General Relativity

General relativity (GR) is a geometrical theory of gravity, extending Newtonian gravity (NG) allowing it to be reconciled with special relativity (SR).

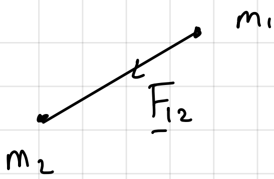
- GR revolutionises one's understanding of space, time and dynamics, generalises the concept of spacetime (Minkowski space) in SR. to some curved spacetime manifold.
- There is a beautiful fit between physics and maths through concepts, like curvature, geodesics, metric, Our treatment of geometry will be as elementary as possible but self-contained.
- Celebrated predictions and applications of GR include existence of blackholes, gravitational waves, etc – it is central to research in e.g. quantum gravity and string theory, through to cosmology and blackholes.

1. Introduction

1.1 Gravity and Relativity

Newton's law of grav. gives the force on a mass m_1 at $\underline{x}_1$ due to a mass m_2 at $\underline{x}_2$;

$$\underline{F}_{12} = -Gm_1m_2 \frac{\underline{x}_1 - \underline{x}_2}{|\underline{x}_1 - \underline{x}_2|^3} . \quad (*)$$



where $G \approx 6.67 \times 10^{-11}\ m^3\,kg^{-1}\,s^{-2}$ is Newton's gravitational const.

<u>Note</u>: We could distinguish between active and passive grav mass by writing $m_1^{(P)} m_2^{(A)}$ in $(*)$, but by N3,

$$\underline{F}_{12} = -\underline{F}_{21}$$

$$\Rightarrow \frac{m_1^{(P)}}{m_1^{(A)}} = \frac{m_2^{(P)}}{m_2^{(A)}} = 1 \quad \text{(WLOG)}$$

i.e. Single concept of grav. mass.

Now consider the force on a mass m at $\underline{x}$ due to matter distribution with density $\rho(\underline{x})$. This can be written

$$\underline{F} = m\underline{g}(\underline{x}),$$

where the gravitational field

$$\underline{g}(\underline{x}) = -\nabla \underline{\Phi}.$$

and the grav. potential $\underline{\Phi}$ satisfies

$$\nabla^2 \underline{\Phi} = 4\pi G \rho$$

N2 gives

$$\underline{F} = m^{(I)} \ddot{\underline{x}} = m\underline{g}(\underline{x}).$$

where $m^{(I)}$ is inertial mass.

Remarkable fact: $m^{(I)}/m = 1$ (WLOG) tested experimentally to order $\mathcal{O}(10^{-12})$. No explanation for this in Newtonian theory.

NG and ND are successful — e.g. describe planetary orbits to high accuracy.

But know ND applies only for $v \ll c$ and beyond this, we need SR.

Recall in both ND and SR, there is a preferred class of inertial frames (IF) within which objects move in straight lines with const. velocities in absence of forces. In ND, assume existence of absolute time.

In SR we assume speed of light const., then IFs related by Lorentz transformations and we have no notion of events being simultaneous in general, so we can't interpret (*).

When do we expect modifications to be important? Consider circular orbit, radius r about mass M:

$$\Phi = -\frac{GM}{r}.$$

and N2 $\Rightarrow \frac{v^2}{r} = \frac{GM}{r^2} \Rightarrow \frac{v^2}{c^2} = \frac{|\Phi|}{c^2}$

$v/c \ll 1$ suggests NG adequate $|\Phi|/c^2 \ll 1$.

In solar sys., $|\Phi|/c^2 < 10^{-5}$.

1.2 Gravitational fields and accelerating frames

Einstein's happiest thought: the grav. field has only a relative existence, because an observer falling freely under gravity detects no field (locally).

Consider a choice of coords $\underline{x}, t$. Call this frame S. Motion of a body in S in grav. field $\underline{g}(\underline{x}, t)$ is given by

$$\ddot{\underline{x}} = \underline{g}(\underline{x}, t).$$

Define new coords $\tilde{\underline{x}}, t$ for frame $\tilde{S}$ with

$$\tilde{\underline{x}} = \underline{x} - \underline{b}(t).$$

for some $\underline{b}(t)$.

$$\Rightarrow \ddot{\tilde{\underline{x}}} = \ddot{\underline{x}} - \ddot{\underline{b}}(t), \quad \text{or} \quad \ddot{\tilde{\underline{x}}} = \tilde{\underline{g}} = \underline{g} - \ddot{\underline{b}}$$

is accel. in $\tilde{S}$. Note $\underline{x} = \underline{b}(t)$ in S is position at origin $\tilde{\underline{x}} = 0$ in $\tilde{S}$.

Consider $\underline{g}$ uniform, indep. of $\underline{x}$, then

- If $\underline{g} = 0$, we can produce $\tilde{\underline{g}} = -\ddot{\underline{b}} \neq 0$ as desired.
- If $\underline{g} \neq 0$, we can choose $\underline{b}$ to make $\tilde{\underline{g}} = 0$ ($\ddot{\underline{b}} = \underline{g}$).

Given interpretation above, refer to $\tilde{S}$ with $\tilde{\underline{g}} = 0$ as a freely falling frame (FFF).

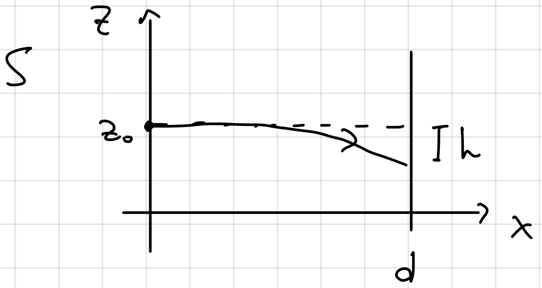
Simple version of Einstein Equivalence Principle:

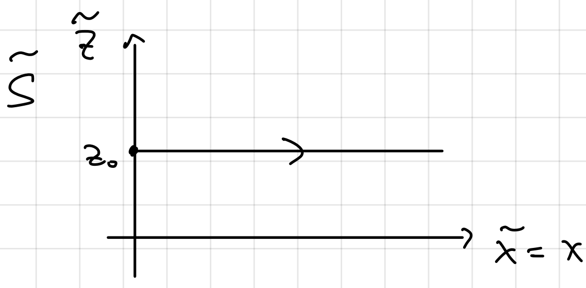
In an isolated lab, there is no expt. that can distinguish between uniform acceleration of lab and a uniform grav. field. Moreover, results of expt. in a FFF are same as in an IF:

(a) <u>Gravitational bending of light</u>

S (lab) e.g. on surface of Earth

$$\underline{g} = -g\,\hat{\underline{z}}$$





choose $\underline{b} = -\frac{1}{2} g t^2 \hat{\underline{z}}$

$\Rightarrow \tilde{\underline{g}} = 0$

$z = z_0 - \frac{1}{2} g t^2 \quad \Leftarrow$

trajectory of light ray

$z = z_0$ const.

In horizontal distance d, light ray drops by distance

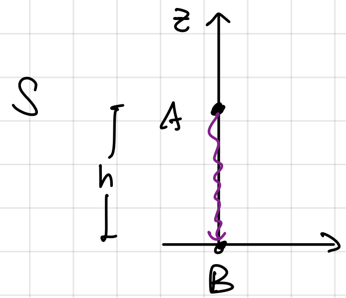
$$h = \frac{1}{2} g t^2 = \frac{g d^2}{2c}.$$

to leading order.

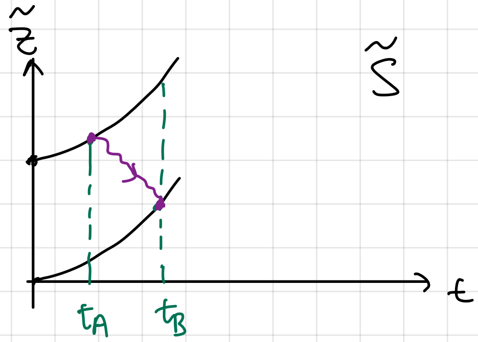
For $g \approx 10\,ms^{-2}$, $d = 1km \Rightarrow h \approx 5\times10^{-11}m$, but experimental verification of light from GR, e.g. total eclipse in 1911.

(b) Gravitational red/blue shift

Consider Alice (A) and Bob (b) separated by height h on surface as shown



$$\underline{g} = -g\hat{\underline{z}} = -\nabla\Phi$$
$$\Phi = gz$$



A sends light signal at time t_A, B receives at t_B.

$z = h$ for A $\qquad\qquad \tilde{z} = h + \frac{1}{2}gt^2$ for A

$z = 0$ for B $\qquad\qquad \tilde{z} = \frac{1}{2}gt^2$ for B.

In $\tilde{S}$, $$h + \frac{1}{2}g\,t_A^2 - \frac{1}{2}g\,t_B^2 = c(t_B - t_A)$$

Suppose signals send and receive repeatedly at small intervals Δt_A, Δt_B, replace $t_A \mapsto t_A + \Delta t_A$, $t_B \mapsto t_B + \Delta t_B$, expand to 1st order and compare, get

$$g t_A \Delta t_A - g t_B \Delta t_B = c(\Delta t_B - \Delta t_A).$$

$$\Rightarrow \frac{\Delta t_B}{\Delta t_A} = \frac{1 + g t_A/c}{1 + g t_B/c} \qquad \text{assuming } g t_A, g t_B \ll c.$$

$$\approx 1 - \frac{g}{c}(t_B - t_A)$$
$$= 1 - \frac{gh}{c^2} \qquad \text{to leading order.}$$

Hence, $$\frac{\Delta t_B}{\Delta t_A} = 1 - \frac{1}{c^2}(\Phi_A - \Phi_B)$$

Note for $|\Phi/c^2| \ll 1$, weak gravity.

Regarding intervals as ticks by clock, clocks run at different rate. depending on gravitational potential.

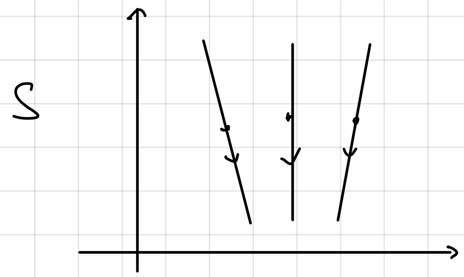
For frequencies $\nu_A = 1/t_A$. $\nu_B = 1/t_B$.

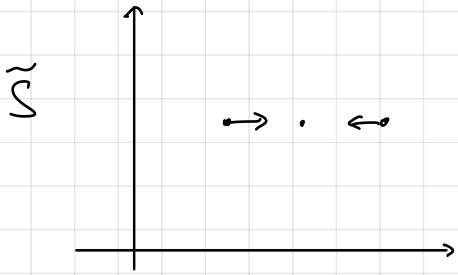
$$\frac{\nu_B}{\nu_A} = 1 + \frac{1}{c^2}(\Phi_A - \Phi_B)$$

- $\Phi_A > \Phi_B \Rightarrow \nu_B > \nu_A$ grav. blue shift.
- $\Phi_A < \Phi_B \Rightarrow \nu_B < \nu_A$ grav. red shift.

Experimental test: as above $h \approx 22.5m \Rightarrow \frac{gh}{c^2} \approx 10^{-15}$, tested to about 1%.

<u>Non-uniform fields</u>





In non-uniform field, best we can do is make $\tilde{g}=0$ at some point only locally a FFF.

Shall see effects of grav field.

Compare neighbouring trajectories

$$\ddot{x}_i = g_i(\underline{x}, t)$$

use coords x_i and set $\partial_i = \partial/\partial x_i$

$$(x_i + h_i)^{\cdot\cdot} = g_i(\underline{x} + \underline{h}, t)$$

Taylor expand:

$$g_i(\underline{x}+\underline{h},t) = g_i(\underline{x},t) + h_j \partial_j g_i(\underline{x},t) + O(|\underline{h}^2|)$$

Combining,

$$\ddot{h}_i + E_{ij} h_j = 0,$$

where

$$E_{ij} = -\partial_j g_i = \partial_i \partial_j \Phi \quad (\underline{g} = -\nabla \Phi)$$
$$= E_{ji}$$

is the tidal tensor.

1.3 Special Relativity Revisited

Consider Minkowski spacetime (IF) with cords

$$x^\mu, \quad \mu = 0,1,2,3, \quad \text{or } (x^0, x^1, x^2, x^3) = (ct, \underline{x}).$$

Inner product or (flat) <u>Minkowski metric</u>

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$$

with <u>line element</u>

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$
$$= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$
$$= -c^2 dt^2 + d\underline{x} \cdot d\underline{x}.$$

Metric gives <u>invariant separations</u> between nearby points or <u>events</u>

$$(\delta s)^2 = -c^2(\delta t)^2 + \delta\underline{x} \cdot \delta\underline{x}$$

generalising Pythagoras.

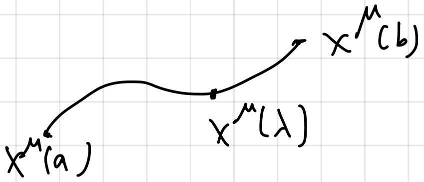
- $(\delta s)^2 > 0$, $|\delta\underline{x}|^2 > c^2(\delta t)^2$: separation is <u>spacelike</u> and δs is <u>proper distance</u>
- $(\delta s)^2 = 0$, $|\delta\underline{x}|^2 = c^2(\delta t)^2$: separation is <u>lightlike</u> / <u>null</u>.
- $(\delta s)^2 < 0$, $|\delta\underline{x}|^2 < c^2(\delta t)^2$: separation is <u>timelike</u>, and $(\delta s)^2 = c^2(\delta\tau)^2$, where $\delta\tau$ is <u>proper time</u>.

Note: Positions of indices important, i.e. x^μ, x^ν on coords, and $\eta_{\mu\nu}$. Summation convention applies when one index is up and one index down. Inverse metric $\eta^{\mu\nu}$ and

$$\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu{}_\nu .$$

(and in this case $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$).

Consider parameterised curve $x^\mu(\lambda)$, $a \leq \lambda \leq b$, with <u>tangent vector</u> $T^\mu = \dfrac{dx^\mu}{d\lambda}$.



To 1st order,

$$\delta x^\mu = T^\mu \delta\lambda$$

Small displacement. Curve is said to be

$$\left.\begin{array}{c}\text{spacelike}\\ \text{lightlike}\\ \text{timelike}\end{array}\right\} \text{ if } \eta_{\mu\nu}T^\mu T^\nu \left\{\begin{array}{l} >0\\ =0\\ <0\end{array}\right.$$

(compare with $(\delta s)^2$ above)

For a spacelike curve, consider

$$S = \int_a^b \mathcal{L}\, d\lambda, \quad \mathcal{L} = (\eta_{\mu\nu}T^\mu T^\nu)^{1/2} = \left(\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2}.$$

total proper distance along curve, and

$$\mathcal{L} = \frac{ds}{d\lambda}.$$

where $s(\lambda)$ is proper distance (arc length) up to some endpoint with $s(a) = 0$.

For a timelike curve, consider

$$cT = \int_a^b \mathcal{L}\, d\lambda , \quad \mathcal{L} = (-\eta_{\mu\nu} T^\mu T^\nu)^{1/2} = \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right)^{1/2}$$
$$= c \frac{d\tau}{d\lambda} ,$$

where $\tau(\lambda)$ is proper time along curve, with $\tau(a) = 0$.

Then T is total proper time along curve.

Under a change of parameterisation $\lambda \to \hat{\lambda}$ with $d\hat{\lambda}/d\lambda > 0$,

new tangent vector

$$\hat{T}^\mu = \frac{dx^\mu}{d\hat{\lambda}} = \frac{dx^\mu}{d\hat{\lambda}} \frac{d\hat{\lambda}}{d\lambda} = \left(\frac{d\hat{\lambda}}{d\lambda}\right) T^\mu$$

change of scale but not direction. In addition,

$$d\hat{\lambda} = \frac{d\hat{\lambda}}{d\lambda}\, d\lambda$$

hence

$$\int_a^b \mathcal{L}(T^\mu)\, d\lambda = \int_{\hat{a}}^{\hat{b}} \mathcal{L}(\hat{T}^\mu)\, d\hat{\lambda} .$$

Consider spacelike curves with specified end points (as above).

Minimum distance S is obtained for a geodesic s.t.

$$\delta S = 0$$

to first order in δx^μ, with $\delta x^\mu(a) = \delta x^\mu(b) = 0$.

Given by Euler-Lagrange eqn

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) = 0 ,$$

with $\mathcal{L} = \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$.

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{1}{2\mathcal{L}}\left(2\eta_{\alpha\nu} \dot{x}^\nu\right)$$

Multiply by $\eta^{\mu\alpha}$ to get

$$\frac{d}{d\lambda}\left(\frac{1}{\mathcal{L}}\dot{x}^\mu\right) = 0.$$

Changing parameter to $s(\lambda)$ simplies eqn to

$$\frac{d^2x^\mu}{ds^2} = 0 \Rightarrow x^\mu = U^\mu s + x^\mu(0).$$

with U^μ const. (tangent vector), i.e. straight line.

From defn of s,

$$\eta_{\mu\nu}U^\mu U^\nu = 1.$$

<u>(a) Motion of Massive particles in Minkowski spacetime</u>

(i) Trajectory of massive particle is a timelike curve, called the <u>world line</u>

(ii) Proper time along any trajectory is physical line measured by a clock moving with particle. (clock postulate)

(iii) A free particle (no focus) In an IF moves in a straight line with constant speed

Recast (iii) using variational principle: extremise

$$S = cT = \int_a^b \mathcal{L}\, d\lambda \quad \text{with} \quad \mathcal{L} = (-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}$$

for trajectories with given endpoints - this gives trajectories

EL:
$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha}\right) = \frac{d}{d\lambda}\left(-\frac{1}{\mathcal{L}}\dot{x}_\alpha\right) = 0$$

Change param. to $\tau(\lambda)$. Since $\mathcal{L} = c\frac{d\tau}{d\lambda}$, EL becomes

$$\frac{d^2}{d\tau^2}(x^\mu) = 0 \Rightarrow x^\mu(\tau) = U^\mu\tau + x^\mu(0).$$

with $\frac{dx^\mu}{d\tau} = U^\mu$ is 4-velocity. $\eta_{\mu\nu}U^\mu U^\nu = -c^2$.

<u>Note</u> : For $x^\mu = (ct, x)$, if we use t as a param, we get tangent vector

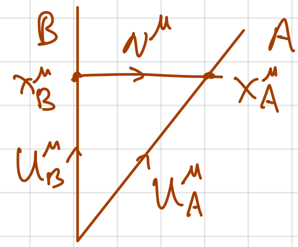
$$V^\mu = \frac{dx^\mu}{dt} = (c, \underline{v}) \quad , \quad \underline{v} = \frac{d\underline{x}}{dt}$$
$$= \frac{d\tau}{dt} \, U^\mu$$

$$\Rightarrow \quad U^\mu = \gamma(\underline{v}) \, V^\mu \quad , \quad \gamma(\underline{v}) = (1 - \underline{v}^2/c^2)^{-1/2} .$$

<u>Example</u> (Time dilation) Consider worldlines for A and B.

$$X_A^\mu = U_A^\mu \tau_A \; , \quad X_B^\mu = U_B^\mu \tau_B$$

coinciding at $\tau_A = \tau_B = 0$.



Writing $(V^\mu)^2 = \eta_{\mu\nu} V^\mu V^\nu$, know

$$(U_A^\mu)^2 = (U_B^\mu)^2 = -c^2 .$$

Compare events shown, separated by

$$y^\mu = N^\mu v \tau_B .$$

where v is relative speed and $\eta_{\mu\nu} N^\mu N^\nu = 1$, $\eta_{\mu\nu} U_B^\mu N^\nu = 0$

$$x_B^\mu + y^\mu = x_A^\mu \quad \Rightarrow \quad (U_B^\mu \tau_B + N^\mu v \tau_B)^2 = (U_A^\mu \tau_A)^2$$
$$\Rightarrow \quad -c^2 \tau_B^2 + v^2 \tau_B^2 = -c^2 \tau_A^2 .$$
$$\Rightarrow \quad \tau_B = \gamma(\underline{v}) \, \tau_A > \tau_A \qquad \text{time dilation.}$$

Note that proper time is <u>maximised</u> for Bob

<u>(b) Extension : describing massless and massive particles</u>

Cannot try approach in (a) for massless particles / light rays.

e.g. no motion of proper time along null trajectory.

Still have version of (iii):

free massless particles move in an IF in straight line with speed c.

There is a modified principle that applies to massless and massive particles: world line $x^\mu(\lambda)$ with fixed end pts $x^\mu(a)$, $x^\mu(b)$ extremises

$$I = \int_a^b L \, d\lambda \quad , \quad \text{where } L = \cdot\, \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \; (= \mathcal{L}^2 \text{, with } \mathcal{L} \text{ as before for massive particles})$$

E-L gives

$$\frac{d}{d\lambda}\left(\eta_{\alpha\nu}\dot{x}^\nu\right) = 0 \Rightarrow \frac{d^2 x^\mu}{d\lambda^2} = 0$$

$$\Rightarrow x^\mu = U^\mu \lambda + c^\mu .$$

with U^μ, C^μ const. and $\eta_{\mu\nu} U^\mu U^\nu = 0$ in massless case.

Automatically get convenient <u>affine</u> parameterisation with

$$\lambda = A\tau + B$$

in massive case.

(c) Lorentz Invariance

Consider coords related by

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu \quad (\Lambda \text{ const. matrix})$$

(can choose $\tilde{x}^\mu = x^\nu = 0$ at a given point by translating in spacetime)

To ensure

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu$$

invariant, we need

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \,.$$

or, as matrices,

$$\Lambda^T \eta \Lambda = \eta$$

The set of all such transformations form the <u>Lorentz group</u> relates IF in SR.

<u>Example</u>

$$\Lambda = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 \\ -\sinh\theta & \cosh\theta & \\ 0 & & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

boost in x^1 dir, with speed $v = c \tanh\theta$

<u>Example</u> $\Lambda = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right)$, R 3x3 rotation acting on x^1, x^2, x^3.

2. Metric and Geodesics on Manifolds

2.1 Manifolds and metrics

(a) Manifolds

A <u>manifold</u> is described <u>locally</u> by a set of coords $\{x^\mu\}$ and no. of coords is the dimension.

<u>Example</u> $\mathbb{R}^3$: dim 3,

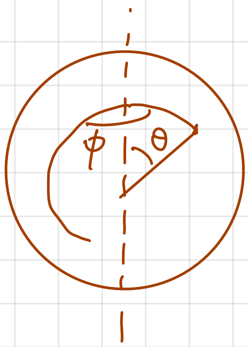
Minkowski spacetime: dim 4

In general, a manifold is <u>not</u> a vector space, so coords x^μ not components of a position vector.

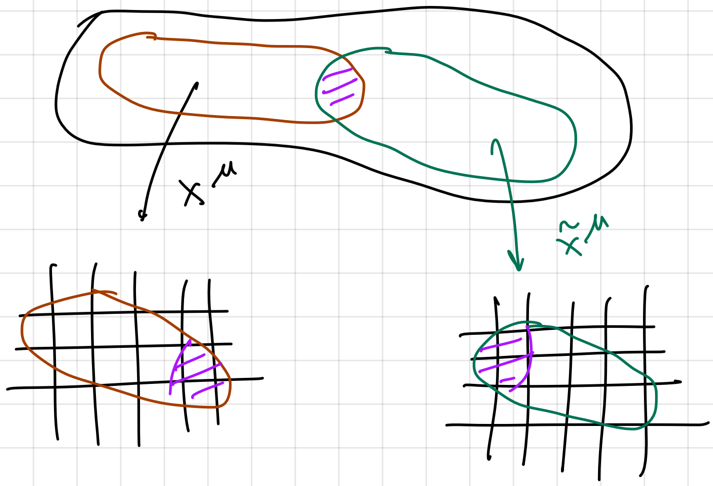
<u>Example</u> 2D sphere with polar coords $x^1 = \theta$, $x^2 = \varphi$.

$0 < \theta < \pi$, $0 < \varphi < 2\pi$. This is well-behaved away from "median" $\varphi = 0$ (or 2π)

For complete description, need additional set of coords $\tilde{x}^1, \tilde{x}^2$. Can choose to be polar angles corresponding to some different median



In general, a smooth manifold $\mathcal{M}$ (of dim n) is defined by a set of coordinate patches or charts as shown



Indices μ, ν, α, β take in values. In overlap region, $\{x^\mu\}$ and $\{\tilde{x}^\alpha\}$ are smooth (infinitely diff'ble) f^n of one another.

Small changes related by chain rule

$$\delta \tilde{x}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \delta x^\mu$$

$$\delta x^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \delta \tilde{x}^\alpha .$$

Matrix of partial derivatives is invertible

$$\det\left(\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\right) \neq 0,$$

and

$$\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\alpha}{\partial x^\nu} = \delta^\mu{}_\nu \quad \left(= \frac{\partial x^\mu}{\partial x^\nu} \right),$$

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tilde{x}^\beta} = \delta^\alpha{}_\beta$$

Example $\mathbb{R}^2$ with $\{x^1, x^2\}$ Cartesian coords, $\tilde{x}^1 = r$, $\tilde{x}^2 = \varphi$ usual polar.

$$x^1 = r\cos\varphi \, , \; x^2 = r\sin\varphi .$$

$$
M = \left(\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\right) = \begin{array}{c|cc} \mu \backslash \alpha & 1 & 2 \\ \hline 1 & & \\ 2 & & \end{array} \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}
$$

and $\det M = r \neq 0$ on overlap $(r > 0)$ and restriction on φ.

(b) Metrics

A metric on a manifold $\mathcal{M}$ specified by a symmetric tensor, with components

$$g_{\mu\nu}(x) = g_{\nu\mu}(x)$$

in each coordinate set $\{x^\mu\}$.

This defines a line element

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \qquad (g_{\mu\nu}\ n \times n \text{ matrix})$$

which gives invariant notion of "separation" on $\mathcal{M}$.

So in coords $\{\tilde{x}^\alpha\}$ we have

$$ds^2 = \tilde{g}_{\alpha\beta}(\tilde{x})\, d\tilde{x}^\alpha d\tilde{x}^\beta$$

ds^2 invariant, so

$$\tilde{g}_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} \qquad \text{(chain rule).}$$

As matrices

$$(\tilde{g}_{\alpha\beta}) = M^T (g_{\mu\nu}) M$$

with $M = \dfrac{\partial x^\nu \leftarrow \text{rows}}{\partial \tilde{x}^\beta \leftarrow \text{cols}}$

Example For $\mathbb{R}^2$, $(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (standard inner prod.)

$$\Rightarrow (\tilde{g}_{\alpha\beta}) = M^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} M = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\Rightarrow ds^2 = (dx^1)^2 + (dx^2)^2 = dr^2 + r^2 d\varphi^2$$

In general, $(g_{\mu\nu})$ is $n \times n$ symmetric, non-sing matrix, and at a given point, can choose $\tilde{x}^\alpha$, and hence M to diagonalise metric with resulting non-zero entries ± 1.

Here consider (mainly)

- $(+1, \dots, +1)$ signature is Euclidean / pos-def – study this mainly for $n = 2, 3$
- $(-1, +1, \dots, +1)$ signature is Lorentzian. Assume this unless otherwise specified, mainly $n = 4$.

In these cases, $(\mathcal{M}, g)$ called Riemannian or pseudo-Riemannian manifold.

At each point, we have an inverse metric $g^{\mu\nu}$ satisfying

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$$

Examples

- n-dim Euclidean space with Cartesian coords $\{x^i\}$. (Sometimes use i, j etc in pos-def case)

$$g_{ij} = \delta_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad \Rightarrow (g^{ij}) = (g_{ij})$$

- n-dim Minkowski space with coords $\{x^\mu\}$

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad \Rightarrow (g^{\mu\nu}) = (g_{\mu\nu})$$

• 2D sphere with $x^1 = \theta$, $x^2 = \varphi$ ($0 < \varphi < 2\pi$, $0 < \theta < \pi$).

$$ds^2 = g_{ij}\, dx^i dx^j = d\theta^2 + \sin^2\theta\, d\varphi^2.$$

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad \text{and} \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{pmatrix}$$

Note: We can think of this metric as inherited from 3D, but our treatment of geometry is intrinsic to 2D.

2.2 Curves and Geodesics

(a) Definitions

A parameterised curve $x^\mu(\lambda)$ on a manifold $\mathcal{M}$ has a tangent vector

$$T^\mu = \frac{dx^\mu}{d\lambda} = \dot{x}^\mu.$$

using coords $\{x^\mu\}$.

Using coords $\{\tilde{x}^\alpha\}$, curve $\tilde{x}^\alpha(\lambda)$ and tangent vec. is

$$\tilde{T}^\alpha = \frac{d\tilde{x}^\alpha}{d\lambda} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} T^\mu .$$

Note

$$g_{\mu\nu} T^\mu T^\nu = \tilde{g}_{\alpha\beta} \tilde{T}^\alpha \tilde{T}^\beta$$

just as

$$g_{\mu\nu}\, \delta x^\mu \delta x^\nu = \tilde{g}_{\alpha\beta}\, \delta\tilde{x}^\alpha \delta\tilde{x}^\beta .$$

for any small changes, and have

$$\delta x^\mu = T^\mu \delta\lambda, \quad \delta\tilde{x}^\alpha = \tilde{T}^\alpha \delta\lambda$$

for small change along curve.

For $g_{\mu\nu} T^\mu T^\nu \begin{Bmatrix} > 0 \\ = 0 \\ < 0 \end{Bmatrix}$, curve is $\begin{cases} \text{spacelike} \\ \text{lightlike / null} \\ \text{timelike} \end{cases}$.

We define an (affinely parameterised) geodesic to be a curve $x^\mu(\lambda)$ with $a \leq \lambda \leq b$ which, for fixed end points $x^\mu(a)$ and $x^\mu(b)$, extremises

$$I = \int_a^b L \, d\lambda,$$

where

$$L = -g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -g_{\mu\nu} T^\mu T^\nu .$$

It follows (see below) that $dL/d\lambda = 0$ along the curve.

(b) Interlude: Euler-Lagrange Equations

Seek to extremise

$$I = \int_a^b L(q^r, \dot{q}^r, \lambda) \, d\lambda .$$

where $q^r(\lambda)$, $r = 1, \ldots, n$, defined for $a \leq \lambda \leq b$, with $q^r(a)$, $q^r(b)$ fixed and $\dot{q}^r = dq^r/d\lambda$.

Working to 1st order in δq^r with $\delta q^r(a) = \delta q(b) = 0$,

$$\delta L = \frac{\partial L}{\partial q^r} \delta q^r + \frac{\partial L}{\partial \dot{q}^r} \delta \dot{q}^r$$

$$\delta I = \int_a^b \left[\frac{\partial L}{\partial q^r} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{q}^r}\right) \right] \delta q^r \, d\lambda = 0$$

for any δq^r iff

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{q}^r}\right) - \frac{\partial L}{\partial q^r} = 0 .$$

- L is indpt of q^r (fixed r), then E-L

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{q}^r}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}^r} = \text{const.}$$

- L has no explicit dependence on λ, then

$$\dot{q}^r \frac{\partial L}{\partial \dot{q}^r} - L = \text{const} .$$

These are examples of first integrals.

(c) Geodesics equations

Apply results in (b) to

$$I = \int_a^b L(x^\mu, \dot{x}^\mu)\, d\lambda \quad , \quad L = -g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = -2 g_{\mu\beta}\, \dot{x}^\beta$$

chain rule for $g_{\mu\beta}(x^\gamma(\lambda))$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = -\left(2 g_{\mu\beta}\, \ddot{x}^\beta + 2\, \partial_\gamma g_{\mu\beta}\, \dot{x}^\gamma \dot{x}^\beta\right)$$

$$= -\left(2 g_{\mu\beta}\, \ddot{x}^\beta + (\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\mu\gamma})\, \dot{x}^\gamma \dot{x}^\beta\right)$$

$$\frac{\partial L}{\partial x^\mu} = -\partial_\mu g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta .$$

E-L $\Rightarrow$

$$2 g_{\mu\beta}\, \ddot{x}^\beta + (\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\beta\mu} - \partial_\mu g_{\beta\gamma})\, \dot{x}^\beta \dot{x}^\gamma = 0.$$

$\times \frac{1}{2} g^{\alpha\mu}$

$$\Rightarrow \quad \boxed{\ddot{x}^\alpha + \Gamma_\beta{}^\alpha{}_\gamma\, \dot{x}^\beta \dot{x}^\gamma = 0}$$

where $\Gamma_\beta{}^\alpha{}_\gamma$ is the Levi-Civita connection or metric connection.

Also written as $\{\beta\, {}^\alpha\, \gamma\}$ Christoffel symbol.

Note that there is a first integral (no explicit λ dependence)

$$\dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu} - L = L$$

const. along a geodesic. Hence $\frac{dL}{d\lambda} = 0$ on geodesics.

Example $\quad (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad x^1 = r, \quad x^2 = \varphi$

$$(g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

$= + g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}$

E-L from $-L = \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\varphi}{d\lambda}\right)^2$ are

$$\ddot{r} - r\dot{\varphi}^2 = 0 \quad , \quad \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0$$

$$\Rightarrow \Gamma_1{}^2{}_2 = \Gamma_2{}^2{}_1 = 1/r, \quad \Gamma_2{}^1{}_2 = -r$$

Note can choose $\lambda = s$ arc length

$$\Rightarrow \dot r^2 + r^2 \dot\varphi^2 = 1$$

(and 1st integral because L indep. of φ)

$$r^2 \dot\varphi = \ell \quad \text{const.}$$

$$\Rightarrow \dot r^2 + \ell^2/r^2 = 1$$

$$\Rightarrow \frac{r}{(r^2-\ell^2)^{1/2}} \frac{dr}{ds} = \pm 1$$

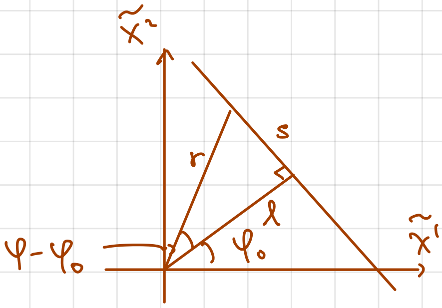
$$\Rightarrow (r^2-\ell^2)^{1/2} = \pm s + \cancel{\text{const}}\ 0$$

$\because$ point from which we measure s.

$$\Rightarrow r = (\ell^2 + s^2)^{1/2}$$

$$\dot\varphi = \ell/r^2 = \frac{\ell}{\ell^2+s^2} \Rightarrow \varphi - \varphi_0 = \tan^{-1}\frac{s}{\ell}.$$

and note $ds^2 = (d\tilde x^1)^2 + (d\tilde x^2)^2$.



(d) Summary and some key principles in GR

Assume spacetime is 4-dimensional (pseudo-) Riemannian manifold with Lorentz - signature metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Invariant separation $(\delta s)^2$ between x^μ and $x^\mu + \delta x^\mu$ can be timelike (<0), null / lightlike $(=0)$ or spacelike (>0)

Consider curve $x^\mu(\lambda)$ with $a \le \lambda \le b$ and tangent vec.

$$T^\mu = \dot x^\mu = \frac{dx^\mu}{d\lambda}.$$

Let $L(x^\alpha, \dot x^\alpha) = -g_{\mu\nu}(x^\alpha)\, \dot x^\mu \dot x^\nu$.

For $L<0$, then

$$\frac{ds}{d\lambda} = (-L)^{1/2}$$

defines proper distance along curve, and

$$S = \int_a^b (-L)^{1/2} d\lambda = s(b) - s(a)$$

is the proper distance along endpoints

For $L>0$, then

$$c\frac{d\tau}{d\lambda} = L^{1/2}$$

defines proper time $\tau(\lambda)$ along curve and

$$cT = \int_a^b L^{1/2}\, d\lambda = c(\tau(b) - \tau(a)).$$

Geodesic with affine parameterisation defined to extremise

$$I = \int_a^b L\, d\lambda = \int_a^b (-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)\, d\lambda$$

and this results in $\frac{dL}{d\lambda} = 0$ for sol^{n}s.

We can choose either $\lambda = \tau$

$$\Rightarrow \quad g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -c^2$$

or $\lambda = s$

$$\Rightarrow \quad g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = +1$$

A geodesic with this def^n also extremises S and T.

Key principle:

- The trajectory or worldline of a massive particle is a time-like curve, and proper time along the worldline is physical time measured by clock moving with particle [clock postulate]
- The worldline of a free massive test particle is a timelike geodesic: it extremises proper time between end points. (test particle means mass small enough that we can neglect its effect on metric)

- The trajectory of light ray (massless particle) is a lightlike or null geodesic.

2.3 Static Spacetimes and Newtonian Limit

Consider a spacetime with coords $x^0 = ct$, x^i, s.t.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$
$$= g_{00} c^2 dt^2 + g_{ij} dx^i dx^j \qquad (g_{0i} = g_{i0} = 0)$$

and

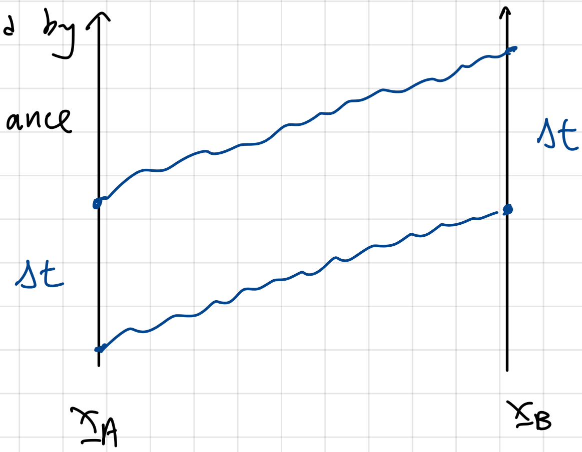
$$\partial_t g_{\mu\nu} = 0 \qquad (\text{metric stationary})$$

Write $g_{00}(\underline{x})$, $g_{ij}(\underline{x})$, for $\underline{x} = (x^1, x^2, x^3)$.

(a) Gravitational Redshift

Observers A (Alice) and B (Bob) at fixed positions $\underline{x}_A, \underline{x}_B$ and A sends light signals, received by B.

Sending and receiving signals separated by Δt for A and B from translation invariance of metric (relating null geodesics)



Change in proper time given by

$$(\Delta \tau_A)^2 = -g_{00}(\underline{x}_A)\,(\Delta t)^2$$
$$(\Delta \tau_B)^2 = -g_{00}(\underline{x}_B)\,(\Delta t)^2$$

$$\Rightarrow \frac{\Delta \tau_B}{\Delta \tau_A} = \left(\frac{-g_{00}(\underline{x}_B)}{-g_{00}(\underline{x}_A)}\right)^{1/2} \quad \text{grav. red / blue shift}$$

To reproduce formula in §1.2 for weak fields, consider

$$-g_{00}(\underline{x}) = 1 + 2\Phi/c^2 \quad \text{with} \quad |\Phi|/c^2 \ll 1$$

$$\Rightarrow (-g_{00}(\underline{x}))^{1/2} = 1 + \Phi/c^2$$

and $$\frac{\Delta\tau_B}{\Delta\tau_A} = 1 + \frac{1}{c^2}\left(\Phi(x_B) - \Phi(x_A)\right)$$

as in § 1.2.

(b) Newtonian Limit

Consider non-relativistic motion of massive particle in weak grav. field. Timelike geodesic parameterised by proper time τ

$$L = -g_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2 - g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = c^2 \quad (1^{st}\text{ integral})$$

General E-L eqn for space coords

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

For t-coord,

$$-g_{00}\, c^2 \frac{dt}{d\tau} = E \qquad \text{const. } (1^{st}\text{ integral})$$

Rearranging, $$\left(\frac{d\tau}{dt}\right)^2 = -g_{00} - \frac{1}{c^2} g_{ij} v^i v^j \; ,$$

where $v^i = \frac{dx^i}{dt} = \frac{dx^i}{d\tau}\cdot\frac{d\tau}{dt}$.

Consider weak field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ small – make to 1^{st} order

and low (non-rel) velocities $v^i \ll c$ – work to 2^{nd} order

From (a), $$g_{00} = -1 + h_{00} = -(1 + 2\Phi/c^2)$$

$$g_{ij} = \delta_{ij} + h_{ij}$$

$$\Rightarrow \left(\frac{d\tau}{dt}\right)^2 = 1 + \frac{2\Phi}{c^2} - \frac{1}{c^2}\delta_{ij} v^i v^j$$

(for $\Phi = 0$ we get $1/\gamma(v)^2$).

t- 1^{st} integral becomes

$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 \left(1 - \frac{\Phi}{c^2} + \frac{1}{2c^2}\delta_{ij}v^iv^j\right) \quad \text{(binomial thm).}$$

$$= \underbrace{c^2}_{\text{rest}} + \underbrace{\tfrac{1}{2}\delta_{ij}v^iv^j}_{\text{KE}} + \underbrace{\Phi}_{\text{PE}} = E \quad \text{const.}$$

This is conservation of energy per unit mass.

Now look at x^i - eqns. Note $\Gamma^i_{\alpha\beta}$ 1^{st} order in (derivatives) of $h_{\mu\nu}$.
(recall $\Gamma = 0$ in Minkowski space).

To order specified,

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}.$$

and $\frac{dt}{d\tau} = 1 + O(\Phi) + O(v^2)$. so

$$\frac{d^2 x^i}{dt^2} = -\Gamma^i_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} = -\Gamma^i_{00}c^2$$

But

$$\Gamma^i_{00} = \frac{1}{2}g^{i\alpha}\left(\cancelto{0}{\partial_0 g_{\alpha 0}} + \cancelto{0}{\partial_0 g_{0\alpha}} - \partial_\alpha g_{00}\right)$$

$$= -\frac{1}{2}g^{ij}\partial_j g_{00}.$$

$g^{ij} = \delta^{ij} + O(h)$.

$$= \frac{1}{2}\delta^{ij}\left(\partial_j\left(\frac{2\Phi}{c^2}\right)\right).$$

Hence.

$$\boxed{\frac{d^2 x^i}{dt^2} = -\delta^{ij}\partial_j\Phi}$$

Newton's 2^{nd} law with gravitational potential Φ.

Note: You have not needed to say what h_{ij} is - return to this shortly.

2.4 Changing Coordinates and Equivalence Principles

For any metric (Lorentz signature)

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \tilde{g}_{\alpha\beta}d\tilde{x}^\alpha d\tilde{x}^\beta.$$

with coords $\{x^\mu\}$, $\{\tilde{x}^\alpha\}$, then

$$\tilde{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\frac{\partial x^\nu}{\partial\tilde{x}^\beta}g_{\mu\nu}.$$

Previously noted, at a given point O, could rearrange

$$g_{\mu\nu} = \eta_{\mu\nu}$$

at O by choice of $\{x^\mu\}$ with $x^\mu = 0$ at O.

This is preserved by $\tilde{x}^\alpha = \Lambda^\alpha{}_\mu x^\mu$ with Λ Lorentz transformation.

Consider behaviour of metric as we move away from O, using Taylor,

$$g_{\mu\nu} = \eta_{\mu\nu} + C_{\mu\nu\rho} x^\rho + \dots \quad \text{(to 1st order)}$$

with coeff. obeying $C_{\mu\nu\rho} = C_{\nu\mu\rho}$.

For a change in coords given by

$$\tilde{x}^\mu = x^\mu + \frac{1}{2} A^\mu{}_{\nu\rho} x^\nu x^\rho + \dots \quad \text{(to 2nd order)}.$$

where coeff. $A^\mu{}_{\nu\rho} = A^\mu{}_{\rho\nu}$.

Note

$$x^\mu = \tilde{x}^\mu - \frac{1}{2} A^\mu{}_{\nu\rho} \tilde{x}^\nu \tilde{x}^\rho + \dots \quad \text{(to 2nd order)}$$

To find new metric components, we need

$$\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} = \delta^\mu{}_\alpha - A^\mu{}_{\alpha\rho} \tilde{x}^\rho + \dots$$

Then

$$\tilde{g}_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta}$$

$$= (\eta_{\mu\nu} + C_{\mu\nu\rho} x^\rho + \dots)(\delta^\mu{}_\alpha - A^\mu{}_{\alpha\sigma} \tilde{x}^\sigma + \dots)(\delta^\nu{}_\beta - A^\nu{}_{\beta\gamma} \tilde{x}^\gamma + \dots)$$

$$= \eta_{\alpha\beta} - C_{\alpha\beta\rho} \tilde{x}^\rho - A_{\beta\alpha\sigma} \tilde{x}^\sigma - A_{\alpha\beta\gamma} \tilde{x}^\gamma + \dots$$

where $A_{\alpha\beta\gamma} = \eta_{\alpha\mu} A^\mu{}_{\beta\gamma}$ (and $\tilde{x}^\mu = x^\mu$ to leading order)

In summary,

$$\tilde{g}_{\alpha\beta} = \eta_{\alpha\beta} + \tilde{C}_{\alpha\beta\rho} \tilde{x}^\rho + .$$

where $\tilde{C}_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} - A_{\beta\alpha\gamma} - A_{\alpha\beta\gamma} = 0$, by choosing

$$A_{\alpha\beta\gamma} = \frac{1}{2}\left(C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha} \right).$$

<u>Conclusion</u>: We can always choose coords to ensure that at a given point O, we have $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\rho g_{\mu\nu} = 0$. Hence $\Gamma^\rho_{\mu\nu} = 0$ at O.

This choice defines a <u>local inertial frame</u>.

[Sol^n above find by cycling indicies

$$
\left.\begin{aligned}
+ \quad C_{\alpha\beta\gamma} &= A_{\alpha\beta\gamma} + A_{\beta\alpha\gamma} \\
- \quad C_{\beta\gamma\alpha} &= A_{\beta\gamma\alpha} + A_{\gamma\beta\alpha} \\
+ \quad C_{\gamma\alpha\beta} &= A_{\gamma\alpha\beta} + A_{\alpha\gamma\beta}
\end{aligned}\right\} \Rightarrow \text{sol}^n
$$

]

Einstein Equivalence Principle

In a local inertial frame, the results of all non-gravitational experiments will be indistinguishable from results of same experiments in an inertial frame in Minkowski space

Contrast this with weak eqv principle : trajectory of a free falling test body depends only on its initial position and velocity, indpt of composition. This is universality of free fall or equality of inertial and gravitational mass.

3. Motion in the Schwarzschild Metric

Saw in §2.3 that a static metric of form

$$ds^2 = -\left(1+\frac{2\Phi}{c^2}\right)c^2dt^2 + (\delta_{ij}+h_{ij})\,dx^i dx^j .$$

with Φ/c^2, $h_{ij} \ll 1$. reproduces N2. for motion in gravitational potential Φ (by considering geodesics).

Special case in NG is $\Phi = -\frac{GM}{r}$. potential for point mass M at $r=0$. $\left(g = -\frac{GM}{r^2}\hat{r}\right)$ with no sources in $r>0$.

Generalisation in GR : Schwarzschild metric

$$ds^2 = -\left(1-\frac{2GM}{c^2 r}\right)c^2dt^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2).$$

This is an exact sol^n of vacuum Einstein eqns, no sources in $r>0$

Φ exact in NG and determined uniquely by symmetry. Same is true for Schwarzchild in GR (Birkoff's thm)

Common properties of metric and Newtonian Potential:

(i) Static – time translation symmetry.

(ii) Sph. sym. (at const t, r, we have 2-sphere with polar angles θ, φ)

(iii) For $r \gg \frac{2GM}{c^2}$, we recover previous case of weak gravity with $\Phi = -\frac{GM}{r}$, and identify M with mass.

(iv) Interesting behaviour as $r \to 0$ but also at $r = r_s = \frac{2GM}{c^2}$, the Schwarzschild radius – what's going on?

Note that for Sun, $r_s \approx 3\text{km}$,

Earth, $r_s \approx 9\text{mm}$

Moon, $r_s \approx 0.1\text{mm}$

Just as in Newtonian gravity, metric must be modified to describe interiors of star, etc.

3.1 Equations for timelike and null geodesics

Take units with $c = G = 1$.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

Geodesic eqn for massive or massless particles with affine param. λ (set $\lambda = \tau$ proper time in massive case)

$$\left.\begin{aligned} &t\text{-eqn:} \quad \left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda} = E \quad \text{const.} \\ &\varphi\text{-eqn:} \quad r^2 \sin^2\theta \, \frac{d\varphi}{d\lambda} = h \quad \text{const.} \end{aligned}\right\} \quad (*)$$

θ- eqn:
$$\frac{d}{d\lambda}\left(r^2 \frac{d\theta}{d\lambda}\right) - r^2 \sin\theta\cos\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0.$$

Instead of using r-eqn, consider standard first integral

$$\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2\left(\frac{d\theta}{d\lambda}\right)^2 - r^2\sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = \kappa = \begin{cases} 1 & \text{massive} \\ 0 & \text{massless.} \end{cases}$$

Note $\theta = \frac{\pi}{2}$ and $d\theta/d\lambda = 0$ satisfies θ-eqn. We can ensure this holds at initial point of geodesic by choosing θ, φ appropriately (and using properties of ODE)

Once this is done, we get, after substituting for $dt/d\lambda$ and $d\varphi/d\lambda$,

$$E^2 - \left(\frac{dr}{d\lambda}\right)^2 - \left(1-\frac{2M}{r}\right)\frac{h^2}{r^2} = \kappa\left(1-\frac{2M}{r}\right)$$

or using dots for $d/d\lambda$,

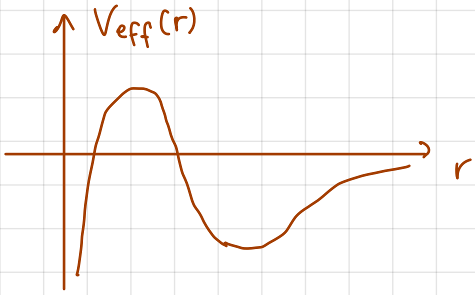
$$\frac{1}{2}\dot{r}^2 + V_{eff}(r) = \frac{1}{2}(E^2 - \kappa) \qquad (*).$$

where $V_{eff}(r) = -\kappa\frac{M}{r} + \frac{h^2}{2r^2} - \frac{Mh^2}{r^3}$, $E = \left(1-\frac{2M}{r}\right)\dot{t}$, $h = r^2\dot{\varphi}$

Mechanical analogy: position r of particle of unit mass moving in potential $V_{eff}(r)$, according to N2, with eqn above conservation energy, e.g. $V'_{eff}(r)$ the effective inward force, and $V'_{eff}(r) = 0$ condition for sol^n $r = $ const.

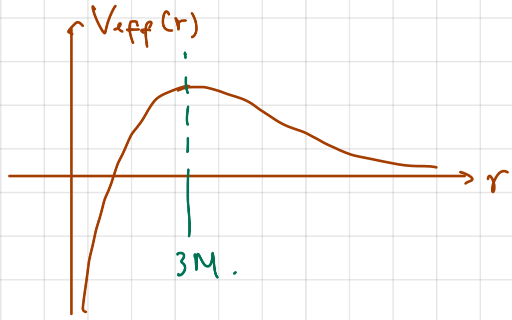
Examples

$\kappa = 1$ (massive) - details depend on values of M, h, but we may have



$$V_{eff}(r) = \underbrace{-\frac{M}{r} + \frac{h^2}{2r^2}}_{NG} \underbrace{- \frac{Mh^2}{r^3}}_{GR\ effect}.$$

$\kappa = 0$ (massless)



$$V_{eff}(r) = \frac{h^2}{2r^2} - \frac{Mh^2}{r^3}$$

To find shape of orbit, convenient to set

$$u(\varphi) = 1/r. \Rightarrow -\frac{1}{r^2}\dot{r} = \frac{du}{d\varphi}\dot{\varphi} \Rightarrow \dot{r} = -hu',$$

where $u' = du/d\varphi$. Then

$$(*) \Rightarrow \frac{1}{2}\left(\frac{du}{d\varphi}\right)^2 + \frac{1}{h^2}V_{eff}\left(\frac{1}{h}\right) = \text{const.}$$

This implies

$$\frac{d^2u}{d\varphi^2} + u = \frac{M\kappa}{h^2} + 3Mu^2$$

$(**)$

3.2 Massive Particles: Circular and near-circular orbits

Set $\kappa = 1$, and get

$$V_{eff}(r) = -\frac{M}{r} + \frac{h^2}{2r^2} - \frac{Mh^2}{r^3}.$$

(a) Newtonian gravitation: for comparison

Standard approach: r, φ coords in plane with grav. potential

$$\Phi = -\frac{M}{r}.$$

quantities depend on absolute time t.

$$\frac{1}{2}\dot{r}^2 + V_{eff}(r) = \text{const.} \quad \text{(conservation of energy)}$$

with

per unit mass

$$V_{eff}(r) = -\frac{M}{r} + \frac{h^2}{2r^2}.$$

$$h = r^2\dot{\varphi} \text{ const.} \quad \text{(cons of ang. mom.)}$$

$h > 0 \Rightarrow$ circular orbits exist

when $V_{eff}'(r) = 0$

$$\Rightarrow r = \frac{h^2}{M} = \ell. \quad \text{length scale}$$

orbits stable when.

$$V_{eff}''(\ell) > 0.$$

General sol^n for orbit shape comes from setting $u = 1/r$, and finding

$$\frac{d^2u}{d\varphi^2} + u = \frac{M}{h^2}$$

Sol^n: $\quad u = \frac{1}{r} = \frac{1}{\ell}(1 + e\cos(\varphi - \varphi_0)) \quad , \ \ell = h^2/M.$

this is conic section with eccentricity e. Bounded orbits for $e < 1$, and ellipses for $0 < e < 1$.

Sol^n for planetary orbits with $e \ll 1$ (exclude Pluto).

(b) Circular orbits in GR.

$$V_{eff}(r) = \underbrace{-\frac{M}{r} + \frac{h^2}{2r^2}}_{\text{Newtonian}} \underbrace{- \frac{Mh^2}{r^3}}_{\text{GR correction}}$$

$$V_{eff}'(r) = \frac{M}{r^2} - \frac{h^2}{2r^3} + \frac{3Mh^2}{r^4} = 0$$

$$\Leftrightarrow \quad r = r_\pm = \frac{h^2}{2M}\left(1 \pm \left(1 - \frac{12M^2}{h^2}\right)^{1/2}\right)$$

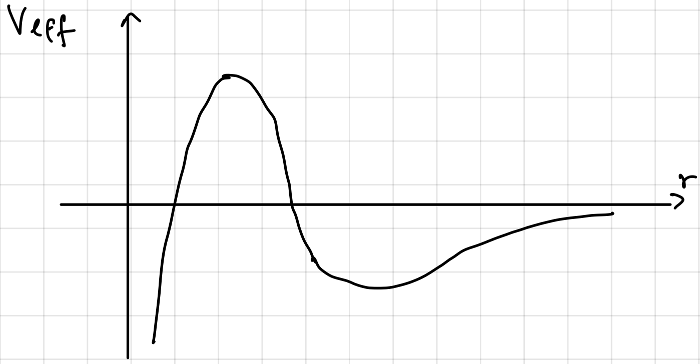
For real sol^n to exist, need

$$h > 2\sqrt{3}\, M.$$

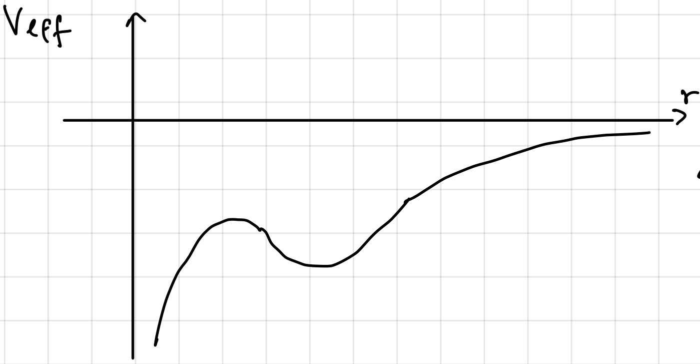
Then

$$h^2 = \frac{Mr^2}{r - 3M}$$

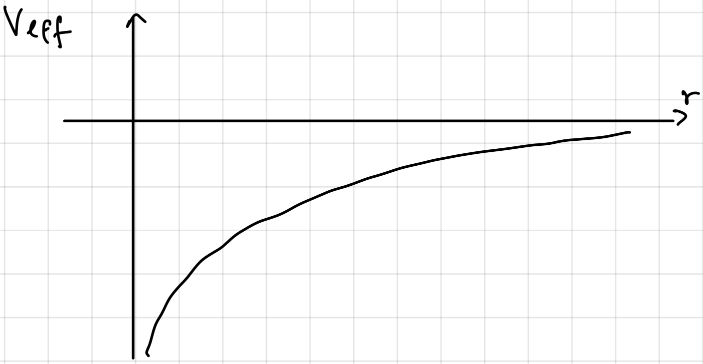
So sol^n require $\quad r > 3M. \ (= \frac{3}{2} r_s).$



$h > 4M$



$4M > h > 2\sqrt{3}M.$



$h < 2\sqrt{3}M$

Checking detail, also find

$$V_{eff}''(r) = \frac{M(r-6M)}{r^3(r-3M)}$$

Conclusion: stable circular orbits for $r > 6M$

unstable " $3M < r < 6M$.

(c) Nearly circular orbits in GR

Consider

$$u'' + u = \frac{M}{h^2} + \underbrace{3Mu^2}_{GR}$$

GR, small when $Mu \ll 1$ or $r \gg r_s$

Look for near circular sol^n in form

$$u = \frac{1}{\ell}(1 + v(\varphi))$$

with v small, $\ell = h^2/M$. GR term then small for $M/\ell \ll 1$.

Substitute in:

$$\frac{1}{\ell}\left(v'' + v + \cancel{1}\right) = \cancel{\frac{M}{h^2}} + \frac{3M}{\ell^2}(1+v)^2$$

$$\Rightarrow \quad v'' + v = \frac{3M}{\ell}(1+2v) \qquad \text{(to } 1^{st} \text{ order in } v\text{)}.$$

or $$v'' + \omega^2 v = \frac{3M}{\ell},$$

with $\omega^2 = 1 - \frac{6M}{\ell}$. Binomial thm $\Rightarrow$ $\omega \simeq 1 - \frac{3M}{\ell}$.

Sol^n:

$$v = \underbrace{\frac{3M}{\ell}\cdot\frac{1}{\omega^2}}_{\text{small}} + e\cos\left(\omega(\varphi - \varphi_0)\right)$$

Hence,

$$u = \frac{1}{\ell}\left(\left(1 + \frac{3M}{\ell}\right) + e\cos\left(\omega(\varphi-\varphi_0)\right)\right).$$

Small change to const. term is to be expected. Compare with

$$r_+ = \frac{h^2}{2M}\left(1 + 1 - \frac{6M^2}{h^2}\right) = \frac{h^2}{M}\left(1 - \frac{3M}{h^2}\right) = \ell\left(1 - \frac{3M}{\ell}\right).$$

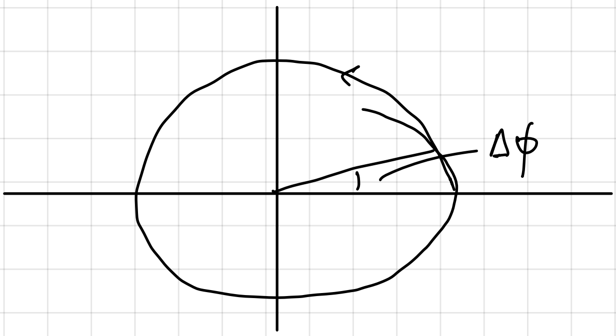
Main pt: sol^n is almost an ellipse but now need to increase φ by

$$\frac{2\pi}{\omega} \simeq 2\pi + \frac{6\pi M}{\ell}$$

to return to same value of u, i.e. ellipse precesses by

$$\Delta\varphi = \frac{6\pi M}{\ell} \quad (\ll 1)$$

on each orbit.



Largest effect for smaller orbits. e.g. Mercury find

$$\Delta\varphi \approx 5\times10^{-7} \text{ rad/orbit}$$

equivalent to 43" of arc per century. This prediction of GR is experimentally confirmed.

3.3 Massless particles and Deflection of light

Refer to (*) in § 3.1 with $\kappa=0$.

$$V_{eff}(r) = \frac{h^2}{2r^2} - \frac{Mh^2}{r^3}$$

$$V'_{eff}(r) = -\frac{h^2}{r^3} + \frac{3Mh^2}{r^4} = 0 \quad \text{for } r=3M.$$

$\kappa=0$ in (**):

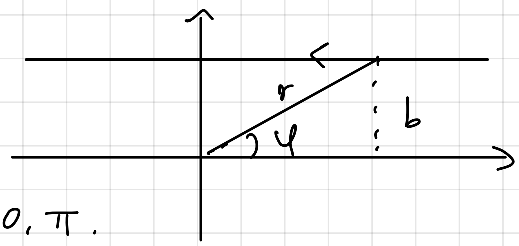
$$u'' + u = 3Mu^2$$

Sol^n $u = \frac{1}{3M} = \frac{1}{r}$ circular orbit (unstable from sketch of $V_{eff}(r)$).

Consider shape of orbits in regime $r \gg M$. To 0-th order,

$$u = \frac{1}{b}\sin\varphi. \quad \text{(straight line)}$$

One integration const. fixed to ensure symmetry under $\varphi \to \pi - \varphi$ and note $u \to 0$ $(r \to \infty)$ as $\varphi \to 0, \pi$.



Other const. of integration b gives "closest approach" or impact parameter. Take $b \gg M$ to ensure $Mu \ll 1$.

Seek sol^n to next order

$$u = \frac{1}{b}\left(\sin\varphi + v(\varphi)\right).$$

with $v(\varphi)$ small ($\ll M/b$)

Sub. in to get

$$v'' + v = \frac{3M}{b}\left(\sin\varphi + \underbrace{\cancel{v}}_{\text{small}}\right)^2$$

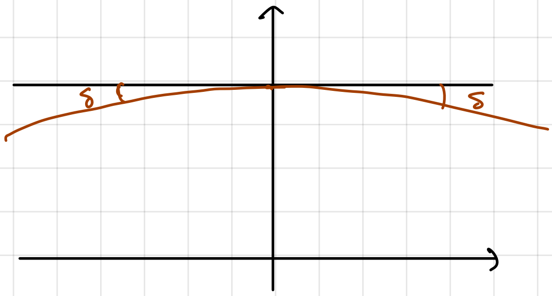
$$= \frac{3M}{b}\sin^2\varphi = \frac{3M}{2b}(1-\cos 2\varphi).$$

Particular sol^n with space symmetry $\varphi \mapsto \pi - \varphi$ is

$$v = \cancel{\alpha \frac{M}{b}\sin\varphi} + \left(\frac{3M}{2b} + \frac{M}{2b}\cos 2\varphi\right)$$

produces small redef. of b — ignore to the order we are interested in

Find values of φ for which $u \to 0$ in new sol^n.



$$\sin\varphi + \frac{3M}{2b} + \frac{M}{2b}\cos 2\varphi = 0$$

But this happens for $\varphi = -\delta$, $\varphi = -\pi + \delta$ $(\delta > 0)$

with δ small,

$$-\delta + \frac{2M}{b} = 0$$

and deflection

$$\Delta\varphi = 2\delta = \frac{4M}{b}.$$

For measurement during solar eclipse, find $\Delta\varphi \approx 1.75''$ in agreement with GR.

One other important experimental test — Shapiro time delay.
Similar effect discussed on Ex Sheet 2.

4. Tensors, Connections and Curvature

We work on a general manifold (as in §2.1).

4.1 Vectors and Tensors

(a) Definitions

Vectors and tensors are defined as objects with components in each coord. system, e.g. $\{x^\mu\}$, or $\{\tilde{x}^\alpha\}$ with a transformation rule relating cpts in different coord systems.

A vector is an object with components V^μ or $\tilde{V}^\alpha$ s.t.

$$\boxed{\tilde{V}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} V^\mu .}$$

This is vector transformation rule.

A covector is an object with component U_μ or $\tilde{U}_\alpha$ s.t.

$$\boxed{\tilde{U}_\alpha = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} U_\mu .}$$

This is covector transformation rule.

Def^n apply at any point. Vector and covector-valued f^n's are called (co)vector fields.

vector $\longleftrightarrow$ contravariant vector

covector $\longleftrightarrow$ covariant vector

Note: Vectors and covectors can be "paired" to give a scaler.

$$\tilde{V}^\alpha \tilde{U}_\alpha = V^\mu U_\mu$$

[Check $\quad LHS = \underbrace{\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} V^\mu \frac{\partial x^\nu}{\partial \tilde{x}^\alpha}}_{\delta_\mu{}^\nu} U_\nu = RHS$]

Examples For a parameterised curve $x^\mu(\lambda)$, or $\tilde{x}^\alpha(\lambda)$, the tangent vec.

$$T^\mu = \frac{dx^\mu}{d\lambda}, \quad \tilde{T}^\alpha = \frac{d\tilde{x}^\alpha}{d\lambda}.$$

Noted previously that these cpts obey the vector transformation rule.

For a scaler f^n $f(x^\mu)$ or $f(\tilde{x}^\alpha)$ (same f^n), we have, generalising the gradient

$$U_\mu = \frac{\partial f}{\partial x^\mu}, \quad \text{or} \quad \tilde{U}_\alpha = \frac{\partial f}{\partial \tilde{x}^\alpha}$$

and these obey the covector transformation rule.

Consider how $f(x^\mu(\lambda))$ changes along curve,

$$\frac{d}{d\lambda} f(x^\mu(\lambda)) = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\lambda} = U_\mu T^\mu \quad \text{invariant}$$

Generalising this, a tensor of type (p,q) has cpts

$$T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \quad \text{or} \quad \tilde{T}^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q}$$

related by

$$\tilde{T}^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} = \frac{\partial \tilde{x}^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \tilde{x}^{\alpha_p}}{\partial x^{\mu_p}} \frac{\partial x^{\nu_1}}{\partial \tilde{x}^{\beta_1}} \cdots \frac{\partial x^{\nu_q}}{\partial \tilde{x}^{\beta_q}} T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}$$

This is tensor transformation rule.

Example
- (co)vector is tensor of type $(1,0)$, or $(0,1)$.
- Scaler is tensor of type $(0,0)$

Note: previously saw invariance of $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ implied transformation of $g_{\mu\nu}$ to $\tilde{g}_{\alpha\beta}$ – now recognise $g_{\mu\nu}$ is a $(0,2)$-tensor.

Also note $\partial_\mu V^\nu$ is not a tensor

(b) Raising and Lowering indices

The metric provides additional structure which allows us to change between vectors and covectors by raising and lowering indices.

e.g. given V^μ, define $V_\mu = g_{\mu\nu} V^\nu$

given U_μ, define $U^\mu = g^{\mu\nu} U_\nu$.

Consistent, since

$$g^{\alpha\mu} g_{\mu\nu} V^\nu = \delta^\alpha{}_\nu V^\nu = V^\alpha .$$

and can check correct transformation rules for all objects above.

Similarly, for tensor of type (p,q), we can choose to lower an index to get a type $(p-1, q+1)$

$$g_{\beta\alpha} T^{\alpha\mu_1 \dots \mu_{p-1}}{}_{\nu_1 \dots \nu_q} = T_\beta{}^{\mu_1 \dots \mu_{p-1}}{}_{\nu_1 \dots \nu_q} .$$

or raise an index similarly.

Note: $V^\mu U_\mu = g_{\mu\nu} V^\mu U^\nu = g^{\mu\nu} V_\mu U_\nu = V_\mu U^\mu$.

and $g^{\mu\nu} g_{\nu\alpha} = \delta^\mu{}_\alpha$ can be interpreted as raising/lowering indices on metric itself.

(c) Tensor algebra

Operations on tensors can be defined on operations on components provided this is respected by tensor transformation rule.

(i) Addition (only of tensors of same type (p,q)) defined by

$$(T+S)^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} = T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} + S^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}$$

(ii) Scaler multiplication

$$(\lambda T)^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} = \lambda \left(T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}\right). \quad \forall \text{ scaler } \lambda.$$

(iii) Tensor product – For T type (p,q), S type (m,n), define $(T \otimes S)$ of type $(p+m, q+n)$ by

$$(T \otimes S)^{\alpha_1 \dots \alpha_p \mu_1 \dots \mu_m}{}_{\beta_1 \dots \beta_q \nu_1 \dots \nu_n} = T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} S^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}.$$

e.g. Using this we can construct a tensor of type $(1,2)$ from a vector V^α and covectors U_μ, W_ν.

$$T = V \otimes U \otimes W, \quad T^\alpha{}_{\mu\nu} = V^\alpha U_\mu W_\nu.$$

Note: For a tensor of rank 2, say, $T^{\alpha\beta}$, it is not true in general that $T^{\alpha\beta} = U^\alpha V^\beta$ for some U^α, V^β. But true that T can be expressed as a sum of such terms.

(iv) Index contraction – Given T type (p,q), can define S of type $(p-1, q-1)$ by contracting on one upper and one lower index

$$T^{\alpha \mu_1 \dots \mu_{p-1}}{}_{\alpha \nu_1 \dots \nu_{q-1}} = S^{\mu_1 \dots \mu_{p-1}}{}_{\nu_1 \dots \nu_{q-1}}$$

respects tensor transformation rule, just as we checked $V^\alpha U_\alpha$ scaler.

(v) (Anti) Symmeterisation on like indices

Defined for any pair of indices of some type.

e.g. $T_{\alpha\beta} = \pm T_{\beta\alpha}$, say T (anti) symmetric

Similarly, $T_\mu{}^{\alpha\beta} = \pm T_\mu{}^{\beta\alpha}$

Notation : define $T_{(\alpha\beta)} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right)$ sym

$T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right)$ anti sym

Clearly $T_{\alpha\beta} = T_{(\alpha\beta)} + T_{[\alpha\beta]}$

Note $T_{\alpha\beta}$ sym $\Leftrightarrow$ $T_{\alpha\beta} = T_{(\alpha\beta)}$ $\Leftrightarrow$ $T_{[\alpha\beta]} = 0$.

Example $g_{\alpha\beta} = g_{(\alpha\beta)}$ by defn.

Example $T^{(\alpha\beta)} S_{\alpha\beta} = T^{\alpha\beta} S_{(\alpha\beta)}$.

$$\begin{aligned}
LHS &= \tfrac{1}{2}\left(T^{\alpha\beta} + T^{\beta\alpha}\right) S_{\alpha\beta} \\
&= \tfrac{1}{2}\left(T^{\alpha\beta} S_{\alpha\beta} + T^{\beta\alpha} S_{\alpha\beta}\right) \\
&= \tfrac{1}{2} T^{\alpha\beta} S_{\alpha\beta} + \tfrac{1}{2} T^{\alpha\beta} S_{\beta\alpha} \\
&= T^{\alpha\beta} \cdot \tfrac{1}{2}\left(S_{\alpha\beta} + S_{\beta\alpha}\right) = T^{\alpha\beta} S_{(\alpha\beta)}
\end{aligned}$$

and $T^{(\alpha\beta)} S_{[\alpha\beta]} = 0$.

We extend this notation to any no. of indices, e.g.

$$T_{\substack{(\alpha\beta\gamma) \\ [\alpha\beta\gamma]}} = \frac{1}{3!}\left(T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} \pm T_{\beta\alpha\gamma} \pm T_{\gamma\beta\alpha} \pm T_{\alpha\gamma\beta}\right)$$

(vi) Quotient Rule.

In §4.1 (a), we noted that $\tilde{V}^{\alpha}\tilde{U}_{\alpha} = V^{\mu} U_{\mu}$ $\forall$ vec and covec.

Conversely, if this is invariant for any vec. V^{μ}, then

$$\left(\frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \cdot V^{\mu}\right) \tilde{U}_{\alpha} = V^{\mu} U_{\mu}$$

True $\forall$ V^{μ} $\Rightarrow$ $$\frac{\partial \tilde{x}}{\partial x^{\mu}} \tilde{U}_{\alpha} = U_{\mu}.$$

which is covector transformation rule. This has clear generalisation.

Example If $R^\alpha{}_{\beta\mu\nu} V^\beta = S^\alpha{}_{\mu\nu}$ is a tensor $\forall V^\beta$, then $R^\alpha{}_{\beta\mu\nu}$ is a tensor.

If $T^{\alpha\beta\mu\nu}{}_{\gamma\delta\rho} S^\rho{}_{\mu\nu}$ is a tensor $\forall S^\rho{}_{\mu\nu}$, then T a tensor.

4.2 Connections and Covariant Derivatives

(a) Motivation and Definitions

Recall: for a scaler f^n f, $\partial_\mu f$ is a covector, but for a vector field V^μ, $\partial_\mu V^\nu$ is not a tensor.

$$\tilde{\partial}_\alpha \tilde{V}^\beta = \left(\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \partial_\mu \right) \left(\frac{\partial \tilde{x}^\beta}{\partial x^\nu} V^\nu \right)$$

$$= \left(\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \right) \partial_\mu V^\nu + \underbrace{\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial^2 \tilde{x}^\beta}{\partial x^\mu \partial x^\nu}}_{\text{spoils tensor trans. rule.}} V^\nu$$

Consider (simpler situation) how V^μ changes along a curve $x^\mu(\lambda)$.

$$\frac{d}{d\lambda} \left(V^\mu(x^\alpha(\lambda)) \right) = \frac{dx^\alpha}{d\lambda} \frac{\partial V^\mu}{\partial x^\alpha} = \underbrace{T^\alpha \partial_\alpha V^\mu}_{\text{translating this as above}}.$$

with $T^\mu = \dfrac{dx^\mu}{d\lambda}$ ~~tgt~~ vec. to curve.

But compare with geodesic eqn in form

$$\frac{dT^\mu}{d\lambda} + \Gamma_\alpha{}^\mu{}_\nu T^\alpha T^\nu = 0.$$

with Γ levi-Civita connection. This takes some form in any coord. system (derived from VP) and we can deduce how Γ must transform.

$$\boxed{\tilde{\Gamma}_\alpha{}^\beta{}_\gamma = \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\rho}{\partial \tilde{x}^\gamma} \Gamma_\mu{}^\nu{}_\rho + \frac{\partial \tilde{x}^\beta}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial \tilde{x}^\alpha \partial \tilde{x}^\gamma}.}$$

This is the connection transformation rule.

This suggests we define covariant derivative of vec. field V^μ along curve to be

$$\frac{D}{D\lambda} V^\mu = \frac{d}{d\lambda} V^\mu + \Gamma_{\alpha}{}^{\mu}{}_{\nu} T^\alpha V^\nu .$$

and this should be a vector (can check). Geodesic eqn become

$$\frac{D}{D\lambda} T^\mu = 0$$

Furthermore, with defn above, can write

$$\frac{D}{D\lambda} V^\mu = T^\alpha (\partial_\alpha V^\mu + \Gamma_{\alpha}{}^{\mu}{}_{\nu} V^\nu) = T^\alpha \nabla_\alpha V^\mu$$

where

$$\boxed{\nabla_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma_{\alpha}{}^{\mu}{}_{\nu} V^\nu}$$

This is covariant derivative of a vector field. Check this is (1,1) tensor.

Already have $\boxed{\nabla_\alpha f = \partial_\alpha f}$ covariant derivative of a scaler fn.

Define similarly

$$\boxed{\nabla_\alpha U_\mu = \partial_\alpha U_\mu - \Gamma_{\alpha}{}^{\rho}{}_{\mu} U_\rho}$$

Covariant derivative of a covector field.

This ensures the following Leibnitz property:

$$\boxed{\nabla_\alpha (V^\mu U_\mu) = (\nabla_\alpha V^\mu) U_\mu + V^\mu (\nabla_\alpha U_\mu)}$$

Check: LHS $= \nabla_\alpha (V^\mu U_\mu) = \partial_\alpha (V^\mu U_\mu) = (\partial_\alpha V^\mu) U_\mu + V^\mu (\partial_\alpha U_\mu)$

$$\text{RHS} = (\partial_\alpha V^\mu + \cancel{\Gamma_{\alpha}{}^{\mu}{}_{\nu} V^\nu}) U_\mu + V^\mu (\partial_\alpha U_\mu - \cancel{\Gamma_{\alpha}{}^{\rho}{}_{\mu} U_\rho})$$

Explicit check that $\nabla_\mu V^\nu$ transforms like a $(1,1)$-tensor.

$$\tilde{\nabla}_\alpha \tilde{V}^\beta = \tilde{\partial}_\alpha \tilde{V}^\beta + \tilde{\Gamma}_\alpha{}^\beta{}_\gamma \tilde{V}^\gamma$$

$$= \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \left(\partial_\mu V^\nu + \Gamma_\mu{}^\nu{}_\rho V^\rho \right)$$

$$+ \underbrace{\left[\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial^2 \tilde{x}^\beta}{\partial x^\mu \partial x^\rho} + \frac{\partial \tilde{x}^\beta}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial \tilde{x}^\alpha \partial \tilde{x}^\gamma} \frac{\partial \tilde{x}^\gamma}{\partial x^\rho} \right]}_{\text{claim } = 0} V^\rho$$

$$= \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \nabla_\mu V^\nu \qquad \text{as desired.}$$

To show $[\cdots] = 0$, consider first term and write it as

$$\frac{\partial}{\partial x^\rho} \Big(\underbrace{\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial x^\mu}}_{= \delta_\alpha{}^\beta} \Big) - \frac{\partial \tilde{x}^\beta}{\partial x^\mu} \frac{\partial}{\partial x^\rho} \left(\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \right).$$

$$= - \frac{\partial \tilde{x}^\beta}{\partial x^\mu} \cdot \frac{\partial \tilde{x}^\gamma}{\partial x^\rho} \frac{\partial^2 x^\mu}{\partial \tilde{x}^\gamma \partial \tilde{x}^\alpha}.$$

We can extend these def^{n}s to any T, type (p,q) to get covariant derivative ∇T, type $(p,q+1)$

Example $(p,q) = (1,1)$

$$\nabla_\alpha W^\mu{}_\nu = \partial_\alpha W^\mu{}_\nu + \Gamma_\alpha{}^\mu{}_\rho W^\rho{}_\nu - \Gamma_\alpha{}^\rho{}_\nu W^\mu{}_\rho.$$

Example $(p,q) = (0,2)$

$$\nabla_\alpha S_{\mu\nu} = \partial_\alpha S_{\mu\nu} - \Gamma_\alpha{}^\rho{}_\mu S_{\rho\nu} - \Gamma_\alpha{}^\rho{}_\nu S_{\mu\rho}.$$

(b) Connections in general and metric compatibility

We can define covariant derivatives using any connection Γ that satisfies the connection transformation rule.

For a general connection.

$$\nabla_\alpha \nabla_\beta f - \nabla_\beta \nabla_\alpha f$$
$$= \nabla_\alpha (\partial_\beta f) - \nabla_\beta (\partial_\alpha f)$$
$$= (\partial_\alpha (\partial_\beta f) - \Gamma_\alpha{}^\gamma{}_\beta \partial_\alpha f) - (a \leftrightarrow b)$$
$$= -T_\alpha{}^\gamma{}_\beta \partial_\gamma f .$$

where $T_\alpha{}^\gamma{}_\beta = \Gamma_\alpha{}^\gamma{}_\beta - \Gamma_\beta{}^\gamma{}_\alpha$ is the torsion, a tensor of type $(1,2)$. For Levi-Civita, $T_\alpha{}^\gamma{}_\beta = 0$ "torsion free".

Fundamental Thm of Riemannian Geometry

For a manifold with metric $g_{\mu\nu}$, there is a unique torsion free connection for which the metric is covariantly const.

$$\nabla_\alpha g_{\mu\nu} = 0 .$$

Pf: $\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma_\alpha{}^\rho{}_\mu g_{\rho\nu} - \Gamma_\alpha{}^\rho{}_\nu g_{\mu\rho} = 0 .$

$$\Rightarrow \quad \partial_\mu g_{\nu\alpha} - \Gamma_\mu{}^\rho{}_\nu g_{\rho\alpha} - \Gamma_\mu{}^\rho{}_\alpha g_{\nu\rho} = 0$$

$$\Rightarrow \quad \partial_\nu g_{\alpha\mu} - \Gamma_\nu{}^\rho{}_\alpha g_{\rho\mu} - \Gamma_\nu{}^\rho{}_\mu g_{\alpha\rho} = 0 .$$

Combine these with $-, +, +$ and $\Gamma_\alpha{}^\mu{}_\beta = \Gamma_\beta{}^\mu{}_\alpha$ to get.

$$-\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - 2\Gamma_\mu{}^\rho{}_\nu g_{\rho\alpha} = 0$$

$$\Rightarrow \quad \Gamma_\mu{}^\rho{}_\nu = \frac{1}{2} g^{\rho\alpha} (\partial_\nu g_{\alpha\mu} + \partial_\mu \partial_{\nu\alpha} - \partial_\alpha g_{\mu\nu}) \qquad \square$$

Note this implies that $\nabla_\alpha g^{\mu\nu} = 0$ and covariant differentiation commutes with raising and lowering indices.

e.g. $\nabla_\alpha (V_\mu) = \nabla_\alpha (g_{\mu\nu} V^\nu) = g_{\mu\nu} \nabla_\alpha V^\nu$,

$$\nabla_\alpha (V^\mu U_\mu) = \nabla_\alpha (g_{\mu\nu} V^\mu U^\nu) = g_{\mu\nu} (\nabla_\alpha V^\mu) U^\nu + g_{\mu\nu} V^\mu (\nabla_\alpha U^\nu).$$

(c) Summary of def^{n}s and properties

For a tensor T of type (p,q), the covariant derivative ∇T is a tensor of type $(p,q+1)$ with

$$\boxed{\begin{aligned}\nabla_\alpha T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} &= \partial_\alpha T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} + \Gamma_\alpha{}^{\mu_1}{}_\rho T^{\rho \mu_2 \dots \mu_p}{}_{\nu_1 \dots \nu_q} \\ &+ \dots + \Gamma_\alpha{}^{\mu_p}{}_\rho T^{\mu_1 \dots \mu_{p-1} \rho}{}_{\nu_1 \dots \nu_q} - \Gamma_\alpha{}^\rho{}_{\nu_1} T^{\mu_1 \dots \mu_p}{}_{\rho \nu_2 \dots \nu_q} \\ &- \dots - \Gamma_\alpha{}^\rho{}_{\nu_q} T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_{q-1} \rho}.\end{aligned}}$$

Covariant derivative of general tensor.

Tensor transformation rule for ∇T follows from transformation rules for T and Γ, but can also be deduced from the following.

Properties:

(i) $\nabla(\alpha T + \beta S) = \alpha \nabla T + \beta \nabla S$, for T, S tensor of type (p,q), α, β const.

(ii) $\nabla_\mu f = \partial_\mu f$ for any scaler f.

(iii) $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$ Leibnitz property.

(iv) ∇ commutes with index contraction (see prev. vecs, covecs).

Above, (i) - (iv) holds for a general connection.

(v) ∇ commutes with raising and lowering indices for ∇, the Levi-Civita connection, satisfying

$$\nabla_\alpha g_{\mu\nu} = 0 \quad , \quad \nabla_\alpha \nabla_\beta f - \nabla_\beta \nabla_\alpha f = 0 \quad \text{(torsion free)}$$

Further def^{n}s:

Given a vec. field V^μ, define the covariant derivative along V^μ

$$\nabla_V = V^\alpha \nabla_\alpha$$

acting on any tensor, and

$$\frac{D}{D\lambda} S = \nabla_T S = T^\alpha \nabla_\alpha S$$

covariant differentiation of tensor S along curve $x^\mu(\lambda)$ with tangent $T^\alpha = dx^\mu / d\lambda$.

(d) Parallel Transport

Consider manifold $\mathcal{M}$ with metric $g_{\mu\nu}$, and a curve $x^\mu(\lambda)$ with $T^\mu = \dot{x}^\mu$. A tensor S is parallelly transported along the curve if

$$\frac{DS}{D\lambda} = \nabla_T S = T^\alpha \nabla_\alpha S = 0$$

Note : If S is defined only on the curve, we must interpret above by taking

$$T^\alpha \partial_\alpha \longrightarrow \frac{d}{d\lambda}.$$

Example
$$\frac{D}{D\lambda} f = \frac{df}{d\lambda} \quad \text{for } f \text{ scaler, being parallelly transported}$$

$$\frac{D}{D\lambda} S^\alpha = \frac{dS^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} T^\beta S^\gamma = T^\beta \nabla_\beta S^\alpha = 0.$$

If curve is an affinely parameterised geodesic, then

$$\frac{D}{D\lambda} T^\alpha = 0 ,$$

tangent vec. parallelly transported along curve.

For vectors U^α, V^α

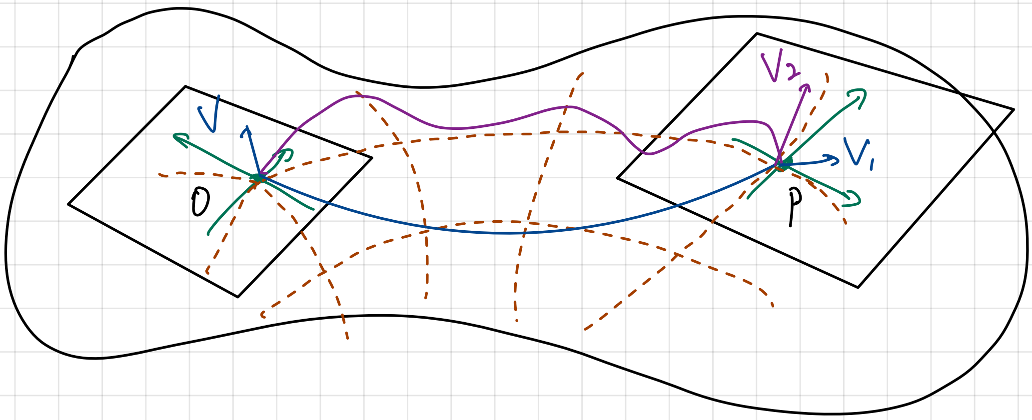
metric covariantly const.

$$\frac{D}{D\lambda}\left(g_{\alpha\beta} U^\alpha V^\beta\right) = g_{\alpha\beta}\left(\frac{D}{D\lambda} U^\alpha\right) V^\beta + g_{\alpha\beta} U^\alpha \left(\frac{D}{D\lambda} V^\beta\right)$$

$= 0$

if U^α, V^α parallely transported, i.e. lengths, angles, etc are preserved.

To expand on this, consider the following sketch:



Vectors V^μ live in tangent space to $\mathcal{M}$ at each point as shown. Basis for each tangent space comes from choice of coords $\{x^\mu\}$ on $\mathcal{M}$.

Without a connection, we know what it means for V^μ to vary smoothly, but can't say whether it's constant.

With connection Γ, we can compare vecs in tangent spaces at nearby points and define covariant derivative.

For any pair of points O, $x^\mu = 0$, and P, we can parallely transport V^μ from O to P along any given curve, with results V_1, V_2 for different curves as shown.

In general, V_1, V_2 different and this corresponds to <u>curvature</u> of metric.

(e) Normal Coordinates and Punctuation Notation

Recall in §2.4, around any point P, we can choose <u>local inertial coords</u> / <u>local inertial frame</u> with

$$\left.\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} \quad (\text{or } \delta_{\mu\nu}) \\ \text{and} \quad \partial_\alpha g_{\mu\nu} &= 0 \end{aligned}\right\} \text{ at } P$$

$$\Rightarrow \Gamma_\alpha{}^\mu{}_\beta = 0 \qquad \text{at } P$$

Coordinates $\{x^\mu\}$ s.t. $\Gamma = 0$ at P are also called <u>normal coords</u>. Since a tensor identity holds in every coord. system if it holds in any chosen coord. system, we can adopt normal coords to simplify calculations

$$\nabla_\alpha T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} \text{ "=" } \partial_\alpha T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}$$

The symbol "=" is sometimes used to denote equality in normal coords at P. Also convenient to denote

$\begin{matrix}\partial_\alpha \\ \nabla_\alpha\end{matrix}$ by subscripts $\begin{matrix},\alpha \\ ;\alpha\end{matrix}$ (or superscripts) e.g. $\nabla_\alpha V^\beta = V^\beta{}_{;\alpha}$. $\quad \partial_\alpha V^\beta = V^\beta{}_{,\alpha}$

So
$$V^\beta{}_{;\alpha} = V^\beta{}_{,\alpha} + \Gamma_\alpha{}^\beta{}_\gamma V^\gamma .$$

and
$$S^\alpha{}_{\beta;\gamma} \text{ "=" } S^\alpha{}_{\beta,\gamma} \qquad \text{(at } P\text{)}$$

Be aware of double derivatives, e.g.
$$\nabla_\alpha \nabla_\beta U_\gamma = U_{\gamma;\beta\alpha} \quad (\text{usually not } U_{\gamma;\beta;\alpha})$$

order is important of α, β indices

$\partial \Gamma \neq 0$ in normal coords., even though $\Gamma = 0$ at P.

4.3 Curvature and the Riemann Tensor.

(a) Definitions

Claim If V^μ is any vec. field, then

Ricci identity.

$$\boxed{(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu}$$

where $R^\mu{}_{\alpha\beta\nu}$ is a $(1,3)$-tensor, the Riemann (curvature) tensor.

Note LHS is a tensor, so if we check RHS does not depend on ∂V or $\partial^2 V$, then R is a tensor by quotient rule.

Pf: $\nabla_\alpha \nabla_\beta V^\mu - (\alpha \leftrightarrow \beta)$

$$= \partial_\alpha (\nabla_\beta V^\mu) - \cancel{\Gamma_\alpha{}^\sigma{}_\beta \nabla_\sigma V^\mu} + \Gamma_\alpha{}^\mu{}_\sigma \nabla_\beta V^\sigma - (\alpha \leftrightarrow \beta)$$

$$= \partial_\alpha (\cancel{\partial_\beta V^\mu} + \Gamma_\beta{}^\mu{}_\nu V^\nu) + \Gamma_\alpha{}^\mu{}_\sigma (\partial_\beta V^\sigma + \Gamma_\beta{}^\sigma{}_\nu V^\nu) - (\alpha \leftrightarrow \beta)$$

$$= (\partial_\alpha \Gamma_\beta{}^\mu{}_\nu) V^\nu + (\Gamma_\alpha{}^\mu{}_\sigma \Gamma_\beta{}^\sigma{}_\nu) V^\nu + \cancel{\Gamma_\beta{}^\mu{}_\sigma \partial_\alpha V^\sigma} + \cancel{\Gamma_\alpha{}^\mu{}_\sigma \partial_\beta V^\sigma} - (\alpha \leftrightarrow \beta)$$

This verifies our claim we deduce

$$\boxed{R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma_\beta{}^\mu{}_\nu + \Gamma_\alpha{}^\mu{}_\sigma \Gamma_\beta{}^\sigma{}_\nu - \partial_\beta \Gamma_\alpha{}^\mu{}_\nu - \Gamma_\beta{}^\mu{}_\sigma \Gamma_\alpha{}^\sigma{}_\nu} \qquad \square$$

There is an equivalent Ricci identity for covector.

$$\boxed{(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) U_\mu = -R^\nu{}_{\mu\alpha\beta} U_\nu .}$$

Similar for general tensors with terms as above for each vector/covector index.

From Riemann tensor, define

Def^n Ricci tensor: $R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$

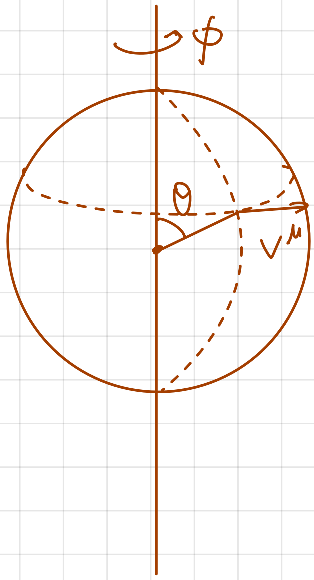
Ricci scaler: $R = g^{\alpha\beta} R_{\alpha\beta}$.

Example 2 sphere with $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$, $g_{\theta\theta} = 1$, $g_{\phi\phi} = \sin^2\theta$.

$\Gamma_\alpha{}^\mu{}_\beta$ has non-zero cpt.

$$\Gamma_\theta{}^\phi{}_\phi = \Gamma_\phi{}^\phi{}_\theta = \cot\theta$$

$$\Gamma_\phi{}^\theta{}_\phi = -\sin\theta\cos\theta .$$



Consider V^α with $V^\theta = 0$, $V^\phi = 1$.

$\nabla_\alpha V^\beta$ has non-zero cpt.

$$\nabla_\phi V^\theta = \Gamma_\phi{}^\theta{}_\alpha V^\alpha = \Gamma_\phi{}^\theta{}_\phi V^\phi = -\sin\theta\cos\theta$$

$$\nabla_\theta V^\phi = \Gamma_\theta{}^\phi{}_\alpha V^\alpha = \Gamma_\theta{}^\phi{}_\phi V^\phi = \cot\theta .$$

$$\nabla_\theta \nabla_\phi V^\theta = \partial_\theta (\nabla_\phi V^\theta) - \Gamma_\theta{}^\alpha{}_\phi (\nabla_\alpha V^\theta) + \cancel{\Gamma_\theta{}^\theta{}_\alpha (\nabla_\phi V^\alpha)}$$

$$= \partial_\theta(-\sin\theta\cos\theta) - \cot\theta(-\sin\theta\cos\theta) = \sin^2\theta .$$

$$\nabla_\phi \nabla_\theta V^\theta = 0 .$$

Hence, $(\nabla_\theta \nabla_\phi - \nabla_\phi \nabla_\theta) V^\theta = \sin^2\theta$ and $R_{\theta\phi\theta\phi} = \sin^2\theta$

$$\overset{\shortparallel}{R^\theta{}_{\alpha\theta\phi} V^\alpha} = R^\theta{}_{\phi\theta\phi}$$

$\Rightarrow R_{\alpha\beta} = g_{\alpha\beta}$, $R = 2$.

(b) Symmetries of the Riemann tensor.

(i) $R^{\mu}{}_{\nu\alpha\beta} = - R^{\mu}{}_{\nu\beta\alpha}$ (from defn)

(ii) $R^{\mu}{}_{[\nu\alpha\beta]} = 0$. (or given (i))

$$R^{\mu}{}_{\nu\alpha\beta} + R^{\mu}{}_{\alpha\beta\nu} + R^{\mu}{}_{\beta\nu\alpha} = 0$$

(iii) $R_{\mu\nu\alpha\beta} = - R_{\nu\mu\alpha\beta}$.

(iv) $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$.

From these symmetries,

(v) $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = R_{\beta\alpha}$.

Taking careful account of symmetries, we have

$$\frac{1}{12} n^2 (n^2 - 1) \quad , \quad \text{so} \quad \underset{n=4}{20}, \ \underset{n=3}{6}, \ \underset{n=2}{1}$$

indpt. cpts in dim n.

To prove symmetries:

(ii) In normal coords,

$$R^{\mu}{}_{\nu\alpha\beta} \text{ "=" } \partial_\alpha \Gamma_{\beta}{}^{\mu}{}_{\nu} - \partial_\beta \Gamma_{\alpha}{}^{\mu}{}_{\nu}$$
$$R^{\mu}{}_{\alpha\beta\nu} \text{ "=" } \partial_\beta \Gamma_{\nu}{}^{\mu}{}_{\alpha} - \partial_\nu \Gamma_{\beta}{}^{\mu}{}_{\alpha}$$
$$R^{\mu}{}_{\beta\nu\alpha} \text{ "=" } \partial_\nu \Gamma_{\alpha}{}^{\mu}{}_{\beta} - \partial_\alpha \Gamma_{\nu}{}^{\mu}{}_{\beta}$$

terms cancel when we add

(iii), (iv): Use

$$g_{\mu\sigma} \Gamma_{\beta}{}^{\sigma}{}_{\nu} = \tfrac{1}{2} (g_{\mu\beta,\nu} + g_{\mu\nu,\beta} - g_{\beta\nu,\mu})$$

$$\partial_\alpha (\text{above}) \text{ "=" } \tfrac{1}{2} (g_{\mu\beta,\nu\alpha} + g_{\mu\nu,\beta\alpha} - g_{\beta\nu,\mu\alpha})$$

Hence $$R_{\mu\nu\alpha\beta} \text{ "=" } \tfrac{1}{2} (g_{\mu\beta,\nu\alpha} - g_{\nu\beta,\mu\alpha} - g_{\mu\alpha,\nu\beta} + g_{\nu\alpha,\mu\beta})$$

(c) Bianchi identity

The Riemann tensor satisfies

$$\nabla_\gamma R^\mu{}_{\nu\alpha\beta} + \nabla_\alpha R^\mu{}_{\nu\beta\gamma} + \nabla_\beta R^\mu{}_{\nu\gamma\alpha} = 0,$$

or

$$R^\mu{}_{\nu[\alpha\beta;\gamma]} = 0 \qquad \text{(Bianchi identity)}$$

Pf: In normal coords,

$$R^\mu{}_{\nu\alpha\beta;\gamma} \text{ "=" } R^\mu{}_{\nu\alpha\beta,\gamma} \qquad \text{at point } P.$$

$$= \Gamma_\nu{}^\mu{}_{\beta,\alpha\gamma} - \Gamma_\nu{}^\mu{}_{\alpha,\beta\gamma} + (\cancel{\Gamma\partial\Gamma} \text{ terms}) \to 0$$

Similarly,

$$R^\mu{}_{\nu\beta\gamma;\alpha} = \Gamma_\nu{}^\mu{}_{\gamma,\beta\alpha} - \Gamma_\nu{}^\mu{}_{\beta,\gamma\alpha}$$

$$R^\mu{}_{\nu\gamma\alpha;\beta} = \Gamma_\nu{}^\mu{}_{\alpha,\gamma\beta} - \Gamma_\nu{}^\mu{}_{\gamma,\alpha\beta}$$

On adding, terms cancel in pairs (equality of mixed partials)

Result follows, since tensor identity true in any coord. system if true in one coord. sys. □

Contracted Bianchi identity

$$\nabla^\rho R_{\alpha\rho} - \tfrac{1}{2}\nabla_\alpha R = 0$$

This follows by contracting Bianchi identity above on μ, γ.

$$\nabla_\gamma R^\gamma{}_{\nu\alpha\beta} - \nabla_\alpha R_{\nu\beta} + \nabla_\beta R_{\nu\alpha} = 0$$

Contract again on ν, β gives

$$\nabla_\gamma R^\gamma{}_\alpha - \nabla_\alpha R + \nabla_\beta R^\beta{}_\alpha = 0$$

and result follows.

(d) Parallel transport and curvature

Parallel transport of a vector field $V^\mu(\lambda) = V^\mu(x^\alpha(\lambda))$ along a curve $x^\mu(\lambda)$ starting at $x^\mu(0) = 0$ is given by

$$\frac{D}{D\lambda} V^\mu = 0$$

$$\Rightarrow \frac{d}{d\lambda} V^\mu(\lambda) = -\left(\Gamma_\alpha{}^\mu{}_\nu \dot{x}^\alpha V^\nu\right)(\lambda).$$

$$\Rightarrow V^\mu(\lambda) = V^\mu(0) - \int_0^\lambda \left(\Gamma_\alpha{}^\mu{}_\nu \dot{x}^\alpha V^\nu\right)(\tau)\, d\tau$$

Use this to generate expansion order-by-order in "size" of curve. corresponding to # factors of x^α, $\dot{x}^\alpha$, etc.

$$V^\mu(\lambda) = V^\mu(0) + [\Delta V^\mu]_1(\lambda) + [\Delta V^\mu]_2(\lambda) + \cdots$$

First iteration:

$$[\Delta V^\mu]_1(\lambda) = -\int_0^\lambda \left(\Gamma_\alpha{}^\mu{}_\nu V^\nu\right)(0)\, \dot{x}^\alpha(\tau)\, d\tau$$

$$= -\left(\Gamma_\alpha{}^\mu{}_\nu V^\nu\right)(0)\, x^\alpha(\lambda).$$

Second iteration: use the above, but also

$$\Gamma_\alpha{}^\mu{}_\nu(\lambda) = \Gamma_\alpha{}^\mu{}_\nu(0) + \Gamma_\alpha{}^\mu{}_{\nu,\beta}(0)\, x^\beta(\lambda) + \cdots$$

$$\Rightarrow \quad [\Delta V^\mu]_2(\lambda) = S^\mu{}_{\nu\alpha\beta}\, V^\nu(0) \int_0^\lambda (x^\beta \dot{x}^\alpha)(\tau)\, d\tau.$$

where

$$S^\mu{}_{\nu\alpha\beta} = \left(-\Gamma_\alpha{}^\mu{}_{\nu,\beta} + \Gamma_\alpha{}^\mu{}_\sigma \Gamma_\beta{}^\sigma{}_\nu\right)(0)$$

Now for a closed curve $0 \le \lambda \le 1$ with $x^\mu(1) = 0 = x^\mu(0)$.

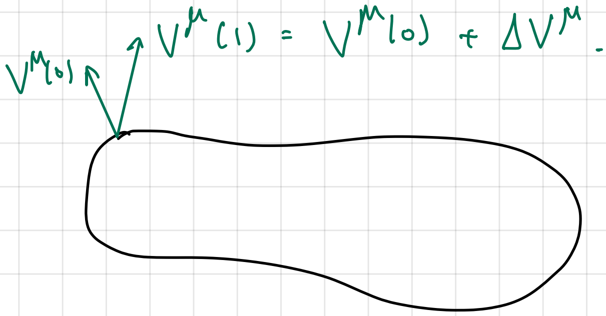
$$[\Delta V^\mu]_1(1) = 0$$

In $[\Delta V^\mu]_2$, have

$$\omega^{\alpha\beta} = -\int_0^1 (x^\beta \dot{x}^\alpha)(\tau)\, d\tau = \frac{1}{2}\int_0^1 (x^\alpha \dot{x}^\beta - x^\beta \dot{x}^\alpha)\, d\tau = -\omega^{\beta\alpha}$$

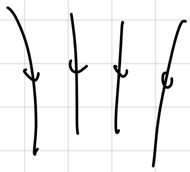
analogous to "area" in x^α, x^β coords.

But
$$S^{\mu}{}_{\upsilon[\alpha\beta]} = \frac{1}{2} R^{\mu}{}_{\nu\alpha\beta}$$
at $x^{\mu} = 0$. Hence,
$$[\Delta V^{\mu}]_2 = -\frac{1}{2} R^{\mu}{}_{\nu\alpha\beta}\, \omega^{\alpha\beta} V^{\nu}(0).$$
Change in vector as a result of parallel transport is given by curvature



(e) Geodesic Deviation

In flat (Minkowski) space, geodesics are straight lines and if initially ∥, then stay ∥.
In curved space, separation of geodesics varies and is controlled by Riemann tensor



To understand this, first consider vec. field T^{α}, S^{α}, V^{α}, and note
$$(\nabla_T \nabla_S - \nabla_S \nabla_T) V^{\mu} = R^{\mu}{}_{\nu\alpha\beta} T^{\alpha} S^{\beta} V^{\nu} + \nabla_{[T,S]} V^{\mu}$$
where $\nabla_T = T^{\alpha} \nabla_{\alpha}$, $\nabla_S = S^{\alpha}\nabla_{\alpha}$, and
$$[T, S]^{\beta} = T^{\alpha} \nabla_{\alpha} S^{\beta} - S^{\alpha} \nabla_{\alpha} T^{\beta} = \nabla_T S^{\beta} - \nabla_S T^{\beta}$$
$$= T^{\alpha}\partial_{\alpha} S^{\beta} - S^{\alpha}\partial_{\alpha} T^{\beta}. \ (\Gamma \text{ terms cancel})$$
is the <u>commutator</u>.

This follows from the Ricci identity, since

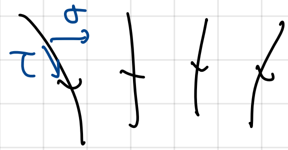
$$(T^\alpha \nabla_\alpha)(S^\beta \nabla_\beta) - (\alpha \leftrightarrow \beta)$$
$$= (T^\alpha \nabla_\alpha S^\beta)\nabla_\beta + T^\alpha S^\beta \nabla_\alpha \nabla_\beta - (\alpha \leftrightarrow \beta)$$

Now apply this. Consider $x^\mu(\tau, \sigma)$ giving geodesic for each fixed σ with τ affine parameter, and

$$T^\mu = \frac{\partial x^\mu}{\partial \tau} \quad \text{tgt vec.} \qquad \nabla_T T^\alpha = 0$$

while

$$S^\mu = \frac{\partial x^\mu}{\partial \sigma} \quad \text{"separation vec"}$$



$$[T, S]^\beta = T^\alpha \partial_\alpha S^\beta - S^\alpha \partial_\alpha T^\beta$$
$$= \frac{\partial S}{\partial \tau} - \frac{\partial T^\beta}{\partial \sigma} = \frac{\partial^2 x^\beta}{\partial \sigma \, \partial \tau} - \frac{\partial^2 x^\beta}{\partial \sigma \, \partial \tau} = 0$$

Then apply identity with $V^\mu = T^\mu$ to get

$$\nabla_T \nabla_S T^\mu - \nabla_S \overbrace{\cancel{\nabla_T T^\mu}}^{0 \text{ geodesic}} = R^\mu{}_{\nu\alpha\beta} T^\alpha S^\beta T^\nu$$

and $\nabla_S T^\mu = \nabla_T S^\mu$

$$\Rightarrow \quad \boxed{\nabla_T^2 S^\mu + E^\mu{}_\beta S^\beta = 0}$$

with $E^\mu{}_\beta = - R^\mu{}_{\nu\alpha\beta} T^\nu T^\alpha$ eqn of geodesic deviation.

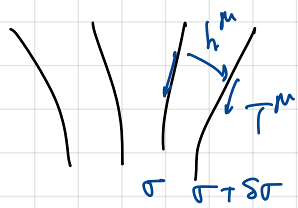
Consider closely separated geodesics

$$x^\mu(\tau, \sigma) \quad \text{and} \quad x^\mu(\tau, \sigma + \delta\sigma) \simeq x^\mu(\tau, \sigma) + h^\mu(\tau),$$

where $h^\mu(\tau) = \delta\sigma \, S^\mu$. Hence,

$$\nabla_T^2 h^\mu + E^\mu{}_\beta h^\beta = 0$$

to leading order in h^μ.



5. Einstein's Equations

5.1 Introduction and Motivation

Compare

	Newtonian gravity	GR
field	potential $\Phi(x,t)$	metric $g_{\alpha\beta}(x)$
EoS for point particle / test mass	$\frac{d^2 x_i}{dt^2} = +g_i = -\frac{\partial \Phi}{\partial x_i}$	$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$. $(\Gamma \sim \partial g)$
Local inertial frame at a point	Set $g_i = 0$	Set $\Gamma = 0$.
Nearby trajectories x_i, $x_i + h_i$	$\frac{d^2 h_i}{dt^2} + E_{ij} h_j = 0$, $E_{ij} = -\partial_j g_i = \partial_i \partial_j \Phi$ tidal tensor	$\nabla_T^2 h^\alpha + E^\alpha{}_\beta h^\beta = 0$, $E^\alpha{}_\beta = R^\alpha{}_{\mu\beta\nu} T^\mu T^\nu$ geodesic deviation
Field eqn	$\nabla^2 \Phi = 4\pi G \rho$, source ρ mass density	Einstein's eqn.?

(a) Statement of Einstein's eqn

(i) Without sources (in vacuum)

e.g. point mass at origin produces $\Phi = -MG/r$ obeying $\nabla^2 \Phi = 0$ for $r > 0$. Comparison above suggests

$$\boxed{R_{\mu\nu} = 0} \qquad \text{(Einstein's eqn in vacuum)}$$

(ii) With sources

Mass in NG replaced by energy and mom. in SR, described by

$$T^{\alpha\beta} = T^{\beta\alpha},$$

the <u>energy mom.</u> or <u>stress energy tensor</u>, obeying

$$\boxed{\partial_\alpha T^{\alpha\beta} = 0}$$

in Minkowski space. (4 conservation laws for $\beta=0$ (energy) and $\beta = 1, 2, 3$ (mom.)).

In curved space, expect this equivalent to

$$\boxed{\nabla_\alpha T^{\alpha\beta} = 0}$$

$T_{\alpha\beta}$ is a candidate for source on RHS of previous eqn, but $\nabla_\alpha R^{\alpha\beta} \neq 0$ in general. But, from contracted Bianchi identities,

$$\boxed{G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}}$$

the Einstein tensor, satisfies

$$\nabla_\alpha G_{\alpha\beta} = \nabla_\alpha R_{\alpha\beta} - \frac{1}{2}\nabla^\beta R = 0$$

This suggests

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}}$$

This is Einstein's eqn with source $T_{\mu\nu}$.

By comparing with Newtonian theory, $\kappa = \frac{8\pi G}{c^4}$.

Note: This is consistent with (i) because contracting with $g^{\mu\nu}$ gives

$$R - \frac{1}{2}(4) R = -R = \kappa T, \quad T = T^\mu{}_\mu.$$

Hence, Einstein eqn can be rewritten

$$\boxed{R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)}$$

(trace-reversed form)

(iii) With cosmological constant

An additional term can be added to Einstein tensor,

$$G_{\mu\nu} + \Lambda g_{\mu\nu},$$

with Λ cosmological const, and this obeys

$$\nabla^{\mu}(G_{\mu\nu} + \Lambda_{\mu\nu}) = 0$$

This results in

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}}$$

This is Einstein's eqn with cos. const.

This is the most general possibility for a sym. tensor that is covariantly const. (Lovelock's thm: $H_{\alpha\beta} = H_{\beta\alpha}$, $\nabla^{\alpha} H_{\alpha\beta}$ and $H_{\alpha\beta}$ depend only on g, ∂g, $\partial^2 g$ at each point)

Observation suggests $|\Lambda|^{-1/2}$ comparable to observable size of universe, so unless we are discussing problems in cosmology, we can safely set $\Lambda = 0$.

(b) Newtonian Limit

Consider

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

with $h_{\mu\nu}$ small (weak gravity) and assume in coord. with $x^0 = t$, that metric "almost static", so we can ignore all t derivatives.

This implies

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}(h_{\mu\beta,\alpha\nu} - h_{\nu\beta,\mu\alpha} - h_{\mu\alpha,\beta\nu} + h_{\nu\alpha,\mu\beta})$$

hence

$$R_{00} = g^{\mu\alpha} R_{\mu 0 \alpha 0} \xrightarrow{R \text{ small}} \eta^{\mu\alpha} R_{\mu 0 \alpha 0} = \delta^{ij} R_{i0j0} = \frac{1}{2}(-h_{00,ij})\delta^{ij}$$

Also assume $T_{00} = \rho$. (rest mass energy) is the only non-negligible cpt of $T_{\alpha\beta}$. and $\kappa\rho$ small.

Then

$$T = g^{\mu\nu} T_{\mu\nu} = \eta^{\mu\nu} T_{\mu\nu} = -\rho.$$

$$\Rightarrow \kappa\left(T_{00} - \tfrac{1}{2} T g_{00}\right) = \tfrac{1}{2}\kappa\rho$$

Hence Einstein eqn imply

$$R_{00} = \kappa\left(T_{00} - \tfrac{1}{2} T(-1)\right), \quad \text{or}$$

$$-\nabla^2 h_{00} = \kappa\rho, \qquad (1)$$

where $\nabla^2 = \delta^{ij}\nabla_i\nabla_j$.

But we found previously that for NR motion $v^i = \frac{dx^i}{dt} \ll 1$ and $\tau = t$,

$$\frac{dv^i}{dt} = -\Gamma^i_{\ 00} = \frac{1}{2}\delta^{ij}(h_{00,j}) \qquad (2)$$

This lead us to identity $h_{00} = -2\Phi$ in (2), then in (1) we recover

$$\nabla^2 \Phi = 4\pi G\rho$$

by choosing $\kappa = 8\pi G$ (with $c=1$).

5.2 Spherically symmetric vacuum sol^n: Schwartzschild metric

We will derive, in outline, the Schwartzschild metric from Einstein's eqn.

Thm (Birkhoff's thm) The most general sph. sym. sol^n of vacuum Einstein eqn is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$, for some const. M. Any such sol^n

is therefore static and stationary and asymptotically flat. (Mink.)

We assume sph. sym. acts on surfaces that are 2-spheres with coords θ, φ and r, t const. (these surfaces are orbits of symmetry group).

* Pf: General Lorentzian metric with sph. sym.

$$ds^2 = -A\,dt^2 + 2B\,dr\,dt + C\,dr^2 + R^2 d\Omega^2.$$

where A, B, C, R f^n of r, t.

Change coords $r \mapsto \tilde{r}(r,t)$, $t \mapsto \tilde{t}(r,t)$, then drop tildes.

First set $\tilde{r} = R(r,t)$. Then let $\tilde{t} = t + f(r,t)$, and check we can choose f to eliminate $dr\,dt$ term in ds^2.

In these coords, we have

$$ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 d\Omega^2$$

for some $\nu(r,t)$, $\lambda(r,t)$. Now compute Ricci / Einstein tensor and set $=0$.

Find $G_{tt} = 0 \Rightarrow -1 + e^{\lambda} + r\lambda' = 0$ (1)

$G_{rr} = 0 \Rightarrow 1 - e^{\lambda} + r\nu' = 0$ (2)

$G_{tr} = 0 \Rightarrow \dot{\lambda} = 0$ (3)

$G_{\phi\phi} = \sin^2\theta\, G_{\theta\theta} = 0$ (4)

All other cpts vanish.

Note (3) $\Rightarrow \lambda(r)$ indpt of t.

(1) + (2) $\Rightarrow (\lambda + \nu)' = 0 \Rightarrow \nu = -\lambda(r) + h(t)$, for some $h(t)$.

By further redefinition, $t \mapsto \tilde{t}$ with $d\tilde{t} = e^{1/2\, h(t)} dt$, we get

$$ds^2 = -e^{-\lambda} dt^2 + e^{\lambda} dr^2 + r^2 d\Omega^2$$

(1) and (2) now coincide and

$$e^{-\lambda}(1 - r\lambda') = \frac{d}{dr}(re^{-\lambda}) = 1$$

$$\Rightarrow \quad e^{-\lambda(r)} = 1 + \frac{K}{r} \quad \text{for some const. } K.$$

Schwartzschild with $K = -2M$. Finally, with simpler form of metric involving only $\lambda(r)$, content of (4) is

$$\lambda'' - (\lambda')^2 + \frac{2\lambda'}{r} = 0$$

which is also satisfied. $\square$

5.3 Matter-energy sources and $T_{\mu\nu}$.

(a) General comments

Conservation eqn in Mink. space for a 4-vector current J^α is

$$\partial_\alpha J^\alpha = 0$$

With choice of time and space coords $(c=1)$, we have

$$x^\alpha = (t, x^i), \quad \partial_\alpha = (\partial/\partial t, \partial/\partial x_i), \quad J^\alpha = (J^0, J^i), \; i=1,2,3.$$

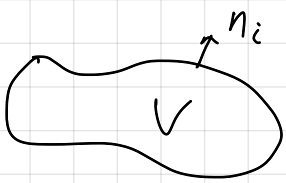
and

$$\frac{\partial J^0}{\partial t} = -\frac{\partial J^i}{\partial x^i}.$$

This implies

$$\frac{d}{dt}\int_V J^0 dV = -\int_V \frac{\partial}{\partial x^i} J^i dV = -\int_{\partial V} J^i n_i \, dS.$$

V some volume in $\{x^1, x^2, x^3\}$ space (indpt. of t) and n_i normal to boundary ∂V.



J^0: density. J^i: flux.

and if $J^\alpha = 0$ outside V (or vanishes in limit $V \to \mathbb{R}^3$), then

$$Q = \int_V J^0 dV$$

conserved.

For energy-momentum, we have tensor $T^{\alpha\beta}$ with conservation eqn

$$\partial_\alpha T^{\alpha\beta} = 0$$

$\beta = 0$: energy. $\beta = i$: momentum.

T^{00}: energy density

T^{i0} : energy flux in i dir

T^{0j} : mom. density, cpt in j dir.

T^{ij} : mom. flux / stress j cpt in i dir.

Generalisation to curved space:

$\nabla_\alpha J^\alpha = 0$ reduces to $\partial_\alpha J^\alpha$ in LIF, and $\nabla_\alpha T^{\alpha\beta} = 0$ similarly, local conservation law.

Scaler conservation law survives in curved space since

$$\nabla_\alpha J^\alpha = \partial_\alpha J^\alpha + \Gamma_\alpha{}^\alpha{}_\beta J^\beta$$
$$= \partial_\alpha J^\alpha + \left(\frac{1}{\sqrt{-g}} \partial_\beta (\sqrt{-g})\right) J^\beta, \qquad g = \det(g_{\alpha\beta}).$$
$$= \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g}\, J^\alpha) = 0.$$

Hence conserved charge of form

$$\int_V \sqrt{-g}\, J^0 \, dV.$$

For $T_{\alpha\beta}$, need additional symmetry of spacetime to be able to construct conserved quantities, given by a covector ξ_α called <u>Killing vector</u>.

$$\nabla_{(\alpha} \xi_{\beta)} = 0.$$

Then $J^\alpha = T^{\alpha\beta} \xi_\beta$ satisfies

$$\nabla_\alpha J^\alpha = (\nabla_\alpha T^{\alpha\beta}) \xi_\beta + T^{\alpha\beta} \nabla_{(\alpha} \xi_{\beta)} = 0.$$

(b) Electromagnetism – Maxwell theory

In SR, $\underline{E}$ and $\underline{B}$ field combined in antisym field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

where A_μ is 4-vec containing potential ϕ and $\underline{A}$

$$\Rightarrow \partial_{[\rho} F_{\mu\nu]} = 0 \qquad \text{(Bianchi identity)}$$

Contains 2 Maxwell's eqn.

Remaining Maxwell eqn

$$\partial_\mu F^{\mu\nu} = (\text{const}) J^\nu \longleftarrow J^\nu = (\rho, \underline{J}).$$

Note $\partial_\nu J^\nu = 0$ for consistency.

Energy momentum tensor

$$T_{\alpha\beta} = (\text{const.}) \left(F_{\alpha\mu} F_\beta{}^\mu - \frac{1}{4} \eta_{\alpha\beta} F_{\gamma\mu} F^{\gamma\mu} \right)$$

is conserved for $J^\mu = 0$:

$$\frac{1}{(\text{const.})} \partial^\alpha T_{\alpha\beta} = (\partial^\alpha F_{\alpha\mu}) F_\beta{}^\mu + F_{\alpha\mu} \partial^\alpha F_\beta{}^\mu - \frac{1}{2} F^{\gamma\mu} \partial_\beta F_{\gamma\mu}$$

$$= F^{\gamma\mu} \left(\partial_\gamma F_{\beta\mu} - \frac{1}{2} \partial_\beta F_{\gamma\mu} \right).$$

$$= -\frac{1}{2} F^{\gamma\mu} \left(\partial_\gamma F_{\beta\mu} + \partial_\mu F_{\beta\gamma} + \partial_\beta F_{\gamma\mu} \right) = 0.$$

Generalisation to conserved space:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (\Gamma \text{ cancel})$$

$$\Rightarrow \nabla_{[\rho} F_{\mu\nu]} = \partial_{[\rho} F_{\mu\nu]} = 0$$

Remaining Maxwell eqn

$$\nabla_\mu F^{\mu\nu} = 0. \qquad (\Gamma \text{ present!})$$

EM tensor

$$T_{\alpha\beta} = (\text{const.}) \left(F_{\alpha\mu} F_\beta{}^\mu - \frac{1}{4} g_{\alpha\beta} F_{\gamma\mu} F^{\gamma\mu} \right)$$

is then covariantly conserved.

$$\nabla^\alpha T_{\alpha\beta} = 0.$$

Example of sol^{n}s with EM source : Reissner–Nordström metric — modifies Schwartzschild to include electric charge Q in addition to M with $T_{\mu\nu}$, incorporating electric field due to point charge, modifying geometry

$$\left(1-\frac{r_s}{r}\right) \longrightarrow \left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right), \quad \text{with} \quad r_Q = (\text{const.})Q.$$

(c) Perfect fluids

In SR, fluid motion is given by velocity field U^α, which is "smoothed out" over any constituent particles. At a given point, we can move to rest frame of fluid with

$$U^\alpha = (1,0,0,0) \qquad (c=1)$$

Consider simple example with constituent particles with n number density measured in rest frame. — defines a scaler.

Then

$$N^\alpha = nU^\alpha = (n,0,0,0)$$

in rest frame, and

$$\partial_\alpha N^\alpha = 0$$

is the conservation of particle number.

If there is no relative motion of particles, then $\rho = mn$ is energy density with m rest mass, so

$$\rho = T^{00}$$

in rest frame.

If no other contribution to energy mom. tensor, then we can infer

$$T^{\alpha\beta} = \rho U^\alpha U^\beta.$$

More generally, we may have, e.g.

$$T^{ij} = \delta^{ij} p.$$ ← isotropic

where p is pressure of fluid

A <u>perfect fluid</u> is characterised by energy density ρ and pressure p, with

$$T^{\alpha\beta} = \text{diag}(\rho, p, p, p)$$

in rest frame, and

$$T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p \eta^{\alpha\beta}$$

in any frame.

Conservation of EM: $\partial_\alpha T^{\alpha\beta} = 0$ ⇒ generalisation to curved space

$$T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p g^{\alpha\beta},$$

and conservation

$$\nabla_\alpha T^{\alpha\beta} = 0$$

<u>Examples</u> (i) <u>Dust</u> : $p = 0$ (no pressure).

(ii) Blackbody radiation (photon gas) : $p = \frac{1}{3}\rho$.

(iii) Vacuum energy (dark energy) : $\rho = -p = \frac{\Lambda}{8\pi G}$, includes Λ in $T_{\alpha\beta}$.

<u>5.4 FLRW Spacetimes</u>

<u>(a) Homogeneous and Isotropic metrics</u>

Models of the universe at largest scales are req'd to be <u>spatially homogeneous</u> — "the same at all points" — Copernican principle, and <u>isotropic</u> — "the same in all directions".
(together called the Cosmological Principle).

Mathematically precise version: spacetime metric

$$ds^2 = -dt^2 + a(t)^2 \, d\Sigma^2$$

is coords $\{t, x^i\}$, with

$$d\Sigma^2 = h_{ij}(x) \, dx^i dx^j.$$

homogeneous and isotropic 3D geometry. It can be shown that this implies

$$d\Sigma_k^2 = \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right),$$

with

$$k = \begin{cases} 1 & \underline{\text{closed}} \\ 0 & \underline{\text{flat}} \\ -1 & \underline{\text{open}} \end{cases}$$

These are the Friedmann-Lemaître-Robertson-Walker (FLRW) metric/spacetime/universes. Easy to check that geodesic eqn have sol^n with $t = \tau$, x^i = const.. These are trajectories of <u>co-moving</u> particles, observers, galaxies, ... ($\perp$ to preferred spacelike slices).

Homogeneity and isotropy are indpt. but not unrelated: an observer who is not co-moving will not see the universe as isotropic.

Forms of metric above are implied by spatial homogeneity and isotropy via concept of manifold of constant curvature.

$$R_{ijkl} = K \left(h_{ik} h_{jl} - h_{il} h_{jk} \right)$$

for some const. K.

Alternative coords for metrics above

$\underline{k=1}$: $\quad d\Sigma_{+1}^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (r = \sin\chi)$.

3-dim sphere, const. +ve curvature

$\underline{k=0}$: $\quad d\Sigma_0^2$ is flat metric in $\mathbb{R}^3$ in sph. polars

$\underline{k=-1}$: $\quad d\Sigma_{-1}^2 = d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (r = \sinh\chi)$

3-dim hyperbolic space, const. -ve curvature,

(b) General Properties

(i) Hubbles Law

Distance between two comoving galaxies is

$$d(t) = a(t)R,$$

where R is the distance measured using t-indpt metric h_{ij}. Then

$$V(t) = \dot{d}(t) = \dot{a}(t)R = \left(\frac{\dot{a}}{a}\right) d(t),$$

where $\dot{} = \frac{d}{dt}$. i.e.

$$V(t) = H(t)\, d(t),$$

where $H(t) = \dot{a}/a$. E.g. $R^2 = \Delta x^i \Delta x^i$ in flat space $(k=0)$.
H_0 is the value of $H(t)$ at present time, is called, the Hubble const.

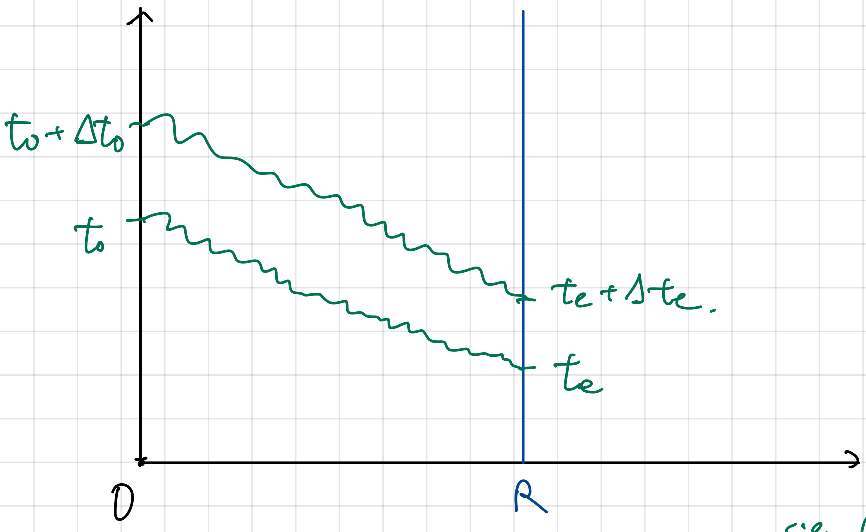
(ii) Cosmological redshift

Previously discussed grav. redshift in static spacetimes. For radial null geodesic in FLRW metric,

$$ds^2 = -dt^2 + \frac{a^2}{1-kr} dr^2 = 0.$$

$$\Rightarrow \quad \frac{dt}{a(t)} = \pm \frac{dr}{\sqrt{1-kr^2}}$$

Observer at $r=0$ (WLOG) sees pulses at t_0 and $t_0+\Delta t_0$ emitted by galaxy at $r=R$ at t_e and $t_e+\Delta t_e$. Both observer and galaxy are co-moving.



$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} = -\int_R^0 \frac{dr}{R\sqrt{1-kr^2}} .$$

sign for ingoing geodesic

If Δt_e and Δt_0 small compared to scale of variation of $a(t)$, then

$$\frac{\Delta t_0}{a(t_0)} - \frac{\Delta t_e}{a(t_e)} = 0$$

Hence,

$$\frac{\Delta t_0}{\Delta t_e} = \frac{\Delta \tau_0}{\Delta \tau_e} = \frac{\nu_e}{\nu_0} = \frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} > 1 \quad \Rightarrow \text{redshift}.$$

(comoving).

Note for <u>nearby</u> galaxies,

$$a(t_e) \approx a(t_0) + (t_e - t_0)\,\dot a(t_0) + \dots$$

$$\Rightarrow \frac{\Delta \tau_0}{\Delta \tau_e} \approx 1 + (t_0 - t_e)\,H(t_0) + \dots$$

5.5 Cosmological models

(a) FLRW dynamics

Einstein tensor for a FLRW metric given by

$$G_{tt} = \frac{3}{a^2}(\dot{a}^2 + k) \quad , \quad G_{ij} = -(2a\ddot{a} + \dot{a}^2 + k)\, h_{ij} .$$

Take a source a <u>comoving</u> perfect fluid

$$T_{tt} = \rho(t) \qquad \text{energy density}$$

$$T_{ij} = \underset{\substack{\uparrow \\ \text{pressure}}}{P(t)}\ a(t)^2\, h_{ij}$$

Einstein eqn $(G = c = 1)$ give

(tt) :
$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}} \qquad \text{(Friedmann eqn)} \qquad (1)$$

(ij) :
$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) \qquad (2).$$

Conservation of EM tensor gives

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 ,$$

or
$$\boxed{\frac{d}{dt}(a^3\rho) = -P\frac{d}{dt}(a^3)} \qquad (3)$$

rate of increase of energy in some fixed comoving volume

rate of working against pressure as volume changes.

Suppose fluid has eqn of state

$$P = w\rho .$$

for some const. w.

- $w = 0$ dust
- $w = 1/3$ radiation
- $w = -1$ vacuum energy.

Substituting in (3) and integrating ($w \geq -1$), we get

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^{3(1+w)}$$

Subscript "0" means present time.

For a given ept, we can sub in (1) and get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} - \frac{k}{a^2}$$

and integrate to find $a(t)$.

Energy density dilutes at different rates depending on w.

- $w = 0 \Rightarrow \rho(t) \propto a(t)^{-3}$
- $w = 1/3 \Rightarrow \rho(t) \propto a(t)^{-4}$
- $w = -1 \Rightarrow \rho$ const.

<u>(b) Some simple solutions</u>

Convenient to use <u>conformal time coords</u>

$$\eta = \int^t \frac{dt'}{a(t')}$$

to ensure

$$d\eta = \frac{dt}{a(t)}.$$

and FLRW metric then

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + d\Sigma^2\right)$$

Friedmann eqn (1) for single cpt fluid source becomes

$$\left(\frac{da}{d\eta}\right)^2 + k a^2 = C^2 a^{1-3w},$$

where $C^2 = \frac{8\pi}{3} \rho_0 a_0^{3(1+w)}$ const.

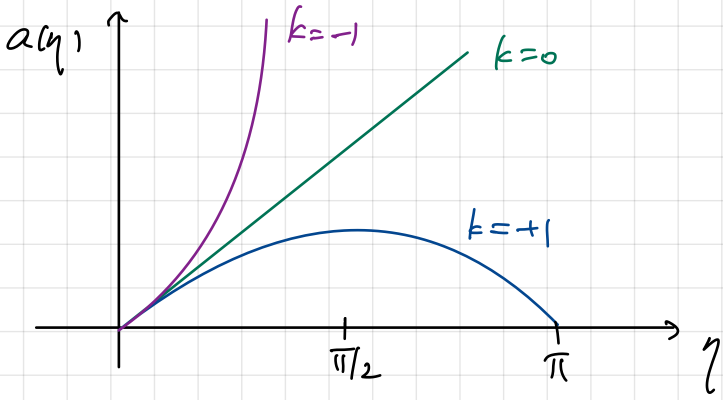
Now find sol^n for various values of w and k

<u>(i) $w = 1/3$ radiation</u>

$$a(\eta) = \begin{cases} C \sin \eta & k=1 \\ C\eta & k=0 \\ C \sinh \eta & k=-1 \end{cases}$$

with choice of integration const : for each sol^n there is a point in the past with $a=0$. shift η to choose this to be at $\eta=0$.

Singularity at $\eta=0$ is Big Bang.



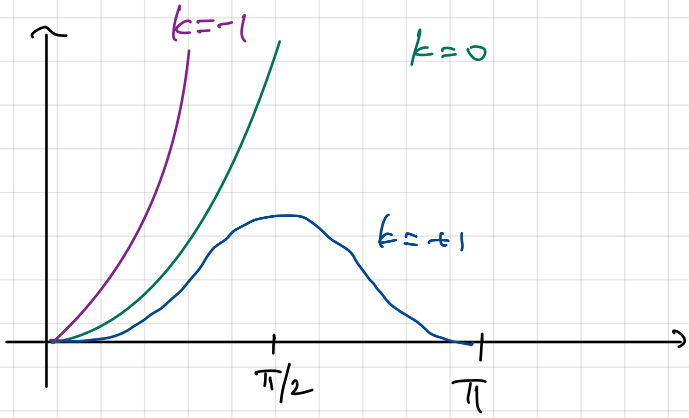
For $k=0, -1$, universe expands indefinitely.

For $k=+1$, it contracts and have big crunch at $\eta=\pi$.

<u>(ii) $W=0$ dust</u>

$$a(\eta) = \begin{cases} \frac{C^2}{2}(1-\cos\eta) & k=1 \\ (C^2/4)\,\eta^2 & k=0 \\ \frac{C^2}{2}(\cosh\eta - 1) & k=-1 \end{cases}$$

with $a=0$ at $\eta=0$ by choice of integration const.



As in (i), indefinite expansion for $k=0,-1$, and Big Crunch for $k=+1$.

(iii) $w=-1$ Vacuum energy / Dark energy

$$\rho = -P = \frac{\Lambda}{8\pi} > 0 \qquad (G=1)$$

Fried. $\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} = \lambda^2$, say

Sol^n:

$$a(t) = \frac{1}{\lambda}\cosh\lambda t \qquad k=+1$$

$$a(t) \propto e^{\pm\lambda t} \qquad k=0$$

$$a(t) = \frac{1}{\lambda}\sinh\lambda t \qquad k=-1$$

In fact, these represents different "slicings" of a maximally symmetric sol^n called <u>de Sitter spacetime</u>. No BB or BC in this model.

(iv) Einstein static universe

Combination of matter ρ_m ($P_m = 0$) and vacuum energy.

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}\left(\rho_m + \frac{\Lambda}{8\pi}\right). \qquad \text{(Fried.)}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) = -\frac{4\pi}{3}\left(\rho_m + \frac{\Lambda}{8\pi} - \frac{3\Lambda}{8\pi}\right).$$

Sol^n with $\dot{a} = \ddot{a} = 0$ for $k=1$ given by

$$\rho_m = \frac{\Lambda}{4\pi} \quad , \quad a^2 = \frac{1}{\Lambda}.$$

However, this sol^n is <u>unstable</u>.

(c) Our universe

(i) Homogeneity and isotropy hold to a good degree on length scales of several hundred Mpc. Our universe then can be regarded as a "borderline" FLRW model ($k \approx 0$) to a good accuracy.

(ii) Universe expanding with Hubble const.

$$H_0 = (\dot{a}/a)_0 \approx 72 \pm 7 \ (\text{km/s}) / \text{Mpc}.$$

(iii) At present, energy of universe consists of

$\sim 75\%$ dark energy

$\sim 25\%$ matter

$\sim$ Small of radiation.

For the matter, only about 4% is normal matter, and rest <u>dark matter</u>, which can be seen to affect motion of galaxies. It's nature is unknown.

6. The Linearized Einstein Equations

6.1 Reduction to the Wave eqn

Consider a perturbation about Mink. space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

and work in 1st order in $|h_{\mu\nu}| \ll 1$. So

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu},$$

where we now raise and lower indices using η:

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}.$$

We will make use of coord. changes

$$x^\alpha \longmapsto \tilde{x}^\alpha = x^\alpha - \xi^\alpha(x).$$

For $|\xi^\alpha(x)| \ll 1$, this produces a change in metric, or <u>gauge transformation</u>.

$$h_{\mu\nu} \longmapsto \tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu.$$

This follows because

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \eta^\alpha{}_\beta - \partial_\beta \xi^\alpha, \quad \frac{\partial x^\beta}{\partial \tilde{x}^\alpha} = \eta^\beta{}_\alpha + \partial_\alpha \xi^\beta.$$

together with

$$(\eta_{\mu\nu} + h_{\mu\nu}) \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} = \eta_{\alpha\beta} + \tilde{h}_{\alpha\beta}.$$

Substituting Einstein's eqn, we need to find

$$G_{\alpha\beta} = \frac{1}{2}\Big[-\partial_\mu \partial^\mu h_{\alpha\beta} - \partial_\alpha \partial_\beta h + \partial_\mu \partial_\alpha h_\beta{}^\mu + \partial_\mu \partial_\beta h_\alpha{}^\mu - \eta_{\alpha\beta}\partial_\mu \partial_\nu h^{\mu\nu} + \eta_{\alpha\beta}\partial_\mu \partial^\mu h \Big],$$

where $h_\mu{}^\mu = h$.

Looks complicated, but we can simplify by using

(i) $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$, or $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}$,

where $\bar{h} = \bar{h}_\mu{}^\mu = -h$.

(ii) Gauge transformation

$$\bar{h}_{\mu\nu} \mapsto \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\alpha \xi^\alpha$$

(a) Summary of some key expressions for weak field metric

With $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and working to 1st order for $h_{\mu\nu}$,

- $\Gamma_\alpha{}^\mu{}_\beta = \frac{1}{2} \eta^{\mu\gamma} \left(h_{\alpha\gamma,\beta} + h_{\beta\gamma,\alpha} - h_{\gamma,\alpha\beta} \right).$

- $R_{\mu\nu\alpha\beta} = \eta_{\mu\sigma} \left(\partial_\alpha \Gamma_\beta{}^\sigma{}_\gamma - \partial_\beta \Gamma_\alpha{}^\sigma{}_\nu \right)$

$$= \frac{1}{2} \left(h_{\mu\beta,\nu\alpha} - h_{\mu\alpha,\nu\beta} + h_{\nu\alpha,\mu\beta} - h_{\nu\beta,\mu\alpha} \right)$$

- $R_{\nu\beta} = \eta^{\mu\alpha} R_{\mu\nu\alpha\beta}$

$$= \frac{1}{2} \left(-\partial_\mu \partial^\mu h_{\alpha\beta} + \partial_\mu \partial_\beta h_\nu{}^\mu + \partial_\mu \partial_\nu h_\beta{}^\mu - \partial_\beta \partial_\nu h_\mu{}^\mu \right)$$

- $R = \eta^{\nu\beta} R_{\nu\beta} = -\partial_\mu \partial^\mu h + \partial_\mu \partial_\nu h^{\mu\nu}$, $h = h_\alpha{}^\alpha$.

(b) Lorentz or De Donder Gauge $\Rightarrow$ Wave eqn

Using (i)., we find

$$G_{\alpha\beta} = \frac{1}{2} \left(-\partial_\mu \partial^\mu \bar{h}_{\alpha\beta} - \eta_{\alpha\beta} \partial_\mu \partial_\nu \bar{h}^{\mu\nu} + \partial_\alpha \partial_\mu \bar{h}_\beta{}^\mu + \partial_\beta \partial_\mu \bar{h}_\alpha{}^\mu \right).$$

First term involves

$$\partial_\mu \partial^\mu = -\left(\frac{\partial}{\partial t}\right)^2 + \frac{\partial}{\partial x^i} \frac{\partial}{\partial x_i} = -\left(\frac{\partial}{\partial t}\right)^2 + \nabla^2$$

the wave operator.

In remaining terms, have expression

$$\partial_\mu \bar{h}_\alpha{}^\mu \longrightarrow \partial_\mu \bar{h}_\alpha{}^\mu + \partial_\mu \partial^\mu \xi_\alpha$$

under a gauge trans^n (ii).

We can then choose ξ^α to make this zero by solving wave eqn for ξ^α with specified choice. With this choice,

$$\boxed{\partial_\mu \bar{h}_\alpha{}^\mu = 0}$$

This is Lorentz / De Donder gauge

$$\Rightarrow G_{\alpha\beta} = -\frac{1}{2} \partial_\mu \partial^\mu \bar{h}_{\alpha\beta}$$

$$\Rightarrow \boxed{\partial_\mu \partial^\mu \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}}$$ linearised Einstein eqn.

[Aside: Comparison with EM.

Maxwell eqn: $\partial^\mu F_{\mu\alpha} = \partial^\mu(\partial_\mu A_\alpha - \partial_\alpha A_\mu) = J_\alpha$ (const.)

Gauge invariance: $A_\mu \mapsto A_\mu + \partial_\mu f$ for any f^n f on spacetime.

and can make $\partial_\mu A^\mu = 0$. (Lorentz gauge)., then

$$\partial^\mu \partial_\mu A_\alpha = (\text{const.}) J_\alpha$$]

6.2 The Newtonian limit revisited

(a). General weak field metric

Consider (as before) weak gravity and low velocities, for sources as well as test particles. Previously found (§2.3) that choosing

$$h_{00} \simeq -2\Phi.$$

implies geodesic motion for low velocities v reproduces N2 in grav. potential Φ. Then §5.1 used Einstein's eqn in form

$$R_{00} = \kappa \left(T_{00} - \frac{1}{2} T g_{00}\right), \quad \text{with } T_{00} = \rho \qquad (c=1)$$

to show that

$$\nabla^2 \Phi = 4\pi\rho \qquad (G=1).$$

and

$$\kappa = 8\pi. \qquad (G=c=1)$$

Now consider all cpts of Einstein's eqn recast as wave eqn (above) and introduce small parameter ε to set scale and organise approximations. (keep $c=1$).

In general, expect time variation of sources to result from motion, so $\partial/\partial t \ll \partial/\partial x$. and

$$\Phi \sim v^2 \sim \varepsilon \ll 1 \qquad (c=1)$$

Wave eqn becomes

$$\nabla^2 \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta},$$

$$T_{00} = \rho + O(\varepsilon^2) \sim O(\varepsilon)$$
$$T_{0i} \sim T_{00} v_i \sim O(\varepsilon^{3/2})$$
$$T_{ij} \sim T_{00} v_i v_j \sim O(\varepsilon^2).$$

consistent with perfect fluid with $p \sim \rho v^2$.

Same order for h_{00}

h_{0i},

h_{ij}.

Note

$$\nabla^2 \bar{h}_{00} = -16\pi T_{00} = -16\pi\rho$$
$$\Rightarrow \bar{h}_{00} = -4\Phi.$$

by comparing with $\nabla^2 \Phi = 4\pi\rho$.

$$\Rightarrow \bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = 4\Phi \sim O(\varepsilon).$$

$$\Rightarrow h_{00} = \bar{h}_{00} - \frac{1}{2}\eta_{00}\bar{h} = -4\Phi - \frac{1}{2}(-1)(4\Phi) = -2\Phi. \quad \text{(as before)}$$

$$h_{0i} = \bar{h}_{0i} \sim O(\varepsilon^{3/2}).$$

$$\begin{aligned} h_{ij} &= \bar{h}_{ij} - \frac{1}{2}\delta_{ij}\bar{h} \\ &= O(\varepsilon^2) - \frac{1}{2}\delta_{ij}\cdot 4\Phi \\ &= -2\Phi\,\delta_{ij} + O(\varepsilon^2) \end{aligned}$$

Weak field metric:

$$\boxed{ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Phi)\,\delta_{ij}\,dx^i dx^j,}$$

with $\nabla^2\Phi = 4\pi\rho$.

(b) Sol^n for point source and Schwarzschild

For mass M at $r=0$, take $\Phi = -M/r$ $(G=1)$ with $\nabla^2\Phi = 0$, $r>0$.

This gives weak field metric

$$ds^2 = (1-\frac{2M}{r})\,dt^2 + (1+\frac{2M}{r})(dr^2 + r^2 d\Omega^2).$$

Not quite Schwarzschild! But can reach standard form by re-definition.

$$R^2 = (1+\frac{2M}{r})\, r^2$$

$$\Rightarrow R = r\left(1+\frac{2M}{r}\right)^{1/2} = r\left(1+\frac{M}{r} + O\left(\left(\frac{M}{r}\right)^2\right)\right).$$

$$\Rightarrow dR = dr\left(1 + o\left(\left(\frac{M}{r}\right)^2\right)\right), \quad \frac{M}{R} = \frac{M}{r} + o\left(\left(\frac{M}{R}\right)^2\right).$$

Hence, to first order in M/R, we get

$$ds^2 = -\left(1-\frac{2M}{R}\right)dt^2 + \left(1+\frac{2M}{R}\right)dR^2 + R^2 d\Omega^2.$$

agrees with Schwarzschild for $R \gg M$.

6.3 Gravitational Waves

(a) Plane Waves Sol^n

In vacuum, linearised Einstein eqn are

$$\partial^\mu \partial_\mu \bar{h}_{\alpha\beta} = 0$$

These admit wave sol^n, conveniently written

$$\bar{h}_{\alpha\beta} = \mathrm{Re}\left(H_{\alpha\beta}\, e^{ik_\rho x^\rho} \right).$$

where k^μ real wave vec., $H_{\alpha\beta} = H_{\beta\alpha}$ complex matrix and we will understand "Re" and work mainly with complex expressions.

This is a sol^n provided

$$k_\mu k^\mu = 0,$$

ie. k^μ null vec.

Lorentz gauge condition

$$\partial^\mu \bar{h}_{\mu\alpha} = 0 \iff k^\mu H_{\mu\alpha} = 0$$

Example $\quad k^\mu = k(1,0,0,1), \quad k_\mu = k(-1,0,0,1)$

$$\exp(ik_\mu x^\mu) = \exp(-ik(t-x^3)), \quad t = x^0$$

get wave travelling in z direction with speed $c=1$.

Gauge condition becomes

$$H_{0\alpha} + H_{3\alpha} = 0 \quad \text{for any } \alpha.$$

Still have gauge freedom, using any ξ satisfying

$$\partial^\mu \partial_\mu \xi_\alpha = 0.$$

preserves Lorentz gauge.

For plane wave sol^n written above, consider

$$\xi_\alpha = -i X_\alpha e^{ik_\rho x^\rho}$$

Then

$$\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \eta_{\alpha\beta} \partial_\gamma \xi^\gamma = (k_\alpha X_\beta + k_\beta X_\alpha - \eta_{\alpha\beta} k_\gamma X^\gamma) e^{ik_\rho x^\rho}.$$

Effect on sol^n is

$$H_{\alpha\beta} \mapsto H_{\alpha\beta} + (k_\alpha X_\beta + k_\beta X_\alpha - \eta_{\alpha\beta} k_\gamma X^\gamma)$$

Check that $k^\beta H_{\beta\alpha} \mapsto k^\beta H_{\beta\alpha}$ follows since $k_\gamma k^\gamma = 0$.

Return to example of wave in z-dir. Set $k=1$, so

$$k^\mu = (1, 0, 0, 1), \qquad k_\mu = (-1, 0, 0, 1)$$

Consider $X_\alpha = (0, C, 0, 0)$. C const.

$$H_{01} \longrightarrow H_{01} + k_0 X_1 + \underbrace{k_1 X_0}_{=0} - \underbrace{\eta_{01} k_\gamma X^\gamma}_{=0} = H_{01} - C.$$

Using this, we can set $H_{01} = -H_{31} = 0$.

Similarly, set $H_{02} = -H_{32} = 0$ by considering $X_\alpha = (0, 0, C', 0)$.

Now choose $X_\alpha = (A, 0, 0, B)$. No effect on H_{01} or H_{02}, but

$$\begin{aligned} H_{00} \longrightarrow & H_{00} + k_0 X_0 + k_0 X_0 - \eta_{00} k^\gamma X_\gamma \\ = & H_{00} - 2A - (-1)(A+B). \\ = & H_{00} + B - A. \end{aligned}$$

and

$$\begin{aligned} H_{03} \longrightarrow & H_{03} + k_0 X_3 + k_3 X_0 - \underbrace{\eta_{03} (A+B)}_{=0} \\ = & H_{03} - B + A \end{aligned}$$

consistent with $H_{00} + H_{03} = 0$. (gauge choice).

Last step, consider H_{ij} with $i, j = 1, 2$, and $X_\alpha = (A, 0, 0, B)$.

$$H_{ij} \to H_{ij} - \delta_{ij} (A+B). \quad \text{for } i, j = 1, 2.$$

Taking account of everything above,

$$H_{0\mu} = H_{3\mu} = 0 \, , \qquad H_{11} = -H_{22} \, .$$

This is the transverse traceless gauge.

The conclusion is that we can take

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 transverse polarisations for a grav. wave, given by constants $H_+, H_\times$.

Note also that this gauge choice $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$.

(b) Effect on Test Particles

Consider grav. wave in z dir. as in (a), with transverse traceless gauge choice.

Note

$$\Gamma^\alpha_{\ 0 0} = \frac{1}{2} \eta^{\alpha\beta} \left(\partial_0 h_{\beta 0} + \partial_0 h_{0\beta} - \partial_\beta h_{00} \right) = 0 \, . \quad (H_{\mu 0} = 0)$$

So for a particle initially at rest with $U^\alpha = (1,0,0,0)$, then

$$\frac{dU^\alpha}{d\tau} = -\Gamma^\alpha_{\ 0 0} = 0 \, . \qquad (\text{at } t=0) \, .$$

Hence particles in x-y (x^1-x^2) plane don't change their coord. positions due to the wave.

However, proper separation does oscillate, as given by metric,

$$ds^2 = -dt^2 + (1+h_+)\, dx^2 + (1-h_-)\, dy^2 + 2h_\times \, dx \, dy + dz^2 .$$

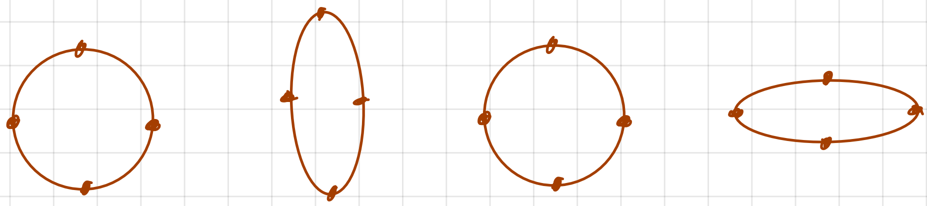
Where $h_{\substack{+\\ \times}} = \mathrm{Re} \left(H_{\substack{+\\ \times}} \, e^{i k_\rho x^\rho} \right)$.

Examples

- $H_+ \neq 0$, $H_\times = 0$. For pairs of particles with coords

$(\pm\delta, 0)$ in x-y plane, $\delta s^2 = 4(1+h_+)\delta^2$ (δ small)

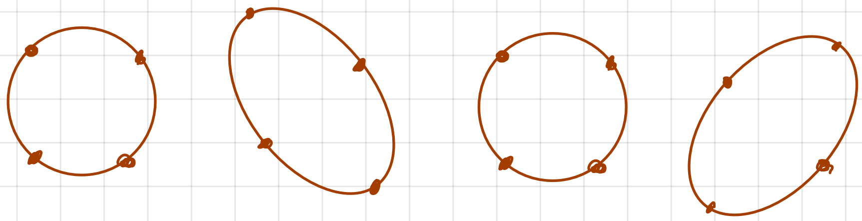
$(0, \pm\delta)$. $\delta s^2 = 4(1-h_+)\delta^2$



- $H_+ = 0$, $H_\times \neq 0$. For pairs with coords

$$\frac{1}{\sqrt{2}}(\pm\delta, \pm\delta), \qquad \delta s^2 = 4(1+h_\times)\delta^2$$

$$\frac{1}{\sqrt{2}}(\pm\delta, \mp\delta) \qquad \delta s^2 = 4(1-h_\times)\delta^2.$$

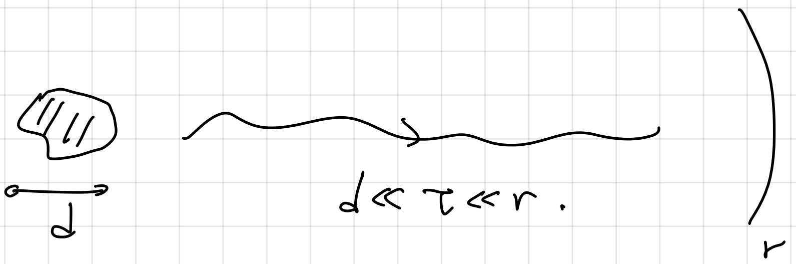


* (c) Detection and Quadrupole Formula

Direct detection in 2015 (Ligo) – accuracy required 10^{-21}.

Previously, indirect evidence from observation of binary pulsar (Hulse-Taylor).

Consider source of lengthscale d and motion within source on time scale τ.



At very large radius r, total energy flux produced, averaged over time scale large compared to τ is

$$\langle P \rangle_t = \frac{1}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle_{t-r}. \qquad (G=c=1)$$

where

$$Q_{ij} = I_{ij} - \frac{1}{3} I_{kk} \delta_{ij}$$

reduced quadrupole tensor, and

$$I_{ij}(t) = \int_{\text{Source}} x_i x_j \rho(\underline{x}, t) \, d^3\underline{x}$$

Metric fluctuations at r depend on $\ddot{I}_{ij}(t-r)$.

Prediction from formula above matches observations for binary pulsar.

Orbital period $\approx 0.05s$, and change due to emission of grav. waves is $\approx 10 \mu s$ per year.

(d) PP-wave metrics

There are some (fairly) simple exact sol^{n}s of vacuum Einstein eqn corresponding to plane fronted waves.

$$ds^2 = H(u, x, y)\, du^2 \underbrace{- du\, dv}_{dt^2 - dz^2} + dx^2 + dy^2.$$

with $u = t - z$, $v = t + z$.

For such a metric, the only cpt of Ricci tensor which is not identically 0 is R_{uu}, and

$$R_{uu} = 0 \iff (\partial_x^2 + \partial_y^2) H = 0.$$

Particular sol^n: $H = H_0 e^{iku}$ for H_0, k const.

7. Black Holes

Schwarzschild metric is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2.$$

appears to have singularities at $r = 2M$ and $r=0$.

Scaler invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6} \to \infty$ as $r \to 0$, but it is finite at $r=2M$. Suggests $r=0$ genuine singularity, but $r=2M$ may not be.

7.1 Radial Geodesics in $r>2M$

$$\left(1-\frac{2M}{r}\right)\dot{t} = E = 1.$$

by choice of affine parameter, and

$$\left(1-\frac{2M}{r}\right)\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 = \kappa = \begin{cases} 1 & \text{timelike} \\ 0 & \text{null.} \end{cases}$$

(a) Timelike geodesics

Motion of an infalling observer

$$1-\dot{r}^2 = 1-\frac{2M}{r} \qquad (E=1 \leftrightarrow \dot{r}=0 \text{ as } r\to\infty).$$

$$\Rightarrow r^{1/2}\dot{r} = -\sqrt{2M} \qquad (- \text{ sign for infalling})$$

$$\Rightarrow \frac{2}{3}\left(r_1^{3/2} - r_0^{3/2}\right) = -\sqrt{2M}\left(\tau_1 - \tau_0\right)$$

From $r_0 = r(\tau_0)$, we reach $r_1 = r(\tau_1) < r_0$ in finite proper time.

Nothing special happens as $r_1 \to 2M$.

In terms of coordinate time t, however,

$$r^{1/2}\frac{dr}{dt} = -\sqrt{2M}\left(1-\frac{2M}{r}\right).$$

$$\Rightarrow -\sqrt{2M}\left(t_1 - t_0\right) = \int_{r_0}^{r_1} \frac{r^{3/2}}{r-2M}\,dr.$$

log divergent as $r_1 \to 2M$.

t is the proper time for a distant observer fixed at $r \gg M$, hence they never see infalling observer reach $r=2M$.

(b) Null geodesics

$\dot{r}=1$ and r is affine parameter.

$$dt = \pm\left(1-\frac{2M}{r}\right)^{-1} dr = \pm\, dr_*.$$

for $\left\{\begin{matrix}\text{outgoing}\\ \text{ingoing}\end{matrix}\right\}$ curves, where

$$\frac{dr_*}{dr} = \left(1-\frac{2M}{r}\right)^{-1} = 1+\frac{1}{\frac{r}{2M}-1}.$$

Choose

$$r_* = r + 2M\log\left|\frac{r}{2M}-1\right|. \quad (\text{for the moment}, r>2M).$$

Null geodesics in $r>2M$ given by

$$t-r_* = \text{const.}\ (\text{outgoing}) \quad \text{or} \quad t+r_* = \text{const.}\ (\text{ingoing})$$

7.2 Causal Structure and Horizon of Schwarzschild Black Hole.

(a) Ingoing Eddington-Finkelstein coords

Take v, r, θ, ϕ as spacetime coords, where

$$v = t + r_*$$

const. on ingoing null geodesics.

$$dv = dt + dr_* = dt + dr\cdot\left(1-\frac{2M}{r}\right)^{-1}$$

Schwarzschild becomes

$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2.$$

Singularity at $r=0$ and non-sing for $0<r<\infty$, $-\infty<v<\infty$ (and θ, ϕ as usual).

Note 2x2 block $\begin{pmatrix} -(1-2M/r) & 1 \\ 1 & 0\end{pmatrix}$ has $\det \neq 0$.

Change in coords using r_* is valid for $r>2M$, but can also be applied for $r<2M$ (same relations) and metrics agree. Hence, these ingoing EF coords allows us to understand from one region to another.

(b) <u>Null geodesic using ingoing EF coords.</u>

$$-\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr = 0$$

Satisfied for

(i) $dr=0 \Rightarrow v = \text{const.}$ now holds for all $r>0$.

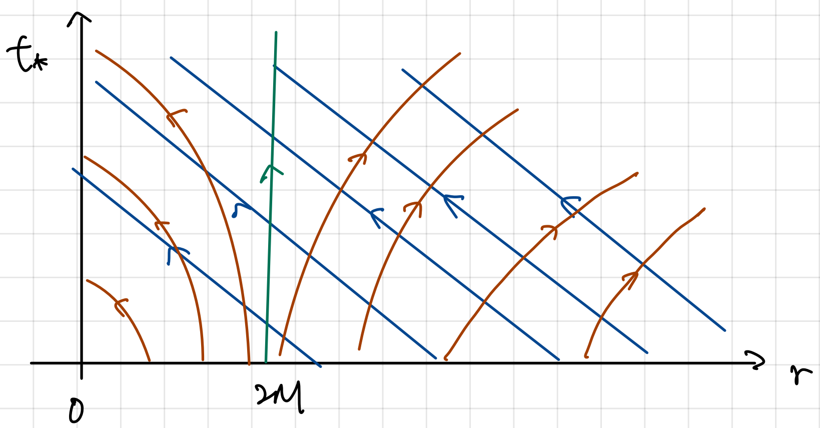
(ii) $\left(1-\frac{2M}{r}\right)dv = 2\,dr \Rightarrow r = 2M$ (new), or

$$\frac{dv}{dr} = 2 + \frac{2}{\frac{r}{2M}-1} \Rightarrow v = 2r + 4M\log\left|\frac{r}{2M}-1\right| + \text{const.}$$

To sketch these sol^{n}s, let

$$t_* = v - r \qquad (t_* + r = t + r_*).$$

and plot this against r.



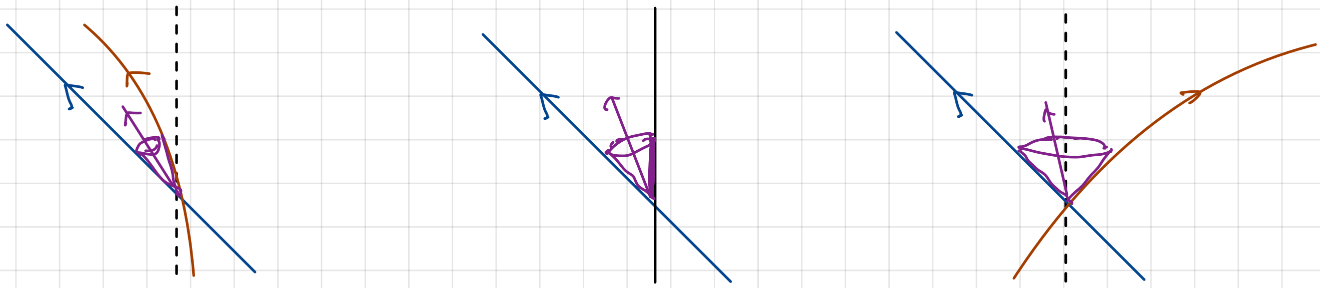
For sketch, note for sol^n in (ii),

$$t_* = r + 4M\log\left|\frac{r}{2M}-1\right| + \text{const.}$$

$$\Rightarrow \frac{dt_*}{dr} = \frac{\frac{r}{2M}+1}{\frac{r}{2M}-1} \gtrless 0 \text{ for } r \gtrless 2M.$$

- In region I : $r > 2M$, have ingoing and outgoing null geodesics — agree with §7.1 (b)
- In region II : $r < 2M$, null geodesics all ingoing and reach $r=0$ in finite affine parameter.

- Tangent vectors to timelike curves (radial) must lie inside forward light cones as shown.



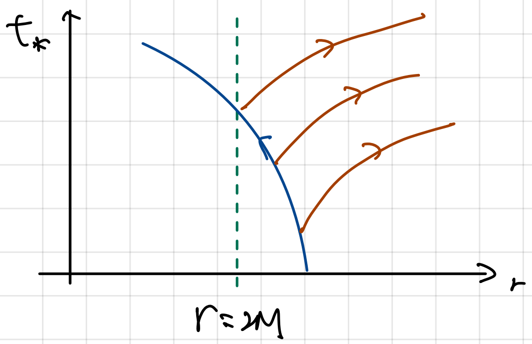
$r < 2M$. $\qquad$ $r = 2M$. $\qquad$ $r > 2M$.

$\Rightarrow$ any massive particles in II reaches $r=0$ in finite proper time.

- Particles can pass from I to II, but no particle or signal can pass from II to I : this is a <u>black hole</u> with <u>event horizon</u> at $r = 2M$.
- $r = 2M$ is a null surface, and area of slice with coords θ, ϕ is $4\pi(2M)^2 = 16\pi M^2$, the area of black hole.
- Now, reconsidering metric in original coords, we see that r is actually a time-like coord. and everything terminates at some fixed value $r=0$

7.3 Gravitational Collapse

- Blackhole forms if a star contract under its own gravity until enough mass concentrate in a small enough region
- Above $\sim 2M_\odot$, end result cannot be white dwarf or neutron star, and either star shed mass → supernova, or it collapses to form a black hole.



- Considering spherical collapse with infalling observer at surface of star, sending signals to distant observer, as shown
- Signals sent will be redshifted and redshift $\to \infty$ as horizon approach. Collapsing star appears to "freeze" and go dark in a short time.

7.4 Kruskal extension

Ingoing EF coords v, r, θ, ϕ, with $v = t + r_*$ used above.

But could use instead outgoing EF coords u, r, θ, ϕ, with

$$u = t - r_*.$$

This allows region I ($r > 2M$) to be extended to a region II with $0 < r < 2M$.

II is different from I because now any null curves in II starts at $r=0$, get opposite of a black hole — a white hole.

To clarify this, introduce Kruskal coords U, V, θ, ϕ. with

$$U = -e^{-u/4M}, \qquad V = e^{v/4M}.$$

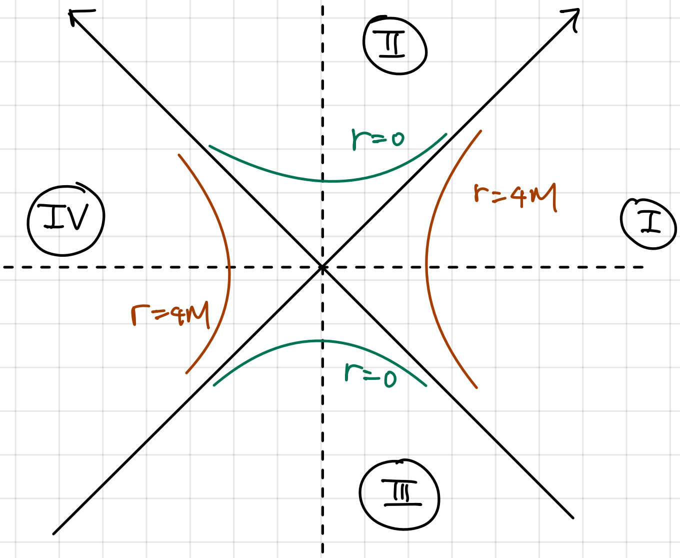
and note

$$UV = -e^{+r_*/2M} = -e^{r/2M}\left(\frac{r}{2M} - 1\right).$$

Using this to define $r(U,V)$ we have

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2 d\Omega^2$$

Metric can be extended to I, II, III, IV.



$r = $ const. $\neq 2M$: hyperbola

$r = 2M$: axes

Only region I and II are relevant for physical applications.
Region IV isometric to I but no signal can pass through them.

7.5 Comments

(a) Singularity Theorems and Geodesic incompleteness

- Original sing. thm. (Penrose) : small departures from sph. sym. in grav. collapse do not affect formation of black hole (w/ sing.)

- Similar approach to expanding universe (with conditions on matter content) imply existence of some initial sing., or <u>Big Bang</u>.
- Signal of a sing. is <u>geodesic incompleteness</u>: cannot extend to arbitrary values of parameters to the future or past. The termination point is the sin. o/w, space is called geodesically complete.

*(b) Black hole thermodynamics

No complete theory of quantum gravity, but Hawking showed using QM in curved space, that black holes should radiate with a thermal spectrum and temp.

$$T = \frac{\hbar c^3}{8\pi G M k_B} \quad \text{(Hawking temp.)}$$

In addition, black hole has an entropy given by

$$S_{BH} = \frac{k_B c^3}{4\hbar G} A \quad \text{(Beckenstein – Hawking).}$$

For a solar mass blackhole, have

$$T \sim 6 \times 10^{-8}\ \mathrm{K}.$$

The black hole will "evaporate" due to loss of energy and for one of this mass, time scale ~ much larger than age of universe.