coulomb gauge,

$$\sqrt{2\omega_p}\, a_r^\dagger(p) \xrightarrow[LSZ]{} i \varepsilon_r^\mu(p) \int d^4x \, e^{-ipx} \partial^2 A_\mu .$$

ingoing photon

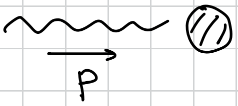
$$\sqrt{2\omega_p}\, a_{r'}(p) \longrightarrow i \varepsilon_{r'}^\mu(p) \int d^4x \, e^{ipx} \partial^2 A_\mu .$$

outgoing photon

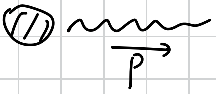
with $\varepsilon^0_r = 0$, $\underline{\varepsilon} \cdot \underline{p} = 0$.

External lines:

1) Photon: we add polarisation vector

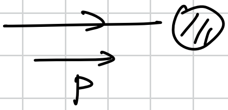


$= \varepsilon^s_\mu(p)$. incoming

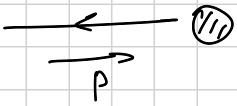


$= \varepsilon^s_\mu(p)$ outgoing

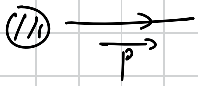
2) Fermions (electrons)



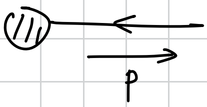
$u^r(p)$. incoming fermion



$\bar{v}^r(p)$ incoming antifermion



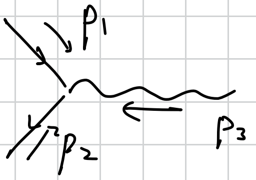
$\bar{u}^r(p)$ outgoing fermion



$v^r(p)$ outgoing antifermion

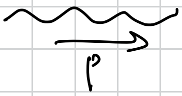
Internal lines

1) Vertex



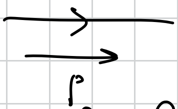
$: -ie\gamma^\mu (2\pi)^4 \delta(p_1+p_2+p_3)$

2) Photon propagator



$$: \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2+i\varepsilon} \underbrace{\left(-\eta_{\mu\nu} - (\alpha-1)\frac{p_\mu p_\nu}{p^2}\right)}_{\tilde{\Pi}_{\mu\nu}}.$$

3) Fermion propagator



$$: \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2+m^2+i\varepsilon} (\not{p}+m).$$

And minus if fermion are exchanged.

Goal: evaluate $\langle f | S - \mathbb{1} | i \rangle$ using pert. thy (Feyn rules / LSZ)

$\Rightarrow$ Restriction on S-matrix (amplitude)

$$\langle f | S - \mathbb{1} | i \rangle = i (2\pi)^4 \, \delta\left(\sum_F p_F - \sum_I p_I \right) \mathcal{A}(p, \cdot) .$$

a) If we have an internal photon,

$$\mathcal{A} = \mathcal{M}_{\mu\nu} \tilde{\Pi}^{\mu\nu}(p) .$$

$\mathcal{A}$ is indpt. of α, since we do not expect S to depend on a choice of gauge.

$$\Rightarrow \boxed{\mathcal{M}_{\mu\nu} p^\mu p^\nu = 0} \qquad \text{(Ward identity).}$$

b) If we have an external photon

$$\mathcal{A} = \varepsilon^\mu(p) . \mathcal{M}_\mu .$$

If we do Lor. transf^n, $\mathcal{A}$ is invar. We know that

$$p'^\mu = \Lambda^\mu{}_\nu p^\nu .$$

expect

$$\mathcal{M}'_\mu = \Lambda^\nu{}_\mu \mathcal{M}_\nu .$$

Does ε^μ_s transform as a vec?

Ex: $p_\mu = (E, 0, 0, E)$, $\varepsilon_1^\mu = (0, 1, 0, 0)$. Consider

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 3/2 & 1 & 0 & -1/2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 1/2 \end{pmatrix}$$

Check $\Lambda^T \eta \Lambda = \eta$., and

$\Lambda^\mu{}_\nu p^\nu = p^\mu$ leaves p invar.

$\Lambda^\mu{}_\nu \varepsilon^\nu{}_1 = \varepsilon^\mu{}_1 + \frac{1}{E} p^\mu$ → ε is not tranforming as a vec.

In general, $\varepsilon'^\mu_s = \Lambda^\mu{}_\nu \varepsilon^\mu_s + c(\Lambda) p^\mu \Rightarrow$ looks like a gauge transf^n $\quad \varepsilon \cdot p = 0$, $\quad p^2 = 0$.

If we do a Lor. transf^n of

$$A = \varepsilon_s^\mu M_\mu \longrightarrow A + c\, p^\mu M_\mu \overset{!}{=} A$$

$$\Rightarrow \boxed{p^\mu M_\mu = 0}$$

Anytime when replace $\varepsilon_s^\mu(p)$ by p^μ, the A should vanish.

Example Electron Scattering $e^- e^- \longrightarrow e^- e^-$. at tree level.

$$\langle f | S - \mathbb{1} | i \rangle = (-ie)^2 \left(\bar{u}^{s'}(p') \gamma^\mu u^s(p) \cdot \frac{i \tilde{\Pi}_{\mu\nu}(p-p')}{(p-p')^2} \bar{u}^{r'}(q') \gamma^\nu u^r(q) \right.$$

$$\left. - (p' \leftrightarrow q', s' \leftrightarrow r') \right) \cdot (2\pi)^4 \delta(p+q-p'-q').$$

Look at one vertex

$$= -\eta_{\mu\nu} - (\alpha - 1) \frac{k_\mu k_\nu}{k^2}. \quad k = p - p' = q - q'.$$

$$\bar{u}^{s'}(p') \gamma^\mu u^s(p) \tilde{\Pi}_{\mu\nu}(p-p') \bar{u}^{r'}(q') \gamma^\nu u^r(q).$$

$$= \frac{1}{k^2} \bar{u}^{s'}(p') k_\mu \gamma^\mu u^s(p) \bar{u}^{r'}(q') k_\nu \gamma^\nu u^r(q).$$

We will use

$$(-\not{p} + m) u(p) = 0. \qquad (*)$$

$$\bar{u}(p) (\not{p} - m) = 0. \qquad (**)$$

and

$$\bar{u}^{s'}(p') \not{k}\ u^{s}(p) = \bar{u}^{s'}(p')\ (\underset{(*)}{\not{p}} - \overset{(**)}{\not{p}'})\ u^{s}(p).$$

$$= \bar{u}^{s'}(p')\ (m-m)\, u^{s}(p) = 0 \quad ,$$

$$\bar{u}^{r'}(q') \not{k}\ u^{r}(q) = 0$$

$\Rightarrow$ $\mathcal{A}$ is indep of α $(M_{\mu\nu} k^{\mu} k^{\nu} = 0)$.

Note. In this example, diagram is α indpt. separately. In general, Ward id. is valid on the sum of diag. to each other in coupling.