General Relativity

Manifolds

The basic object in GR is a smooth, n-dim manifold.

This is a set M together with

- a collection of coordinate charts $\mathcal{O}_\alpha \subset M$ s.t. $M = \bigcup_\alpha \mathcal{O}_\alpha$.
- a bijection $\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha \subset \mathbb{R}^n$ open.
- if $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \phi$, then

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \to \phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$$

is smooth.

Rmk:

- Can replace "smooth" by C^k (k-times diff)
- Don't want def^n of M to depend on particular choice of $\{(\mathcal{O}_\alpha, \phi_\alpha)\}$.

We say a collection of $\{(\mathcal{O}_\alpha, \phi_\alpha)\}_\alpha$ form an atlas for M. Two atlases $\{(\mathcal{O}_\alpha, \phi_\alpha)\}$, $\{(\mathcal{O}'_\alpha, \phi'_\alpha)\}$ are compatible if there union is also an atlas.

A maximal atlas is the union of all compatible atlas on M.

Examples

1) If $\mathcal{U} \subset \mathbb{R}^n$ is open, can take $\mathcal{O} = \mathcal{U}$ and $\phi = id$.

$$\phi : (x^1, \dots, x^n) \mapsto (x^1, \dots, x^n).$$

so our (non-maximal) atlas has only one chart.

In particular, suppose $U = (-1,1) \subset \mathbb{R}$ and consider

- $((-1,1), \mathrm{id}) = (O_\alpha, \phi_\alpha)$
- $((-1,1), \psi)$ where $\psi : x \mapsto x^3$.

However, $\psi \circ \mathrm{id}^{-1} = \psi \circ \mathrm{id} : x \mapsto x^3$ smooth ✓.

$\mathrm{id} \circ \psi^{-1} : y \mapsto \sqrt[3]{y}$ not smooth

2) $S^1 = \{p \in \mathbb{R}^2 \mid p \cdot p = 1\}$.

- If $p \in S^1 \setminus \{(-1,0)\}$, then $\exists!\ \theta_1 \in (-\pi, \pi)$ s.t. $p = (\cos\theta_1, \sin\theta_1)$
- If $p \in S^1 \setminus \{(1,0)\}$, then $\exists!\ \theta_2 \in (0, 2\pi)$ s.t. $p = (\cos\theta_2, \sin\theta_2)$.

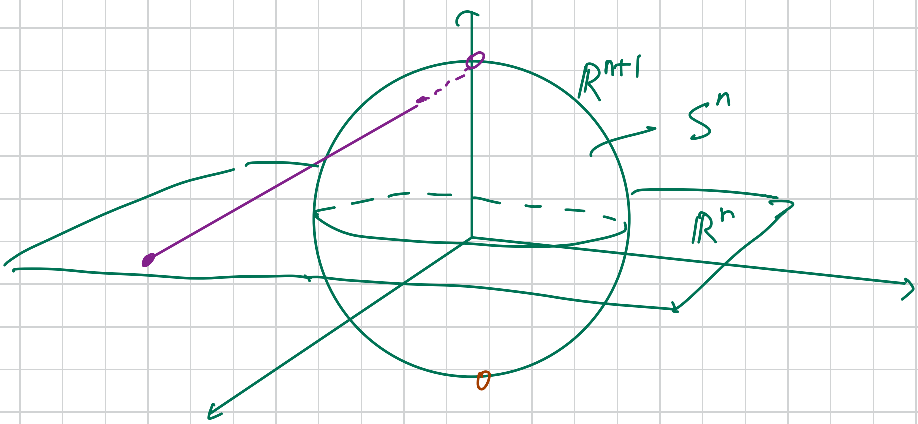
We have $\phi_1(O_1 \cap O_2) = (-\pi, 0) \cup (0, \pi)$, and

$$\phi_2 \circ \phi_1^{-1} : \theta_1 \mapsto \begin{cases} \theta_1 & \theta_1 \in (0, \pi) \\ \theta_1 + 2\pi & \theta_1 \in (-\pi, 0) \end{cases}.$$

3) $S^n = \{p \in \mathbb{R}^{n+1} \mid p \cdot p = 1\}$.

Can define coords by <u>stereographic projections</u>.

Let $\{\underline{E}_1, \dots, \underline{E}_{n+1}\}$ be usual Cartesian basis of $\mathbb{R}^{n+1}$ and $\{\underline{e}_1, \dots, \underline{e}_n\}$ be Cartesian for $\mathbb{R}^n$.



We pick $O_1 = S^n \setminus \{\underline{E}_{n+1}\}$ and for $p \in O_1$, let

$$\phi_1(p) = \frac{1}{1 - p^{n+1}} (p^1 \underline{e}_1 + \dots + p^n \underline{e}_n) \in U_1 \subset \mathbb{R}^n$$

$O_2 = S^n \setminus \{-\underline{E}_{n+1}\}$ and for $p \in O_2$, let

$$\phi_2(p) = \frac{1}{1+p^{n+1}} (p^1 \underline{e}_1 + \dots + p^n \underline{e}_n) \in U_2 \subset \mathbb{R}^n.$$

We have $\phi_1(O_1 \cap O_2) = \mathbb{R}^n \setminus \{\underline{0}\}$, and

$$\phi_2 \circ \phi_1^{-1} : \underline{x} \mapsto \frac{\underline{x}}{\|\underline{x}\|^2} \quad \text{for} \quad \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}\}.$$

This is ∞-diff'ble on $\mathbb{R}^n \setminus \{\underline{0}\}$, so S^n is a manifold.

4) If M and N are each manifolds of dim m, n resp., then $M \times N = \{(p, q) \mid p \in M, q \in N\}$ is a manifold of dim $m+n$.

If (O_α, ϕ_α) is a chart on M, (O'_β, ϕ'_β) a chart on N, then $O_{\alpha\beta} = O_\alpha \times O'_\beta$ and

$$\phi_{\alpha\beta}(p, q) = (\phi_\alpha(p), \phi'_\beta(q))$$

is a coord chart of product. As α, β vary, get an atlas on $M \times N$.

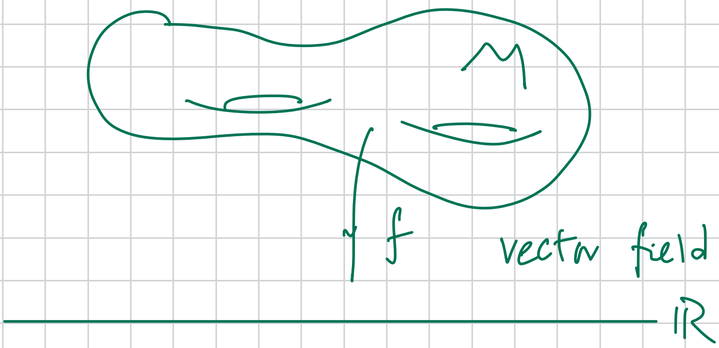
Smooth functions

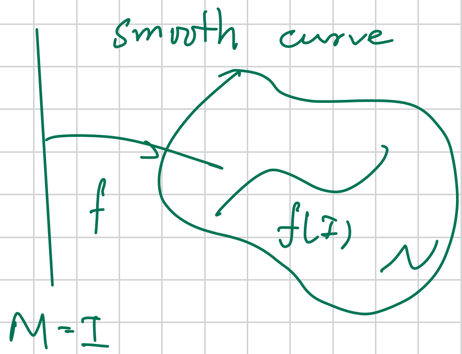
Def^n A smooth f^n is a map $f: M \to N$ between mflds M, N, of dim m, n resp. s.t. for any coord charts (O, ϕ) on M and (O', ϕ') on N, the composition $\phi' \circ f \circ \phi$ is a smooth map from $U \subset \mathbb{R}^m$ to $U' \subset \mathbb{R}^n$.

A smooth map $f: M \to N$ which has a smooth inverse $f^{-1}: N \to M$ is called a diffeomorphism.

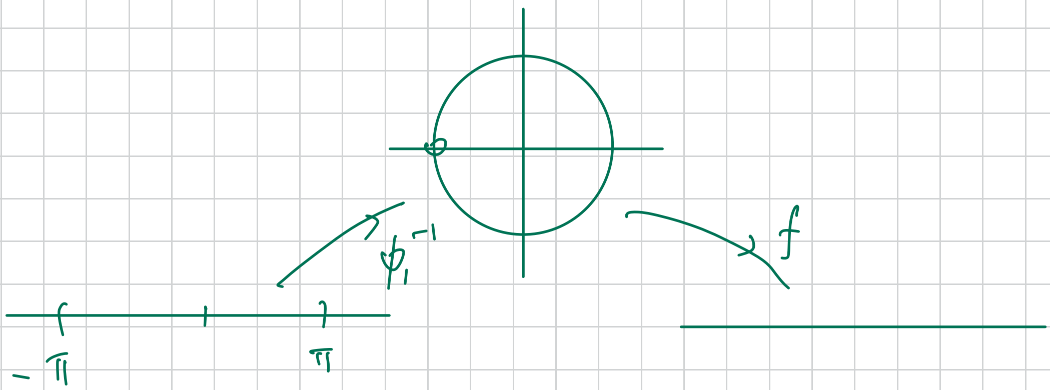
Example Suppose $N = \mathbb{R}$. Denote the space of all smooth f^ns $f: M \to \mathbb{R}$ by $C^\infty(M; \mathbb{R})$, or $C^\infty(M)$. In physics, these are often called scaler fields.

Example Suppose $M = I = (a,b) \subset \mathbb{R}$. Then the smooth map $f: I \to N$ is called a smooth curve in N.





Example Recall $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
Let $f(x,y) = x$. Using our charts $f \circ \phi^{-1}: (-\pi, \pi) \to \mathbb{R}$ is given by $f \circ \phi_1^{-1}: \theta_1 \mapsto \cos\theta_1$ — smooth ✓



Similarly, on O_2, have $f \circ \phi_2^{-1}: (0, 2\pi) \to \mathbb{R}$ — smooth ✓
$\theta_2 \mapsto \cos\theta_2$.

For any other chart $(O_\alpha', \varphi_\alpha)$, we have

$$f \circ \varphi_\alpha^{-1} = f \circ (\phi_1^{-1} \circ \phi_1) \circ \varphi_\alpha^{-1}$$

$$= \underbrace{(f \circ \phi_1^{-1})}_{\text{smooth}} \circ \underbrace{(\phi_1 \circ \phi_\alpha^{-1})}_{\substack{\text{smooth by} \\ \text{def}^n \text{ of } S^1 \\ \text{as mfld}}}$$

So smooth for any compatible atlas

<u>Example</u> let (O, ϕ) be a chart on M and for $p \in O$, write $\phi(p) = (x^1(p), \dots, x^m(p))$, Then for $1 \le i \le m$, the $x^i(p)$ are smooth $\mathbb{R}$-valued f^ns on M. Again if (O', ϕ') is another chart and $p \in O \cap O'$, then $x^i \circ (\phi')^{-1}$ is the i-th cpt of $\phi \circ (\phi')^{-1}$, so is smooth. These are called the <u>coordinate f^ns</u> of $p \in M$ in the chart (O, ϕ).

We sometimes write $F_\alpha = f \circ \phi_\alpha^{-1}$ where $f : M \to \mathbb{R}$, then $F_\alpha : U_\alpha \to \mathbb{R}$, where $U \subset \mathbb{R}^m$. Then f^n f is smooth if F_α is smooth $\forall \alpha$ in our atlas and

$$F_\alpha \big|_{U_\alpha \cap U_\beta} = F_\beta \big|_{U_\alpha \cap U_\beta}$$

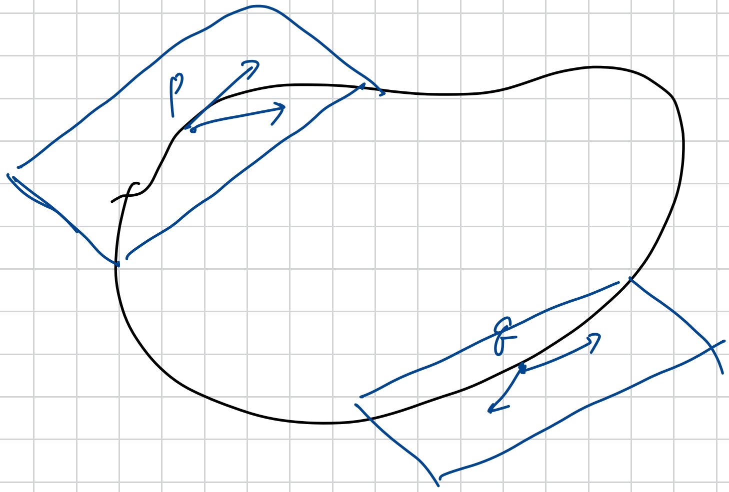
on overlaps.

<u>Curves and Tangent Vectors</u>

For a surface $M \subset \mathbb{R}^3$, we have a notion of the tangent space T_pM at $p \in M$. This consists of all $\underline{v} \in \mathbb{R}^3$ that are tangent to M at p.

However, there's no natural identification between

$\underline{v} \in T_pM$ and $\underline{u} \in T_qM$ for $p \neq q$.

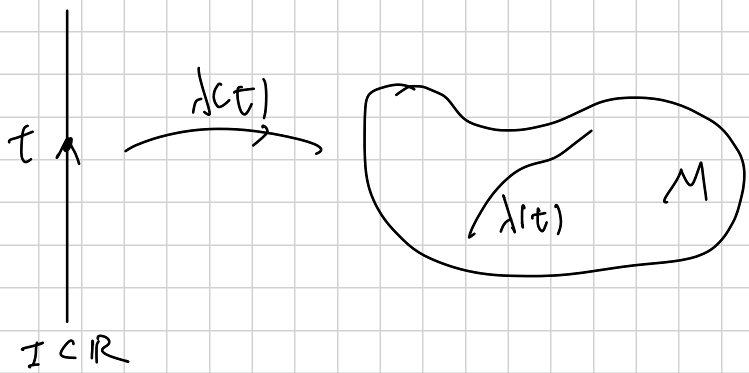


To have a def^n of T_pM that's intrinsic to M, i.e. doesn't rely on embedding $M \subset \mathbb{R}^{m+k}$, use curves.

Suppose $\underset{\sim}{\lambda}(t)$ is a smooth curve in M, define a tangent vector to M at $p = \lambda(0)$ by taking the derivative

$$\frac{d}{dt} f(\underset{\sim}{\lambda}(t))\Big|_{t=0}$$

for any smooth f^n $f: M \to \mathbb{R}$.



Example If $M = \mathbb{R}^m$, then $\underset{\sim}{\lambda}(t) = (x^1(t), \ldots, x^m(t))$ and

$$X_p f = \frac{df}{dt}(\underline{x}(t))\Big|_{t=0} = \frac{dx^\mu(t)}{dt} \frac{\partial f}{\partial x^\mu}\Big|_{t=0}$$

the usual directional derivative of f along the curve.

Thus, the tangent vector to M at p is a linear map $C^\infty(M) \to \mathbb{R}$.

Note that

(i) X_p is linear:

$$X_p(\alpha f + \beta g) = \alpha X_p f + \beta X_p g$$

$\forall \alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(M)$

(ii) X_p obeys Leibniz rule:

$$X_p(f \cdot g) = [f X_p(g) + g X_p(f)]_p$$

If (O, ϕ) is a chart, and $p \in O$ with $\phi(p) = (x^1, \dots, x^m)$, then $F = f \circ \phi^{-1}$ represents our f^n f in this chart, then

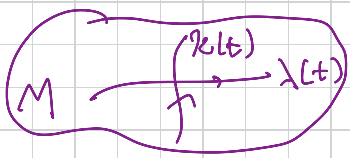
$$f \circ \lambda(t) = f \circ \phi^{-1} \circ \phi \circ \lambda(t) = F \circ \phi \circ \lambda(t)$$

So if $\lambda(0) = p$,

$$\frac{d}{dt} f(\lambda(t))\Big|_{t=0} = \underbrace{\frac{\partial F}{\partial x^\mu}\Big|_{\phi(p)}}_{\text{depends on } F, \phi} \underbrace{\frac{d x^\mu(\lambda(t))}{dt}\Big|_{t=0}}_{\text{depends on } \lambda, \phi}$$

<u>Prop</u> The set of tangent vecs at $p \in M$ forms a vector space $T_p(M)$, the <u>tangent space</u> to M at p.

Pf: Given tangent vecs X_p, Y_p, must show $\alpha X_p + \beta Y_p$ is a tangent vec $\forall \alpha, \beta \in \mathbb{R}$. Let λ, κ be smooth curves with $\lambda(0) = \kappa(0) = p$



and let X_p, Y_p be the tangent vectors at p.

Also suppose $(\mathcal{O}, \phi)$ is a coordinate chart with $p \in \mathcal{O}$ and $\phi(p) = 0$.

Consider the smooth curve $\nu(t) = \phi^{-1}\left[\alpha\, \phi(\lambda(t)) + \beta\, \phi(\kappa(t))\right]$.

Have $\nu(0) = p$. Let $Z_p = \alpha X_p + \beta Y_p$, then

$$Z_p(f) = \frac{d}{dt} f(\nu(t)) \Big|_{t=0}$$

$$= \frac{\partial F}{\partial x^\mu}\Big|_{\phi(p)} \frac{\partial}{\partial t} \left(\alpha x^\mu(\lambda(t)) + \beta x^\mu(\kappa(t))\right)\Big|_{t=0}$$

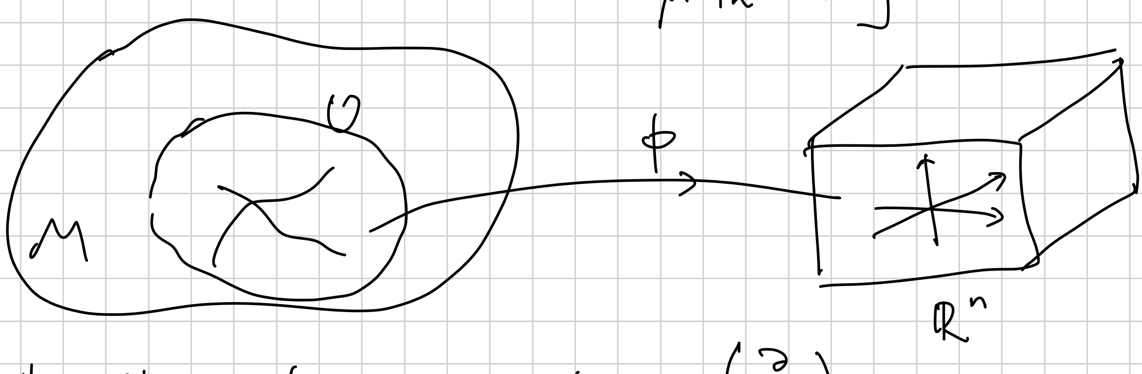
$$= \frac{\partial F}{\partial x^\mu}\Big|_{\phi_p} \left[\alpha \frac{\partial x^\mu(\lambda(t))}{\partial t} + \beta \frac{\partial x^\mu(\kappa(t))}{\partial t}\right]_{t=0}$$

$$= \alpha X_p(f) + \beta Y_p(f)$$

Hence, $T_p(M)$ is a vector space. $\square$

To show $\dim T_p(M) = n = \dim(M)$, consider the curves

$$\lambda_\mu(t) = \phi^{-1}(0, \ldots, 0, \underset{\uparrow \ \mu\text{-th entry}}{t}, 0, \ldots, 0)$$



We call these tangent vectors $\left(\frac{\partial}{\partial x^\mu}\right)_p$

This name is because

$$\left(\frac{\partial}{\partial x^\mu}\right)_p f = \frac{d F(x^\mu(t))}{dt} = \frac{\partial F}{\partial x^\nu} \frac{\partial x^\nu(\lambda_\mu(t))}{\partial t}$$
$$= \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}$$

These vectors are linearly indpt. Suppose not, $\exists \alpha^\mu \in \mathbb{R}$, not all 0, s.t. $\alpha^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = 0$.

$$\Rightarrow \alpha^\mu \left(\frac{\partial F}{\partial x^\mu}\right)_p = 0 \qquad \forall \text{ smooth } f^n\ F.$$

But pick $F = x^\nu$ to find $\alpha^\nu = 0 \ \forall \nu$. $\#$.

Furthermore, they form a basis, since if λ is any smooth curve with tangent X_p at p, then

$$X_p(f) = \left.\frac{\partial F}{\partial x^\mu}\right|_{\phi(p)} \frac{d x^\mu(\lambda(t))}{dt} = \left.\frac{d x^\mu}{dt}(\lambda(t))\right|_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p f.$$
$$= X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p f.$$

where $X^\mu = \left.\frac{d x^\mu}{dt}(\lambda(t))\right|_{t=0}$ are the <u>cpts</u> of X_p in this basis.

Suppose $(\mathcal{O}, \phi)$ and $(\mathcal{O}', \phi')$ are two charts with $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$ and $p \in \mathcal{O} \cap \mathcal{O}'$ and set $\phi(p) = (x^1(p), \ldots, x^n(p))$, $\phi'(p) = (x'^1(p), x'^2(p), \ldots, x'^n(p))$, and let $F = f \circ \phi^{-1}$,

and $F' = f \circ (\phi')^{-1} = f \circ \phi^{-1} \circ \phi \circ (\phi')^{-1}$, i.e. $F(x) = F'(x'(x))$

Then

$$\left(\frac{\partial}{\partial x^\mu}\right)_p f = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}$$

$$= \left(\frac{\partial}{\partial x^\mu}\left(F'(x'(x))\right)\right)_{\phi(p)}$$

$$= \left(\frac{\partial F'}{\partial x'^\nu}\right)_{\phi(p)} \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)}$$

$$= \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p f$$

Hence,

$$\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p$$

Similarly. for a general tangent vector X_p at p.
Can expand

$$X_p = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = X'^\nu \left(\frac{\partial}{\partial x'^\nu}\right)_p.$$

$$= X^\mu \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p$$

So we conclude

$$\boxed{X'^\nu = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} X^\mu}$$

Covectors

Recall: if V is a vector space, $\dim V = n < \infty$, then the dual space to V is the space V^* of linear maps $V \to \mathbb{R}$, $\dim V^* = \dim V$

Suppose $\{e_\mu\}$ for $\mu = 1, \dots, n$ is a basis for V, the <u>dual basis</u> is the basis $\{f^\nu\}$ for $\nu = 1, \dots, n$ of V^* defined by

$$f^\nu(e_\mu) = \delta^\nu{}_\mu .$$

We define the cotangent space at $p \in M$ is the dual space $T_p^*(M)$. An elt. $\eta \in T_p^*(M)$ is a covector (or 1-form) at p. In particular, if $\{e_\mu\}$ basis for T_pM and $\{f^\nu\}$ the dual basis, can expand any $\eta = \eta_\nu f^\nu$ in this basis

- $\eta(e_\mu) = \eta_\mu$
- $\eta(X) = \eta_\nu f^\nu(X^\mu e_\mu) = \eta_\mu X^\mu$.

<u>Def^n</u> Given a smooth f^n $f : M \to \mathbb{R}$, define $(df)_p$ to be the covector at p which maps

$$(df)_p : X_p \longmapsto X_p(f) \qquad \forall X_p \in T_pM$$

We call $(df)_p$ the <u>gradient</u> of f at p.

<u>Note</u> If $f : M \to \{\text{const.}\}$, then

$$(df)_p(X_p) = X_p(f) = 0 \qquad \forall X_p$$

and we set $(df)_p = 0 \in T_p^*(M)$

<u>Example</u> Choose $f = x^\nu$ in some chart (O, ϕ). Then

$$(dx^\nu)_p \left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial}{\partial x^\mu}\right) x^\nu \Big|_{\phi(p)} = \delta^\nu{}_\mu$$

So $(dx^\nu)_p$ is dual basis to $(\partial/\partial x^\mu)_p$.

We've seen that under a change of basis / coord. that

$$\left(\frac{\partial}{\partial x^\mu}\right)_p \mapsto \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p .$$

So dually,

$$(dx^\mu)_p \mapsto \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi'(p)} (dx'^\nu)_p$$

for the basis of co-vectors.

More generally, suppose $\{f^\mu\}$ and $\{f'^\mu\}$ are <u>any</u> pair of bases for T_p^*M, and $\{e_\mu\}$ and $\{e'_\mu\}$, the corresponding dual basis of T_pM. Then $\exists$ matrices A, B s.t.

$$f'^\mu = A^\mu{}_\nu f^\nu \quad \text{and} \quad e'_\mu = B^\nu{}_\mu e_\nu .$$

However, $\delta^\mu{}_\nu = f'^\mu(e'_\nu) = A^\mu{}_\rho f^\rho (B^\sigma{}_\nu e_\sigma)$

$$= A^\mu{}_\rho B^\sigma{}_\nu f^\rho(e_\sigma)$$

$$= A^\mu{}_\rho B^\sigma{}_\nu \delta^\rho{}_\sigma$$

$$= A^\mu{}_\rho B^\rho{}_\nu .$$

Hence, $B^\mu{}_\nu = (A^{-1})^\mu{}_\nu$.

Indeed,

$$\left.\frac{\partial x'^\nu}{\partial x^\mu}\right|_{\phi(p)} \left.\frac{\partial x^\mu}{\partial x'^\rho}\right|_{\phi'(p)} = \delta^\nu{}_\rho .$$

for the coord bases.

<u>The tangent and Cotangent bundles</u>

We can "glue" together all the tangent spaces to create a new manifold called the <u>tangent bundle</u> TM.

We define

$$TM := \bigcup_{p\in M} \{p\} \times T_pM .$$

i.e. set of all pairs (p, X_p) with $p\in M$, $X_p \in T_pM$.

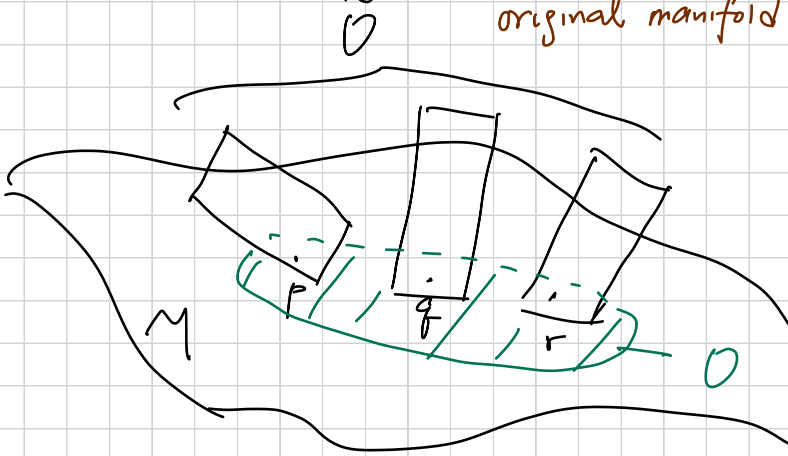
If $\{O_\alpha, \phi_\alpha\}$ is an atlas for M, we can obtain an atlas for TM by setting

$$\tilde{O}_\alpha = \bigcup_{p\in O_\alpha} \{p\} \times T_pM \subset TM$$

and

$$\tilde{\phi}_\alpha : \tilde{O}_\alpha \to \tilde{U}_\alpha \subset \mathbb{R}^{2n} \quad (\text{where } \dim M = n) .$$

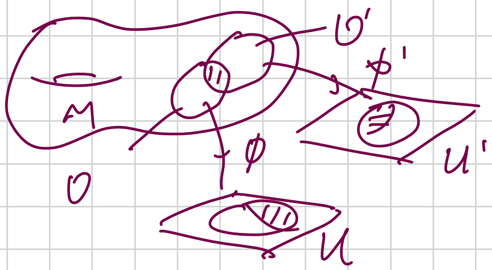
$\tilde{\phi}_\alpha$ is given by $\tilde{\phi}_\alpha : (p, T_pM) \to (\underbrace{\phi_\alpha(p)}_{\in\mathbb{R}^n,\ \text{coords of original manifold}}, \underbrace{X_p^\mu}_{\text{cpts of } X_p \in \mathbb{R}^n \text{ in the } \phi \text{ coord basis}})$



Note As yet, no way to compare $X_p \in T_pM$ with $Y_q \in T_qM$ for $p \neq q$.

Exercise suppose (O, ϕ) and (O', ϕ') are charts on M with $\phi' \circ \phi^{-1}(x) = x'(x)$. Show that on $(\phi' \circ \phi^{-1} U) \cap U'$, we have

$$\tilde{\phi}' \circ \tilde{\phi}^{-1} : (x, X^\mu) \mapsto (x'(x), \underbrace{\frac{\partial x'^\mu}{\partial x^\nu} X^\nu}_{= X'^\nu})$$



and hence deduce that TM is a smooth $2n$-dim manifold.

Similarly, define cotangent bundle T^*M by

$$T^*M = \bigcup_{p\in M} \{p\} \times T_p^*M.$$

Exercise Show that T^*M also a $2n$-dim manifold.

Exercise We have projection maps $\pi_1 : TM \to M$, $(p, X_p) \mapsto p$

and $\pi_2 : T^*M \to M$, $(p, \eta_p) \mapsto p$. Show that these are smooth.

Tensors

Often in physics, we have to deal with multi-index objects, e.g. stress-tensor $T_{\mu\nu}$, EM field strength $F_{\mu\nu}$, Riemann curvature $R^\mu{}_{\nu\alpha\beta}$. We'd now like to treat these geometrically.

Def^n A tensor of type (r,s) at a point $p\in M$ is a multilinear map

$$T : \underbrace{T_p^*M \times \dots \times T_p^*M}_{r \text{ copies}} \times \underbrace{T_pM \times \dots \times T_pM}_{s \text{ copies}} \to \mathbb{R}.$$

(multilinear = linear in each entry).

Example • A tensor of type $(0,1)$ $T: T_pM \to \mathbb{R}$ linear, is a covector.

• A tensor of type $(1,0)$ $T: T_p^*M \to \mathbb{R}$ linear, i.e. an elt of $(T_p^*M)^* = T_pM$., so it's a vector.

<u>Example</u> We could define a tensor of type $(1,1)$ by

$$\delta(\eta, X) = \eta(X)$$

for any $\eta \in T_p^*M$, $X \in T_pM$.

Suppose $\{e_\mu\}$ a basis for T_pM and $\{f^\nu\}$ the corresponding dual basis of T_p^*M. The <u>cpts</u> of a tensor T in this basis are the numbers

$$T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = T\left(f^{\mu_1}, \dots, f^{\mu_r}, e_{\nu_1}, \dots, e_{\nu_s}\right)$$

<u>Example</u> For δ, have cpts $\delta^\mu{}_\nu = \delta(f^\mu, e_\nu) = f^\mu e_\nu = \delta^\mu{}_\nu$.
above

<u>Example</u> Suppose T is a $(2,1)$-tensor, and let $\omega, \eta \in T_p^*M$, $X \in T_pM$, then

$$\begin{aligned} T(\omega, \eta, X) &= T(\omega_\mu f^\mu, \eta_\nu f^\nu, X^\rho e_\rho) \\ &= \omega_\mu \eta_\nu X^\rho \, T(f^\mu, f^\nu, e_\rho) \quad (T \text{ multilinear}) \\ &= \omega_\mu \eta_\nu X^\rho \cdot T^{\mu\nu}{}_\rho \end{aligned}$$

Abstract Index Notation

We often need complicated compositions of maps, e.g. may wish to consider

$$T^\mu{}_{\nu\kappa} S^\kappa{}_{[\lambda} \nabla_{\nu]} \phi \, R^\sigma{}_{\nu\alpha\beta}.$$

To keep track of this geometrically, in abstract index notation, we write a tensor of type (r,s) as

$$T^{a_1 \dots a_r}{}_{b_1 \dots b_s}\,.$$

Latin indices (not Greeks)

This does <u>not</u> mean cpts in any basis, just a name for our tensor that help us keep track of contractions.

For example, for a $(1,1)$-tensor,

$$\begin{aligned} T'^{\mu}{}_{\nu} &= T(f'^{\mu}, e'_{\nu}) \\ &= T(A^{\mu}{}_{\rho} f^{\rho}, (A^{-1})^{\sigma}{}_{\nu} e_{\sigma}) \\ &= A^{\mu}{}_{\rho} (A^{-1})^{\sigma}{}_{\nu}\, T(f^{\rho}, e_{\sigma}) \\ &= A^{\mu}{}_{\rho} (A^{-1})^{\sigma}{}_{\nu}\, T^{\rho}{}_{\sigma} \end{aligned}$$

In particular, for a coord basis

$$e'_{\mu} = \frac{\partial}{\partial x'^{\mu}} = \left.\frac{\partial x^{\nu}}{\partial x'^{\mu}}\right|_{\phi'(p)} \left.\frac{\partial}{\partial x^{\nu}}\right|_{p}$$

$$f'^{\mu} = (dx')^{\mu} = \left.\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right|_{p} (dx^{\nu})_{p}\,.$$

<u>Operations on Tensors</u>

<u>a) Contraction</u>

Given a tensor of type (r,s) with $r,s>0$, can form a new tensor S of type $(r-1, s-1)$ by contraction.

E.g. if $(r,s) = (2,2)$. We define

$$S(\eta, X) = T(\eta, f^{\mu}, X, e_{\mu})$$

To see this is indpt of our basis choice, note

$$T(\eta, f'^{\mu}, X, e'_{\mu}) = T(\eta, A^{\mu}{}_{\nu} f^{\nu}, X, (A^{-1})^{\rho}{}_{\mu} e_{\rho})$$
$$= A^{\mu}{}_{\nu} (A^{-1})^{\rho}{}_{\mu} T(\eta, f^{\nu}, X, e_{\rho})$$
$$= \delta^{\rho}{}_{\nu} T(\eta, f^{\nu}, X, e_{\rho})$$
$$= T(\eta, f^{\nu}, X, e_{\nu})$$

b) Tensor product

Given a tensor of type (p,q) and T of type (r,s). Define their tensor product to be $S \otimes T$ of type $(p+r, q+s)$ given by

$$S \otimes T(\omega^1, \dots, \omega^p, \eta^1, \dots, \eta^r, X_1, \dots, X_q, Y_1, \dots, Y_s)$$
$$= S(\omega^1, \dots, \omega^p, X_1, \dots, X_q)\, T(\eta^1, \dots, \eta^r, Y_1, \dots, Y_s).$$

$\forall\, \omega^i, \eta^j \in T_p^* M, \quad X_i, Y_j \in T_p M.$

Exercise Show that any $(1,1)$ tensor T can be written as
$$T = T^{\mu}{}_{\nu}\, e_{\mu} \otimes f^{\nu}.$$

c) Symmeterisation / Antisymmeterisation

For any tensor T of type $(0,2)$ define symmeterised / antisymmeterised version by

$$S(X,Y) = \frac{1}{2}\left(T(X,Y) + T(Y,X)\right)$$
$$A(X,Y) = \frac{1}{2}\left(T(X,Y) - T(Y,X)\right)$$

In abstract index notation, if T_{ab}, then
$$S_{ab} = \frac{1}{2}(T_{ab} + T_{ba}) = T_{(ab)}$$
$$A_{ab} = \frac{1}{2}(T_{ab} - T_{ba}) = T_{[ab]}$$

These (anti)symmeterisation can apply to many indices of the same type, e.g.

$$T^{(abc)}{}_{d} = \frac{1}{3!}\left(T^{abc}{}_{d} + T^{bac}{}_{d} + T^{bca}{}_{d} + T^{acb}{}_{d} + T^{cba}{}_{d} + T^{bac}{}_{d}\right)$$

$$T^{[abc]}{}_{d} = \frac{1}{3!}\left(T^{abc}{}_{d} + T^{bac}{}_{d} + T^{bca}{}_{d} - T^{acb}{}_{d} - T^{cba}{}_{d} - T^{bac}{}_{d}\right)$$

i.e. average over all (signed) permutations of indices enclosed in brackets. We can also exclude indices by vertical bars

$$T^{(a|b|c)}{}_{d} = \frac{1}{2}\left(T^{abc}{}_{d} + T^{cba}{}_{d}\right).$$

Tensor Bundles

The spaces $(T^r{}_s)_p M$ of tensors of type (r,s) at p fit together to form a tensor bundle

$$T^r{}_s M = \bigcup_{p\in M} \{p\} \times (T^r{}_s)_p M.$$

If (O,ϕ) is a coord chart in M, we set

$$\tilde{O} = \bigcup_{p\in O} \{p\} \times (T^r{}_s)_p M$$

and

$$\tilde{\phi}(p, T_p) = (\phi(p), T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s})$$

and $(\tilde{O}, \tilde{\phi})$ is a coord. chart on $T^r{}_s M$.

Exercise Show that $T^r{}_s M$ is a smooth manifold using $(\tilde{O}, \tilde{\phi})$ and calculate $\dim T^r{}_s M$.

There's a smooth projection map $\pi : T^r{}_s M \to M$ given by

$$\pi(p, (T^r{}_s)_p M) \mapsto p.$$

A <u>tensor field</u> is a smooth map $T : M \to T^r{}_s M$ s.t. the $\pi \circ T = id_M$.

If (U, ϕ) is a coord chart on M, then

$$\tilde{\phi} \circ T \circ \phi^{-1}(x^\mu) = (x^\mu, T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(x)).$$

and the cpts $T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(x)$ are smooth f^{n}s of x.

<u>Example</u> $T^1{}_0 M = TM$. tangent bundle, and a tensor field of type $(1,0)$ is a vector field. We can write

$$X(p) = (p, X_p).$$

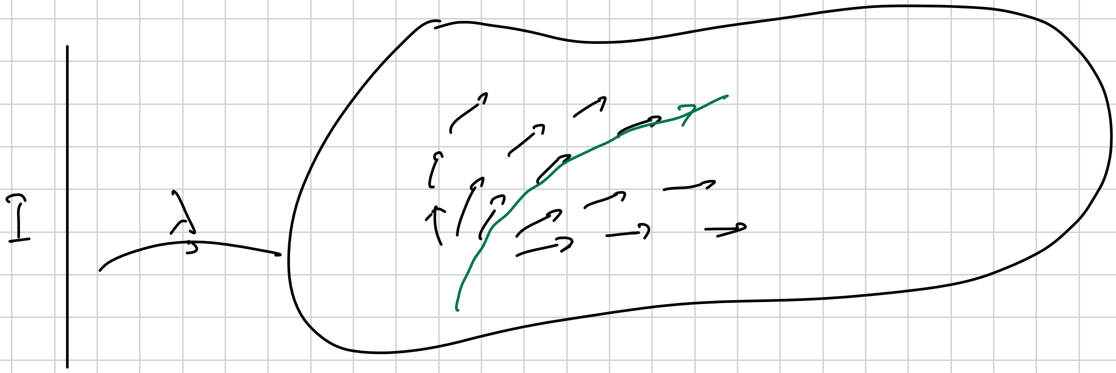
with $X_p = X^\mu(x(p)) \left(\frac{\partial}{\partial x^\mu}\right)$ in coord basis.

A vector field can act on a smooth f^n $f : M \to \mathbb{R}$ to give a new smooth f^n $X(f) : M \to \mathbb{R}$ defined by $X(f)|_p = X_p(f)$. If f is represented in some coord patch by $F = f \circ \phi^{-1}$, then $X(f)|_p = X^\mu(\phi(p)) \frac{\partial F}{\partial x^\mu}(\phi(p))$.

<u>Exercise</u> Show the vec. field X is smooth iff f^n $X(f)$ is smooth $\forall$ smooth f^n f.

<u>Integral Curves</u>

Given a vec. field X on M, we say a curve $\lambda : I \to M$ is an <u>integral curve of X</u> if its tangent at every point p on the curve is X_p.



i.e., $\dfrac{d\lambda(t)}{dt} = X_{\lambda(t)} \quad \forall\, t \in I. \quad (\dagger)$

Given a vec. field and a curve s.t. $\lambda(0) = p$, there is a unique integral curve through p (unique up to reparameterisation $t \mapsto t'(t)$ and extension beyond $\lambda(I)$).

To see this, pick a chart $(\mathcal{O}, \phi)$ with $\phi = (x^1, \dots, x^n)$ with $\phi(p) = 0$. Then $(\dagger)$ becomes

$$\frac{dx^\mu(t)}{dt} = X^\mu(x(t))$$

in this chart. This is a first order ODE for $x^\mu(t)$ with $x^\mu(0) = 0^\mu$, so sol^n unique within $\mathcal{O}$.

Example Suppose $X = \dfrac{\partial}{\partial x^1} + x^1 \dfrac{\partial}{\partial x^2}$ in some coord patch and $\phi(p) = \underline{0}$. Then integral curve of X through p obeys

$$\dot{x}^1 = 1, \quad \dot{x}^2 = x^1, \quad \dot{x}^i = 0$$

with $x^i(0) = \underline{0}$. This has sol^n

$$x^1(t) = t, \quad x^2(t) = t^2/2, \quad x^i(t) = 0, \; i \neq 1, 2$$

So the integral curve is $\phi^{-1}(t, t^2/2, 0, \dots, 0)$

Commutators

If X and Y are two vector fields, then

$$X : C^\infty(M) \to C^\infty(M)$$
$$f \mapsto X(f)$$

as does Y, and also $XY : C^\infty(M) \to C^\infty(M)$ with

$$XY : f \mapsto X(Y(f))$$

However, XY is not itself a vec. field because the Leibniz property fails:

$$XY(fg) = X(f\,Y(g) + Y(f)\,g)$$

$$= X(f)\,Y(g) + f\,XY(g) + X(g)\,Y(f) + XY(f)\,g$$

Since the failure is sym. in $X \leftrightarrow Y$, this motivates us to define the commutator

$$[X, Y] : C^\infty(M) \to C^\infty(M)$$
$$f \mapsto X(Y(f)) - Y(X(f)),$$

$\forall f \in C^\infty(M)$. We have

$$[X,Y](fg) = f\big([X,Y]g\big) + \big([X,Y]f\big)\,g.$$

In a coord basis, we have for $F = f \circ \phi^{-1}$,

$$[X,Y]f = X\Big(Y^\nu \frac{\partial F}{\partial x^\nu}\Big) - Y\Big(X^\mu \frac{\partial F}{\partial x^\mu}\Big)$$

$$= X^\mu \frac{\partial}{\partial x^\mu}\Big(Y^\nu \frac{\partial F}{\partial x^\nu}\Big) - Y^\nu \frac{\partial}{\partial x^\nu}\Big(X^\mu \frac{\partial F}{\partial x^\mu}\Big)$$

$$= X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} + X^\mu Y^\nu \frac{\partial^2 F}{\partial x^\mu \partial x^\nu}$$

$$- Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial F}{\partial x^\mu} - X^\mu Y^\nu \frac{\partial^2 F}{\partial x^\nu \partial x^\mu}.$$

$$= \Big(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\Big)\frac{\partial F}{\partial x^\nu} = [X,Y]^\nu \frac{\partial F}{\partial x^\nu},$$

where $[X,Y]^\nu = X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}$ are the cpts of $[X,Y]$ in the coord spaces.

Note We derived this using $\frac{\partial^2}{\partial x^\mu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\mu}$ so holds only in coord basis.

Exercise Suppose $X_1, \dots, X_n$ are vec. fields on M that are LI at each $p \in M$ and that $[X_i, X_j] = 0 \ \forall i,j$ throughout some patch $\mathcal{O} \in M$, Then can introduce a chart $\phi : \mathcal{O} \to U \subset \mathbb{R}^n$ s.t.

$$X_1 = \frac{\partial}{\partial y^1}, \quad \dots, \quad X_n = \frac{\partial}{\partial y^n}$$

with $(y^1, \dots, y^n)$ coords on this chart. This says we can use integral curves of these $\{X_i\}$ as coord axes.

The metric tensor

We're familiar in Eucld/Mink. space that distances are measured by an inner product.

e.g. $x \cdot y = x^1 y^1 + \dots + x^n y^n$ Euclid inner product in $\mathbb{R}^n$

$X \cdot Y = -X^0 Y^0 + X^1 Y^1 + \dots + X^3 Y^3$ Mink inner product in $\mathbb{R}^{1,3}$.

Def^n A metric at a point $p \in M$ is a $(0,2)$-tensor g obeying

- $g(X,Y) = g(Y,X) \quad \forall$ vector $X, Y \in T_pM$. (sym)
- $g(X,Y) = 0 \quad \forall \ Y \in T_pM$ iff $X = 0$.

We sometimes write $g(X,Y) = \langle X, Y \rangle = \langle X, Y \rangle_g = X \cdot Y$.

By adapting the Gram-Schmidt orthogonalisation, we can always construct a basis $\{e_\mu\}$ of T_pM (where $\mu = 1, \dots, n$) s.t.

$$g_{\mu\nu} = g(e_\mu, e_\nu) = \begin{cases} 0 & \mu \neq \nu \\ \pm 1 & \mu = \nu . \end{cases}$$

The numbers (r, s) of $-$ signs and $+$ signs is basis indpt. This is called the Signiture of g.

A metric of signiture $(0, n)$ is a Riemannian metric.
A metric of signiture $(1, n-1)$ is Lorentzian. (Special case of pseudo-Riemannian).

Defn A (pseudo-) Riemannian Manifold is a pair of (M, g) where M is a smooth manifold and g is a (pseudo-) Riemannian metric.

On a Riemannian manifold, the norm of a vector $X \in T_pM$ is $|X| = \sqrt{g(X, X)}$ and the angle between two vectors is $\theta = \cos^{-1}\left(\frac{g(X, Y)}{|X||Y|}\right)$.

The length of a curve $\lambda : (a, b) \to M$ is defined to be

$$l(\lambda) = \int_a^b |\dot{\lambda}| \, dt$$

Exercise Check that this is invar. under reparam of λ with $t \mapsto t'(t)$ and $\frac{dt'}{dt} > 0$.

In a coord basis, we can write

$$g = g_{\mu\nu} \, dx^\mu \otimes dx^\nu = ds^2$$

where $g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$ and $\{dx^\mu\}$ is the dual basis of T_p^*M at $p \in O$.

Example • $(\mathbb{R}^n, \delta)$ has $g = ds^2 = (dx^1)^2 + \dots + (dx^n)^2$

$$= \delta_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

• $(\mathbb{R}^{1,3}, \eta)$ has $g = ds^2 = -(dx^0)^2 + (dx^1)^2 + \dots + (dx^3)^2$

$$= \eta_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

Minkowski metric cpts.

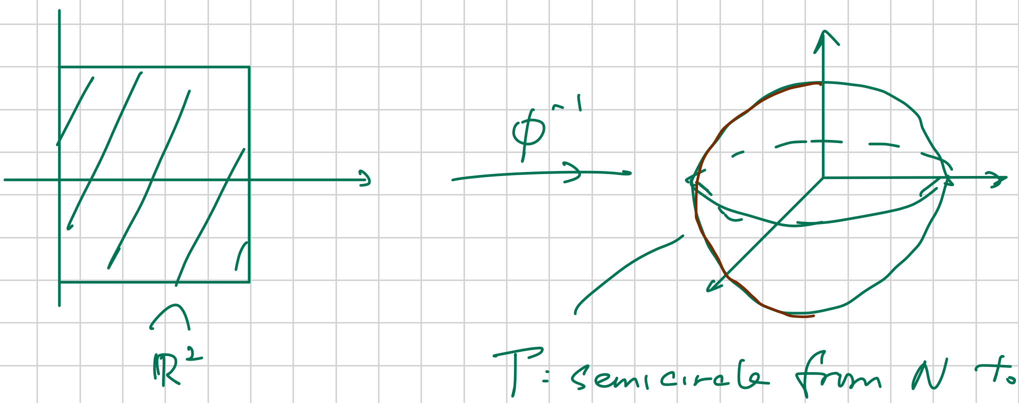
• on $S^2 = \{\underline{x}\cdot\underline{x} = 1 \mid \underline{x} \in \mathbb{R}^3\}$, define a chart by

$$\phi^{-1} : (0, \pi) \times (-\pi, \pi) \to S^2$$

$$(\theta, \varphi) \mapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

In this chart, the round metric is

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$



T: semicircle from N to S in plane $y = 0$

Have $O = S^2 \setminus T$. The round metric is induced from standard Euclidean δ on $\mathbb{R}^3$ when constrained to $|\underline{x}| = 1$.

Because g non-degenerate, its cpt $g_{\mu\nu} = g(e_\mu, e_\nu)$ form an invertible matrix (in any basis). The inverse is a symmetric tensor field of type $(2,0)$, g^{-1}.

In A.I.N., we write

$$\underset{g^{-1}}{g^{ab}}\ \underset{g}{g_{bc}} = \delta^a{}_c$$

Note: g^{ab}, not $(g^{-1})^{ab}$.

Example For metric on $\mathcal{O} \subset S^2$ before, g^{-1} has cpts

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{pmatrix}.$$

The metric allows us to canonically identify $T^*M \underset{g}{\cong} TM$: for any $X \in TM$, we get a covector field $X^\flat = g(X, \cdot)$ or in A.I.N.

the "flat" sign ⇓ drop.

$$X_a = g_{ab} X^b$$

and similarly, for any $\omega \in T^*M$, define $\omega^\# = g^{-1}(\cdot, \omega) \in TM$
In AIN,

$$\omega^a = g^{ab}\omega_b.$$

To extend this to other tensors, we first generalise our def^n slightly and allow a tensor of type (r,s) at $p \in M$ to be multilinear map

$$T: T_p^*M \times T_pM \times T_pM \times T_p^*M \times \dots \times T_pM \to \mathbb{R}.$$

with r copies of T_p^*M and s copies of T_pM in any order, allowing us to write, for example,

$$T^a{}_{bc}{}^{de}{}_f$$

with upstairs/downstairs indices in any order.

Then using the metric, can obtain new tensors by "raising / lowering" indices.

Example If $T^{ab}{}_c$ type $(2,1)$ tensor, then define a $(1,2)$ tensor $T^a{}_{bc} = g_{bd} T^a{}_{dc}$.

Lorentzian Signature

If (M, g) is a Lorentzian manifold, at each $p \in M$, can find a basis $\{e_\mu\}$ of T_pM s.t.

$$g_{\mu\nu} = g(e_\mu, e_\nu) = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

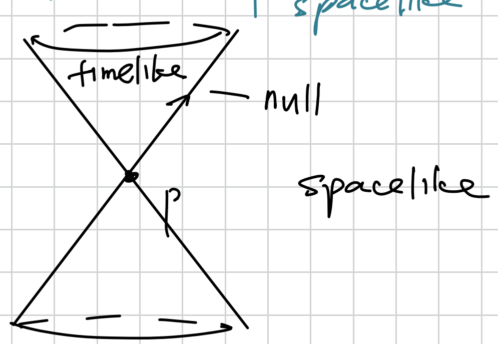
This basis is not unique. Suppose $\{e'_\mu\}$ is another such basis and $e'_\mu = (A^{-1})^\nu{}_\mu e_\nu$, then

$$\eta_{\mu\nu} = g(e'_\mu, e'_\nu) = g\left((A^{-1})^\rho{}_\mu e_\rho, (A^{-1})^\sigma{}_\nu e_\sigma\right)$$
$$= (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu \, g(e_\rho, e_\sigma)$$
$$= (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu \, \eta_{\rho\sigma}.$$

So any two such basis are related by a Lorentz trans^n at p.

In this basis, T_pM has $\eta_{\mu\nu}$ as its metric, so has the structure of Mink. space.

Def^n A vec. $X \in T_pM$ is $\left\{\begin{array}{l} \text{timelike} \\ \text{null} \\ \text{spacelike} \end{array}\right\}$ if $g(X, X)|_p \left\{\begin{array}{l} < 0 \\ = 0 \\ > 0 \end{array}\right\}$.



A curve $\lambda : I \to M$ in a Lorentzian mfld is spacelike/null/timelike if its tangent vector field X is sp/null/timelike at each $p \in \lambda(I)$.

Curves of Extremal Proper Time

For λ spacelike curve, its length defined as on a Riemannian mfld. However, for a timelike curve, we instead use a proper time

$$\tau(\lambda) = \int_a^b \sqrt{-g(X,X)}\, du$$

where X is the tangent vector to λ, and $u \in I$ a param.

In a coord basis

$$X = X^\mu(u) \frac{\partial}{\partial x^\mu} = \frac{d x^\mu(u)}{du} \frac{\partial}{\partial x^\mu} .$$

So

$$\tau(\lambda) = \int_a^b \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, du$$

with $\dot{x}^\mu = dx^\mu / du$.

Suppose $\lambda : (0,1) \to M$ is a smooth timelike curve with $\lambda(0) = p$, $\lambda(1) = q$ and $\lambda(u)$ extremises $\tau(\lambda)$, then treating

$$L = \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} ,$$

this curves obeys the E-L eqn

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu} .$$

We compute
$$\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{1}{L} g_{\mu\nu} \dot{x}^\nu ,$$
$$\frac{\partial L}{\partial x^\mu} = -\frac{1}{2L} (\partial_\mu g_{\rho\sigma}) \dot{x}^\rho \dot{x}^\sigma$$

It's helpful to parameterise λ using proper time itself. Since
$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{du} \frac{du}{d\tau} ,$$
this gives
$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \left(\frac{du}{d\tau}\right)^2 = -1 .$$
(note $d\tau^2|_{p\in\lambda(I)} = g(X,X)|_p$)
$$\Rightarrow \frac{du}{d\tau} = \frac{1}{L} \quad \text{and} \quad \frac{1}{L}\frac{d}{du} = \frac{d}{d\tau} .$$

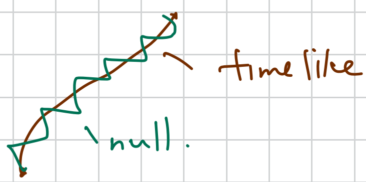
Consequently, our E-L involves
$$\frac{d}{d\tau}\left(g_{\mu\nu} \frac{dx^\nu}{d\tau} \right) = \frac{1}{2} (\partial_\mu g_{\rho\sigma}) \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}$$

$$\Rightarrow \quad g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} + (\partial_\rho g_{\mu\nu}) \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} = \frac{1}{2} (\partial_\mu g_{\rho\sigma}) \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}$$

Since g invertible, this is
$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma_\mu{}^\nu{}_\rho \frac{dx^\mu}{d\tau} \frac{dx^\rho}{d\tau} = 0 ,$$
where $\Gamma_\mu{}^\nu{}_\rho = \frac{1}{2} g^{\nu\sigma} (\partial_\mu g_{\sigma\rho} + \partial_\rho g_{\sigma\mu} - \partial_\sigma g_{\mu\rho})$.

<u>Note</u> : $\Gamma_\mu{}^\nu{}_\rho = \Gamma_\rho{}^\nu{}_\mu$, but <u>not</u> cpt. of a tensor in any basis.

<u>Rmks</u> • Timelike extremal curves are <u>maxima</u> of proper time



- Can obtain same E-L using

$$L' = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

parameterised by τ.

Covariant Derivatives

We know how to differentiate a f^n along a curve

$$X[f] = X^\mu \partial_\mu f$$

where X is the tangent vector to a curve. However, we can't diff. vec. fields (or other tensor fields) because they live in a diff. space T_pM at each point $p \in M$.

Also, while $\partial_\mu f$ are cpts of a covec. (df) in a coord basis, the naive derivatives $\partial_\mu X^\nu$ of the cpts of a vec. field X are not the cpts of any tensor.

Def^n A covariant derivative ∇ on M is a map sending any pair of vec. fields X, Y to a new vec. field $\nabla_X(Y)$ s.t.

(i) $\nabla_{fX+gY}(Z) = f\nabla_X(Z) + g\nabla_Y(Z)$ $\forall$ vec. field X, Y, Z, and smooth f^n f, g

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.

(iii) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

(where $X(f) = X^\mu \partial_\mu F$ in a coord basis).

We call $\nabla_X Y$ the covar. deriv. of Y along X. Note that (i) implies $\nabla Y : X \mapsto \nabla_X Y$ is a tensor of type $(1,1)$, i.e. linear over f^ns in its arguments.

In A.I.N., write

$$(\nabla Y)^a{}_b = \nabla_b Y^a = Y^a{}_{;b}$$

<u>Def^n</u> In any basis $\{e_\mu\}$, the <u>connection component</u> $\Gamma^\mu_{\nu\rho}$ is defined by

$$\nabla_{e_\rho}(e_\nu) = \Gamma^\mu_{\nu\rho}\, e_\mu.$$

<u>Note</u> $\Gamma^\mu_{\nu\rho}$ not cpt of tensor, and are not necc. related to the Christoffel symbol

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}\left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\rho\nu}\right).$$

Then for general vec. $X = X^\mu e_\mu$, $Y = Y^\nu e_\nu$, we have

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^\mu e_\mu}(Y^\nu e_\nu) \\
&= X^\mu \nabla_{e_\mu}(Y^\nu e_\nu) \\
&= X^\mu e_\mu(Y^\nu)\, e_\nu + X^\mu Y^\nu \nabla_{e_\mu}(e_\nu) \\
&= X^\mu e_\mu(Y^\nu)\, e_\nu + X^\mu Y^\nu \Gamma^\sigma_{\nu\mu} e_\sigma \\
&= X^\mu\left(e_\mu(Y^\sigma) + Y^\nu \Gamma^\sigma_{\nu\mu}\right) e_\sigma.
\end{aligned}$$

$$\Rightarrow (\nabla_X Y)^\sigma = X^\mu e_\mu(Y^\sigma) + \Gamma^\sigma_{\nu\mu} X^\mu Y^\nu,$$

are the cpt. of the vec. $\nabla_X Y$ in this basis.

If $e_\mu = \partial/\partial x^\mu$, then

sometimes write $Y^\sigma{}_{,\mu}$.

$$(\nabla_X Y)^\sigma = X^\mu (\partial_\mu Y^\sigma) + \Gamma^\sigma_{\nu\mu} X^\mu Y^\nu$$

We extend this to more general tensors by setting $\nabla_X f = X(f)$ and then using Leibniz prop.

Example If η covec. field, then we define

$$(\nabla_X \eta)(Y) = \nabla_X(\eta(Y)) - \eta(\nabla_X Y),$$

i.e. $\nabla_X(\eta(Y)) = (\nabla_X \eta)(Y) + \eta(\nabla_X Y)$. $\forall$ vec. Y.

In cpts,

$$\begin{aligned}(\nabla_X \eta)(Y) &= X^\mu e_\mu(\eta_\nu Y^\nu) - \eta_\nu (\nabla_X Y)^\nu \\ &= X^\mu (e_\mu \eta_\nu) Y^\nu + \cancel{X^\mu \eta_\nu e_\mu(Y^\nu)} \\ &\qquad - \eta_\nu \left(\cancel{X^\mu e_\mu(Y^\nu)} + \Gamma^\nu_{\rho\mu} X^\mu Y^\rho\right) \\ &= X^\mu Y^\nu \left(e_\mu(\eta_\nu) - \Gamma^\rho_{\nu\mu} \eta_\rho\right).\end{aligned}$$

$$\Rightarrow \quad (\nabla \eta)_{\mu\nu} = e_\mu(\eta_\nu) - \Gamma_\nu{}^\rho{}_\mu \eta_\rho.$$

and we often write $(\nabla \eta)_{ab} = \nabla_a \eta_b = \eta_{b;a}$.

Exercise Show that for general type (r,s) T, $\nabla_X T$ has cpt

$$\begin{aligned}\nabla_\rho T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} &= \partial_\rho T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \\ &\quad + \Gamma^{\mu_1}_{\kappa\rho} T^{\kappa \mu_2 \dots \mu_r}{}_{\nu_1 \dots \nu_s} + \dots + \Gamma^{\mu_r}_{\kappa\rho} T^{\mu_1 \dots \mu_{r-1} \kappa}{}_{\nu_1 \dots \nu_s} \\ &\quad - \Gamma^{\kappa}_{\nu_1 \rho} T^{\mu_1 \dots \mu_r}{}_{\kappa \nu_2 \dots \nu_s} - \dots - \Gamma^{\kappa}_{\nu_s \rho} T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_{s-1} \kappa}.\end{aligned}$$

Since $\nabla_c T^a{}_b$ is tensor field, we can take further covar. deriv. $\nabla_d \nabla_c T^a{}_b = (T^a{}_{b;c})_{;d} = T^a{}_{b;cd}$.

There's no guarantee that $\nabla_a \nabla_b = \nabla_b \nabla_a$.

If f is a f^n, then $\nabla_\mu f = \partial_\mu f$ is (the cpts of) a covec., so in a coord. basis

$$\nabla_\nu \nabla_\mu f = \nabla_\nu (\partial_\mu f)$$
$$= \partial_\nu \partial_\mu f - \Gamma^\rho_{\mu\nu} \partial_\rho f.$$

and hence $f_{;[\mu\nu]} = -\Gamma^\rho_{[\mu\nu]} \partial_\rho f$.

We say a connection is <u>torsion free</u> iff

$$\nabla_a \nabla_b f - \nabla_b \nabla_a f = 0 \quad \forall f \in C^\infty(M).$$

In a coord basis, torsion free implies $\Gamma^\rho_{\nu\mu} = \Gamma^\rho_{(\nu\mu)}$.

<u>Lem</u> If ∇ torsion free, the torsion tensor T vanishes, where T is $(1,2)$-tensor

$$T(\cdot, X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$\forall$ vec. field X, Y.

<u>Pf</u>: In a coord basis, we have

$$(\nabla_X Y - \nabla_Y X)^\mu = X^\sigma Y^\mu{}_{;\sigma} - Y^\sigma X^\mu{}_{;\sigma}$$
$$= X^\sigma(\partial_\sigma Y^\mu + \Gamma^\mu_{\rho\sigma} Y^\rho) - Y^\sigma(\partial_\sigma X^\mu - \Gamma^\mu_{\rho\sigma} X^\rho)$$
$$= (X^\sigma \partial_\sigma Y^\mu - Y^\sigma \partial_\sigma X^\mu) + X^\sigma Y^\rho (\Gamma^\mu_{\rho\sigma} - \Gamma^\mu_{\sigma\rho}).$$
$$= [X, Y]^\mu.$$

Since $\nabla_X Y$, $\nabla_Y X$ and $[X, Y]$ are all genuine vecs, this holds as an eqn among tensors, i.e. $T(\cdot, X, Y) = 0 \ \forall X, Y$
$\Rightarrow T = 0$ $\square$

Rmk Even if $\nabla_a \nabla_b f - \nabla_b \nabla_a f = 0$, no guarantee that $\nabla_a \nabla_b = \nabla_b \nabla_a$ when acting on more general tensors.
Here, by $\nabla_a \nabla_b f = (\nabla\nabla f)_{ab}$, we mean a tensor of type $(0,2)$ defined by

$$\boxed{\nabla_X \nabla_Y f = X(df(Y)) - df(\nabla_X Y).}$$

(Recall $\nabla_Y f = df(Y) = Y(f) \ (= \partial_\mu F \, Y^\mu)$ and so $\nabla f = df$ is a covector) and that for any covec η defined by $\nabla_X \eta$ by

$$(\nabla_X \eta)(Y) = X(\eta(Y)) - \eta(\nabla_X Y).$$

Hence, by $\nabla_X \nabla_Y f - \nabla_Y \nabla_X f$, we mean

$$\begin{aligned} & X(df(Y)) - df(\nabla_X Y) - Y(df(X)) + df(\nabla_Y X) \\ = \ & X(Y(f)) - Y(X(f)) - df(\nabla_X Y - \nabla_Y X) \\ = \ & ([X, Y] - \nabla_X Y - \nabla_Y X) f. \end{aligned}$$

On a (pseudo-) Riemannian manifold, a connection is <u>metric compatible</u> if

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

$\forall$ vec. field X, Y, Z, (i.e. no "$(\nabla_Z g)(X, Y)$" term).

On any (M, g), $\exists!$ connection ∇ which is metric compatible and torsion-free.

证：

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \qquad (1)$$

$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X). \qquad (2)$$

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad (3)$$

(1) + (2) − (3):

$$Z(g(X,Y)) + Y(g(Z,X)) - X(g(Y,Z)) \circledast g(\nabla_Z X, Y) - g(\nabla_X Z, Y)$$
$$+ g(\nabla_Y X, Z) - g(\nabla_X Y, Z)$$
$$+ g(\nabla_Z Y, X) + g(\nabla_Y Z, X)$$

Since torsion free, $\nabla_Z X - \nabla_X Z = [Z, X]$ and similarly

$\nabla_Y X - \nabla_X Y = [Y, X]$, so

$$\circledast\, g([Z,X], Y) + g([Y,X], Z) + g(\nabla_Z Y, X)$$
$$+ g(\nabla_Z Y, X) - g([Z,Y], X)$$

Hence,

$$g(\nabla_Z Y, X) = \frac{1}{2}\Big(Z(g(X,Y)) + Y(g(X,Z)) - X(g(Y,Z))$$
$$- g([Z,X],Y) - g([Y,X],Z) + g([Z,X],Y)\Big)$$

Now, RHS indpt of ∇, and since g non-degenerate, this defines $\nabla_Z Y$.

We can also check that this defines a covar. deriv.

For example,

$$g(\nabla_{(fZ)} Y, X) = f\, g(\nabla_Z Y, X) + \frac{1}{2}\Big[Y(f)\, g(X,Z) - X(f)\, g(Y,Z)$$
$$+ X(f)\, g(Z,Y) - Y(f)\, g(Z,X) \Big] \square$$

This torsion-free, metric compatible connection is called the Levi-Civita connection.

In a coord basis, we have

$$g(\nabla_{e_\mu} e_\nu, e_\sigma) = \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

$$= g(\Gamma^\rho_{\mu\nu} e_\rho, e_\sigma)$$

$$= \Gamma^\rho_{\mu\nu} g_{\rho\sigma}$$

So the Levi-Civita connection cpts are just Christoffel symbol.

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).$$

Geodesics

Earlier, found a curve of extremal length / proper time obeyed

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0. \qquad (*)$$

when parameterised by τ, where $X^\mu = \frac{dx^\mu}{d\tau}$ is the tangent vector to the curve. By chain rule,

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{d}{d\tau}(X^\mu) = \frac{\partial X^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau} = X^\nu X^\mu{}_{,\nu}$$

(after expanding X^μ to a vec. field in any smooth way)

Hence $(*)$ becomes

$$0 = X^\nu (X^\mu{}_{,\nu} + \Gamma^\mu_{\nu\sigma} X^\sigma) = X^\nu X^\mu{}_{;\nu} = X^\nu \nabla_\nu X^\mu$$

Def^n Let M be manifold and ∇ any connection. An <u>affinely parameterised geodesic</u> of ∇ is a curve whose tangent vec. X obeys $\nabla_X X = 0$.

<u>Note</u> If we change param. $\tau \to \tau(u)$ with $h = \frac{d\tau}{du} > 0$, then

$$Y^\mu = \frac{dx^\mu}{du} = \frac{d\tau}{du} \cdot \frac{dx^\mu}{d\tau} = h X^\mu$$

so $X \to hX = Y$ with $h>0$. Hence

$$\begin{aligned}\nabla_Y Y &= \nabla_{hX}(hX) \\ &= h \nabla_X (hX) \\ &= h^2 \nabla_X(X) + h\, X(h)\, X = X(h)\, Y.\end{aligned}$$

where $X(h) = \frac{d}{d\tau}\left(\frac{d\tau}{du}\right) = \frac{1}{h}\frac{d}{du}\left(\frac{d\tau}{du}\right) = \frac{1}{h}\frac{d^2\tau}{du^2}$.

So $\nabla_Y Y = 0$ iff $\frac{d^2\tau}{du^2} = 0$, ie. $\tau = \alpha u + \beta$ with $\alpha > 0, \beta \in \mathbb{R}$.

<u>Thm</u> Given $p \in M$ and $X \in T_pM$, $\exists!$ affinely parameterised geodesic $\lambda_{X_p} : I \to M$ which obeys $\lambda_{X_p}(0) = p$, $\dot\lambda_{X_p}(0) = X_p$.

<u>Pf</u>: Choose coords s.t. $\phi(p) = (0, \dots, 0)$ and $\phi(\lambda_{X_p}(t)) = (x^1(t), x^2(t), \dots)$. Then $x^\mu(t)$ obeys

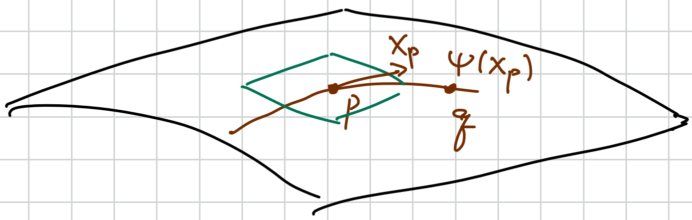
$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0.$$

and $x^\mu(0) = 0$. $\frac{dx^\mu}{dt}(0) = X_p^\mu$. and sol^n is unique $\square$

<u>Postulate</u>: In GR, free particles move along geodesics of the Levi-civita ∇. For massive particles, these geodesics are <u>timelike</u>, while for massless, <u>null</u>.

Normal Coordinates

There's a useful set of coords we can build using affinely param. (a.p.) geodesics. Fix $p \in M$ and define a map $\psi : T_pM \to M$ by $\psi(X_p) = \lambda_{X_p}(1)$ where λ_{X_p} is the a.p. geodesic with $\lambda_{X_p}(0) = p$. $\dot{\lambda}_{X_p}(0) = X_p$.



Suppose $\tilde{\lambda}(t) = \lambda_{X_p}(\alpha t)$ for some $\alpha \neq 0$. Then $\tilde{\lambda}(0) = p$ and $\dot{\tilde{\lambda}}(0) = \alpha \dot{\lambda}_{X_p}(0) = \alpha X_p$, so $\lambda_{X_p}(\alpha t) = \lambda_{\alpha X_p}(t)$, and moreover the curve $\lambda : \alpha \mapsto \psi(\alpha X_p)$ is a.p. geodesic.

Claim If $U \subset T_pM$ is sufficiently small nbd of $0 \in T_pM$, the map $\psi : U \to M$ is 1-to-1 and onto locally.
ψ is called the exponential map ($\psi = \mathrm{Exp.}$).

Def^n Suppose $\{e_\mu\}$ basis of T_pM. We construct normal coords at p as follows. For any $q \in \psi(U) \subset M$, define $\phi(q) = (X^1, \ldots, X^n)$ the cpts of X_p in $\{e_\mu\}$ basis, where $\psi(X_p) = q$.

By previous calculation, the curve given in normal coords by $X^\mu(t) = tY^\mu$ for const. Y^μ is a.p. geodesic. Hence, since Y^μ const.,

$$\Gamma^\mu_{\nu\sigma}(tY^\rho)\, Y^\nu Y^\sigma = 0,$$

so at p $(t=0)$, $\Gamma^\mu_{\nu\sigma}(0)\, Y^\nu Y^\rho = 0$ $\forall\, Y^\mu$, which implies

$T_{(\nu}{}^{\mu}{}_{\rho)}(0) = 0$, or $T_\nu{}^\mu{}_\rho(0) = 0$ if ∇ torsion-free.

If ∇ is Levi-Civita connection on (pseudo-) Riemannian manifold (M, g), then since

$$g_{\mu\nu,\rho} = \frac{1}{2}\left(g_{\mu\nu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu}\right) + \frac{1}{2}\left(g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\rho\nu,\mu}\right)$$

$$= T_\mu{}^\sigma{}_\rho\, g_{\sigma\nu} + T_\nu{}^\sigma{}_\rho\, g_{\sigma\mu}.$$

we have $g_{\mu\nu,\rho}|_p = 0$. Since we can always choose $\{e_\mu\}$ orthonormal at p, we have

<u>Lem</u> On any Riemannian / Lorentzian (M, g), we can choose (normal) coords at $p \in M$ s.t.

$$g_{\mu\nu}|_p = \begin{cases} \delta_{\mu\nu} & \text{(Riemannian)} \\ \eta_{\mu\nu} & \text{(Lorentzian)} \end{cases}$$

and $g_{\mu\nu,\rho}|_p = 0$.

Curvature

Suppose $\lambda: I \to M$ is a curve with tangent $\dot\lambda(t)$, we say a tensor T is <u>parallel transported / propagated</u> along λ if $\nabla_{\dot\lambda} T = 0$.

<u>Example</u> If λ a.p. geod., and $T = \dot\lambda$, then $\nabla_{\dot\lambda}\dot\lambda = 0$, so $\dot\lambda$ is parallel propagated along its own a.p. geod.

Any parallelly propagated tensor field T is determined along λ by its value T_p at any point $p \in \lambda(t)$.

In a coord basis, $\nabla_{\dot\lambda} T = 0$ (e.g. T has type (1,1))

becomes

$$0 = \frac{dx^\mu}{dt} \partial_\mu T^\nu{}_\rho + \Gamma^\nu{}_{\sigma\mu} T^\sigma{}_\rho \frac{dx^\mu}{dt} - \Gamma^\kappa{}_{\rho\mu} T^\nu{}_\kappa \frac{dx^\mu}{dt}$$

$$= \frac{d}{dt}(T^\nu{}_\rho) + \Gamma^\nu{}_{\sigma\mu} T^\sigma{}_\rho \frac{dx^\mu}{dt} - \Gamma^\kappa{}_{\rho\mu} T^\nu{}_\kappa \frac{dx^\mu}{dt}$$

This is a 1st order ODE for $T^\nu{}_\rho$, so $\exists!$ sol^n (locally) if $T^\nu{}_\rho(0)$ given. Hence parallel transport along curves gives iso^m between

$$(T^r{}_s)_p \underset{\lambda}{\cong} (T^r{}_s)_q$$

depends on λ

In general, this identification depends on the curve.

The Riemann Tensor

The Riemann Tensor encodes the extent to which parallel propagation along different curves differ.

Given X, Y, Z smooth vec. fields, and connection ∇, define a new vector field

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Lem This defines a tensor of type (1,3) $R^a{}_{bcd}$ by

$$(R(X,Y)Z)^a = R^a{}_{bcd} X^c Y^d Z^b.$$

Pf: Suppose $f: M \to \mathbb{R}$ smooth f^n.

$$R(fX, Y)Z = \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z$$

$$= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X} Z$$

$$= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X(Z)$$
$$- f\nabla_{[X,Y]} Z + Y(f)\nabla_X Z.$$

$$= f\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z\right)$$

Also, $R(X, fY)Z = -R(fY, X)Z$
$$= -fR(Y,X)Z = fR(X,Y)Z$$

<u>Exercise</u>: $R(X,Y)f(Z) = fR(X,Y)Z$ □

In a basis $\{e_\mu\}$ and dual basis $\{f^\nu\}$, we have cpts

$$R^\mu{}_{\nu\rho\sigma} = f^\mu(R(e_\rho, e_\sigma)e_\nu)$$

and in a coord basis, $[e_\nu, e_\sigma] = [\partial_\nu, \partial_\sigma] = 0$, so

$$R(e_\rho, e_\sigma)e_\nu = \nabla_{e_\rho}(\nabla_{e_\sigma} e_\nu) - \nabla_{e_\sigma}(\nabla_{e_\rho} e_\nu)$$
$$= \nabla_{e_\rho}(\Gamma^\kappa_{\nu\sigma} e_\kappa) - \nabla_{e_\sigma}(\Gamma^\kappa_{\nu\rho} e_\kappa)$$

$$= \partial_\rho \Gamma^\tau_{\nu\sigma} e_\tau + \Gamma^k_{\nu\sigma}\Gamma^\tau_{k\rho} e_\tau - \partial_\sigma \Gamma^\tau_{\nu\rho} e_\tau$$
$$- \Gamma^k_{\nu\rho}\Gamma^\tau_{k\sigma} e_\tau.$$

$$\Rightarrow \quad R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^k_{\nu\sigma}\Gamma^\mu_{k\rho} - \Gamma^k_{\nu\rho}\Gamma^\mu_{k\sigma}$$

In particular, suppose ∇ has vanishing torsion tensor. Then

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$
$$= \nabla_X(Y^b\nabla_b Z) - \nabla_Y(X^b\nabla_b Z) - [X,Y]^b\nabla_b Z$$
$$= X^a Y^b(\nabla_a\nabla_b Z - \nabla_a\nabla_b Z) +$$
$$\big(\underbrace{\nabla_X Y^b - \nabla_Y X^b - [X,Y]^b}_{T(\cdot,X,Y)=0}\big)\nabla_b Z.$$

So

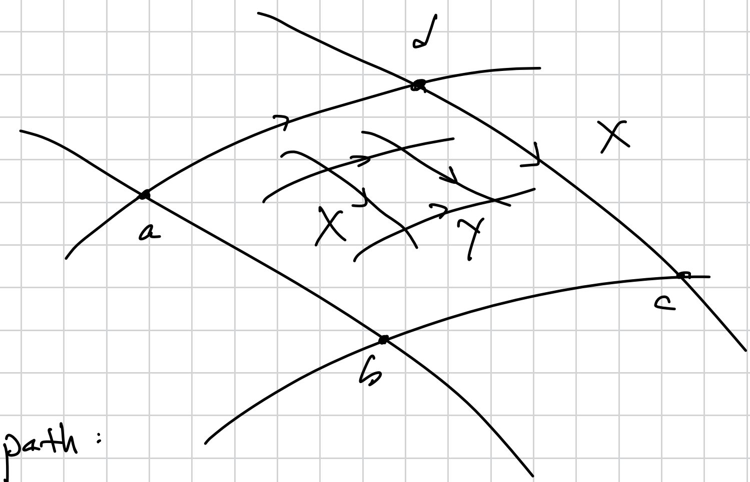
$$\boxed{R^a{}_{bcd} Z^d = (\nabla_a \nabla_b Z - \nabla_b \nabla_a Z)^c}$$

This is Ricci identity

Example For Levi-Civita connection of Mink. space in Cartesian coord frame, $T^\lambda_{\mu\nu} = 0$, so $R^\lambda{}_{\kappa\mu\nu} = 0$. This is a tensor eqn, so $R^a{}_{bcd} = 0$ in any frame, and we call the space flat. Conversely, if the Levi-Civita connection of any metric has $R^a{}_{bcd} = 0$, then space is flat and $\exists$ coords s.t. $g = \eta$ (at least locally).

Visualising Curvature.

Suppose X, Y s.t. $[X, Y] = 0$

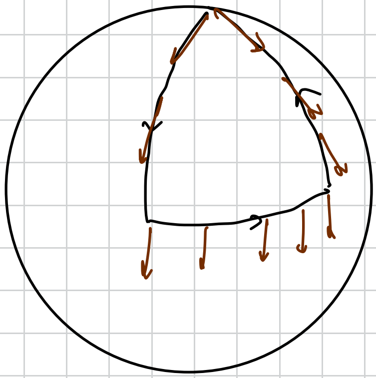


We follow the path:

- $a \to b$ by flowing distance $\varepsilon \ll 1$ along X.
- $b \to c$ $\varepsilon \ll 1$ Y
- $c \to d$ " $-X$
- $d \to a$ $-Y$.

Claim If a vector Z is parallel transported around this quadrilateral to yield a vector Z', then

$$(Z - Z')^\mu = \varepsilon^2 R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma + O(\varepsilon^3).$$



Geodesic Deviation

In Mink. space, geodesics that initially point in the same direction remain parallel forever. There's no notion of "parallel" on a general (M, g) and we'd like to measure whether nearby geodesics get closer / further as one flows along the geodesics.

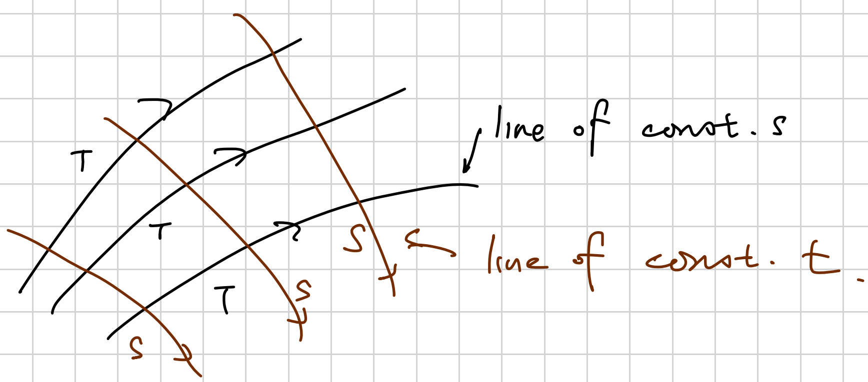
Let M be manifold with connection ∇. A <u>1-param family of geodesics</u> is a map

$$\gamma : I \times I' \to M,$$

where I, I' open intervals in $\mathbb{R}$ s.t.

- $(s,t) \mapsto \gamma(s,t)$ is smooth, 1-to-1 and has smooth inverse on $\gamma(I \times I')$.
- for fixed s. $\gamma(s,t)$ is geodesic with affine param. t.

Then $\gamma(I \times I') = \Sigma \subset S$ is a smooth surface



Let T be a tangent vec. field to the geodesics, and let S be tangent to curves of const. t. In a coord. chart we have $\phi(\Sigma) \ni x^\mu(s,t)$ with $S^\mu = \left.\frac{\partial x^\mu}{\partial s}\right|_t$, so

$$x^\mu(s+\delta s, t) = x^\mu(s,t) + \delta s\ S^\mu(s,t) + O(\delta s^2).$$

S is called the <u>deviation vector</u>.

Extend $(s,t) \in \phi(\Sigma)$ to coords on nbd $O \subset M$ with $\Sigma \subset O$, so $\phi(O) \ni (\underbrace{s,t}_{\text{along } \Sigma}, \underbrace{\ldots}_{\text{transverse to } \Sigma})$. Then $T = \frac{\partial}{\partial t}$ and

$S = \frac{\partial}{\partial s}$ in these coords, and $[S,T] = 0$.

Prop. If ∇ has vanishing torsion, then $\nabla_T(\nabla_T S) = R(T,S)T$.

Pf: Torsion free, so $\nabla_T S - \nabla_S T - [T,S] = 0$, and since $[T,S]=0$, $\nabla_T S = \nabla_S T$. Hence

$$\begin{aligned}
\nabla_T(\nabla_T S) &= \nabla_T(\nabla_S T) \\
&= \nabla_S(\underbrace{\nabla_T T}_{=0}) + (\nabla_T(\nabla_S T) - \nabla_S(\nabla_T T)) \\
&= \nabla_T(\nabla_S T) - \nabla_S(\nabla_T T) - \nabla_{\underbrace{[T,S]}_{=0}} T \\
&= R(T,S)T.
\end{aligned}$$

$\square$

In index notation,

$$(\nabla_T \nabla_T S)^a = R^a{}_{bcd} T^b T^c S^d$$

This is <u>geodesic deviation eqn.</u>

Sometimes, write

$$\left(\frac{D^2 S}{Dt^2}\right)^a = R^a{}_{bcd} T^b T^c S^d$$

Exercise If ∇ torsion-free, show

$$R^a{}_{bcd} = \frac{2}{3}\left(R^a{}_{(bc)d} - R^a{}_{(bd)c}\right)$$

Hence, the full Riemann tensor can be measured by looking at geodesic deviation for all family of geodesics. The geodesic deviation is called a <u>tidal gravitational</u> effect in GR.

Symmeteries of Riemann Tensor

$$R^a{}_{bcd} X^c Y^d Z^b = \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right)^a$$

It follows that $R^a{}_{bcd} = -R^a{}_{bdc}$ for any connection ∇. However, if ∇ torsion-free, then $R^a{}_{bcd}$ has additional sym.

Prop. Let ∇ be a torsion-free connection. Then

(i) $R^a{}_{[bcd]} = 0$

(ii) $R^a{}_{b[cd;e]} = 0$ (Bianchi identity)

Pf: To prove this, helpful to use normal coord.
Torsion-free $\Rightarrow$ $\Gamma^\mu{}_{\nu\lambda} = \Gamma^\mu{}_{\lambda\nu}$ everywhere.

(i) $R^\mu{}_{\nu\rho\sigma}|_p = (\partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho})|_p$ and $R^\mu{}_{[\nu\rho\sigma]}|_p = 0$.
But p was arbitrary, so $R^\mu{}_{[\nu\rho\sigma]} = 0$ everywhere, and since this is tensor eqn, $R^a{}_{[bcd]} = 0$.

(ii) In normal coords,

$$R^\mu{}_{\nu\rho\sigma;\tau}|_p = R^\mu{}_{\nu\rho\sigma,\tau}|_p = \partial_\tau\left(\partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \cancel{\Gamma\Gamma}\right)\Big|_p.$$

and so $R^\mu{}_{\nu[\rho\sigma;\tau]}|_p = 0$ and hence $R^a{}_{b[cd;e]} = 0$. $\square$.

From now on, let ∇ be Levi-Civita connection on (M,g) then we can "lower index" to define

$$R_{abcd} = g_{ae} R^e{}_{bcd}$$

Prop $R_{abcd} = R_{cdab}$ (hence $R_{(ab)cd} = 0$).

Pf: Pick normal coords at p. so $\partial_\mu g_{\nu\rho}|_p = 0$ and $g_{\nu\rho}|_p = \eta_{\nu\rho}$ (if Lorentzian)

Note $0 = \partial_\mu(\delta^\nu{}_\sigma) = \partial_\mu(g^{\nu\tau} g_{\tau\sigma}) = (\partial_\mu g^{\nu\tau}) g_{\tau\sigma} + g^{\nu\tau}(\partial_\mu g_{\tau\sigma})$,

hence

$$\partial_\mu g^{\nu\tau}|_p = 0$$

and consequently

$$\partial_\rho \Gamma^\mu_{\nu\sigma}|_p = \frac{1}{2}\partial_\rho\left(g^{\mu\tau}(\partial_\nu g_{\sigma\tau} + \partial_\sigma g_{\nu\tau} - \partial_\tau g_{\nu\sigma})\right)\Big|_p$$

$$= \frac{1}{2} g^{\mu\tau}(\partial_\rho\partial_\nu g_{\sigma\tau} + \partial_\rho\partial_\sigma g_{\nu\tau} - \partial_\rho\partial_\tau g_{\nu\sigma})|_p.$$

Therefore,

$$R_{\mu\nu\rho\sigma}|_p = g_{\mu\kappa}(\partial_\rho \Gamma^\kappa_{\nu\sigma} - \partial_\sigma \Gamma^\kappa_{\nu\rho})|_p$$

$$= \frac{1}{2}(\partial_\nu\partial_\rho g_{\mu\sigma} + \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\mu\rho})|_p.$$

$$= R_{\rho\sigma\mu\nu}|_p$$

Since this relates cpts of tensors and p arbitrary, must have $R_{abcd} = R_{cdab}$. $\square$.

There are several other tensors associated to the Riemann tensor by contraction.

Define Ricci tensor

$$R_{ab} = R^c{}_{acb}$$

the Ricci scaler

$$R = g^{ab} R_{ab}$$

and the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

It follows that Ricci tensor is sym.

Pf: $R_{ab} = g^{cd} R_{cadb} = g^{cd} R_{dbca} = R_{ba}$. □

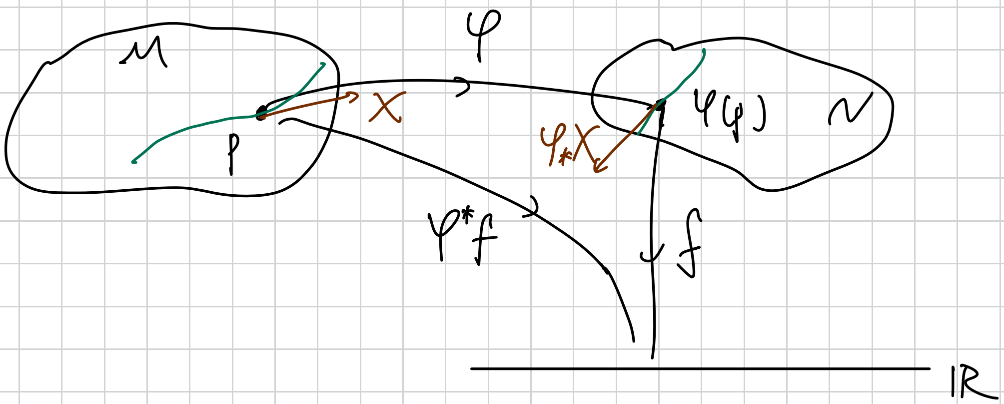
Also, the Bianchi identity implies

$$\nabla_a G^a{}_b = 0$$

Diffeomorphisms and the Lie Derivative

If $\varphi : M \to N$ is a smooth map, then φ induces maps between tangent and cotangent bundles on M and N.

Defn Given $f : N \to \mathbb{R}$, the pullback of f by φ is the map $\varphi^* f : M \to \mathbb{R}$ defined by $\varphi^* f(p) = f(\varphi(p))$ $\forall p \in M$.



Defn Given $X \in T_p M$, define the pushforward of X by φ to be the vector $\varphi_* X \in T_{\varphi(p)} N$ as follows. Let $\lambda : I \to M$ be a smooth curve in M with $\lambda(0) = p$ and $\dot{\lambda}(0) = X$, then $\tilde{\lambda} = \varphi \circ \lambda : I \to N$ is a curve with $\tilde{\lambda}(0) = \varphi(p)$.

We set $\varphi_* X = \dot{\tilde{\lambda}}(0)$

If $f: N \to \mathbb{R}$, then

$$\varphi_* X(f) = \frac{d}{dt}\left(f \circ \tilde{\lambda}(t)\right)\Big|_{t=0}$$
$$= \frac{d}{dt}\left(f \circ \varphi \circ \lambda(t)\right)\Big|_{t=0}.$$
$$= \frac{d}{dt}\left(\varphi^* f \circ \lambda(t)\right)\Big|_{t=0} = X(\varphi^* f)$$

Exercise If x^μ coords on M near p and y^α coords on N near $\varphi(p)$, then φ gives a map $y^\alpha(x^\mu)$. Show that in a coord basis,

$$(\varphi_* X)^\alpha = \frac{\partial y^\alpha}{\partial x^\mu} X^\mu,$$

or

$$\varphi_*\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial}{\partial y^\alpha}.$$

We can pullback general covectors too.

Def^n If $\eta \in T^*_{\varphi(p)} N$, define the pullback $\varphi^*\eta \in T^*_p M$ by

$$\varphi^*\eta(X) = \eta(\varphi_* X) \quad \forall\, X \in T_p M.$$

In particular, if $f: N \to \mathbb{R}$, then

$$\varphi^*(df)(X) = df(\varphi^* X)$$
$$= \varphi_* X(f)$$
$$= X(\varphi^* f) = d(\varphi^* f)(X) \quad \forall X \in T_p M.$$

So for any smooth $f: N \to \mathbb{R}$, have $\varphi^*(df) = d(\varphi^* f)$.

We can likewise extend the pullback to tensors of type $(0,s)$. Let T be a $(0,2)$-tensor at $\varphi(p) \in N$, Define $\varphi^* T$ to be type $(0,2)$ tensor at $p \in M$ given by

$$\varphi^* T(X_1, \dots, X_s) = T(\varphi_* X_1, \dots, \varphi_* X_s) \quad \forall X_i \in T_p M.$$

Similarly, we can pushforward a tensor S of type $(r,0)$ at $p \in M$ to give an $(r,0)$ tensor $\varphi_* S$ defined by

$$\varphi_* S(\eta_1, \dots, \eta_r) = S(\varphi_* \eta_1, \dots, \varphi^* \eta_s) \quad \forall \eta_i \in T^*_{\varphi(p)} N.$$

However, for general smooth $\varphi: M \to N$, no natural way to pushforward / pullback tensors of mixed type (r,s) with $rs \neq 0$.

If $\varphi: M \to N$ has the property that $\varphi_*: T_p M \to T_{\varphi(p)} N$ is injective, then we say φ is an <u>immersion</u> of M in N (this requires $\dim M \leq \dim N$)

If (N, g) is (pseudo-) Riemannian mfld and $\varphi: M \to N$ an immersion, consider $\varphi^* g$. If g Riemannian, $\varphi^* g$ also Riemannian (+ve def.) and is the induced metric on M.

<u>Example</u> Let $(N, g) = (\mathbb{R}^3, \delta)$ and $M = S^2$ with

$$\varphi: (\theta, \phi) \longmapsto (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) = (x^1, x^2, x^3)$$

then

$$\varphi^* \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right) = d\theta^2 + \sin^2\theta \, d\phi^2.$$

If φ is a bij with smooth inverse, it's called a diffeomorphism. We can both pushforward or pullback tensors of arbitrary type using diffeos. Given tensor T of type (r,s) at $p \in M$, define $\varphi_* T$ to be tensor of type (r,s) at $\phi(p)$ given by

$$(\varphi_* T)_{\phi(p)}(\eta_1, \dots, \eta_r, X_1, \dots, X_s) = T_p(\varphi^* \eta_1, \dots, \varphi^* \eta_r ; (\varphi^{-1})_* X_1, \dots, (\varphi^{-1})_* X_s)$$

$\forall\ \eta_i \in T^*_{\phi(p)} N,\ X_i \in T_{\phi(p)} N$.

Clearly need $\dim(N) = \dim(M)$ for diffeo $\varphi : M \to N$ to exist, and we often don't distinguish mfl'ds related by diffeo (ie. write $\varphi : M \to M$). A diffeo is a symmetry of a tensor field if

$$(\varphi_* T_{\varphi^{-1}(p)})_p = (T)_p \quad \forall\ p \in M.$$

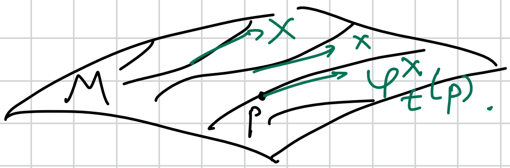
If φ is a sym. of the metric g, it's called an isometry.

Example In $(\mathbb{R}^{1,n}, \eta)$, the map

$$\varphi(x^0, x^1, \dots, x^n) \mapsto (x^0 + c, x^1, \dots, x^n)$$

is an isometry of η.

An important class of diffeo are those generated by a vec. field. If X is a smooth vec. field, we associate to each $p \in M$ the point $\varphi_t^X(p) \in M$ given by flowing param. distance t along integral curves of X.



Suppose $\varphi_t^X(p)$ is defined $\forall\, t \in I \subseteq \mathbb{R}$ for each $p \in M$. Then $\varphi_t^X : M \to M$ is a diffeo $\forall\, t \in I$. Furthermore, if $t+s \in I$, then $\varphi_t^X \circ \varphi_s^X = \varphi_{t+s}^X$, and $\varphi_0^X = id$ $(*)$

So if $I = \mathbb{R}$, this gives $\{\varphi_t^X\}$ the structure of an Abelian group. $\left((\varphi_t^X)^{-1} = \varphi_{-t}^X\right)$.

If φ_t is <u>any</u> smooth family of diffeo obeying $(*)$, then we can define vec. field X by

$$X_p = \frac{d}{dt}\varphi_t(p)\Big|_{t=0} \quad \text{and} \quad \varphi_t = \varphi_t^X$$

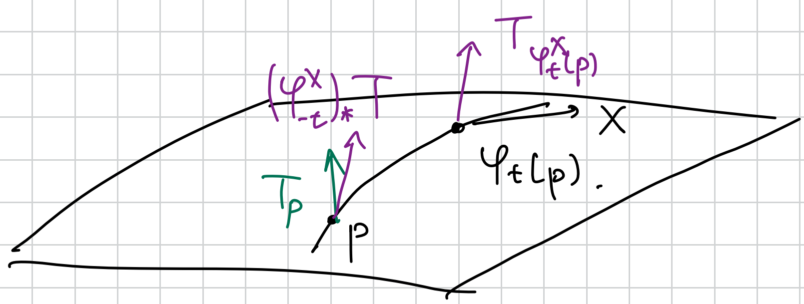
We can use this to compare tensors at different points.

<u>The Lie Derivative</u>

Let $\varphi_t^X : M \to M$ be a smooth 1-param family of diffeo generated by vec. field X,

<u>Defn</u> The <u>Lie derivative</u> of a tensor field T of type (r,s) along X is the tensor field $(\mathcal{L}_X T)$ of type (r,s) given by

$$(\mathcal{L}_X T)_p = \lim_{t\to 0} \frac{((\varphi_{-t}^X)_* T)_p - T_p}{t}$$

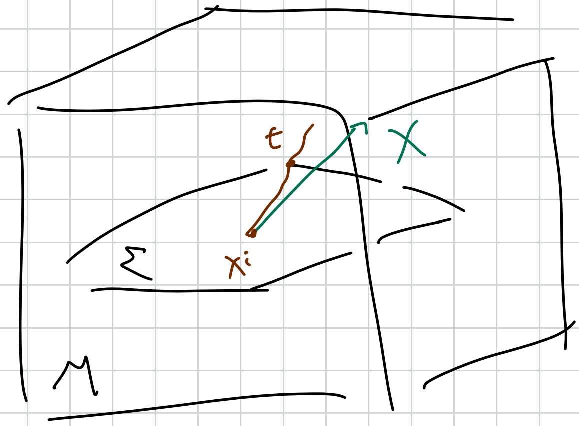


<u>Rmk</u> • Since $(\varphi_{-t}^X)_* = (\varphi_t^X)^*$, we could also define $\mathcal{L}_X T$ using pullbacks.

• $\mathcal{L}_X(\alpha S + \beta T) = \alpha(\mathcal{L}_X S) + \beta(\mathcal{L}_X T)$ for <u>const</u>. α, β.

To understand $\mathcal{L}_X T$, let Σ be any hypersurface transverse to X, i.e. not tangential to X.

Pick coords x^i, ($i = 1, \dots, \dim(M)-1$) along Σ and assign the coords (t, x^i) to point $\in M$ param. distance t along integral curve of X starting from $x^i \in \Sigma$.



In these coords, $X = \frac{\partial}{\partial t}$, and φ^X_t sends $p \mapsto \varphi^X_t(p)$.

$$(t_p, x_p^i) \mapsto (t + t_p, x_p^i).$$

Hence, $\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu{}_\nu$ and so

$$\left((\varphi^X_{-t})_* T_{\varphi^*_t(p)}\right)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = \left[(T)_{\varphi^*_t(p)}\right]^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}.$$

so in <u>these</u> coords, $\mathcal{L}_X$ acts on cpts of a tensor just by $\frac{\partial}{\partial t}$.

$$(\mathcal{L}_X T)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\Big|_p = \lim_{t \to 0} \frac{1}{t}\Big(T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(t + t_p, x_p^i) - T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(t_p, x_p^i)\Big)$$

Consequently, for tensors S, T,

- $\mathcal{L}_X(S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_X T)$.
- $\mathcal{L}_X$ commutes with contraction.

To find coord indpt formula, start with smooth $f: M \to \mathbb{R}$, we have

$$\mathcal{L}_X(f) = \frac{\partial f}{\partial t} = X(f).$$

For a vec. field Y, have

$$(\mathcal{L}_X Y)^\mu = \frac{\partial Y^\mu}{\partial t} = [X, Y]^\mu$$

So we identify

$$\mathcal{L}_X Y = [X, Y]$$

Rmk $\nabla_X Y$ and $[X, Y]$ are both vec. field. $\nabla_X Y = X^a \nabla_a Y$ behaves like a tensor in X (i.e. only need to know value of X at any point p, and choice of X). However, $\mathcal{L}_X Y = [X, Y]$ knows about both X and its derivative at p.

Exercise If ω covec. field., show

$$(\mathcal{L}_X \omega)_a = (\nabla_X \omega)_a + \omega_b \nabla_a X^b.$$

and

$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a,$$

where $X_a = g_{ac} X^c$, and where ∇ is the Levi-Civita connection.

If φ_t is a 1-param family of isometries of (M, g) generated by X, then

$$\mathcal{L}_X g = 0$$

Conversely, if $\mathcal{L}_X g = 0$, then X generates a 1-param family of isometries.

Def^n A vec. field obeying $\mathcal{L}_K g = 0$ is called a *Killing vector*.
It obeys
$$\nabla_{(a} K_{b)} = 0$$
where ∇ is Levi-Civita connection. This is *Killing's eqn*.

Lem Suppose K is killing and $\lambda : I \to M$ a geodesic of Levi-Civita ∇, then $g(\dot\lambda, K)$ is const along λ.

Pf:
$$\frac{d}{dt} g(\dot\lambda, K) = \dot\lambda^a \nabla_a (K_b \dot\lambda^b)$$
$$= \underbrace{\dot\lambda^a \dot\lambda^b \nabla_a K_b}_{=0 \text{ by Killing}} + K_b \underbrace{(\dot\lambda^a \nabla_a \dot\lambda^b)}_{=0 \text{ by geodesic eqn}} = 0.$$

General Relativity

In Einstein's GR, we postulate that our spacetime is a Lorentzian mfd (M, g) with $\dim M = 4$. We also require a theory of *matter* on M, consisting of

- some *fields* Φ^A (e.g. f^{n}s $\Phi : M \to \mathbb{R}$, gauge field $A_a, \ldots$)
- some e.o.m. $E_i(\Phi^A, g, \nabla, \ldots) = 0$ that are *geometric*, i.e. if $\varphi : M \to M$ diffeo, then
$$E_i(\varphi^* \Phi^A, \varphi^* g, \ldots) = \varphi^* E_i(\Phi^A, g, \ldots)$$
- an energy-mom. tensor $T_{ab} = T_{ba}$ obeying $\nabla^a T_{ab} = 0$ (∇ Levi-Civita)

This matter should reduce to a non-gravitational matter theory when $(M, g) = (\mathbb{R}^{1,3}, \eta)$

The dynamics of the metric g itself are governed by Einstein's eqn

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi G\, T_{ab} \qquad (c=1) \quad (*)$$

$\Lambda > 0$ in our universe, but small.

Einstein's eqn + e.o.m. of matter are a coupled system of non-linear PDEs for (Φ^A, g) and should be solved together (in GR).

[Geodesic postulate: test particles move along timelike $(m>0)$ or null $(m=0)$ geodesics of the metric g solving $(*)$.]

So "matter tells space how to curve, while space tell matter how to move."

Physics in Curved Spacetime

We often construct our matter model on (M, g) by starting in $(\mathbb{R}^{1,3}, \eta)$ and "covariantizing".

Example (Klein - Gordon eqn)

$$\partial^\mu \partial_\mu \Phi - m^2 \Phi = 0 \quad \text{on } (\mathbb{R}^{1,3}, \eta) \qquad (+)$$

Note that in inertial coords $\nabla_\mu = \partial_\mu$. So we can write KG in covariant manner as

$$\nabla^a \nabla_a \Phi - m^2 \Phi = 0$$

using L.C. ∇ of any g. However, this procedure is ambiguious - we were also free to write, e.g.

$$\nabla^a \nabla_a \Phi - m^2 \Phi + \xi R \Phi = 0 \text{ for any const } \xi..$$

Associate to our scaler field is the stress tensor (a.k.a. energy-mom tensor)

$$T_{\mu\nu} = \partial_\mu \Phi \, \partial_\nu \Phi - \frac{1}{2}\eta_{\mu\nu}\left(\partial^k \Phi \, \partial_k \Phi + m^2 \Phi\right) \quad \text{on } (\mathbb{R}^{1,3}, \eta)$$

and is conserved ($\partial^\mu T_{\mu\nu} = 0$) wherever (†) holds. Then on general (M, g), we have

$$T_{ab} = \nabla_a \Phi \, \nabla_b \Phi - \frac{1}{2} g_{ab}\left(\nabla^c \Phi \, \nabla_c \Phi + m^2 \Phi^2\right)$$

(We'll see later how these arise from an action principle $S[\Phi, g]$).

<u>Example</u> (Maxwell eqn) The EM field strength $F_{\mu\nu} = -F_{\nu\mu}$ has cpts $F_{i0} = E_i$, $F_{ij} = \varepsilon_{ijk} B_k$.

If j_μ is the current density $(\rho, \vec{j})$, Maxwell's eqn are

$$\partial^\mu F_{\mu\nu} = 4\pi j_\nu, \qquad \partial_{[\mu} F_{\nu k]} = 0.$$

on $(\mathbb{R}^{1,3}, \eta)$. then

secretly still $\partial_{[a} F_{bc]} = 0$.

$$\nabla^a F_{ab} = 4\pi j_b, \quad \nabla_{[a} F_{bc]} = 0 \qquad (‡).$$

The energy-mom tensor of EM field is

$$T_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}\eta_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

$$\longleftrightarrow \quad T_{ab} = g^{cd} F_{ac} F_{bd} - \frac{1}{4} g_{ab} F^{cd} F_{cd}.$$

which obeys $T_{ab} = T_{ba}$, and $\nabla^a T_{ab} = 0$ when (‡) holds. (with $j_a = 0$)

<u>Example</u> (Perfect fluid) This is described by a local velocity field u^μ obeying $u^\mu u_\mu = -1$, together with a pressure p

and a density ρ. These obey

$$u^\mu \partial_\mu \rho + (p+\rho)\, \partial_\mu u^\mu = 0 \qquad \text{(mass conservation)}$$

$$\leftrightarrow \quad u^a \nabla_a \rho + (\rho + p)\, \nabla_a u^a = 0.$$

and

$$(p+\rho)\, u^\mu \partial_\mu u^\nu + \partial^\nu p + (u^\mu \partial_\mu p) u^\nu = 0 \qquad \text{(Euler's eqn)}$$

$$\leftrightarrow \quad (p+\rho)\, u^a \nabla_a u^b + \nabla^b p + (u^a \nabla_a p)\, u^b = 0.$$

We also have the fluid's energy-momentum tensor

$$T_{ab} = (p+\rho)\, u_a u_b + p\, g_{ab}$$

which obeys $T_{ab} = T_{ba}$ and $\nabla^a T_{ab} = 0$ whenever Euler's eqn + mass conservation hold.

Consider normal coords for g, near any $p \in M$ the physics will be approx. Minkowskian with corrections O(Riemann tensor)

Gauge Redundancy

We'd like to solve Einstein's eqn + matter model by prescribing some appropriate initial data (e.g. on some co-dim 1 surface $\Sigma \subset M$) and evolving forwards. However, this cannot be done uniquely because of gauge redundancy.

Recall that the source-free Maxwell eqn

$$\partial^\mu F_{\mu\nu} = 0, \qquad \partial_{[\mu} F_{\nu\rho]} = 0$$

can be solved by introducing a 4-vec. potential A_μ s.t.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Then $\partial_{[\mu} F_{\nu\rho]} = 0$ trivially. The remaining eqn is then

$$\partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu = 0. \qquad (*)$$

We'd like to solve this given initial data on $\Sigma = \{x^0 = 0\} \subset \mathbb{R}^{1,3}$

However, this eqn can't have a good evolution problem, since if $\chi : \mathbb{R}^{1,3} \to \mathbb{R}$ any smooth f^n with $\chi(\Sigma) = 0$, then

$$\tilde{A}_\mu = A_\mu - \partial_\mu \chi$$

will also be a sol^n and $\tilde{F}_{\mu\nu} = \partial_{[\mu} \tilde{A}_{\nu]} = \partial_{[\mu} A_{\nu]} = F_{\mu\nu}$.

To resolve this, we must fix a gauge.

Notice that $(*)$ is <u>not</u> a wave eqn: in terms of (t, x^i), we have for the 0-cpt

$$-\partial_0^2 A_0 + \partial_i^2 A_0 + \partial_0^2 A_i - \partial_i \partial_0 A_i = -\partial_i (\underbrace{\partial_0 A_i - \partial_i A_0}_{= E_i}) = -\nabla \cdot \underline{E} = 0$$

In particular, our initial data $(A_\mu|_\Sigma, \partial_0 A_\mu|_\Sigma)$ will be <u>constrained</u> by the Gauss law.

To handle this, we must fix a gauge. For example, let $\chi : \mathbb{R}^{1,3} \to \mathbb{R}$ (smooth) be chosen s.t. $\tilde{A}_\mu = A_\mu + \partial_\mu \chi$ obeys $\partial^\mu \tilde{A}_\mu = 0$ everywhere. Then $(*)$ becomes

$$\partial^\mu \partial_\mu \tilde{A}_\nu = 0 \qquad (+)$$

which is just a standard wave eqn for cpts $\tilde{A}_\nu$.

In particular, $(+)$ has unique sol^n given $\tilde{A}_\mu|_\Sigma$ and $\partial_0 \tilde{A}_\mu|_\Sigma$ on $\Sigma = \{x^0 = 0\} \subset \mathbb{R}^{1,3}$. Furthermore, if our initial data obeys $\partial^\mu \tilde{A}_\mu|_\Sigma = 0$ and $\partial_0(\partial^\mu \tilde{A}_\mu)|_\Sigma = 0$, then

$\partial^\mu \tilde{A}_\mu = 0$ everywhere (since $\partial^\mu \partial_\mu (\partial^\nu \tilde{A}_\nu) = 0$) and $\tilde{A}_\mu$ solves the original Maxwell eqn (*).

[Also note that $\partial^\mu \partial_\mu A_\nu = 0$ implies

$$\nabla \cdot \underline{E}\big|_\Sigma = \partial_i (\partial_0 A_i - \partial_i A_0)\big|_\Sigma$$
$$= \partial_0 (\partial_i A_i - \partial_0 A_0)\big|_\Sigma$$
$$= \partial_0 (\partial^\mu A_\mu)\big|_\Sigma$$

so initial data $\partial^\mu A_\mu = 0$ and $\partial_0 (\partial^\mu A_\mu)|_\Sigma = 0$ ensures Gauss law obeyed.]

Gauge Redundancy in GR

Einstein's eqn are geometric: if (M, g) solves then for some stress-energy tensor T, $\varphi^* g$ is a sol^n with source $\varphi^* T$ for any diffeo $\varphi: M \to M$. In a local coord chart, this manifests itself as general coordinate invariance. It's a gauge redundancy of Einstein's eqn, so to solve we must first fix a gauge, i.e. fix some coord.

Lem In any local coord chart,

$$R_{\rho\sigma\mu\nu} = \frac{1}{2}\left(g_{\rho\nu,\mu\sigma} + g_{\sigma\mu,\nu\rho} - g_{\rho\mu,\nu\sigma} - g_{\sigma\nu,\mu\rho}\right)$$
$$- \Gamma_{\mu\lambda\rho} \Gamma_\nu{}^\lambda{}_\sigma + \Gamma_{\nu\lambda\rho} \Gamma_\mu{}^\lambda{}_\sigma.$$

Note: involves $\Gamma^\mu_{\mu\alpha}$

$$R_{\sigma\nu} = -\frac{1}{2} g^{\mu\rho} g_{\sigma\nu,\mu\rho} + \frac{1}{2}\underbrace{\left(\partial_\sigma \Gamma^\mu_{\nu\mu} + \partial_\nu \Gamma^\mu_{\sigma\mu}\right)}_{\partial^2 g} - \Gamma^\mu_{\mu\lambda} \Gamma^\lambda_{\nu\sigma}$$
$$+ \Gamma_{\lambda\tau\nu} \Gamma^{\lambda\tau}{}_\sigma + \Gamma_{\lambda\tau\nu} \Gamma^{\tau\lambda}{}_\sigma + \Gamma_{\lambda\tau\sigma} \Gamma^{\tau\lambda}{}_\nu$$

To pick a gauge that's well-adapted to (e.g.) gravitational wave. Suppose we have a coord system where the coords x^μ themselves obey wave eqn.

$$0 = \nabla^\mu \nabla_\mu x^{\bar{\nu}} = \nabla^\mu (\partial_\mu x^{\bar{\nu}}) = \partial^\mu \partial_\mu x^{\bar{\nu}} - \Gamma^\mu_{k\,\mu}\, \partial^k x^{\bar{\nu}}$$

not treating as a vector

breaks general coord inv.

$$\Rightarrow \; 0 = \Gamma_k{}^\mu{}_\mu\, \partial^k x^{\bar{\nu}} \quad \text{in these coords.}$$

$$\Rightarrow \; 0 = \Gamma^\mu_{k\,\mu} = \frac{1}{2} g^{\mu\sigma} (2 g_{\mu k,\sigma} - g_{\mu\sigma,k}).$$

Hence, in this coord system,

$$R_{\sigma\nu} = -\frac{1}{2} g^{\mu\rho} g_{\sigma\nu,\mu\rho} + \underbrace{\text{“}\Gamma\Gamma + \Gamma\Gamma + \Gamma\Gamma\text{”}}_{= P(g, \partial g)}$$

where $P(g, \partial g)$ quadratic in ∂g but indpt. of $\partial^2 g$.

Consequently, $R_{\sigma\nu} = 0$ (vacuum Einstein eqn) is equiv. to a sys. of <u>non-linear</u> wave eqn in this gauge.

In principle / numerically, we can always solve this at least locally. Further, can show that $\nabla^\mu \nabla_\mu x^{\bar{\nu}} = 0$ holds initially, holds everywhere in our local patch.

Since $R_{\sigma\nu} = 0$ is a hyperbolic system of (non-linear) PDEs for g, we expect wave-like sol^{n}s $\Rightarrow$ gravitational waves.

Linearised Theory

Suppose the grav. field is weak. We expect to able to describe the metric as a perturbation of Mink. metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} + \mathcal{O}(\varepsilon^2)$$

If $g_{\mu\nu}$ takes this form, we say we have almost inertial coords. Can easily check

$$g^{\mu\nu} = \eta^{\mu\nu} - \varepsilon h^{\mu\nu} + \mathcal{O}(\varepsilon^2).$$

where

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}.$$

Suppose the metric is in harmonic / wave gauge, then

$$0 = 2\Gamma^{\mu}_{\mu\kappa} = g^{\mu\sigma}\left(g_{\mu\kappa,\sigma} - \tfrac{1}{2} g_{\mu\sigma,\kappa}\right)$$

$$= \varepsilon\, \partial^{\mu}\left(h_{\mu\kappa} - \tfrac{1}{2}\eta_{\mu\kappa} h\right)$$

where $\partial^{\mu} = \eta^{\mu\sigma}\partial_{\sigma}$, $h = \eta^{\mu\nu} h_{\mu\nu}$.

Then

$$R_{\mu\nu} = -\frac{\varepsilon}{2}\eta^{\sigma\tau}\partial_{\sigma}\partial_{\tau} h_{\mu\nu}$$

For EE to hold, must also have

$$T_{\mu\nu} = \epsilon \tau_{\mu\nu} + \mathcal{O}(\varepsilon^2).$$

Then to $\mathcal{O}(\varepsilon)$, we have

$$-\frac{1}{2}\eta^{\sigma\tau}\partial_{\sigma}\partial_{\tau} h_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}\eta^{\sigma\tau}\partial_{\sigma}\partial_{\tau} h = 8\pi\tau_{\mu\nu} \qquad (G_N = 1)$$

or equivalently

$$\Box \bar{h}_{\mu\nu} = -16\pi\tau_{\mu\nu} \qquad (*)$$

$$\partial^{\mu}\tau_{\mu\nu} = 0$$

where $\bar{h}^{\mu}{}_{\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}.$

and impose gauge condition

$$\partial^\mu \bar{h}_{\mu\nu} = 0 \qquad (+)$$

We can solve $(*)$ if given initial data $\bar{h}_{\mu\nu}|_0$, $\partial_0 \bar{h}_{\mu\nu}$ on $\Sigma = \{x^0 = 0\} \subset \mathbb{R}^{1,3}$. Furthermore, this unique sol^n will obey $(+)$ in addition provided our initial data obeys

$$\partial^\mu \bar{h}_{\mu\nu}|_\Sigma = 0 = \partial_0 (\partial^\mu \bar{h}_{\mu\nu})|_\Sigma$$

This is because $\partial^\mu \bar{h}_{\mu\nu}$ solves

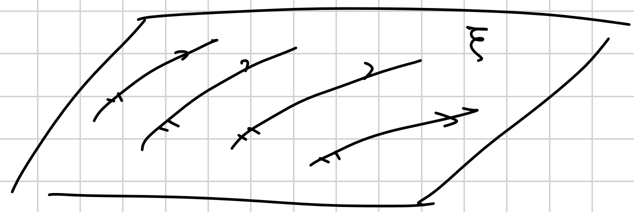
$$\Box (\partial^\mu \bar{h}_{\mu\nu}) = -16\pi \, \partial^\mu T_{\mu\nu} = 0$$

as T conserved to order ε.

So gauge condition holds everywhere, and $(*) + (+) \Leftrightarrow$ sol^n to linearised EE.

<u>Linearised Gauges</u>

Physically equiv. sol^n are related by diffeo $\varphi : M \to M$. Suppose this diffeo reduces to the identity in the case (M, g) reduces to $(\mathbb{R}^{1,3}, \eta)$. This means φ must be of form φ_ε^ξ, i.e. generated by flow along, i.e. of vec. field ξ an amount ε.



By def^n, if S is any tensor, then

$$(\varphi_\varepsilon^\xi)^* S = S + \varepsilon \mathcal{L}_\xi S + O(\varepsilon^2).$$

In particular, if S is already $O(\varepsilon)$, ie. S would vanish on $(\mathbb{R}^{1,3}, \eta)$, e.g. $R^\mu{}_{\nu\kappa\lambda}$. then S must be invar. at leading order along any such diffeo. Such tensors are gauge invar. (around flat space).

In particular, $R^\mu{}_{\nu\kappa\lambda}$, $T_{\mu\nu}$, are gauge invar. to leading order. However,

$$[(\varphi^\xi_\varepsilon)^* \eta]_{\mu\nu} = \eta_{\mu\nu} + \varepsilon(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) + \mathcal{O}(\varepsilon^2).$$

So this generates an

$$h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Such an $h_{\mu\nu}$ represents a gauge transf^n of the flat η metric.

Recall that <u>before</u> fixing the gauge, the linearised Ricci tensor is

$$R_{\mu\nu} = \varepsilon\left(\partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho \partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu h\right)$$

Substituting $h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, find $R_{\mu\nu} = 0$, so vacuum EE obeyed. We call such sol^n <u>pure gauge</u>.

(c.f. If set $A_\mu = \partial_\mu \chi$, then $F_{\mu\nu} = 0$, so $\partial^\mu F_{\mu\nu} = 0$, and $\partial_{[\mu} F_{\kappa\lambda]} = 0$).

<u>The Newtonian Limit</u>

We expect GR to reduce to Newtonian gravity

$$\frac{d^2 x^i}{dt^2} = g^i, \qquad g^i = -\nabla_i \Phi, \qquad \nabla^2 \Phi = 4\pi\rho.$$

in limit where grav. field in weak, and matter is slowly moving ($|v| \ll c$).

To see this, suppose we model our matter as perfect fluid with density ρ, pressure p, velocity field u^a.

Typical fluids have $p/\rho \sim v_s^2 \ll c^2$.
speed of sound in fluid

Choose coords s.t. $u^a = \frac{\partial}{\partial t}$. The condition of NR, weak field for the flow is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu} \sim \mathcal{O}(\varepsilon)$, and for this to be consistent, need $\rho \sim \mathcal{O}(\varepsilon)$. Then linearised EE become

$$\partial^\mu \partial_\mu \bar{h}_{\sigma\tau} = -16\pi T_{\sigma\tau}$$

where $T_{00} = \rho$, $T_{0i} = 0 = T_{ij}$ (neglect p v.s. ρ).

We thus expect $\bar{h}_{0i} = 0 = \bar{h}_{ij}$.

The gauge condition then becomes

$$0 = \partial^\mu \bar{h}_{\mu 0} = -\partial_0 \bar{h}_{00}$$

So $\bar{h}_{00}(x^\mu) = \bar{h}_{00}(x)$. Hence, the linearised EE reduces to

$$\nabla^2 \bar{h}_{00} = -16\pi\rho.$$

Since $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -\bar{h}_{00}$, we have $h_{00} = \bar{h}_{00}/2$.

So if we identify $-h_{00}/2 \equiv \Phi$, we recover Poisson's law (in static case).

Also, consider a test particle moving in this background metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the geodesic Lagrangian is

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

if choose $\dot{} = \frac{\partial}{\partial\tau}$.

$$= -\dot{t}^2 + |\dot{\underline{x}}|^2 + h_{00}\dot{t}^2 = -1.$$

If velocity $|\dot{\underline{x}}|^2 \sim \mathcal{O}(\varepsilon)$, then

$$-\dot{t}^2 = -1 + \mathcal{O}(\varepsilon)$$

So $t = \tau + \mathcal{O}(\varepsilon)$.

$\underline{x}$ eqn:

$$2\frac{d}{d\tau}(\dot{x}^i) = 2\ddot{x}^i = \partial_i h_{00}\dot{t}^2,$$

but since $\dot{t} = 1 + \mathcal{O}(\varepsilon)$, have

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} = -\partial_i\Phi.$$

i.e. $m\underline{a} = m\underline{g}$.

Gravitational Waves

In Rutherford's atom, electron orbiting a nucleus would radiate EM energy, making atom unstable.

In Einstein theory, accelerating masses also radiate grav. energy but this effect is tiny in the solar system.

About 1.3 bn yrs ago, 2 BHs of $m = 36\,M_\odot$, $26\,M_\odot$, so

$$r_{sch} \approx 90\text{ km} \approx \frac{2}{3}\,|\text{Cambridge} - \text{London}|$$

attracted into a merged BH. Just before they merged, they orbited at $\omega = 75\,\text{Hz}$. Their collision released as much energy during last 1 sec., as all the radiation from all stars in the visible universe. These ripples reached Earth on 15 Sep 2015, measured at LIGO. They were weak by time reached Earth.

We seek a sol^n to $\Box \bar{h}_{\mu\nu} = 0$, $\partial^\mu \bar{h}_{\mu\nu} = 0$ representing a propagating wave.

Let $\bar{h}_{\mu\nu} = \mathrm{Re}\left(H_{\mu\nu} e^{ik_\lambda x^\lambda} \right)$, where $H_{\mu\nu} = H_{\nu\mu}$ const., $O(\varepsilon)$. This is sol^n provided

$$k^\lambda k_\lambda = 0,$$

ie k^λ null, and set $k^\lambda = \omega(1,0,0,1)$ WLOG., and

$$k^\mu H_{\mu\nu} = 0,$$

ie. wave is transversely polarised.

We still have some residual gauge freedom. Under a diffeo, have

$$\bar{h}_{\mu\nu} \mapsto \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial^\lambda \xi_\lambda$$

and so

$$\partial^\mu \bar{h}_{\mu\nu} \mapsto \partial^\mu \bar{h}_{\mu\nu} + \partial^\mu \partial_\mu \xi_\nu$$

Hence, if our diffeo has $\Box \xi_\nu = 0$, we still obey the gauge condition.

Let $\xi_\nu = \mathrm{Re}\left(-i X_\nu e^{ik_\lambda x^\lambda} \right)$ for some fixed X_ν, then

$$H_{\mu\nu} \mapsto H_{\mu\nu} + k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k_\lambda X^\lambda.$$

Exercise Show you can use this residual freedom to set

$$H_{0\mu} = 0, \quad H^\mu{}_\mu = 0$$

With this form,

$$0 = k^\mu H_{\mu\nu} = \omega \left(H_{0\nu} + H_{3\nu} \right).$$
$$= \omega H_{3\nu}$$

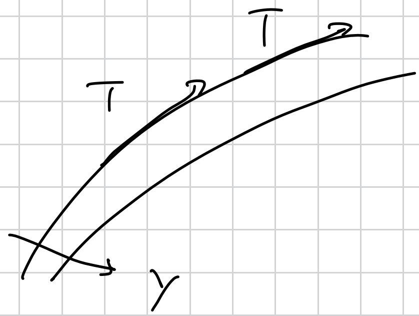
So $H_{3\nu} = 0 \quad \forall \; \nu = 0,1,2,3$. Hence the polarisation tensor takes the form

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, similar to a photon, have 2 indpt polarisations $H_\times$, H_+.
(Can also check $\bar{h}_{\mu\nu} = h_{\mu\nu}$ since $H^\mu{}_\mu = 0$).

To understand the effect of these grav. waves, recall that if T is a vec-field to geodesics and Y join nearby geodesics., the geodesic eqn is

$$T^a \nabla_a \left(T^b \nabla_b Y^c \right) = R^c{}_{dab} T^d T^a Y^b$$



Suppose a freely falling observer sets up inertial frame e_α, $\alpha = 0, \ldots, 3$ s.t. $e_0 = T$. and e_i three spacelike vectors which are parallel transported along each geodesic.

$$T^a \nabla_a (e_\alpha) = 0.$$

If initially orthonormal, then this frame stays orthonormal

$$\frac{d}{d\lambda} g(e_\alpha, e_\beta) = g(\nabla_T e_\alpha, e_\beta) + g(e_\alpha, \nabla_T e_\beta) = 0$$

Then geodesic deviation becomes

$$\nabla_T \nabla_T \underbrace{\left(g(Y, e_\alpha) \right)}_{= Y_\alpha} = g(\nabla_T \nabla_T Y, e_\alpha) = e^a_\alpha R_{abcd} e^b_0 e^c_0 Y^d \quad (*)$$

$= Y_\alpha$. (cpt of Y in e_α basis)

For our grav. wave, Riemann $\sim \mathcal{O}(\varepsilon)$ so to lowest order can assume $e_0 = \partial_t$, $e_i = \partial_i$, so $(*)$ becomes

$$\frac{d^2 Y_\alpha}{dt^2} = R_{\alpha 0 0 \beta} Y^\beta + \mathcal{O}(\varepsilon^2).$$

For the given grav. wave metric, $\bar{h}_{\mu\nu} = \mathrm{Re}(H_{\mu\nu} e^{-ik_\lambda x^\lambda})$ one finds

$$R_{\alpha 0 0 \beta} = \frac{1}{2} h_{\alpha\beta,00} = \frac{1}{2} \frac{d^2 h_{\alpha\beta}}{dt^2}.$$

$$\Rightarrow \frac{d^2 Y_\alpha}{dt^2} = \frac{1}{2}\left(\frac{d^2 h_{\alpha\beta}}{dt^2}\right) Y^\beta.$$

Let's consider a wave with polarisation H_+, then

$$\frac{d^2 Y^0}{dt^2} = \frac{d^2 Y^3}{dt^2} = 0,$$

so no relative accel. of nearby geodesics in direction along wave.

$$\frac{d^2 Y^1}{dt^2} = -\frac{1}{2}\omega^2 |H_+| \cos(\omega t - \phi) Y^1(t)$$

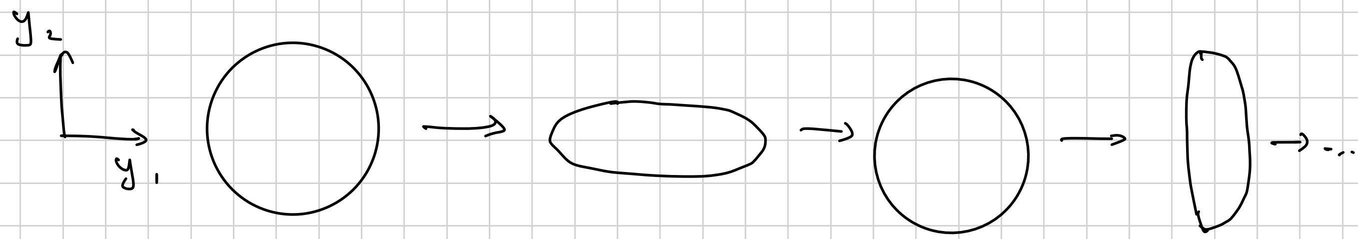
$$\frac{d^2 Y^2}{dt^2} = -\frac{1}{2}\omega^2 |H_+| \cos(\omega t - \phi) Y^2(t).$$

where $\arg(H_+) = \phi$.

Since $|H_+| \ll 1$, can solve perturbatively to find

$$Y^1(t) = Y^1(0)\left(1 + \frac{1}{2}|H_+| \cos(\omega t - \phi)\right) + \dots$$

$$Y^2(t) = Y^2(0)\left(1 - \frac{1}{2}|H_+| \cos(\omega t - \phi)\right) + \dots$$



In LIGO, the detector

Max stretch is bounded by

$$\frac{\delta L}{L} = \frac{|H_+|}{2}$$

For LIGO, $L \approx 3\,km$. $\delta L \sim 10^{-18}\, m \ll r_{proton}$.

The Field Far from the Source

In the presence of a source, the linearised EE become

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \quad (c=1,\ G_N=1) \qquad (\dagger)$$

Just as for EM, this looks like a sourced wave eqn in $(\mathbb{R}^{1,3}, \eta)$, so is solved using a retarded Green's f^n as

$$\bar{h}_{\mu\nu}(t, \underline{x}) = 4 \int d^3x' \, \frac{T_{\mu\nu}(t - |\underline{x} - \underline{x}'|, \underline{x}')}{|\underline{x} - \underline{x}'|} \qquad (\ddagger)$$

where $|\underline{x} - \underline{x}'|$ is calculated using flat Euclidean metric on $\mathbb{R}^{1,3}$.

Suppose the matter $dist^n$ is contracted within distance d of $\underline{0} \in \mathbb{R}^3$ (so $T_{\mu\nu} = (t', \underline{x}') = 0$ for $|\underline{x}'| > d$). Then we expand the metric fluctuation $h_{\mu\nu}$ at distance $r \gg d$ from source using

$$|\underline{x} - \underline{x}'|^2 = r^2 - 2\underline{x}\cdot\underline{x}' + \underline{x}'^2$$
$$= r^2\left(1 - \frac{2}{r}\,\hat{\underline{x}}\cdot\underline{x}' + O(d^2/r^2)\right)$$

where $r = |\underline{x}|$.

So

$$T_{\mu\nu}(t - |\underline{x} - \underline{x}'|, \underline{x}') = T_{\mu\nu}(t', \underline{x}') + \hat{\underline{x}} \cdot \underline{x}' \partial_0 T_{\mu\nu}(t', \underline{x}') + \dots \quad (*)$$

with $t' = t - r$.

If we also assume $T_{\mu\nu}$ is varying on a timescale τ, so

$$\partial_0 T_{\mu\nu} \sim \frac{1}{\tau} T_{\mu\nu},$$

then 2nd term in $(*)$ is of order d/τ which we can neglect if matter is moving non-relativistically. Hence, in these approximations,

$$\bar{h}_{ij} = \frac{4}{r} \int d^3x' \, T_{ij}(t', \underline{x}'),$$

where $t' = t - |\underline{x}|$.

To find the remaining cpts, we can use the gauge condition

$$\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji} \quad (1)$$

and then

$$\partial_0 \bar{h}_{00} = \partial_j \bar{h}_{j0}. \quad (2)$$

(First solve (1), then (2))

To do this, it helps to simplify our expansion for $\bar{h}_{ij}$ using the fact that $\partial^\mu T_{\mu\nu} = 0$ (stress tensor is conserved).

Dropping primes, we have

$$\int d^3x \; T^{ij}(t, \underline{x}) \overset{\circleddash}{=} \int d^3x \left(\underbrace{\partial_k (T^{ik} x^j)}_{=0 \text{ since } T_{ij} \text{ compact support}} - x^j \partial_k T^{ik} \right).$$

Using $\partial_0 T^{i0} + \partial_k T^{ik} = 0$,

$$\overset{\circleddash}{=} \int d^3x \; x^j \partial_0 T^{i0}$$

$$= \partial_0 \left(\int d^3x \; \frac{1}{2} \left(x^j T^{i0} + x^i T^{j0} \right) \right)$$

$$= \frac{1}{2} \partial_0 \left[\int d^3x \; \underbrace{\partial_k (T^{0k} x^i x^j)}_{=0} - (\partial_k T^{0k}) x^i x^j \right]$$

$$= \frac{1}{2} \partial_0 \partial_0 \int d^3x \; T^{00} x^i x^j.$$

Defining the quadruple moment of T^{00} (energy density) as

$$I^{ij} = \int d^3x \; T^{00} x^i x^j,$$

this says

$$\bar{h}_{ij}(t, x) = \frac{2}{r} \ddot{I}_{ij}(t-r).$$

where we've used

- linearised EE
- far field approx $r \gg d$
- slowly moving matter $\frac{d}{c} \ll c$.

Now use (1) to find $\bar{h}_{0i}$.

$$\partial_0 \bar{h}_{0i} = \partial_j h_{ji} = 2 \partial_j \left(\frac{1}{r} \ddot{I}_{ij}(t-r) \right)$$

const. of integration

$$\Rightarrow \bar{h}_{0i} = 2 \partial_j \left(\frac{1}{r} \dot{I}_{ij}(t-r) \right) + k_i(\underline{x})$$

$$= \underbrace{-2 \frac{\hat{x}_j}{r^2} \dot{I}_{ij}(t-r)}_{\text{subleading}} - \frac{2 \hat{x}_j}{r} \ddot{I}_{ij}(t-r) + k_i(\underline{x})$$

$$\approx -2\frac{\hat{x}_j}{r^2}\ddot{I}_{ij}(t-r) + k_i(\underline{x}).$$

Likewise, solving (2) gives

$$\partial_0 \bar{h}_{00} = \partial_i \bar{h}_{0i} = -2\partial_i\left[\frac{\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + k_i(\underline{x})\right]$$

$$\Rightarrow \bar{h}_{00}(t,\underline{x}) = -2\partial_i\left[\frac{\hat{x}_j}{r}\dot{I}_{ij}(t-r)\right] + f(\underline{x}) - 2t\partial_i k_i(x)$$

$$\approx 2\frac{\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r) + f(\underline{x}) - 2t\partial_i k_i(x)$$

To fix k_i, f, note from (‡) that to leading order, we'll have

$$\bar{h}_{00} \approx \frac{4E}{r},$$

where

$$E = \int d^3x'\, T_{00}(t', x')$$

while

$$\bar{h}_{0i} = -\frac{4P_i}{r}$$

where

$$P_i = \int d^3x'\, T_{0i}(t', x')$$

(E, P_i) are the total energy, momentum of our matter dist^n.

In fact, we have

$$\dot{E} = \int d^3x'\, \partial_0 T_{00}(t', x') = -\int d^3x'\, \partial_i T_{i0} = 0$$

Similarly, $\dot{P}_i = 0$, so total energy and mom. of matter appears to be conserved. This holds only in linearised approximation. For our current purpose, this is useful.

<u>Exercise</u> Show that the gauge transformation generated by $\xi^\mu = (\underline{P}\cdot\underline{x}, -\underline{P}t)$ may be used to set $P_i = 0$. Physically, this can be used to boosting to centre of momentum frame.

In this frame, we have

$$h_{00}(t, x) = \frac{4M}{r} + 2\frac{\hat{x}_i \hat{x}_j}{r} \dot{I}_{ij}(t-r).$$

where $E = M$ in c.o.m. frame.

$$h_{0i}(t, x) = -2\frac{\hat{x}_j}{r} \ddot{I}_{ij}(t-r)$$

$$h_{ij}(t, x) = \frac{2}{r} \ddot{I}_{ij}(t-r) \qquad (\ddagger).$$

Note that the metric fluctuation always comes from an accelerating $\underline{\text{quadruple moment}}$ of T_{00} (ρ energy) (similar formulae in EM, except they involve accel. of $\underline{\text{dipole}}$ moment of J_0) (ρ charge)

Energy in Gravitational Waves

For a massless free scaler in $(\mathbb{R}^{3,1}, \eta)$, we have

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu} \partial^\rho \phi \partial_\rho \phi \sim (\partial\phi)^2$$

Similarly, for EM field

$$T_{00} \approx \frac{1}{2}(E^2 + B^2) \sim \frac{1}{2}(\partial A)^2$$

Unfortunately, there's no hope to define a similar local energy density for the grav. field, since we can always choose (normal) coords s.t. $\partial_\mu g_{\kappa\lambda}|_p = 0$ at any $p \in M$.

(We can find meaningful quantities "at infinity" in asy. flat space-times, such as ADM energy, Bondi energy $\Rightarrow$ in BH course).

We can define a form of $t_{\mu\nu}^{(\text{grav.})}$ working perturbatively around flat space.

Since we expect $t_{\mu\nu} \sim \varepsilon^2$, we should expand

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + \dots$$

Observe

$$R_{\mu\nu}[\eta + \varepsilon h^{(1)}] = \varepsilon R^{(1)}_{\mu\nu}[h^{(1)}] + \varepsilon^2 R^{(2)}_{\mu\nu}[h^{(1)}].$$

where

$$R^{(1)}_{\mu\nu}[h] = -\frac{1}{2}\partial^\rho \partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu h + \partial^\rho \partial_{(\mu} h_{\nu)\rho}.$$

as before, and

$$\begin{aligned} R^{(2)}_{\mu\nu}[h] &= \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma} + \partial^\sigma h^\rho{}_\nu \partial_{[\sigma}h_{\rho]\mu} \\ &+ \frac{1}{2}\partial_\sigma\left(h^{\sigma\rho}\partial_\rho h_{\mu\nu}\right) - \frac{1}{4}\partial^\rho h \partial_\rho h_{\mu\nu} - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)\partial_{(\mu}h_{\nu)\rho} \quad (8.84) \end{aligned}$$

quadratic in h. Therefore,

$$R_{\mu\nu}[\eta + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)}] = \varepsilon R_{\mu\nu}[h^{(1)}] + \varepsilon^2\left(R^{(2)}_{\mu\nu}[h^{(1)}] + R^{(1)}_{\mu\nu}[h^{(2)}]\right) + \mathcal{O}(\varepsilon^3)$$

and consequently,

$$\begin{aligned} G_{\mu\nu}[\eta + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)}] = \varepsilon G^{(1)}_{\mu\nu}[h^{(1)}] + \varepsilon^2\Big(& G^{(1)}_{\mu\nu}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}] \\ & - \frac{1}{2}\eta_{\mu\nu} h^{\sigma\tau} R^{(2)}_{\sigma\tau}[h^{(1)}] + \frac{1}{2}\eta_{\mu\nu} h^{\sigma\tau} R^{(1)}_{\sigma\tau}[h^{(1)}] \\ & - \frac{1}{2} h_{\mu\nu}\eta^{\sigma\tau} R^{(1)}_{\sigma\tau}[h^{(1)}]\Big) \end{aligned}$$

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} R_{\rho\sigma}$

If $G_{\mu\nu}[\eta + \varepsilon h^{(1)} + \varepsilon h^{(2)}]$ solves vacuum EE (to order ε^2), we must have

$\mathcal{O}(\varepsilon)$: $\quad G^{(1)}_{\mu\nu}[h^{(1)}] = 0. \quad \Rightarrow \quad R^{(1)}_{\mu\nu}[h^{(1)}] = 0$

$\mathcal{O}(\varepsilon^2)$: $\quad G^{(1)}_{\mu\nu}[h^{(2)}] = -R^{(2)}_{\mu\nu}[h^{(1)}] + \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau} R^{(2)}_{\sigma\tau}[h^{(1)}] =: 8\pi t^{(\text{grav.})}_{\mu\nu}$

where we define $t^{(\text{grav.})}_{\mu\nu}$ by RHS. To simplify a little further, recall Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$.

vanish if $\mathcal{O}(\varepsilon)$ EE obeyed

$$\Rightarrow 0 = \varepsilon \partial^\mu G^{(1)}_{\mu\nu}[h^{(1)}] + \varepsilon^2\left(\partial^\mu G^{(1)}_{\mu\nu}[h^{(2)}] - 8\pi \partial^\mu t^{(\text{grav.})}_{\mu\nu} + h^{(1)} \partial^\mu R^{(1)}_{\mu\nu}[h^{(1)}]\right)$$

Hence, $\partial^\mu G^{(1)}_{\mu\nu}[h] = 0$ for <u>any</u> perturbation, and so

$$\partial^\mu t^{(grav.)}_{\mu\nu} = 0$$

on-shell. We identify $t^{(grav.)}_{\mu\nu}$ as an effective energy-mom. tensor for the grav. field itself, but it is <u>not</u> gauge invar.

If $h^{(1)}_{\mu\nu}$ vanishes suff. fast as $|x| \to \infty$, then $\int_{\mathbb{R}^3} t^{(grav.)}_{00} \, d^3x$ is gauge invar., but there's no <u>local</u> cons. of energy.

To pretend to do better, we can average a tensor as follows: let $W(x,t)$ be a bump f^n (i.e. W smooth, ≥ 0, $\int_{\mathbb{R}^4} W(x,t) d^4x = 1$, $W = 0$ for $|x|^2 + t^2 > a^2$) and define

$$\langle X_{\mu\nu}(x) \rangle_W = \int_{\mathbb{R}^4} W(x-y) X_{\mu\nu}(y) \, d^4x .$$

In the far field regime, with radiation of wavelength $\lambda \ll a$, we have $\partial_\mu W \sim W/a$, so

$$\partial_\mu \langle X_{\rho\sigma}(x) \rangle_W = \int_{\mathbb{R}^4} (\partial_\mu W)(x-y) X_{\rho\sigma}(y) \, d^4x$$

$$\sim \frac{\langle X_{\rho\sigma}(x) \rangle_W}{a} \sim \frac{\lambda}{a} \langle \partial_\mu X_{\rho\sigma} \rangle_W$$

This allows us to drop total deriv. when taking averages, so

$$\langle A \partial B \rangle_W = \langle \partial(AB) \rangle_W - \langle B \partial A \rangle_W \sim -\langle B \partial A \rangle_W$$

Using this, can show

- If h solves linearised EE, $\langle \eta^{\mu\nu} R^{(2)}_{\mu\nu}[h] \rangle_W = 0$.
- $\langle t^{(grav.)}_{\mu\nu} \rangle = \frac{1}{32\pi} \langle \partial_\mu \bar{h}_{\rho\sigma} \partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2} \partial_\mu \bar{h} \partial_\nu \bar{h} - 2 \partial_\sigma \bar{h}^{\rho\sigma} \partial_{(\mu} \bar{h}_{\nu)\rho} \rangle_W$
- $\langle t^{(grav.)}_{\mu\nu} \rangle$ invar. under linearised gauge transf^n.

The Quadrupole Formula

Using $\langle t_{\mu\nu}^{(grav.)} \rangle$ in far field approx., we can calculate energy lost by a system emitting gravitational radiation. The average energy flux is

$$S_i = -\langle t_{0i}^{(grav)} \rangle$$

and average energy flux / power radiated across sphere radius $r \gg a$ centred on source is

$$\langle P \rangle = -\int r^2 d\Omega \, \langle t_{0i}^{(grav.)} \rangle \, \hat{x}^i$$

In wave gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$,

$$\langle t_{0i}^{(grav.)} \rangle = \frac{1}{32\pi} \langle \partial_0 \bar{h}_{\rho\sigma} \partial_i \bar{h}^{\rho\sigma} - \frac{1}{2} \partial_0 \bar{h} \, \partial_i \bar{h} \rangle$$

$$= \frac{1}{32\pi} \langle \partial_0 \bar{h}_{jk} \partial_i \bar{h}^{jk} - 2 \partial_0 \bar{h}_{0j} \partial_i \bar{h}_{0j} + \partial_0 \bar{h}_{00} \partial_i \bar{h}_{00} - \frac{1}{2} \partial_0 \bar{h} \, \partial_i \bar{h} \rangle$$

Look at 1st term, using (‡), have

$$\partial_0 \bar{h}_{jk} = \frac{2}{r} \dddot{I}_{jk}(t-r)$$

neglect

$$\partial_i \bar{h}_{jk} = \left(-\frac{2}{r} \dddot{I}_{jk}(t-r) - \cancel{\frac{2}{r^2}} \ddot{I}_{jk}(t-r) \right) \hat{x}^i .$$

Hence,

$$-\frac{1}{32\pi} \int r^2 d\Omega \, \langle \partial_0 \bar{h}_{jk} \partial_i \bar{h}_{jk} \rangle \hat{x}^i = \frac{1}{2} \langle \dddot{I}_{ij} \dddot{I}_{ij} \rangle .$$

Including all terms, one finds

$$\langle P \rangle_t = \frac{1}{5} \langle \dddot{I}_{jk} \dddot{I}_{jk} - \frac{1}{3} \dddot{I}_{jj} \dddot{I}_{kk} \rangle_{t-r}$$

$$= \frac{1}{5} \langle \dddot{Q}_{jk} \dddot{Q}_{jk} \rangle_{t-r} ,$$

where $Q_{jk} = I_{jk} - \frac{1}{3} I_{ii} \delta_{jk}$ is the trace-free part of quadrupole moment.

Differential Forms

We'd like to derive EE as the E-L eqn of some action. To do this, we need to understand how to integrate over a mfld M, and for this, we need differential forms.

Defn A p-form is a totally antisym tensor field of type $(0,p)$ on M.

If $\{dx^\mu\}$ are a coord basis of T_p^*M, we can write a p-form ω as

$$\omega = \frac{1}{p!} \omega_{[\mu_1 \dots \mu_p]} dx^{\mu_1} \dots dx^{\mu_p} = \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} .$$

where $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$.

Example A 1-form is a covec-field, while a 0-form is a f^n. There are no p-forms with $p > n = \dim M$. The Maxwell field strength tensor $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$ is a 2-form.

The space of all p-forms on M is often denoted $\Omega^p(M)$, and is an ∞-dim space.

We have a natural product on forms: if α a p-form, β a q-form, then

$$\alpha \wedge \beta = (\alpha\wedge\beta)_{\mu_1 \dots \mu_p \mu_{p+1} \dots \mu_{p+q}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+q}}$$

$$= \frac{1}{(p+q)!} (\alpha \wedge \beta)_{[\mu_1 \dots \mu_{p+q}]} dx^{\mu_1} \otimes \dots \otimes dx^{\mu_{p+q}}$$

where

$$(\alpha\wedge\beta)_{[\mu_1\dots\mu_{p+q}]} = \alpha_{[\mu_1\dots\mu_p}\beta_{\mu_{p+1}\dots\mu_{p+q}]}$$

is a $(p+q)$-form. The product obeys

- $\alpha\wedge\beta = (-1)^{pq}\beta\wedge\alpha$.

  (so in particular $\alpha\wedge\alpha = 0$ if p odd).

- $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$

Example If $\alpha\in\Omega^1(M)$, $\beta\in\Omega^2(M)$, then

$$(\alpha\wedge\beta)_{\mu\nu\kappa} = \alpha_\mu\beta_{\nu\kappa} + \alpha_\nu\beta_{\kappa\mu} + \alpha_\kappa\beta_{\mu\nu}.$$

This gives

$$\Omega^*(M) = \bigoplus_{p=0}^{n}\Omega^p(M)$$

the structure of an assoc. but non-commutative algebra.

An important property of forms is that we can define a derivative $d:\Omega^p(M)\to\Omega^{p+1}(M)$ as follows

Def^n If $\omega\in\Omega^p(M)$, then in a coord basis, the exterior derivative of ω is the $(p+1)$-form $d\omega$ with cpts

$$(d\omega)_{\mu_1\dots\mu_{p+1}} = (p+1)\,\partial_{[\mu_1}\omega_{\mu_2\dots\mu_{p+1}]}$$

Example In EM, often write $F = dA$, where $A = A_\mu(x)\,dx^\mu$ is the gauge field, so $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$

This ext. deriv. is defined w/o any connection/metric: suppose ∇ is any torsion free connection, so

$$\nabla_{\mu_1}\omega_{\mu_2\dots\mu_{p+1}} = \partial_{\mu_1}\omega_{\mu_2\dots\mu_{p+1}} - \Gamma^\nu_{\mu_2\mu_1}\omega_{\nu\mu_3\dots\mu_{p+1}} - \dots - \Gamma^\nu_{\mu_{p+1}\mu_1}\omega_{\mu_2\dots\mu_p\nu}$$

Antisymmetrising,

$$\nabla_{[\mu_1}\omega_{\mu_2\cdots\mu_{p+1}]} = \partial_{[\mu_1}\omega_{\mu_2\cdots\mu_{p+1}]}$$

since $\Gamma^{\nu}_{[\mu_i\,\mu_k]} = 0$. Therefore, $d\omega$ is well-defined as an antisym $(0, p+1)$ tensor field, i.e. $\omega \in \Omega^{p+1}(M)$

<u>Exercise</u> The ext. deriv has some important properties:

- $d(d\omega) = d^2\omega = 0$
- $d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^p \alpha\wedge(d\beta)$ if $\alpha\in\Omega^p(M)$, $\beta\in\Omega^q(M)$
- If $\phi: M\to N$ and $\omega\in\Omega^p(N)$, then $\phi^*(d\omega) = d(\phi^*\omega)$

  $\Rightarrow$ If $\phi = \varphi_t^*$ a diffeo assoc. to vec. field X, then $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega)$.

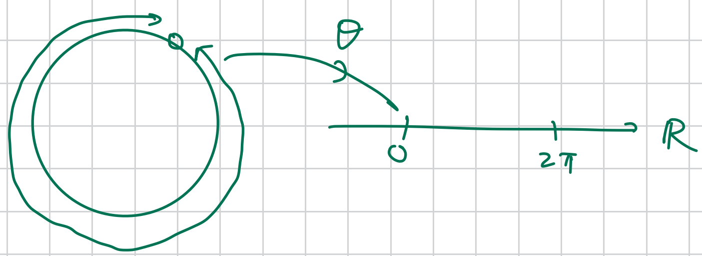
We say ω is <u>closed</u> if $d\omega = 0$. (so every $\omega\in\Omega^n(M)$ is closed)

and ω is <u>exact</u> if $\exists$ some $(p-1)$-form α s.t. $\omega = d\alpha$.

Every exact form is closed $(d^2=0)$, but the converse is true only locally.

<u>Lem</u> (Poincaré lemma) If ω is a closed p-form on M (with $p\geq 1$) then for every point $m\in M$, $\exists$ open nbd $U_m \in M$ with $m\in U_m$ s.t. $\omega = d\alpha$ throughout U_m.

<u>Example</u> On $M = S^1$, all 1-forms are closed. Locally, $\exists$ coord f^n $\theta: U\subset S^1 \to (0, 2\pi)$ and $d\theta$ is a closed form.



However, $\nexists$ global coord f^n on S^1, so globally there's no <u>exact</u> 1-form on S^1.

In general, the space

$$H^p(M) = \{\omega \in \Omega^p(M) \mid d\omega = 0\} \Big/ \{\alpha \in \Omega^{p-1}(M) \mid d\alpha = \omega\}$$

is called the de Rham cohomology group of M and "measures" the non-trivial topology of M.

The Tetrad Formalism

It's often useful to use a basis of vec. fields $\{e_\mu\}$ with the property

$$g(e_\mu, e_\nu) = \eta_{\mu\nu}$$

at every $p \in O \subset M$.

(the cpts of Mink. metric in Cart. basis, i.e. $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.)

Example The Schr. metric is

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

$$= -f^2 dt^2 + \frac{1}{f^2} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

where $f(r) = \sqrt{1 - \frac{2M}{r}}$ and can choose

$$e_0 = \frac{1}{f}\frac{\partial}{\partial t}, \quad e_1 = f\frac{\partial}{\partial r}, \quad e_2 = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad e_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}.$$

have $g(e_0, e_0) = -1$.

In general, $[e_\mu, e_\nu] \neq 0$, so they don't form a coord basis. Any such basis is called an orthonormal frame (or a set of tetrads). The basis is not unique

$$\tilde{e}_\nu = \Lambda_\nu{}^\mu(x)\, e_\mu$$

is also an orthonormal frame if

$$\Lambda^\mu{}_\nu(x)\Lambda^\kappa{}_\rho\,\eta_{\mu\kappa} = \eta_{\nu\rho}$$

at each $x \in \mathcal{O}$. Thus we have the freedom to make <u>local</u> Lorentz transf^n: $\Lambda(x) \in SO(1,3)$ at each x.

"GR arises from <u>gauging</u> Lorentz transf^n."

Given $\{e_\mu\}$, define the dual basis $\{\theta^\mu\}$ in standard way

$$\theta^\nu(e_\mu) = \delta^\nu{}_\mu.$$

<u>Claim</u> $\theta^\mu = \eta^{\mu\kappa} g(e_\kappa, \cdot)$

<u>Pf</u>: $\theta^\nu(e_\mu) = \eta^{\nu\kappa} g(e_\kappa, e_\mu) = \eta^{\nu\kappa}\eta_{\kappa\mu} = \delta^\nu{}_\mu.$ $\square$

<u>Exercise</u> Check $\eta_{\mu\nu}\theta^\mu_a\theta^\nu_b = g_{ab}$ (or $g = \eta_{\mu\nu}\theta^\mu \otimes \theta^\nu$), and $\theta^\mu{}_a \theta_\mu{}^b = \delta^b{}_a$.

<u>Connection 1-forms</u>

Let ∇ be the Levi-Civita connection. The connection 1-forms are defined to be

$$(\omega^\mu{}_\nu)_a = \theta^\mu{}_b \nabla_a e^b{}_\nu$$

in any basis. Recall that $(\nabla_\rho e_\nu)^b = \Gamma_\nu{}^\sigma{}_\rho\, e^b_\sigma = e^a_\rho \nabla_a e^b_\nu$

$$\Rightarrow \nabla_a e^b{}_\nu = \theta^\rho_a e^c_\rho \nabla_c e^b{}_\nu = \theta^\rho{}_a \Gamma_\nu{}^\sigma{}_\rho e^b{}_\sigma$$

$$\Rightarrow (\omega^\mu{}_\nu)_a = \theta^\mu{}_b \nabla_a e_\nu{}^b = \theta^\mu{}_b\theta^\rho{}_a \Gamma_\nu{}^\sigma{}_\rho e^b{}_\sigma = \theta^\rho{}_a \Gamma_\nu{}^\mu{}_\rho.$$

So $\omega^\mu{}_\nu$ encloses the connection cpt. $\Gamma_\nu{}^\mu{}_\rho\,\theta^\rho$ thought of as a 1-form (here in our orthonormal basis)

Lem $(\omega_{\mu\nu})_a = \eta_{\mu k} (\omega^k{}_\nu)_a = -(\omega_{\nu\mu})_a$.

Pf: $(\omega_{\mu\nu})_a = \eta_{\mu k}(\omega^k{}_\nu)_a$

$$= \eta_{\mu k}\, \theta^k{}_b \nabla_a e^b{}_\nu$$

$$= \nabla_a(\eta_{\mu k}\theta^k{}_b e^b{}_\nu) - e^b{}_\nu \nabla_a(\eta_{\mu k} e^k{}_b).$$

$$= -e^b{}_\nu \nabla_a (\underbrace{\eta_{\mu k}\eta^{k\nu}}_{=\delta_\mu{}^\nu} g_{bc} e^c{}_\nu).$$

$$= -g_{bc}\, e^b{}_\nu \nabla_a e^c{}_\mu$$

$$= -\theta_{c\nu} \nabla_a e^c{}_\mu$$

$$= -\eta_{\nu k}\theta_c{}^k \nabla_a e^c{}_\mu = -(\omega_{\nu\mu})_a. \qquad \square$$

Lem θ^μ obeys Cartan's 1st structure eqn

$$d\theta^\mu - \omega^\mu{}_\nu \wedge \theta^\nu = 0.$$

Pf: $(d\theta^\mu)_{ab} = \nabla_a \theta_b{}^\mu - \nabla_b \theta_a{}^\mu = 2\nabla_{[a}\theta^\mu{}_{b]}$,

but note $(\omega^\mu{}_\nu)_a e^b{}_\nu = \nabla_a e^b{}_\mu$. Hence,

$$\nabla_a \theta^\mu_b = (\omega_\nu{}^\mu)_a \theta^\nu{}_b = -(\omega^\mu{}_\nu)_a \theta^\nu{}_b$$

and therefore

$$(d\theta^\mu)_{ab} = 2\nabla_{[a}\theta^\mu{}_{b]}$$

$$= -2(\omega^\mu{}_\nu)_{[a}\theta^\nu{}_{b]} = -(\omega^\mu{}_\nu \wedge \theta^\nu)_{ab}. \qquad \square$$

This is often a convenient way to compute $\omega \to T$.

Example Schr. metric

$$ds^2 = -f^2 dt^2 + \frac{1}{f^2} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2. \quad , \quad f = \sqrt{1-\frac{2M}{r}}$$

and choose

$$\theta^0 = f\,dt, \quad \theta^1 = \frac{1}{f} dr, \quad \theta^2 = r\,d\theta, \quad \theta^3 = r\sin\theta\, d\phi.$$

Then $d\theta^0 = f' \, dr \wedge dt = f' \, \theta^1 \wedge \theta^0$

$d\theta^1 = 0$

$d\theta^2 = dr \wedge d\theta = \frac{f}{r} \, \theta^1 \wedge \theta^2$

$d\theta^3 = \sin\theta \, dr \wedge d\phi + r\cos\theta \, d\theta \wedge d\phi$

$= \frac{f}{r} \, \theta^1 \wedge \theta^3 + \frac{\cot\theta}{r} \, \theta^2 \wedge \theta^3.$

This suggests we take

$$\omega^1{}_\mu = 0, \quad \omega^2{}_1 = \frac{f}{r}\theta^2, \quad \omega^3{}_1 = \frac{f}{r}\theta^3, \quad \omega^3{}_2 = \frac{\cot\theta}{r}\theta^3.$$

Therefore,

$$\omega_{01} = -\omega_{10} = f' \, \theta^0$$
$$\omega_{21} = -\omega_{12} = \frac{f}{r}\theta^2$$
$$\omega_{31} = -\omega_{13} = \frac{f}{r}\theta^3$$
$$\omega_{32} = -\omega_{23} = \frac{\cot\theta}{r}\theta^3$$

, with other $\omega_{\mu\nu} = 0$.

Curvature 2-form

Let's compute $d\omega^\mu{}_\nu$. Have

$$\begin{aligned}
(d\omega^\mu{}_\nu)_{ab} &= \nabla_a (\omega^\mu{}_\nu)_b - \nabla_b (\omega^\mu{}_\nu)_a \\
&= \nabla_a (\theta^\mu{}_c \nabla_b e^c{}_\nu) - \nabla_b (\theta^\mu{}_c \nabla_a e^c{}_\nu). \\
&= \theta^\mu{}_c \left(\nabla_a \nabla_b e^c{}_\nu - \nabla_b \nabla_a e^c{}_\nu \right) \\
&\qquad + (\nabla_a \theta^\mu{}_c)(\nabla_b e^c{}_\nu) - (\nabla_b \theta^\mu{}_c)(\nabla_a e^c{}_\nu) \\
&= \theta^\mu{}_c R^c{}_{dab} e^d{}_\nu + (e^d{}_\sigma \nabla_a \theta^\mu_d)(\theta^\sigma{}_c \nabla_b e^c{}_\nu) \\
&\qquad - (e^d{}_\sigma \nabla_b \theta^\mu{}_d)(\theta^\sigma{}_c \nabla_a e^c{}_\nu) \\
&= (\Omega^\mu{}_\nu)_{ab} + (\omega_\sigma{}^\mu \wedge \omega^\sigma{}_\nu)_{ab}.
\end{aligned}$$

Where $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\sigma\tau} \, \theta^\sigma \wedge \theta^\tau$ are the curvature 2-forms.

We have <u>Cartan's 2nd Structure eqn</u>:

$$d\omega^\mu{}_\nu + \omega^\mu{}_\sigma \wedge \omega^\sigma{}_\nu = \Theta^\mu{}_\nu$$

This is often an efficient way to compute Riemann tensor.

<u>Volume Form and the Hodge *</u>

A mfld M, $\dim M = m$, is orientable if $\exists$ nowhere-vanishing m-form ε. ε is called a orientation. Two such orientations are equivalent if $\varepsilon' = f\varepsilon$ where f^n $f>0$ at each $p \in M$. A basis of vectors $\{e_\mu\}_{\mu=1,\dots,m}$ is right-handed if $\varepsilon(e_1,\dots,e_m) > 0$. A coord system is right-handed if $\{\partial/\partial x^\mu\}$ right handed.

An oriented mfld with metric g has a preferred normalisation for ε. Define the volume form to be ε s.t.

$$\varepsilon(e_1,\dots,e_m) = 1$$

where $\{e_\mu\}$ are our orthonormal frame (right-handed).

In a right-handed coord sys., have

$$\frac{\partial}{\partial x^\alpha} = \delta^\beta{}_\alpha \frac{\partial}{\partial x^\beta} = \theta^\mu{}_\alpha e_\mu{}^\beta \frac{\partial}{\partial x^\beta} = \theta^\mu{}_\alpha e_\mu$$

So

$$\varepsilon\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^m}\right) = \varepsilon\left(\theta^{\mu_1}{}_1 e_{\mu_1}, \dots, \theta^{\mu_m}{}_m e_{\mu_m}\right)$$

$$= \sum_{\pi \in \mathrm{Sym}(m)} \sigma(\pi)\, \theta_1^{\pi(1)} \cdots \theta_m^{\pi(m)}$$

$$= \det(\theta^\mu{}_\alpha)$$

Also, since $\eta_{\kappa\lambda}\Theta^{\kappa}{}_{\alpha}\Theta^{\lambda}{}_{\beta} = g_{\alpha\beta}$, have $\det(\Theta^{\mu}{}_{\alpha}) = \sqrt{|g|}$.

$$\Rightarrow \quad \varepsilon = \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^m .$$

or equiv,

$$\varepsilon_{12\dots m} = \sqrt{|g|}.$$

Lem $\nabla \varepsilon = 0$

Pf: In normal coord, have $\partial_\alpha g_{\beta\gamma}|_p = 0$, $\Gamma^{\alpha}_{\beta\gamma}|_p = 0$.

$$\Rightarrow \nabla_\nu \varepsilon_{\mu_1\dots\mu_m} = 0.$$

But this is tensor eqn, so true in any basis. □

Lem $\varepsilon^{a_1\dots a_p c_{p+1}\dots c_m}\varepsilon_{b_1\dots b_p c_{p+1}\dots c_{m}} = \pm\, p!(m-p)!\, \delta^{a_1}_{[b_1}\delta^{a_2}_{b_2}\cdots\delta^{a_p}_{b_p]}$

where $+$: Riemannian, $-$: Lorentzian.

We can use ε to relate p-forms to $(m-p)$-forms.

Define Hodge star $\ast : \Omega^p(M) \to \Omega^{m-p}(M)$ by

$$(\ast\omega)_{a_1\dots a_{m-p}} = \frac{1}{p!}\varepsilon_{a_1\dots a_{m-p} b_1\dots b_p}\omega^{b_1\dots b_p}.$$

for any $\omega \in \Omega^p(M)$, where $\omega^{b_1\dots b_p} = g^{b_1c_1}\cdots g^{b_pc_p}\omega_{c_1\dots c_p}$.

Lem • $\ast(\ast\omega) = \pm(-1)^{p(m-p)}\omega$ $\quad$ $+$: Riem, $-$: Lor.

• $$(\ast d \underbrace{(\ast\omega)}_{\substack{m-p+1 \\ m-(m-p+1)=p-1}})_{a_1\dots a_p} = \pm(-1)^{p(m-p)}\nabla^b\omega_{ba_1\dots a_{p-1}}.$$

Example If $(M,g) = (\mathbb{R}^3, \delta)$, can identify vec fields with 1-forms using δ. Then $\mathrm{grad}(f) = df$, $\mathrm{div}(X) = \ast d(\ast X^\flat)$, $\mathrm{curl}(X) = \ast dX$.

$d^2 = 0 \Rightarrow \mathrm{curl}(\mathrm{grad}(f)) = 0 \,, \quad \mathrm{div}(\mathrm{curl}(X)) = 0.$

<u>Example</u> Maxwell eqn

$$\nabla^a F_{ab} = -4\pi j_b \,, \qquad \nabla_{[a} F_{bc]} = 0$$

$$\Rightarrow \quad d * F = -4\pi * j \qquad dF = 0.$$

(Poincare lem : $F = dA$ at least locally).

Integration on Manifolds

Suppose we have a right-handed coord chart $\phi : \mathcal{O} \to \mathcal{U}$ on M, with coords x^μ. If ω is an m-form vanishing outside $\mathcal{O}$, then we can write

$$\omega = \omega_{1\dots m}(x) \, dx^1 \wedge \dots \wedge dx^m$$

If $\psi : \mathcal{O} \to \mathcal{U}$ is another RH chart with coords y^ν, then also

$$\begin{aligned} \omega &= \tilde{\omega}_{1\dots m}(y) \, dy^1 \wedge \dots \wedge dy^m \\ &= \tilde{\omega}_{1\dots m}(y(x)) \frac{\partial y^1}{\partial x^{\mu_1}} \cdots \frac{\partial y^m}{\partial x^{\mu_m}} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m} . \\ &= \tilde{\omega}_{1\dots m}(y(x)) \det\left(\frac{\partial y^\nu}{\partial x^\mu}\right) dx^1 \wedge \dots \wedge dx^m . \end{aligned}$$

So

$$\omega_{1\dots n}(x) = \tilde{\omega}_{1\dots n}(y(x)) \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right).$$

Consequently,

$$\int_{\mathcal{U}_1} \omega_{1\dots m}(x) \, dx^1 \wedge \dots \wedge dx^m = \int_{\mathcal{U}_2} \tilde{\omega}_{1\dots m}(y) \, dy^1 \wedge \dots \wedge dy^m ,$$

and we then define $\int_{\mathcal{O} \subset M} \omega$ to be its integral over any coord patch $\mathcal{U} \subset \mathbb{R}^m$ corresponding to some RH chart.

To integrate over all of M, recall we can always find a countable atlas of charts $(\mathcal{O}_i, \phi_i)$. Pick smooth f^n's $\chi_i : \mathcal{O}_i \to [0,1]$ s.t. χ_i vanishes outside $\mathcal{O}_i$, and $\sum_i \chi_i(p) = 1$ $\forall\, p \in M$. The χ_i are often called a partition of unity.

Then for any $\omega \in \Omega^m(M)$, we define

$$\int_M \omega = \sum_i \int_{\mathcal{O}_i} \chi_i \omega .$$

Claim This does not depend on the choice of partition of unity.

Rmks: • Coordinate invariance $\Rightarrow$ if $\varphi : M \to M$ diffeo, then

$$\int_M \omega = \int_M \varphi^*(\omega).$$

• If (M, g) is mfld with metric and $f : M \to \mathbb{R}$ a scaler field, then $f\varepsilon$ is a m-form and define

$$\int_M f = \int_M f\varepsilon = \int_M *f.$$

In local coords, if f vanishes outside some chart $\mathcal{O}$, this is

$$\int_M f = \int_{\mathcal{O}} f(x) \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^m = \int_{\mathcal{O}} f(x)\, d\,\mathrm{vol}_g .$$

Submanifolds and Stoke's theorem

Def^n Suppose S, M are mflds with $s = \dim S < m = \dim M$. A smooth map $\iota : S \to M$ is an embedding if it is an imersion, ie. $\iota_*(TS)_p \to T_{\iota(p)}M$ injective, and is itself injective, ie. $\iota(p) = \iota(q)$ if $p = q$.

If ι is an embedding and $\iota(S) \subset M$ an embedded mfld, we define an s-form ω over $\iota(S)$ by

$$\int_{\iota(S)} \omega = \int_S \iota^*(\omega) \qquad \forall\, \omega \in \Omega^s(M).$$

Note that if $\omega = d\alpha$, then

$$\int_{\iota(S)} d\alpha = \int_S \iota^*(d\alpha) = \int_S d(\iota^*\alpha).$$

<u>Def^n</u> A <u>mfld with boundary</u> M is defined just as for an ordinary mfld, except charts are maps $\phi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $R^m_{\leq 0} = \{(x^1, \dots, x^m) \in \mathbb{R}^m \mid x^1 \leq 0\}$. The boundary ∂M is the set of points mapped in any chart to $\{x^1 = 0\}$. ∂M is naturally an $(m-1)$-dim mfld with an embedding $\iota : \partial M \to M$.

If M is oriented, ∂M inherits an orientation by requiring $(x^2, \dots, x^m)$ is a r.h. chart on ∂M whenever $(x^1, \dots, x^m)$ is r.h. on M.

<u>Thm</u> (Stokes theorem) If M is an oriented m-dim mfld with boundary, and $\omega \in \Omega^{m-1}(M)$, then

$$\int_M d\omega = \int_{\partial M} \omega$$

<u>Example</u> Let Σ be a hypersurface (co-dim 1) with boundary in a 4d space-time M and suppose F is a 2-form on M obeying Maxwell eq^n $dF = 0$, $d*F = -4\pi *j$. Then

$$\frac{1}{4\pi}\int_{\partial\Sigma} *F = \frac{1}{4\pi}\int_\Sigma d*F = -\int_\Sigma *j = Q[\Sigma]$$

This is Gauss law.

Stoke's thm is the basis of all IBP arguments on M.

If M has an metric, we formulate Stoke's thm into divergence thm.

Let X vec. field on M, and recall

$$(X \lrcorner \varepsilon)_{a_2 \dots a_m} = X^b \varepsilon_{b a_2 \dots a_m}$$

One can check $d(X \lrcorner \varepsilon) = (\nabla_a X^a)\varepsilon$.

If we define the <u>flux</u> of X through an embedded co-dim 1 hypersurface $S \subset M$ by

$$\int_{\iota(S)} X \cdot dS = \int_{\iota(S)} X \lrcorner \varepsilon$$

Then Stoke's

$$\Rightarrow \int_{\partial M} X \cdot dS = \int_M d(X \lrcorner \varepsilon) = \int_M (\nabla_a X^a)\varepsilon$$

Which is div. thm.

Recall that $\iota(S)$ is

- spacelike if $h = \iota^* g$ Riemannian
- timelike if $h = \iota^* g$ Lorentzian

In this case, we can relate $\iota^*(X \lrcorner \varepsilon_g)$ to the vol. form ε_h on (S, h). Pick an r.h. orthonormal basis $\{b_2, \dots, b_m\}$ on S, then $\{\iota_* b_2, \dots, \iota_* b_m\}$ still o.n. on M. The <u>unit normal</u> to S in M is the unique vector n that is orthogonal to all $\iota_* b_2, \dots, \iota_* b_m$, with

$$\varepsilon(n, \iota_* b_2, \dots, \iota_* b_m) = g(n,n) = \pm 1.$$

Then

$$(X \lrcorner \varepsilon)(\iota_* b_2, \dots, \iota_* b_m) = X^a n_a = g(X, n).$$

and so

$$\iota^*(X \lrcorner \varepsilon_g) = \iota^*(X^a n_a)\, \varepsilon_h.$$

Consequently,

$$\int_S X^a n_a \, d\,\mathrm{vol}_h = \int_M (\nabla_a X^a)\, \varepsilon_g$$

where $d\,\mathrm{vol}_g = \varepsilon_g$ etc.

n^a points out of M for ∂M timelike, g Lorentzian (or whenever g Riemannian), and points into M for ∂M spacelike, g Lorentzian.

The Einstein–Hilbert Action

We can derive EE from an action principle using the Einstein–Hilbert action

Ricci Scaler

$$S[g] = \frac{1}{16\pi G_N} \int_M R[g]\, \varepsilon_g$$

Note that $R \sim \partial\Gamma + \Gamma\Gamma$ and $\Gamma \sim g^{-1}(\partial g + \partial g - \partial g)$, so EH Lagrangian is quadratic in ∂g, but non-linear in g itself.

To derive vacuum EE, vary $g \mapsto g + \delta g$, where δg vanishes outside a compact set. Expanding to first order, we want to compute $S_{EH}[g+\delta g] - S_{EH}[g]$.

First, consider $\varepsilon_g = (d\,\mathrm{vol})_g = \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^m$, we have

$$\delta\sqrt{|g|} = \delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}(-\delta g),$$

where $g = \det(g_{\mu\nu})$. For any matrix A, have

$$\log(\det A) = \operatorname{tr}(\log A).$$

$$\Rightarrow \quad \frac{1}{\det A}\,\delta(\det A) = \operatorname{tr}(A^{-1}\delta A).$$

In our case, this gives

$$\delta\sqrt{|g|} = -\frac{1}{2\sqrt{-g}}\, g\, g^{ab}\,\delta g_{ab} = \frac{1}{2}\sqrt{|g|}\, g^{ab}\,\delta g_{ab}$$

To compute δR, first consider $\delta\Gamma^{\mu}_{\nu\rho}$. The difference of two connections is a tensor $\delta\Gamma^{b}_{ac}$. To compute this, use normal coords at p, where $\partial_\mu g_{\nu\rho}|_p = 0$. Then

$$\delta\Gamma^{\mu}_{\nu\rho}|_p = \frac{1}{2} g^{\mu\sigma}\left(\delta g_{\sigma\nu,\rho} + \delta g_{\rho\sigma,\nu} - \delta g_{\rho\nu,\sigma}\right)|_p$$

$$= \frac{1}{2} g^{\mu\sigma}\left(\delta g_{\sigma\nu;\rho} + \delta g_{\rho\sigma;\nu} - \delta g_{\rho\nu;\sigma}\right)|_p.$$

This holds for any $p \in M$ and is a relation among tensors, so we have

$$\delta\Gamma^{a}_{bc} = \frac{1}{2} g^{ad}\left(\delta g_{bd;c} + \delta g_{cd;b} - \delta g_{bc;d}\right).$$

in any basis.

Next, consider $\delta R^{\mu}{}_{\nu\rho\sigma}$ again and work in normal coords at p. We have

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma\Gamma,$$

so

$$\delta R^{\mu}{}_{\nu\rho\sigma}|_p = \partial_\rho(\delta\Gamma^{\mu}_{\nu\sigma}) - \partial_\sigma(\delta\Gamma^{\mu}_{\nu\rho})|_p.$$

$$= \nabla_\rho\,\delta\Gamma^{\mu}_{\nu\sigma} - \nabla_\sigma\,\Gamma^{\mu}_{\nu\sigma}|_p.$$

$$\Rightarrow \delta R^a{}_{bcd} = \nabla_c (\delta\Gamma^a_{bd}) - \nabla_d (\delta\Gamma^a_{bc})$$

and likewise

$$\delta R_{ab} = \delta R^c{}_{acb} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}.$$

for the variation of Ricci tensor.

Note that $\delta(g^{ab} g_{bc}) = 0$, so $\delta g^{ab} = -g^{ac} g^{bd} \delta g_{cd}$.

Therefore,

$$\begin{aligned} \delta R &= \delta(g^{ab} R_{ab}) \\ &= \delta g^{ab} R_{ab} + g^{ab} \delta R_{ab} \\ &= -R^{ab} \delta g_{ab} + g^{ab} (\nabla_c(\delta\Gamma^c_{ab}) - \nabla_b(\delta\Gamma^c_{ac})). \\ &= -R^{ab} \delta g_{ab} + \nabla_c X^c , \end{aligned}$$

where $X^c = g^{ab} \delta\Gamma^c_{ab} - g^{ac} \delta\Gamma^b_{ab}$

Altogether, we have

$$\begin{aligned} 16\pi G \, \delta S_{EH} &= \int_M (\delta R \, \varepsilon_g + R \, \delta\varepsilon_g) \\ &= \int_M (\delta R + \frac{R}{2} g^{ab} \delta g_{ab}) \, \varepsilon_g \end{aligned}$$

$$\Rightarrow \delta S_{EH} = \frac{1}{16\pi G} \int_M (-R^{ab} + \tfrac{1}{2} g^{ab} R) \, \delta g_{ab} \, \varepsilon_g + \frac{1}{16\pi G} \int_M \nabla_c X^c \, \varepsilon_g$$

Dropping the total divergence, (since δg_{ab} has compact support), we see $S_{EH}[g]$ is extremised for metrics obeying

$$R_{ab} - \tfrac{1}{2} g_{ab} R = 0$$

the vacuum EE. If we also have some matter field Φ, the full action will be

$$S[g,\Phi] = S_{EH}[g] + S_{matter}[g,\Phi] ,$$

with $S_{\text{matter}} = \int_M \mathcal{L}(g, \Phi)\, \varepsilon_g$. By defⁿ,

$$\delta_g S_{\text{matter}} = \frac{1}{2} \int T^{ab}\, \delta g_{ab}\, \varepsilon_g,$$

where T_{ab} is the stress tensor, then EL becomes

$$R_{ab} - \frac{1}{2} g_{ab} R = 16\pi G\, T_{ab}.$$

<u>Example</u> Suppose Φ is a scaler field, and

$$\mathcal{L}(g, \Phi) = -\frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi + V(\Phi)$$

Then

$$\delta_g S_{\text{matter}} = \int_M \left(-\frac{1}{2}\, \delta g^{ab}\, \partial_a \Phi\, \partial_b \Phi + \frac{\mathcal{L}}{2} g^{ab} \delta g_{ab} \right) \varepsilon_g$$

$$= \int_M \left(\frac{1}{2} \partial^a \Phi\, \partial^b \Phi - g^{ab} \frac{\mathcal{L}}{2} \right) \delta g_{ab}\, \varepsilon_g$$

So

$$T_{ab} = \frac{1}{2} \partial_a \Phi\, \partial_b \Phi - \frac{1}{2} g_{ab} \mathcal{L}.$$

Note also that if we chooce to vary the metric by some diffeo, so $\delta g_{ab} = 2 \nabla_{(a} \xi_{b)}$, then we have

$$\delta S = \frac{1}{16\pi G} \int_M -2 G^{ab} \nabla_{(a} \xi_{b)}\, \varepsilon_g + \int_M T^{ab} \nabla_{(a} \xi_{b)}\, \varepsilon_g$$

$$= \frac{1}{8\pi G} \int_M \big(\underbrace{\cancel{\nabla_a G^{ab}}}_{\to\, 0 \text{ Bianchi id.}} \big) \xi_b\, \varepsilon_g - \int_M (\nabla_a T^{ab}) \xi_b\, \varepsilon_g$$

So the full action is diffeo inv. (even off support of EE) provided $\nabla^a T_{ab} = 0$.

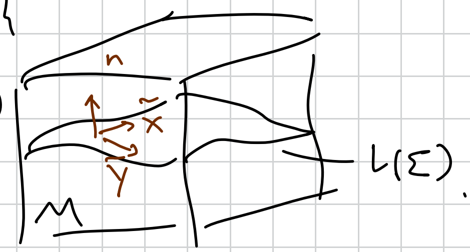
<u>The Cauchy Problem for EE</u>

We expect EE can be solved given initial data on some spacelike hypersurface Σ. What data should we give?

Suppose $\iota : \Sigma \to M$ is an embedding s.t. $\iota(\Sigma)$ is spacelike, then $h = \iota^*(g)$ is Riemannian. Let n be the unit normal to $\iota(\Sigma)$. For any vec. field X, Y on Σ, with $\tilde{X} = \iota_* X$, $\tilde{Y} = \iota_* Y$, define the extrinsic curvature of $\iota(\Sigma)$ as

$$k(X,Y) = \iota^*\left(g\left(n, \nabla_{\tilde{X}} \tilde{Y}\right)\right)$$

Pick local coords $\{y^i\}$ on Σ and $\{x^\mu\}$ on M s.t. $\iota : (y^1, y^2, \dots) \mapsto (0, y^1, y^2, \dots)$



If $X = X^i \frac{\partial}{\partial y^i}$, then $\tilde{X} = X^i(y) \frac{\partial}{\partial x^i}$, and $n_\mu \propto \delta^0{}_\mu$. Then

$$\begin{aligned} k(X,Y) &= g_{\mu\nu} n^\mu (\tilde{X}^\sigma \nabla_\sigma \tilde{Y}^\nu) \\ &\propto \delta^0{}_\nu \left(\tilde{X}^\sigma \partial_\sigma \tilde{Y}^\nu + \tilde{X}^\sigma \Gamma^\nu_{\sigma\tau} \tilde{Y}^\tau \right) \\ &= \tilde{X}^\sigma \tilde{Y}^\tau \Gamma^0_{\sigma\tau} = k(Y,X). \end{aligned}$$

One can show that if (M,g) solves vacuum EE $G_{ab} = 0$, then the constraint eqns

$$\left.\begin{aligned} {}^{(h)}\nabla_i k^i{}_j - {}^{(h)}\nabla_j k^i{}_i &= 0 \\ {}^{(h)}R - k^i{}_j k^j{}_i + (k^i{}_i)^2 &= 0 \end{aligned}\right\} (+)$$

hold on Σ, where ${}^{(h)}\nabla$ is the Levi-Civita connection of h, ${}^{(h)}R$ Ricci scaler of h.

Conversely, if given (Σ, h_{ij}, k_{ij}) with Σ an $(m-1)$ Riem. mfld, metric h_{ij} and sym. $(0,2)$-tensor k_{ij}, s.t. $(+)$ hold, then $\exists$ sol^n (M,g) of vac. EE and an embedding

$\iota : \Sigma \to M$ s.t. $h = \iota^* g$ is an induced metric, k the extrinsic curvature.

Finally, it turns out that

$$\Pi_{ij} = \frac{\sqrt{h}}{16\pi G} \left(k_{ij} - h_{ij} k^m{}_m \right)$$

is the momentum canonically conjugate to h_{ij} in EH action.
The second eqn in (+) becomes, using $\Pi_{ij} \to -i \frac{\delta}{\delta h_{ij}}$,

$$\frac{(16\pi G)^2}{\det h} \left(\hat{\Pi}_{ij} \hat{\pi}^{ij} - (\hat{\Pi}^k{}_k)^2 - R \right) \Psi[h] = 0 ,$$

which is the Wheeler-de Witt eqn.