Fluid Dynamics

1. Revision of IB materials

Continuum hypothesis

By averaging over a small volume, define "continuum" properties

- density $\rho(\underline{x}, t)$
- velocity $\underline{u}(\underline{x}, t)$
- pressure $p(\underline{x}, t)$.

Time derivatives

A fluid particle has position $\underline{x}(t)$ s.t.

$$\dot{\underline{x}} = \underline{u}(\underline{x}, t)$$

The rate of change of a quantity seen by a fluid particle is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla .$$

This is the material derivative. In particular, the acceleration of a fluid particle is

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u}$$

Mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

or equivalently,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0$$

For incompressible fluid, $D\rho / Dt = 0 \Rightarrow \nabla \cdot \underline{u} = 0$

We restrict to incompressible fluids with uniform density.

So ρ indpt. of x, t.

Kinetic boundary condition

Applying conservation of mass to a region close to boundary S,

$$\rho \underline{u}^- \cdot \underline{n} = \rho \underline{u}^+ \cdot \underline{n} .$$

At a fixed boundary,

$$\underline{u} \cdot \underline{n} = 0$$

For moving boundary $F(\underline{x},t) = 0$, the surface consist of material points so

$$DF/Dt = 0$$

for free surface problems.

Momentum conservation

Assume the only surface force on a material surface is given by pressure as $-p\,\underline{n}\,dS$, then Newton's eqn of motion is

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \underline{F}(\underline{x},t),$$

where $\underline{F}(\underline{x},t)$ is the force per unit volume, e.g. ρg.

Dynamic Boundary Condition

Momentum conservation appled near S gives

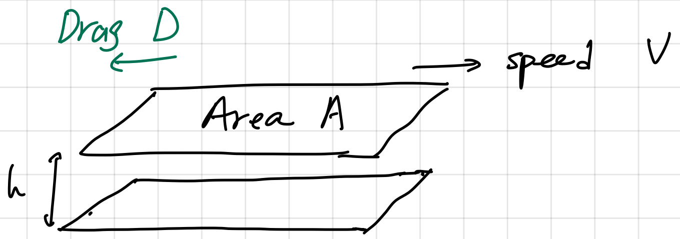
$$-p^- \underline{n} = -p^+ \underline{n},$$

i.e. pressure is continuous across S.

2. Newtonian viscous fluids

2.1 Dynamic viscosity (see IB)

Measurement using Couette viscometer



Empirical observation gives

$$D \propto \frac{AV}{h}.$$

Note the linearity on V.

Tangential stress

$$\tau \equiv \frac{D}{A} = \mu \frac{V}{h}$$

Const. of proportionality is the dynamic viscosity.

Units : $[\mu] = \frac{N}{m^2} \cdot \frac{m}{m\,s^{-1}} = kg\ m^{-1} s^{-1}. = Pa\ s.$

Typical values :

	μ
water	10^{-3} Pa s
air	10^{-5} Pa s
syrup	10^{2} Pa s

Note μ_{syrup} depend strongly on temperature.

2.2 Velocity gradient

Rate-of-strain tensor and vorticity consider a real field $\underline{u}(\underline{x})$ in the neighbourhood of a fixed point.

Taylor expanding,

$$\underline{u}(\underline{x}) = \underline{u}(\underline{x}_0) + (\underline{x} - \underline{x}_0) \cdot \nabla \underline{u}|_{\underline{x}_0} + \cdots$$

WLOG, set $\underline{x}_0 = \underline{0}$. In suffix notation,

$$u_i(\underline{x}) = u_i(\underline{0}) + x_j \left.\frac{\partial u_i}{\partial x_j}\right|_{\underline{0}} + \cdots$$

Decompose the velocity gradient $\nabla \underline{u} = \underline{\underline{e}} + \underline{\underline{\Omega}}$, where $\underline{\underline{e}}$ symmetric, $\underline{\underline{\Omega}}$ antisymmetric.

$$\underline{\underline{e}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right), \quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\underline{\underline{\Omega}} = \frac{1}{2}\left(\nabla \underline{u} - \nabla \underline{u}^T\right), \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right).$$

Define vorticity

$$\underline{\omega} = \nabla \wedge \underline{u}.$$

Exercise Show $(\underline{\omega} \wedge \underline{x})_i = 2\Omega_{ij} x_j$.

So

$$u_i(\underline{x}) = u_i(\underline{0}) + e_{ij} x_j + \Omega_{ij} x_j ,$$

or

$$\underline{u}(\underline{x}) = \underline{u}(\underline{0}) + \underline{\underline{e}}\,\underline{x} + \frac{1}{2}\underline{\omega} \wedge \underline{x}$$

$\underline{\underline{e}}$ is real symmetric, so has real, orthogonal evec with respect to $\underline{\underline{e}} = \mathrm{diag}(e_1, e_2, e_3)$.

Incompressibility $\Rightarrow$ $\nabla \cdot \underline{u} = 0 \Rightarrow \dfrac{\partial u_k}{\partial x_k} = 0.$

$$\Rightarrow \mathrm{tr}(\underline{\underline{e}}) = 0$$

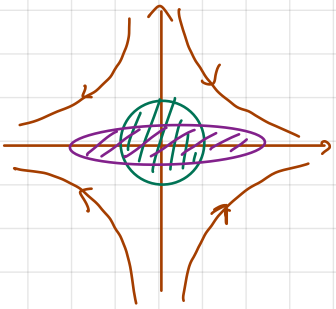
$$\Rightarrow e_1 + e_2 + e_3 = 0$$

$\underline{\underline{e}}$ is called the rate-of-strain tensor.

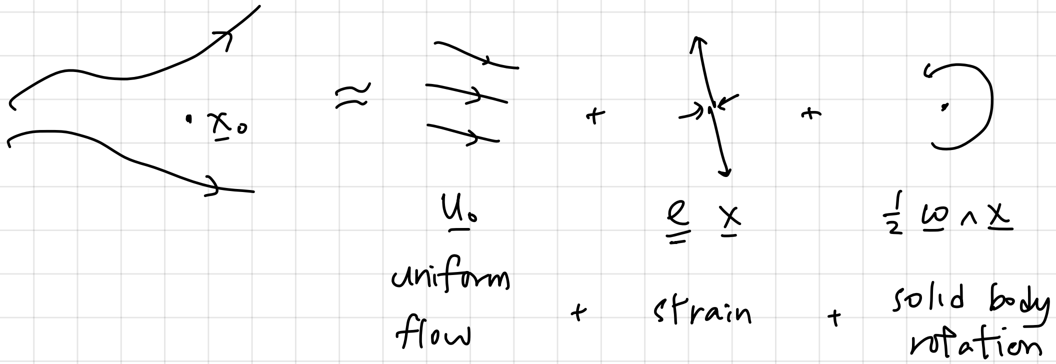
Example $e_2 = -e_1$, $e_3 = 0$, flow $\underline{u} = \underline{\underline{e}}\,\underline{x} = (x, -y, 0)$

Has streamf^n $\psi = xy$.

Pure straining flow :

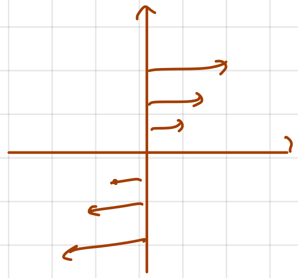


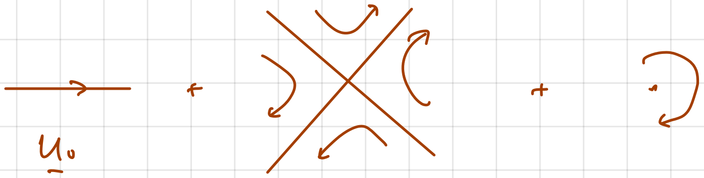
Taylor Series in pictures



Example (see Ex 1 #2)

$\underline{u} = (\gamma y, 0, 0)$





2.3 Stress Tensor

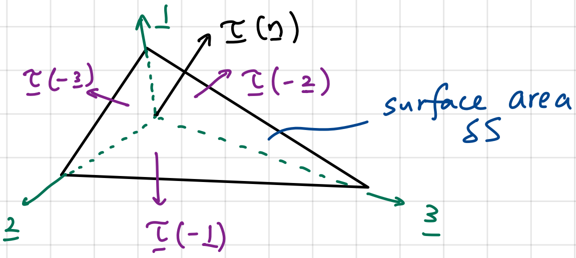
2.3.1 Stress tensor

In continuum mechanics, we consider two types of forces:

(i) body forces $\underline{f}$ that act per unit volume, e.g. gravity, $\underline{f} = \rho \underline{g}$.

(ii) surface forces $\underline{\tau}$ that act per unit area, e.g. pressure p, viscous stress.

Denote by $\underline{\tau}(\underline{n})$ the stress acting on a surface with normal $\underline{n}$. Consider all the forces balance on an infinitestimal tetrahedron of characteristic size L, with three sides coinciding with coordinate planes.



$\underline{1}, \underline{2}, \underline{3}$ are unit vectors forming an orthogonal basis.

Note that $\dfrac{\text{volume forces}}{\text{surface forces}} \sim \dfrac{O(L^3)}{O(L^2)} = O(L) \to 0$ as $L \to 0$.

So the surface forces must balance by themselves.

$$\underline{\tau}(\underline{n})\,\delta S + \underline{\tau}(-\underline{1})\,\delta S_1 + \underline{\tau}(-\underline{2})\,\delta S_2 + \underline{\tau}(-\underline{3})\,\delta S_3 = 0$$

But $\delta S_i = n_i\,\delta S$, so

$$\underline{\tau} = -\underline{\tau}(-\underline{1})\,n_1 - \underline{\tau}(-\underline{2})\,n_2 - \underline{\tau}(-\underline{3})\,n_3.$$

$$\Rightarrow \quad \underline{\tau} = \begin{pmatrix} | & | & | \\ -\underline{\tau}(-\underline{1}) & -\underline{\tau}(-\underline{2}) & -\underline{\tau}(-\underline{3}) \\ | & | & | \end{pmatrix} \underline{n}$$

$$\Rightarrow \quad \boxed{\underline{\tau} = \underline{\underline{\sigma}} \cdot \underline{n}}$$

Note from N3 that $\underline{\tau}(-\underline{j}) = -\underline{\tau}(\underline{j}) \Rightarrow \sigma_{ij} = \tau_i(\underline{j})$ is equal to the i-th component of the force per unit area acting on a plane with normal $\underline{j}$.

2.3.2 Symmetry

Consider the angular momentum balance on a small (so surface force balance) arbitrary volume V.

$$0 = \int_{\partial V} \underline{x} \wedge (\sigma \cdot \underline{n})\, dS \qquad \text{torque}$$

$$= \int_{\partial V} \varepsilon_{ijk}\, x_j\, \sigma_{kl}\, n_l \, dS$$

$$= \int_V \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk}\, x_j\, \sigma_{kl} \right) dS \qquad \text{(div. thm.)}$$

$$= \int_V \varepsilon_{ijk} \Big(\underbrace{\delta_{jl}\, \sigma_{kl}}_{O(1)} + \underbrace{x_j\, \partial_l\, \sigma_{kl}}_{O(L)} \Big)\, dS \quad \text{provided } \underline{\underline{\sigma}} \text{ differentiable}$$

$$\sim \int_V \varepsilon_{ijk}\, \sigma_{kj}\, dV.$$

This is true for abitrary V, so $\varepsilon_{ijk}\, \sigma_{kj} = 0$, so σ_{ij} symmetric.

2.4 Constitutive equation for a Newtonian fluid.

Def^n A fluid is a material that flows under (anisotropic) stress.

The stress tensor has two parts:

(i) isotropic pressure, acting normal to surface, indpt of direction.

(ii) deviatoric part, which acts tangentially and normally.

Decompose $\sigma_{ij} = -p\,\delta_{ij} + \sigma_{ij}^{dev}$, with $\sigma_{jj}^{dev} = 0$

$$\Rightarrow p = -\frac{1}{3}\sigma_{jj}$$

Relative motion in a nbd of a point is proportional to the velocity gradient $\nabla \underline{u}$, so expect $\underline{\underline{\sigma}}^{dev} = \underline{\underline{f}}(\nabla \underline{u})$.

Define a Newtonian fluid by two properties:

(i) $\underline{\underline{\sigma}}^{dev}$ is linear and instantaneous in $\nabla \underline{u}$.

$$\Rightarrow \sigma_{ij}^{dev} = A_{ijkl}\frac{\partial u_k}{\partial x_l}.$$

where A_{ijkl} is a material parameter.

(ii) The fluid is locally isotropic. (e.g. not true for a fibre suspension)

$$\Rightarrow A_{ijkl} = \mu'\,\delta_{ij}\delta_{kl} + \mu''\,\delta_{ik}\delta_{jl} + \mu'''\,\delta_{il}\delta_{jk}.$$

Combining,

$$\sigma_{ij}^{dev} = \mu'\,\delta_{ij}\underbrace{\frac{\partial u_k}{\partial x_k}}_{=0} + \mu''\frac{\partial u_i}{\partial x_j} + \mu'''\frac{\partial u_j}{\partial x_i}$$

σ_{ij} sym, so $\mu'' = \mu''' = \mu$. thus

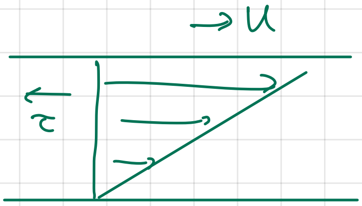
$$\sigma_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\boxed{\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + 2\mu\,\underline{\underline{e}}.}$$

Notes : • many common fluids, e.g. air, water, oil, syrup, are Newtonian to a good approximation.

• The stress tensor does not depend on vorticity $\underline{\Omega}$, only on $\underline{e}$. Solid body rotation does not stress the fluid.

• Consider Couette flow (Ex 1 #2).



$\underline{u} = \left(\frac{Uy}{h}, 0, 0 \right)$

Applied stress $\tau = \sigma_{xy} = 2\mu e_{xy} = 2\mu \left(\frac{1}{2} \frac{U}{h} \right) = \mu \frac{U}{h}$.

So μ introduced here is the dynamic viscosity.

2.5 Momentum Equation

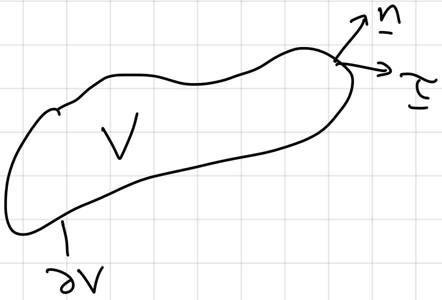
2.5.1 Cauchy equation (valid for any fluid)

Momentum inside a fixed arbitrary (smooth) volume V changes in time for various reasons.

(i). momentum carried across the boundary ∂V (momentum flux)

(ii) volume (body) forces

(iii) surface forces.



From N2,

$$\frac{d}{dt} \int_V \rho u_i \, dV = -\int_{\partial V} \rho u_i (u_j n_j) \, dS + \int_V F_i \, dV + \int_{\partial V} \sigma_{ij} n_j \, dS$$

$$\Rightarrow \int_V \left(\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) \right) dV = \int_V \left(F_i + \frac{\partial \sigma_{ij}}{\partial x_j} \right) dV.$$

V arbitrary, so mass conservation ⇒ = 0.

$$\rho \frac{\partial u_i}{\partial t} + \underline{u_i \frac{\partial \rho}{\partial t}} + \rho u_j \frac{\partial u_i}{\partial x_j} + \underline{u_i \frac{\partial}{\partial x_j} (\rho u_j)} = F_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$

$$\Rightarrow \quad \rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = F_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$

$$\Rightarrow \quad \rho \frac{D\underline{u}}{Dt} \equiv \rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla \underline{u}\right) = \underline{F} + \nabla\cdot\underline{\underline{\sigma}}$$

2.5.2 Navier-Stokes Equation

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu e_{ij}$$

Assuming $\mu =$ uniform

$$\Rightarrow \quad \partial_j \sigma_{ij} = -\partial_i p + \mu\left(\partial_j\partial_j u_i + \partial_i \underbrace{\partial_j u_j}_{=\nabla\cdot\underline{u}\,=\,0}\right)$$

Thus

$$\boxed{\rho\frac{D\underline{u}}{Dt} = \underline{F} - \nabla p + \mu\nabla^2\underline{u}}$$

with $\nabla\cdot\underline{u} = 0$.

Navier

Stokes

Euler eqn.

Stokes eqn.

2.5.3 Boundary Conditions

(i) Mass conservation. $\Rightarrow$ $\underline{u}\cdot\underline{n}$ cts

(ii) $\nabla\underline{u}$ finite (or else $\underline{\underline{\sigma}}$ unbounded)

$$\Rightarrow \quad \underline{u}\cdot\underline{t} = \underline{u} - (\underline{u}\cdot\underline{n})\,\underline{n} \quad \text{cts}$$

tangent vector

(iii) Balanced forces (conserve momentum)

$$(\rho u_i u_j - \sigma_{ij})\,n_j \quad \text{cts}$$

$$\Rightarrow \quad \underline{\underline{\sigma}}\cdot\underline{n} \quad \text{cts}$$

For parallel flows,

$$\mu\frac{\partial u}{\partial y} \quad \text{cts and} \quad p \quad \text{cts}$$

for rigid boundary

no penetration

$\underline{u}\cdot\underline{t} = 0$ (no slip)

2.5.4 Navier-Stokes equation in different coord systems

Cartesian: $\underline{u} = (u, v, w)$, $\nabla^2 \underline{u} = (\nabla^2 u, \nabla^2 v, \nabla^2 w)$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

and similar eqn for v and w.

Rectlinear

Either use $\nabla \wedge (\nabla \wedge \underline{u}) = \nabla\left(\cancelto{0}{\nabla \cdot \underline{u}}\right) - \nabla^2 \underline{u}$.

$$\Rightarrow \quad \nabla^2 \underline{u} = -\nabla \wedge (\nabla \wedge \underline{u})$$

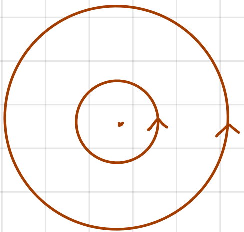
or use diadics

$$\nabla = \frac{\underline{e}_1}{h_1}\frac{\partial}{\partial \xi_1} + \frac{\underline{e}_2}{h_2}\frac{\partial}{\partial \xi_2} + \frac{\underline{e}_3}{h_2}\frac{\partial}{\partial \xi_3}.$$

Example Cylindrical Polars (r, θ, z) with $\underline{u} = v(r)\,\underline{e}_\theta$.

Either

$$\nabla \wedge \underline{u} = \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ 0 & rv & 0 \end{vmatrix} = \left(0, 0, \frac{1}{r}\frac{\partial}{\partial r}(rv)\right)$$



$$\nabla^2 \underline{u} = -\frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ 0 & 0 & \frac{\partial v}{\partial r} + \frac{v}{r} \end{vmatrix} = \left(0, \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}, 0\right).$$

Notice that $\nabla^2 \underline{u} = \left(0, \nabla^2 v - \frac{v}{r^2}, 0\right) \neq (0, \nabla^2 v, 0)$.

Or, start with $\underline{u} = v(r)\,\underline{e}_\theta$

$$\nabla \underline{u} = \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) v(r)\,\underline{e}_\theta$$

Recall $\frac{\partial}{\partial \theta}\underline{e}_r = \underline{e}_\theta$, $\frac{\partial}{\partial \theta}\underline{e}_\theta = -\underline{e}_r$, so

$$\nabla \underline{u} = \underline{e}_r \frac{\partial v}{\partial r}\underline{e}_\theta - \frac{\underline{e}_\theta}{r} v\,\underline{e}_r = \begin{pmatrix} 0 & \partial v/\partial r \\ -v & 0 \end{pmatrix}$$

$$\Rightarrow \quad \underline{\underline{e}} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) & 0 \end{pmatrix}$$

$$\nabla\nabla \underline{u} = \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial v}{\partial r}\underline{e}_r\underline{e}_\theta - \frac{v}{r}\underline{e}_\theta\underline{e}_r\right)$$

$$= \underline{e}_r\left(\frac{\partial^2 v}{\partial r^2}\underline{e}_r\underline{e}_\theta - \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\underline{e}_\theta\underline{e}_r\right)$$

$$+ \frac{\underline{e}_\theta}{r}\frac{\partial v}{\partial r}\left(\underline{e}_\theta\underline{e}_\theta - \underline{e}_r\underline{e}_r\right) - \frac{v\,\underline{e}_\theta}{r^2}\left(-\underline{e}_r\underline{e}_r + \underline{e}_\theta\underline{e}_\theta\right)$$

$$\nabla^2\underline{u} = \nabla\cdot\nabla\underline{u}$$

$$= \frac{\partial^2 v}{\partial r^2}\underline{e}_\theta + \frac{1}{r}\frac{\partial v}{\partial r}\underline{e}_\theta - \frac{v}{r^2}\underline{e}_\theta .$$

and

$$\underline{u}\cdot\nabla\underline{u} = -\frac{v^2}{r}\underline{e}_r .$$

So steady N-S eqn give (for thin case)

$$-\rho\frac{v^2}{r} = -\frac{\partial p}{\partial r}$$

centripetal force

inward pressure force

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right)$$

See Sheet 1 # 6.

2.6 Energy equation – dissipation

Take the scaler product with $\underline{u}$ of the Cauchy momentum eqn.

$$u_i\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = u_i F_i + u_i\frac{\partial \sigma_{ij}}{\partial x_j} \qquad (*)$$

$$\text{LHS}: \quad \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2\right) - \tfrac{1}{2}u^2\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\rho u^2 u_j\right) - \tfrac{1}{2}u^2\frac{\partial}{\partial x_j}(\rho u_j)$$

= 0 by mass conservation.

$$\text{RHS} = u_i F_i + \frac{\partial}{\partial x_j}(u_i\sigma_{ij}) - \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

Integrate $(*)$ over a fixed volume V with boundary ∂V and use div. thm.

$$\underbrace{\frac{d}{dt}\int_V \tfrac{1}{2}\rho u^2\,dV}_{(1)} + \underbrace{\int_{\partial V}\tfrac{1}{2}\rho u^2\,\underline{u}\cdot\underline{n}\,dS}_{(2)} = \underbrace{\int_V \underline{u}\cdot\underline{F}\,dV}_{(3)} + \overbrace{\int_{\partial V} u_i\sigma_{ij}n_j\,dS}^{(4)}$$

$$\underbrace{-\int_V \sigma_{ij}e_{ij}\,dV}_{(5)}$$

Note $\sigma_{ij}\Omega_{ij} = 0$, since σ_{ij} sym, Ω_{ij} anti-sym.

(1): Rate of change of KE inside V due to

(2): Flux of KE across boundary into V

(3): Rate of work done by body forces inside V

(4): Rate of work done by boundary stress $\underline{\tau} = \underline{\underline{\sigma}} \cdot \underline{n}$ on the boundary ∂V

(5): Rate of work done against internal stresses.

For Newtonian fluid, $\sigma_{ij} = -p\,\delta_{ij} + 2\mu e_{ij}$

$$\Rightarrow \sigma_{ij} e_{ij} = -p \underbrace{e_{jj}}_{\nabla \cdot \underline{u} = 0} + 2\mu e_{ij} e_{ij} \geqslant 0$$

So (5) is a sink of energy. It represents energy dissipation by the action of viscosity and deformation.

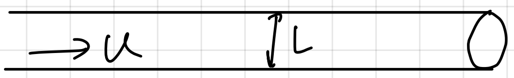
Total dissipation rate

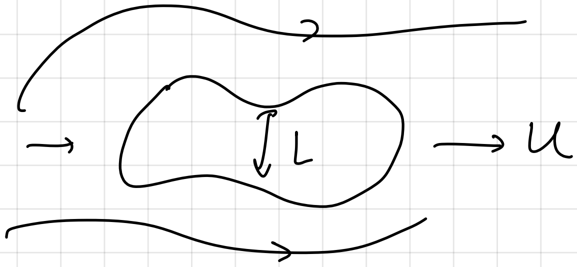
$$\boxed{D = 2\mu \int_V e_{ij} e_{ij} \, dV}$$

2.7 Scaling (Read P. 45-50 of UFF, or IB notes)

2.7.1 Non-dimensionalisation.

Suppose that flow has a characteristic speed U extrinsic, cross-stream lengthscale L and varies on a timescale T.





Suppose that pressure <u>differences</u> have scale P

Define dimensionless variables

$$\underline{u}^* = \frac{\underline{u}}{U}, \quad \underline{x}^* = \frac{\underline{x}}{L}, \quad t^* = \frac{t}{T}. \quad p^* = \frac{p}{P}$$

Then N-S with no body force becomes

$$\rho \frac{U}{T} \frac{\partial \underline{u}^*}{\partial t^*} + \rho \frac{U^2}{L} \underline{u}^* \cdot \nabla^* \underline{u}^* = -\frac{P}{L} \nabla^* p^* + \mu \frac{U}{L^2} \nabla^{*2} \underline{u}^*$$

Pressure is always req'd to enforce incompressiblity and P is chosen (determined) internally to balance dominant terms in N-S.

For now, choose $P = \frac{\mu U}{L}$ to balance viscous term

$$\Rightarrow Re \left(St \frac{\partial \underline{u}^*}{\partial t^*} + \underline{u}^* \cdot \nabla^* \underline{u}^* \right) = -\nabla^* p^* + \nabla^{*2} \underline{u}^*.$$

where $Re = \frac{UL}{\nu}$ is the Reynold's number

$St = \frac{L}{UT}$ is the Strouhal number

$\nu = \mu/\rho$ is the kinematic viscosity.

2.7.2 Dynamical Similarity

Whereas $\underline{u} = \underline{u}(\underline{x}, t ; \rho, \mu, U, L, T, \text{shape})$

$\underline{u}^* = \underline{u}^*(\underline{x}^*, \underline{t}^* ; Re, St, \text{shape})$

2.7.3 Approximations

$St \ll 1 \Rightarrow$ quasi-steady, ignore $\frac{\partial \underline{u}^*}{\partial t^*}$

$St \gg 1 \Rightarrow$ rapidly oscillating, ignore $\underline{u} \cdot \nabla \underline{u}$

$Re \ll 1 \Rightarrow$ viscously dominated. ignore LHS $\Rightarrow$ Stokes eqn

$Re \gg 1 \Rightarrow$ inertially dominated on extremise lengthscales $\Rightarrow$ Euler eqn.

BUT viscous terms can be important on small lengthscales. ⇒ boundary layer-equation (§ 7).

Often T is set by advection time $T \sim \frac{L}{U}$, in which case $St = 1$, and only important parameter is Re.

The kinematic viscosity $\nu = \mu/\rho$ determines how far motions are communicated by viscous stresses between fluid layers.

Typical values :

$$\nu_{water} \approx 10^{-6}\, m^2 s^{-1}$$
$$\nu_{air} \approx 10^{-5}\, m^2 s^{-}$$

3. Unidirectional Flows (see IB)

$$\underline{u} = U\, \underline{e}_x$$

Incompressibility $\nabla \cdot \underline{u} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0.$

$$\Rightarrow \underline{u} \cdot \nabla \underline{u} = 0.$$

Exercise Momentum eqn reduces to

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + F_x + \mu \nabla_\perp^2 \underline{u}$$
$$0 = -\nabla_\perp p + F_\perp$$

where $\perp$ signifies the space $\perp$ to $\underline{e}_x$.

3.1 Plane Couette flow — driven by boundary motion (see IB)

Rate of dissipation = rate of working by boundary forces (tangential stress).

3.2 Plane Poiseuille flow - driven by pressure gradient

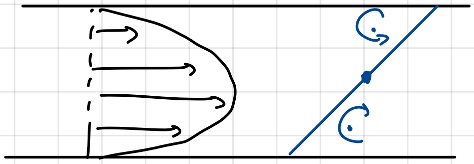
Rate of dissipation = rate of working by external pressure forces.

Key results : $u = -\frac{1}{2\mu} \frac{\partial p}{\partial x} y(h-y).$

Volume flux (per unit length into the page)

$$q = \int_0^h u \, dy = -\frac{h^3}{12\mu} \frac{\partial p}{\partial x} .$$

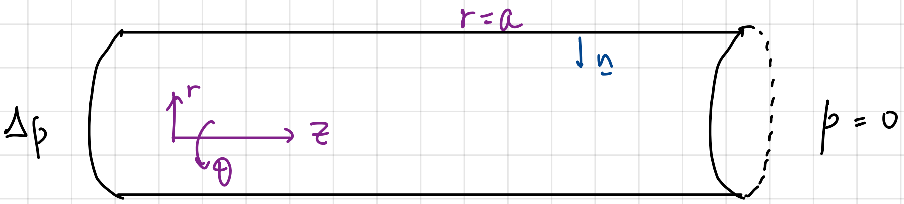
Vorticity $\quad \underline{\omega} = \nabla \wedge \underline{u} = \left(0,\ 0,\ \frac{1}{2\mu} \frac{\partial p}{\partial x} \left(\frac{h}{2} - y\right)\right)$



Strain rate

$$\underline{\underline{e}} = -\frac{1}{2\mu} \frac{\partial p}{\partial x} \begin{pmatrix} 0 & \frac{h}{2} - y \\ \frac{h}{2} - y & 0 \end{pmatrix}$$

3.3 Circular Poiseuille flow (pipe flow)



In cylindrical polars (r, θ, z).

$$\underline{u} = (0,\ 0,\ w(r))$$

N-S give

$$\left.\begin{aligned} 0 &= -\frac{\partial p}{\partial r} \\ 0 &= -\frac{1}{r} \frac{\partial p}{\partial \theta} \end{aligned}\right\} \quad (1)$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) \qquad (2)$$

$\nabla \cdot \underline{u} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial z} = 0$, so

$$(2) \Rightarrow \qquad \frac{\partial^2 p}{\partial z^2} = 0$$

$$\Rightarrow \qquad -\frac{\partial p}{\partial z} = G \qquad \text{const.}$$

$$\Rightarrow \quad \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) = -\frac{G}{\mu} r$$

$$\Rightarrow \quad \frac{\partial w}{\partial r} = -\frac{G}{2\mu} r + \underline{\frac{A}{r}}$$

$A = 0$ by regularity (finite sheer).
at $r = 0$

$$\Rightarrow \quad w = \frac{G}{4\mu}(a^2 - r^2)$$

using no-slip at $r = a$ $(w(r=a) = 0)$.

Volume flux is

$$q = \int \underline{u} \cdot d\underline{A}$$

$$= \int_0^{2\pi} d\theta \int_0^a dr \quad wr$$

$$= \frac{\pi G}{8\mu} a^4.$$

Double radius $\Rightarrow$ 16 times flow rate for given pressure gradient.

To double flow rate, increase radius by ~20%.

Pipe of length L, then $G = \frac{\Delta p}{L}$.

Viscous force on wall due to pipe is

$$F = \text{area} \times \text{stress}$$

$$= 2\pi a L \quad \left(-\mu \frac{\partial w}{\partial r}\Big|_{r=a} \right)$$

$$= 2\pi a L \left(-\mu \frac{G}{4\mu}(-2a) \right)$$

$$= \pi a^2 \Delta p.$$

$$= \text{net pressure force.}$$

Dissipation is $2\mu \int_V \underline{\underline{e}} : \underline{\underline{e}} \, dV$

$$\nabla \underline{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial r} & 0 & 0 \end{pmatrix} \Rightarrow \underline{\underline{e}} = \begin{pmatrix} 0 & 0 & -\frac{Gr}{4\mu} \\ 0 & 0 & 0 \\ -\frac{Gr}{4\mu} & 0 & 0 \end{pmatrix}$$

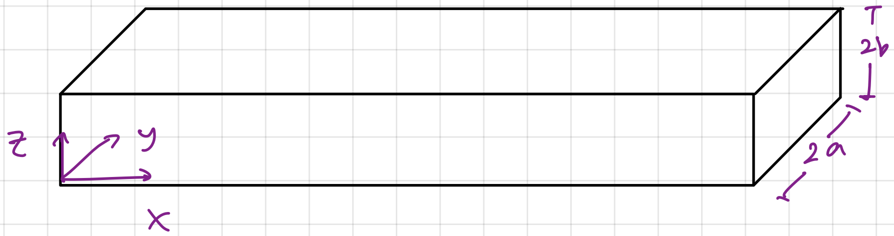
So

$$D = 2\mu \int_V \underline{\underline{e}} : \underline{\underline{e}} \, dV$$

$$= 2\mu \cdot 2 \frac{G^2}{16\mu^2} \cdot 2\pi L \int_0^a r^3 \, dr$$

$$= \frac{\pi G^2 L}{8\mu} a^4 = q \, \Delta p. = \text{rate of working by external pressure force}$$

3.4 Pressure-driven flow in a rectangular channel



Solve $\mu \nabla^2 u(y,z) = -G$ (const.) with no slip

$$u(\pm a, z) = u(y, \pm b) = 0$$

Write u as a superposition of plane Poiseuille flow about z to absorb the pressure gradient, plus a flow u^*.

$$u = \frac{G}{2\mu}(b^2 - z^2) + u^* .$$

Then $\nabla^2 u^* = 0$, $u^*(y, \pm b) = 0$, $u^*(\pm a, z) = -\frac{G}{2\mu}(b^2 - z^2)$.

Using separation of variables,

$$u^* = -\frac{G}{2\mu} b^2 \cdot 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n^3} \frac{\cosh(\lambda_n y/b)}{\cosh(\lambda_n a/b)} \cos\left(\lambda_n \frac{z}{b}\right)$$

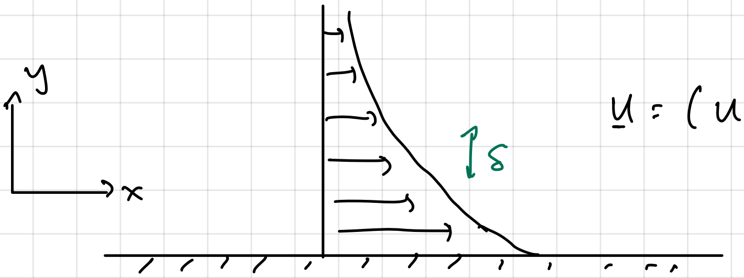
where $\lambda_n = (n + \frac{1}{2})\pi$.

Flow rate is

$$q = \int_{-a}^{a} dy \int_{-b}^{b} dz \; u(y,z)$$

$$= \frac{4\Delta p}{3\mu L} a b^3 \left(1 - \frac{6b}{a} \sum_{n=0}^{\infty} \frac{\tanh\left(\lambda_n \frac{a}{b}\right)}{\lambda_n^5} \right) .$$

3.5 Impulsively started plate



$$\underline{u} = (u(y,t), 0, 0)$$

Fluid in $y>0$ set in motion by rigid plate at $y=0$, which begins to move with fixed speed U at $t=0$.

Decay length (boundary-layer width)

$$\delta \sim \sqrt{\nu t}$$

Similarity sol^n $\quad u = U\,\mathrm{erf}(\eta) \quad . \quad \eta = \frac{y}{2\sqrt{\nu t}}$

Sheer stress on plate

$$\mu \frac{\partial u}{\partial y} = -\mu \frac{U}{\sqrt{\pi \nu t}} = -\tau \quad \text{(say)}$$

where τ is the external force per unit area applied to plate.

Rate of strain $\quad \underline{\underline{e}} = \frac{U e^{-\eta^2}}{2\sqrt{\pi \nu t}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Dissipation rate per unit area

$$D = 2\mu \int_0^\infty \underline{\underline{e}} : \underline{\underline{e}} \; dy$$

$$= 2\mu \frac{U^2}{4\pi \nu t} (1^2 + 1^2) \int_0^\infty e^{-2\eta^2} \, 2\sqrt{\nu t} \, d\eta$$

$$= \frac{2\mu U^2}{\pi \sqrt{\nu t}} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{2}} \neq \tau U$$

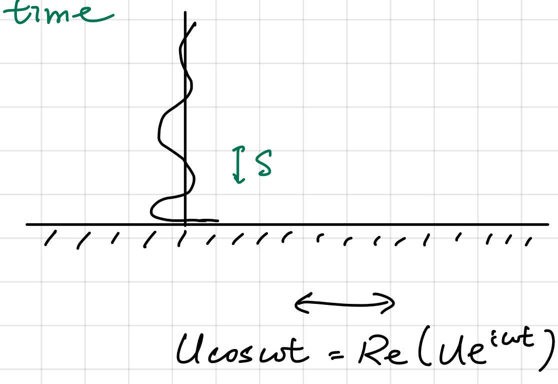
rate of working by boundary $\searrow$

$$\tau U = D + \frac{dK}{dt} \quad \text{— RoC of KE}$$

3.6 Oscillating flat plate (Sheet 1, #7)

Note: in complex representation, the time average of quadratic quantities

$$\langle f g \rangle = Re\left(\tfrac{1}{2} f g^*\right)$$



4. Stokes Flow $(Re \equiv UL/\nu \ll 1)$

Either $U.L$ small : colloids (mist), bacteria, or

ν large : crude oil, lava, glaciers.

N-S become

Stokes eqn $$\begin{cases} 0 = -\nabla p + \mu \nabla^2 \underline{u} + \underline{F} \\ \nabla \cdot \underline{u} = 0 \end{cases}$$

Write $p = p_H + p'$. where $\nabla p_H = \underline{F}$ defines the hydrostatic pressure, then

$$-\nabla p' + \mu \nabla^2 \underline{u} = 0$$

we drop the prime, i.e.

stress tensor

$$\nabla \cdot \underline{\underline{\sigma}} = 0$$

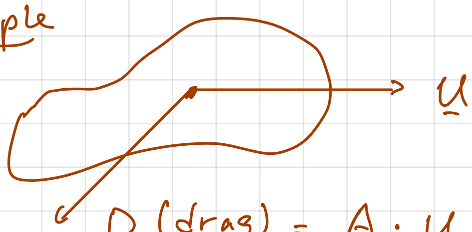
4.1 Simple Properties

(i) Instantaneous — No $\frac{\partial}{\partial t}$ term, always at terminal velocity.

Flow can vary in time parametrically (quasi-steady) if the boundary conditions depend on time.

(ii) $\underline{u}, p, \underline{\underline{\sigma}}$ are all linear in applied forces.

Example



$\underline{D}$ (drag) $= \underline{\underline{A}} \cdot \underline{U}$

$\underline{\underline{A}}$ depends only on geometry.

For a sphere (isotropic), radius a, $(\underline{\underline{A}} = \lambda \underline{\underline{I}})$,

$$\underline{\underline{\sigma}} \propto \mu \frac{\underline{U}}{a}.$$

$$\Rightarrow \underline{D} \propto \mu a \underline{U}.$$

See later for $\underline{D} = -6\pi\mu a \underline{U}$.

(iii) Time reversible

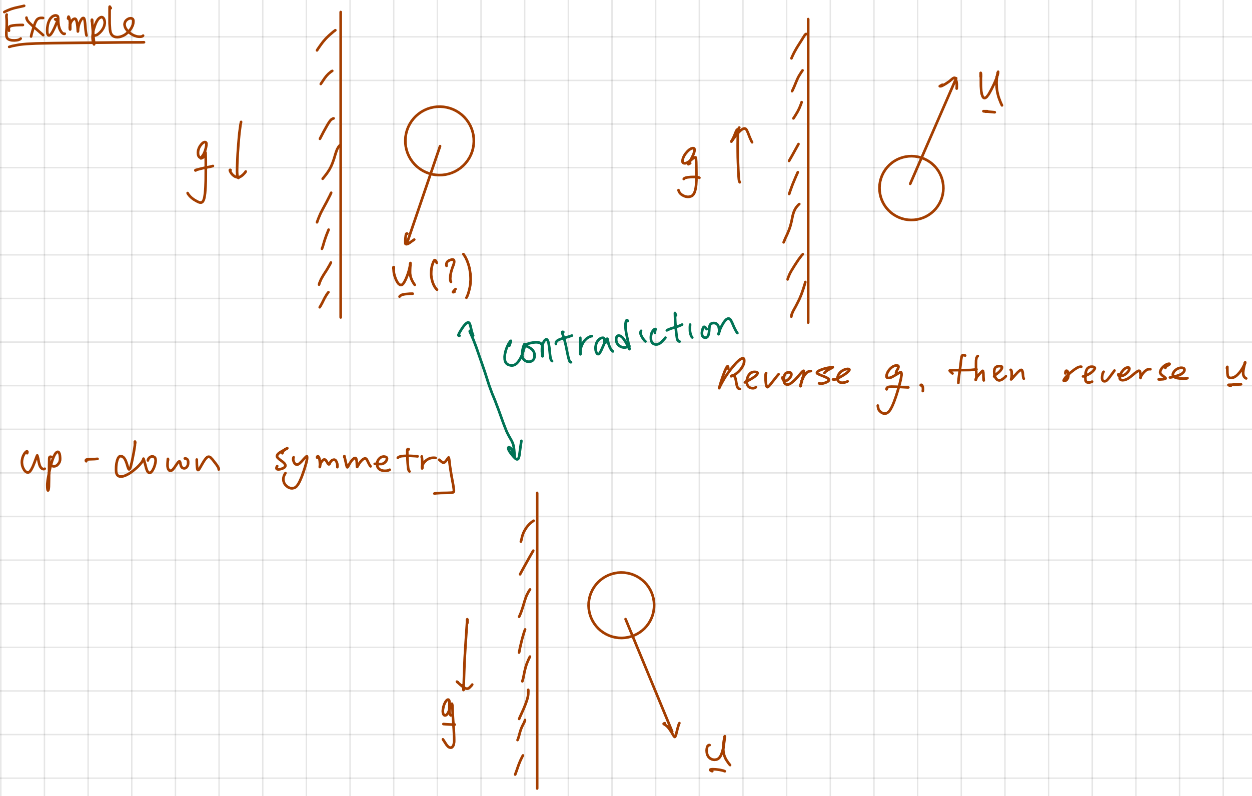
Apply force $\underline{F}(t)$ in $0<t<t_1$.

Reverse force $-\underline{F}(2t_1-t)$ in $t_1<t<2t_1$.

Then flow reverses its history previously.

(iv) Reversible in space (linearity + symmetry).

Example

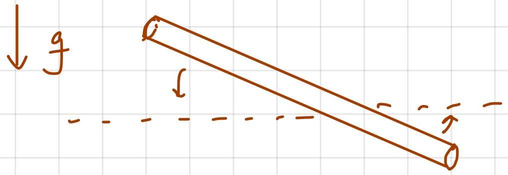


⇒ sphere falls parallel to the wall.

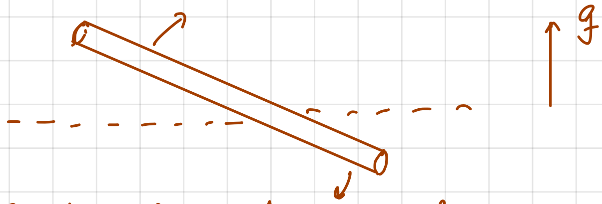
Example Sedimenting rod does not rotate.

Suppose

Reverse g:

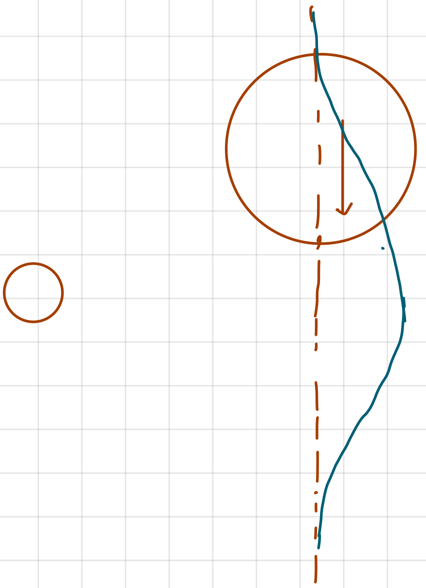


Rotates towards horizontal



Rotates towards vertical.

Example



No net deflection of passing sedimenting spheres.

(v) Harmonic functions

$$0 = -\nabla p + \mu \nabla^2 \underline{u} \qquad (*)$$

$$\nabla\cdot(*) \Rightarrow \nabla^2 p = 0 \Rightarrow p \text{ harmonic}$$

$$\nabla\wedge(*) \Rightarrow \nabla^2 \underline{\omega} = 0 \Rightarrow \underline{\omega} \text{ harmonic}$$

$$\nabla\wedge(\nabla\wedge(*)) \Rightarrow \nabla^2\nabla^2 \underline{u} \equiv \nabla^4 \underline{u} = 0 \Rightarrow \underline{u} \text{ biharmonic}$$

ψ biharmonic in 2D.

4.2 Exploit $\nabla\cdot\underline{\underline{\sigma}} = 0$ for Stokes Flow.

4.2.1 Uniqueness

Suppose $\underline{u}^1, \underline{u}^2$ are Stokes flow in domain V with same boundary condition on ∂V. Let $\underline{u} = \underline{u}^1 - \underline{u}^2$ etc.

$$\int_V 2\mu\, e_{ij} e_{ij}\, dV = \int_V (\sigma_{ij} + \underbrace{p\,\delta_{ij})\, e_{ij}}_{e_{jj}=0 \ \because\ \nabla\cdot\underline{u} = 0}\, dV$$

$$= \int_V \sigma_{ij} \left(\frac{\partial u_i}{\partial x_j} - \Omega_{ij} \right) dV$$

antisym

$$= \int_V \frac{\partial}{\partial x_j}(\sigma_{ij} u_i) - u_i \underbrace{\frac{\partial}{\partial x_j}\sigma_{ij}}_{\nabla\cdot\underline{\underline{\sigma}} = 0}\, dV$$

$$= \int_{\partial V} n_j \sigma_{ij} u_i \, dS$$

$$= \int_{\partial V} \underline{u} \cdot (\underline{\underline{\sigma}} \cdot \underline{n}) \, dS$$

$= 0$ if either $\underline{u}$ or $\underline{\underline{\sigma}} \cdot \underline{n}$ is prescribed on boundary

$\Rightarrow \underline{\underline{e}} = 0 \Rightarrow \underline{\underline{e}}^1 = \underline{\underline{e}}^2$ through V.

$\Rightarrow \underline{u} = 0$ up to solid body motion.

(i) $\underline{u}$ prescribed $\Rightarrow \underline{u}^1 = \underline{u}^2$ in V

(ii) $\underline{\underline{\sigma}} \cdot \underline{n}$ prescribed $\Rightarrow \underline{u}^1 = \underline{u}^2 + \underbrace{\underline{U}}_{\uparrow} + \underbrace{\underline{\Omega}}_{\uparrow} \wedge \underline{x}$ in V

const.

4.2.2 Minimum dissipation theorem (Sheet 2 #4)

Let $\underline{u}^s$ be a Stokes flow, and $\underline{u}$ be any incompressible flow with same velocity on ∂V. Then

$$2\mu \int \underline{\underline{e}}^s : \underline{\underline{e}}^s \, dV \leq 2\mu \int \underline{\underline{e}} : \underline{\underline{e}} \, dV$$

Pf: Consider $0 \leq 2\mu \int_V (\underline{\underline{e}} - \underline{\underline{e}}^s) : (\underline{\underline{e}} - \underline{\underline{e}}^s) \, dV$

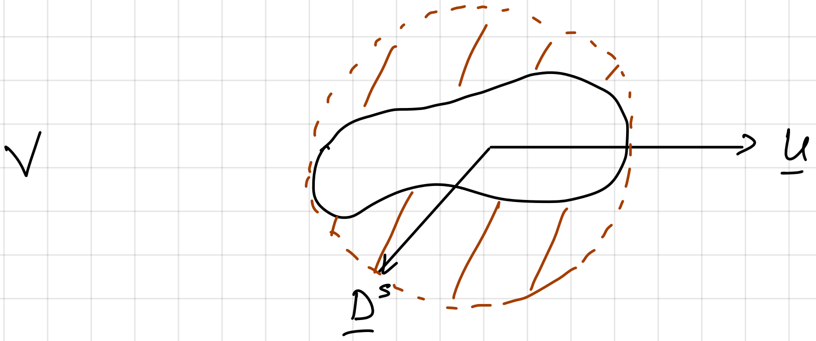
$$= 2\mu \int_V \underline{\underline{e}} : \underline{\underline{e}} \, dV - 2\mu \int_V \underline{\underline{e}}^s : \underline{\underline{e}}^s$$

$$+ 4\mu \int_V \underline{\underline{e}}^s : (\underline{\underline{e}}^s : \underline{\underline{e}}) \, dV$$

Final term $= 2\int_V \underline{\underline{\sigma}}^s : (\underline{\underline{e}}^s - \underline{\underline{e}}) \, dV$

$$= 2\int_{\partial V} \underline{n} \cdot \underline{\underline{\sigma}}^s \cdot (\underline{u}^s - \underline{u}) \, dS = 0. \qquad \square$$

4.2.3 Geometric Bounding



Consider an arbitrary solid body with exterior V, translating with velocity $\underline{U}$ through unbounded viscous fluid, resulting in a drag $\underline{D}^s$.

$$-\underline{D}^s \cdot \underline{U} = 2\mu \int_V \underline{\underline{e}}^s : \underline{\underline{e}}^s \, dV.$$

Now consider the flow $\underline{u}$ outside the body consisting of Stoke's flow outside a circumscribing sphere S of radius b and rigid motion between the body and the sphere.

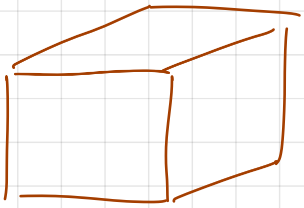
Then
$$\begin{aligned} -\underline{D} \cdot \underline{U} &= 2\mu \int_V \underline{\underline{e}} : \underline{\underline{e}} \, dV \\ &\geq 2\mu \int_V \underline{\underline{e}}^s : \underline{\underline{e}}^s \, dV \\ &= -\underline{D}^s \cdot \underline{U}. \end{aligned}$$

But $\underline{D} = -6\pi\mu b \, \underline{U}$

$$\Rightarrow \quad -\underline{D}^s \cdot \underline{U} \leq 6\pi\mu b U^2.$$

Similarly, (exercise), for an inscribed sphere of radius a,

$$-\underline{D}^s \cdot \underline{U} \geq 6\pi\mu a U^2.$$



Example cube, side $2L$.

$a = L, \quad b = \sqrt{3} L$

So $\quad 6\pi\mu L U^2 \leq -\underline{D}^s \cdot \underline{U} \leq 6\sqrt{3}\,\pi\mu L U^2.$

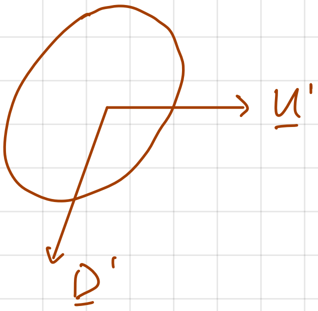
4.2.4 Reciprocal Theorem

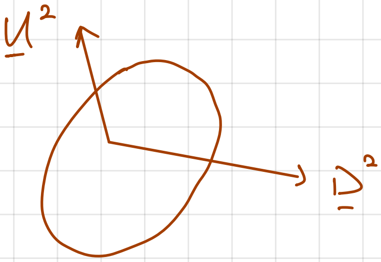
Let $\underline{u}^1$ and $\underline{u}^2$ be two Stokes flows in the same domain V with different b.c.s on ∂V. Then

$$\int_{\partial V} \underline{u}^1 \cdot \underline{\underline{\sigma}}^2 \cdot \underline{n} \, dS = \int_{\partial V} \underline{u}^2 \cdot \underline{\underline{\sigma}}^1 \cdot \underline{n} \, dS$$

Pf: Exercise. Start with $2\mu \int_V \underline{\underline{e}}^1 : \underline{\underline{e}}^2 \, dV = 2\mu \int_V \underline{\underline{e}}^2 : \underline{\underline{e}}^1 \, dV$. and proceed with proof of uniqueness.

Example

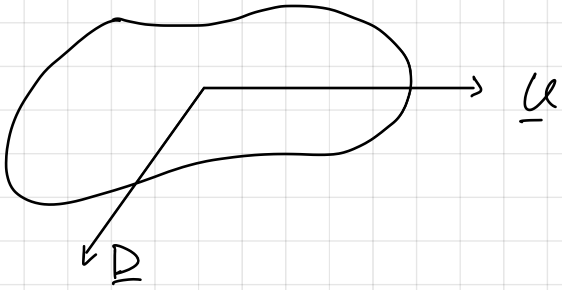




$$\underline{D}^1 \cdot \underline{U}^2 = \underline{D}^2 \cdot \underline{U}^1.$$

Application of Reciprocal theorem

- Linearity of Stokes flow.



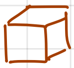
$$D_i = A_{ij} U_j$$

From reciprocal thm.

$$A_{ij} U_j^{(1)} U_i^{(2)} = A_{ij} U_j^{(2)} U_i^{(1)}$$

$$\Rightarrow A_{ij} = A_{ji}$$

So $\underline{\underline{A}}$ is real and symmetric, so diag.

Example Cube  Symmetry ⇒ equal eval ⇒ isotropic

So cube sediments vertically and at the same rate regardless of orientation.

In general, for a solid body with translation $\underline{U}$, rotation $\underline{\Omega}$,

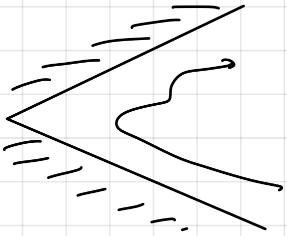
$$\text{Drag} \quad \underline{F} = \underline{\underline{A}} \cdot \underline{U} + \underline{\underline{B}} \cdot \underline{\Omega}$$

$$\text{Torque} \quad \underline{G} = \underline{\underline{C}} \cdot \underline{U} + \underline{\underline{D}} \cdot \underline{\Omega}$$

Exercise Show that $\underline{\underline{A}}, \underline{\underline{D}}$ symmetric, $\underline{\underline{B}} = \underline{\underline{C}}^T$.

4.3 Corner Flows

Let L be the distance to corner, then any flow has $Re_L \ll 1$ sufficiently close to corner.



In 2D, introduce a streamf^n ψ s.t. $\underline{u} = \nabla \wedge (0, 0, \psi)$

Use polar coords

$$\underline{u} = \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta} , -\frac{\partial \psi}{\partial r} , 0 \right) .$$

Vorticity $\underline{\omega} = \nabla \wedge \underline{u} = (0, 0, -\nabla^2 \psi)$. In Stokes flow,

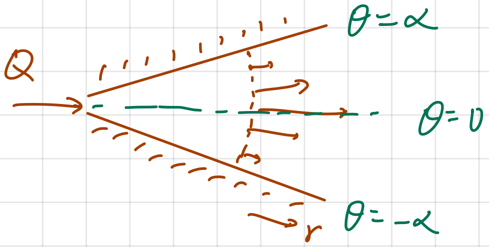
$$\nabla^2 \underline{\omega} = 0 \quad \Rightarrow \quad \boxed{\nabla^4 \psi = \nabla^2 \nabla^2 \psi = 0.}$$

biharmonic eqn

Note $r^4 \nabla^4 \psi = 0$, and r^λ is a ef^n of $r^4 \nabla^4$.

Example (Source flow)

Assume radial flow, i.e. $u_\theta = 0$.



From mass conservation, $u_r \propto Q/r$

$$\Rightarrow \psi = Q f(\theta).$$

Recall that $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$,

$$\Rightarrow \nabla^2 \psi = Q \frac{f''}{r^2}$$

$$\Rightarrow \nabla^4 \psi = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)\left(Q \frac{f''}{r^2} \right) = \frac{4Qf''}{r^4} + \frac{Q}{r^4} f^{IV} = 0.$$

$$\Rightarrow f^{IV} + 4f'' = 0.$$

$$\Rightarrow f = A \cos 2\theta + B \sin 2\theta + C + D\theta .$$

If flow is symmetric, then u_r is even, and so f is odd,

So $A = C = 0$.

Apply no slip : $u_r = 0$ on $\theta = \pm\alpha$

$$\Rightarrow f'(\pm\alpha) = 0$$

$$\Rightarrow 2B\cos 2\alpha + D = 0 \qquad (1).$$

Volume flux :

$$Q = \int_{-\alpha}^{\alpha} u_r \cdot r \, d\theta$$

$$= \int_{-\alpha}^{\alpha} \frac{\partial\psi}{\partial\theta} \, d\theta$$

$$= Q\left[f(\alpha) - f(-\alpha)\right]$$

$$\Rightarrow f(\alpha) = B\sin 2\alpha + D\alpha = \frac{1}{2} \qquad (2).$$

Solving (1) and (2),

$$f(\alpha) = \frac{1}{2} \, \frac{\sin 2\theta - 2\theta\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}$$

Then

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{Q}{r} f'$$

$$= \frac{Q}{r} \, \frac{\cos 2\theta - \cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}$$

Note that if $\alpha \ll 1$, $f' \propto u_r$ parabolic. (as expected)

Also, purely divergent, i.e. $f' > 0$ for all θ only for $\alpha < \pi/2$

<u>Example</u> (Scraper problem, Sheet 2 #7)

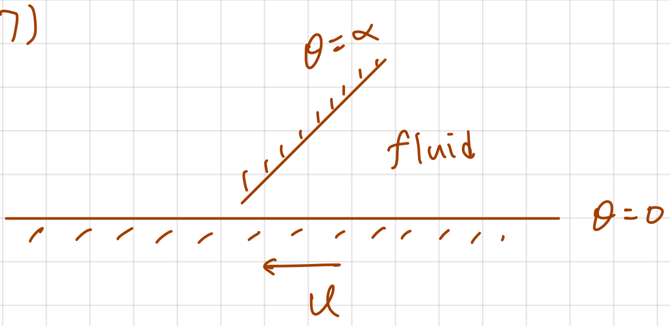
$\nabla^4 \psi = 0$ in fluid

no slip → $u = v = 0$ on $\theta = \alpha$

↖ no penetration ↙

$v = 0$ on $\theta = 0$

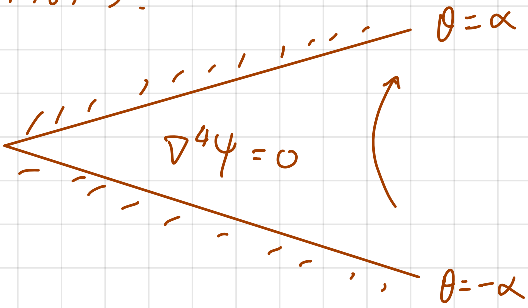
$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = u = -U$ on $\theta = 0$



Then $\psi \propto U r f(\theta)$ by linearity.

Example (Moffatt vortices, Moffatt 1964).

Flow in corner forced by far-field disturbance.



$$\psi = r^{\lambda} f(\theta)$$

$$\Rightarrow \quad \left(D^2+(\lambda-2)^2\right)\left(D^2+\lambda^2\right) f = 0 ,$$

where $D \equiv \frac{d}{d\theta}$, with b.c.

$$f = f' = 0 \quad \text{on} \quad \theta = \pm\alpha$$

$$\Rightarrow \quad f = A\cos\lambda\theta + B\sin\lambda\theta + C\cos(\lambda-2)\theta + D\sin(\lambda-2)\theta .$$

Consider, for example, anti-symmetric flow ⇒ ψ even in θ ⇒ $B = D = 0$.

Then no penetration gives

$$f(\alpha) = 0 \quad \Rightarrow \quad f = E\left(\cos(\lambda-2)\alpha\cos\lambda\theta - \cos\lambda\alpha\cos(\lambda-2)\theta\right).$$

and no slip gives

$$f'(\alpha) = 0 \quad \Rightarrow \quad -\lambda\cos(\lambda-2)\alpha\sin\lambda\alpha + (\lambda-2)\cos\lambda\alpha\sin(\lambda-2)\alpha - 0$$

This is an eigenvalue problem for λ.

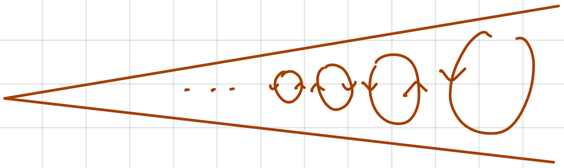
For $2\alpha \lesssim 146°$, no real roots. All roots are complex $\lambda = a+ib$, say, so

$$\psi = r^{a+ib} f(\theta) + (\text{complex conjugate})$$

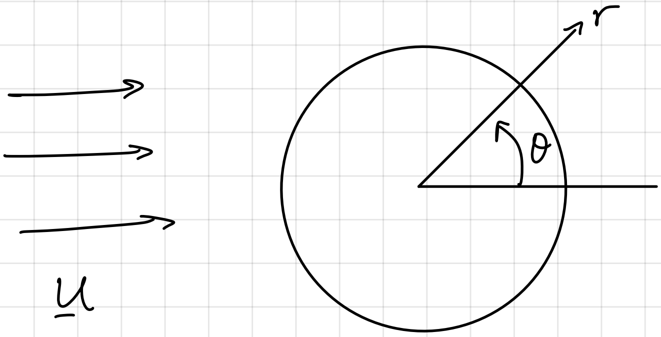
So

$$\psi = r^a\left[\cos(b\log r)\,\mathrm{Re}(f) - \sin(b\log r)\,\mathrm{Im}(f)\right]$$

⇒ ψ oscillates infinitely often as $r \to 0$



4.4 Stokes flow past a sphere



Axisymmetric, indpt of φ in sph. polar coordinates (r, θ, φ)

$$\underline{u} \to (U\cos\theta, -U\sin\theta, 0) \quad \text{as} \quad r \to \infty.$$

4.4.1 Method 1 (Classical)

Solve $\nabla^4 \underline{\psi}^S = 0$ using separation of variables in sph. polar, with

$$\underline{\psi}^S = \left(0, 0, \frac{\psi}{r\sin\theta}\right)$$

ψ is Stokes streamf^n. Noting $\underline{u} = \nabla \wedge \underline{\psi}^S$

$$\underline{u} = \left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, 0\right)$$

$$\psi \sim \frac{1}{2}Ur^2\sin^2\theta \quad \text{as} \quad r \to \infty$$

4.4.2 Method 2 (semi-classical, hybrid)

Note, decaying sol^n of $\nabla^2 = 0$ are $\frac{1}{r}$, $\nabla(\frac{1}{r})$, $\nabla\nabla(\frac{1}{r})$ etc.

We have $\nabla^2 p = 0$, p linear in $\underline{U}$. The only scaler can be formed is

$$p = A\,\underline{U}\cdot\nabla\left(\frac{1}{r}\right) = -A\frac{\underline{U}\cdot\underline{x}}{r^3}.$$

Recall $\nabla r = \hat{\underline{r}}$. We also know $\nabla^2\underline{\omega} = 0$, $\underline{\omega}$ linear in $\underline{U}$, so.

$$\underline{\omega} = B\underline{U} \wedge \nabla\left(\frac{1}{r}\right) = -B\frac{\underline{U}\wedge\underline{x}}{r^3}$$

Note, from Stoke's eqn,

$$\nabla p = -\mu \nabla \wedge \underline{\omega} \quad \Rightarrow A = \mu B.$$

Now $\underline{u} = \left(\frac{1}{r^2\sin\theta} \frac{\partial\psi}{\partial\theta} , -\frac{1}{r\sin\theta} \frac{\partial\psi}{\partial r} , 0 \right)$

$$\Rightarrow \underline{\omega} = \nabla\wedge\underline{u} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\right), -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right] \underline{e}_\varphi$$

$$= -\frac{BU}{r^2}\sin\theta \, \underline{e}_\varphi .$$

$$\Rightarrow \quad \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = \frac{BU}{r}\sin^2\theta$$

Note angular dependence and write

$$\psi = U f(r) \sin^2\theta .$$

$$\Rightarrow \quad f'' - \frac{2}{r^2} f = \frac{B}{r} .$$

$$\Rightarrow \quad f = -\frac{1}{2}Br + \frac{\alpha}{r} + \beta r^2 .$$

Far field, $r\to\infty$ $\Rightarrow$ $\beta = \frac{1}{2}$.

On $r = a$,

- no penetration $\Rightarrow$ $f(a) = 0$ $\Rightarrow$ $-\frac{1}{2}Ba + \frac{\alpha}{a} + \frac{1}{2}a^2 = 0.$
- no slip $\Rightarrow$ $f'(a) = 0$ $\Rightarrow$ $-\frac{1}{2}B - \frac{\alpha}{a^2} + a = 0$

Combining, $\alpha = \frac{a^3}{4}$, $B = \frac{3a}{2}$. So

$$\psi = Ur^2\sin^2\theta\left(\frac{1}{2} - \frac{3}{4}\frac{a}{r} + \frac{1}{4}\frac{a^3}{r^3}\right)$$

$$\Rightarrow \quad u_r = U\cos\theta\left(1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\frac{a^3}{r^3}\right)$$

$$u_\theta = U\sin\theta\left(-1 + \frac{3}{4}\frac{a}{r} + \frac{1}{4}\frac{a^3}{r^3}\right).$$

$$p = \frac{3}{2}\mu a U \frac{\cos\theta}{r^2} .$$

Note that $|\underline{u} - \underline{U}| = O\left(\frac{1}{r}\right)$ as $r\to\infty$. Slow decay.

4.4.3 Method 3 (Semi-modern) (See handout)

Exploit linearity (only) to deduce

$$\underline{u}(\underline{x}) = \underline{U} f(r) + \underline{x}(\underline{U}\cdot\underline{x})\, g(r).$$

$$p(\underline{x}) = \mu (\underline{U}\cdot\underline{x})\, h(r).$$

$$\Rightarrow \underline{u} = \underline{U}\left(1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\frac{a^3}{r^3}\right) + \frac{(\underline{U}\cdot\underline{x})\,\underline{x}}{r^2}\left(-\frac{3}{4}\frac{a}{r} + \frac{3}{4}\frac{a^3}{r^3}\right)$$

$$p = -\frac{3}{2}\mu a \,\frac{\underline{u}\cdot\underline{x}}{r^3}$$

whence

$$\underline{\underline{\sigma}}\cdot\underline{n} = \frac{3}{2}\frac{\mu}{a}\,\underline{u}.$$

Therefore, the force on the sphere is

$$\int_{r=a} \underline{\underline{\sigma}}\cdot\underline{n}\, dS = 4\pi a^2 \cdot \frac{3}{2}\frac{\mu}{a}\,\underline{u} = 6\pi\mu a \underline{U}.$$

4.4.4 Method 4 (Modern) (See Part Ⅲ).

Note: force on sphere can be split between a "form drag"

$$\int_{r=a} (\underline{n}\cdot\underline{\underline{\sigma}}\cdot\underline{n})\,\underline{n}\, dS = 2\pi\mu a \underline{U}$$

and a "skin friction"

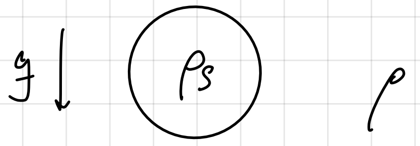
$$\int_{r=a} (\underline{t}\cdot\underline{\underline{\sigma}}\cdot\underline{n})\,\underline{t}\, dS = 4\pi\mu a \underline{U}.$$

For a bubble, form drag is $4\pi\mu a \underline{U}$.

Sedimenting Sphere

$$\text{Weight} - \text{buoyancy} = \frac{4}{3}\pi a^3 \rho_s g - \frac{4}{3}\pi a^3 \rho g$$

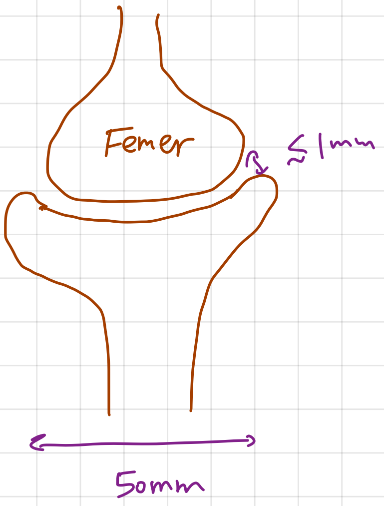
$$= 6\pi\mu a U$$

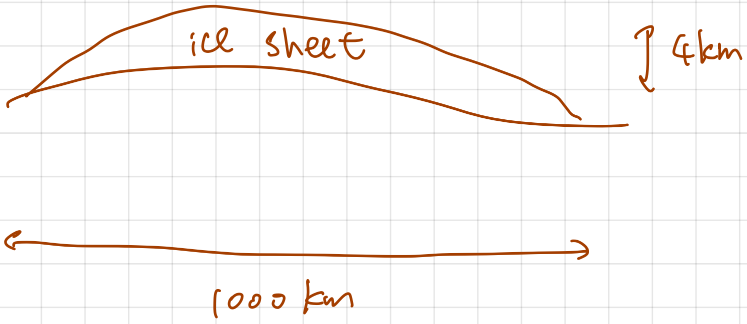


$$\Rightarrow \underline{U} = \frac{2}{9}\frac{(\rho_s - \rho)}{\mu}\, g$$

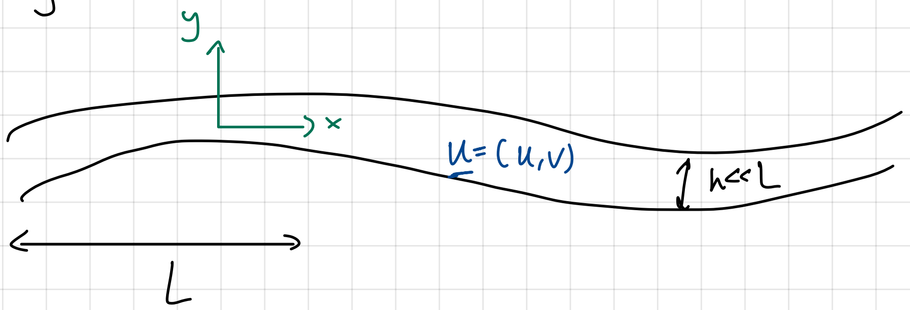
5. Flows in thin layers

Examples





5.1 Scaling



5.1.1 Quasi-parallel

$$\nabla \cdot \underline{u} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{U}{L} \sim \frac{V}{h}$$

$$\Rightarrow V \sim \frac{h}{L} U \ll U$$

So flow is almost uni-directional. (quasi-parallel)

5.1.2 Momentum eqn

x-dir: $$\frac{\partial u}{\partial t} + u \frac{\partial x}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2}$$

Scale time with $\frac{L}{U}$.

$$\frac{U}{L/U} : U \frac{U}{L} : \frac{h}{L} U \cdot \frac{U}{h} : \frac{1}{\rho} \frac{\Delta p}{L} : \nu \frac{U}{L^2} : \nu \frac{U}{h^2} .$$

Note that $\frac{U}{L^2} \ll \frac{U}{h^2}$ ⇒ ignore $\frac{\partial^2 u}{\partial x^2}$

$\frac{U^2}{L} \ll \nu \frac{U}{h^2}$ provided the reduced Reynolds number

$$Re_R = \frac{h}{L} \cdot \frac{Uh}{\nu} \ll 1 .$$

⇒ ignore inertial terms.

The leading order eqn is

$$\boxed{0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} . + f_x}$$

Note that pressure differences $\Delta p \sim \frac{\mu U L}{h^2}$ to balance the largest (only remaining) term in eqn.

y-dir: $$\frac{Dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial y^2} . + \cancel{\nu \frac{\partial^2 v}{\partial x^2}}$$

$$\frac{h}{L}U \cdot \frac{1}{L/U} \; : \; \frac{\mu}{\rho}\frac{UL}{h^2}\cdot\frac{1}{h} \; : \; \nu \frac{h}{L}U \cdot \frac{1}{h^2}$$

$$\left(\frac{h}{L}\cdot\frac{Uh}{\nu}\right)\frac{h^2}{L^2} \; : \; 1 \; : \; \frac{h^2}{L^2}$$

So leading order eqn is

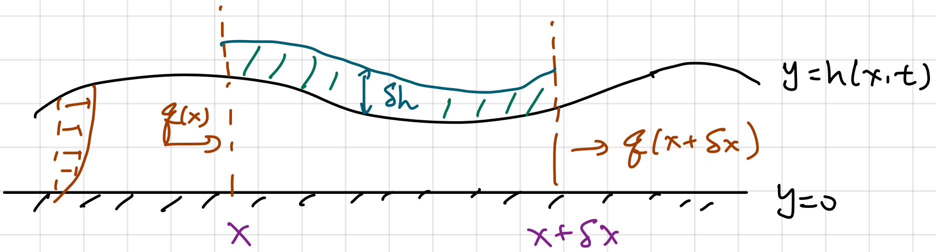
$$\boxed{0 = -\frac{\partial p}{\partial y} + f_y}$$

So to a good approximation of

$$\frac{h}{L}\cdot\frac{Uh}{\nu} \ll 1 , \quad \frac{h}{L} \ll 1$$

the flow satisfies the parallel-flow eqns.

5.2 Conservation of mass



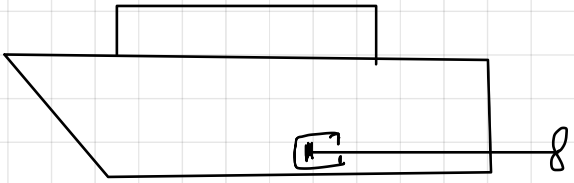
$$[q(x)-q(x+\delta x)]\,\delta t = \delta h \cdot \delta x$$

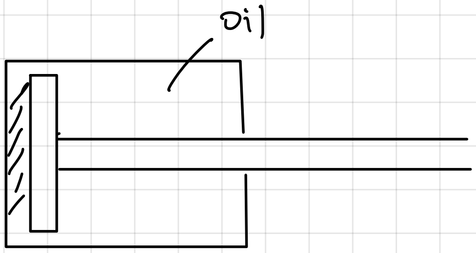
Divide by $\delta x\,\delta t$, take limit.

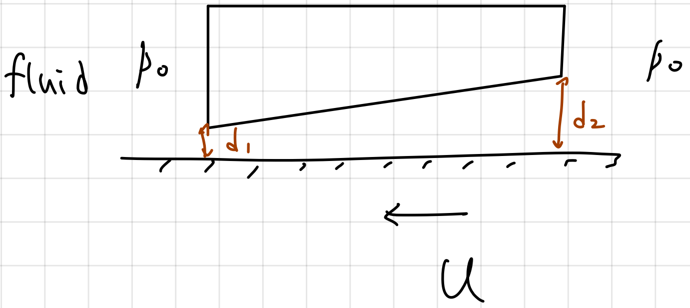
$$-\frac{\partial q}{\partial x} = \frac{\partial h}{\partial t}$$

$$\Rightarrow \boxed{\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0}$$

5.3 Thrust bearing







Always easier to work in frame of reference in which geometry is fixed.

$$h(x) = d_1 + \alpha x \,, \qquad \text{where} \quad \alpha = \frac{d_2-d_1}{L}.$$

Thin film approx valid if $\frac{d_2}{L} \ll 1$ and $\frac{d_2}{L} \cdot \frac{U d_2}{\nu} \ll 1$.

Then

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{\partial p}{\partial y}$$

$$u = -U \quad \text{on} \quad y = 0$$

$$u = 0 \quad \text{on} \quad y = h(x).$$

$$\Rightarrow \quad u = -\frac{1}{2\mu} \frac{\partial p}{\partial x} y(h-y) - U \frac{h-y}{h}.$$

$$\Rightarrow \quad q = \int_0^h u \, dy = -\frac{h^3}{12\mu} \frac{\partial p}{\partial x} - \frac{1}{2} U h.$$

From mass conservation,

$$\frac{\partial h}{\partial t} = 0 \Rightarrow \frac{\partial q}{\partial t} = 0 \Rightarrow q = q_0 \text{ const.}$$

$$\Rightarrow \quad p = p_0 + \frac{6\mu q}{\alpha}\left(\frac{1}{h^2} - \frac{1}{d_1^2}\right) + \frac{6\mu U}{\alpha}\left(\frac{1}{h} - \frac{1}{d_1}\right).$$

But $p = p_0$ also at $x = L$, so

$$q = -U \frac{d_1 d_2}{d_1 + d_2}.$$

Note: $\frac{\partial p}{\partial x} = \alpha \frac{\partial p}{\partial h}$.

On bottom plate,

normal force $F_N = \int_0^L (p - p_0) \, dx = \frac{6\mu U}{\alpha^2}\left[\ln \frac{d_2}{d_1} - 2 \frac{d_2 - d_1}{d_2 + d_1}\right]$

Skin force $F_S = \int_0^L \mu \left.\frac{\partial y}{\partial y}\right|_{y=0} dx = \frac{4\mu U}{\alpha}\left[\ln \frac{d_2}{d_1} - 3 \frac{d_2 - d_1}{d_2 + d_1}\right]$

Exercise If $\alpha \ll 1$, then

$$F_N \sim \frac{\mu U \alpha}{2}\left(\frac{L}{d_1}\right)^3, \qquad F_S \sim \mu U \frac{L}{d_1}$$

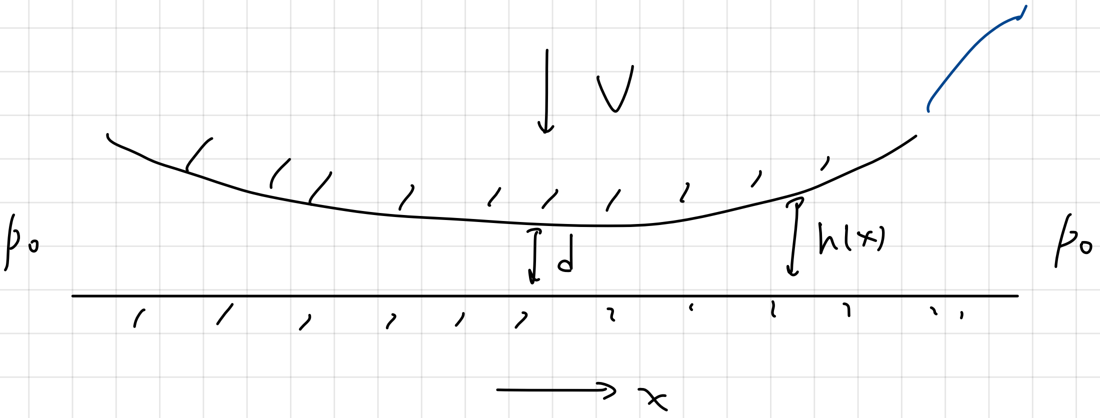
$$\Rightarrow \frac{F_S}{F_N} \sim O\left(\left(\frac{d_1}{L}\right)^2\right) \frac{1}{\alpha}.$$

So we can support a large load (thrust) while keeping friction modest or low.

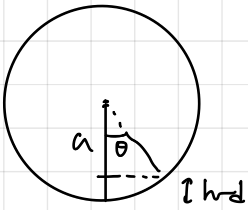
Note $d_1 \sim \mu^{1/3} \left(\frac{U\alpha}{2F_N}\right)^{1/3} L$. So choose oil sufficiently viscous to keep $d_1 >$ roughness of plates.

Need heavier, i.e. more viscous. oil to support a larger thrust.

5.4 Cylinder Approaching a Wall



a) Geometry of thin gap



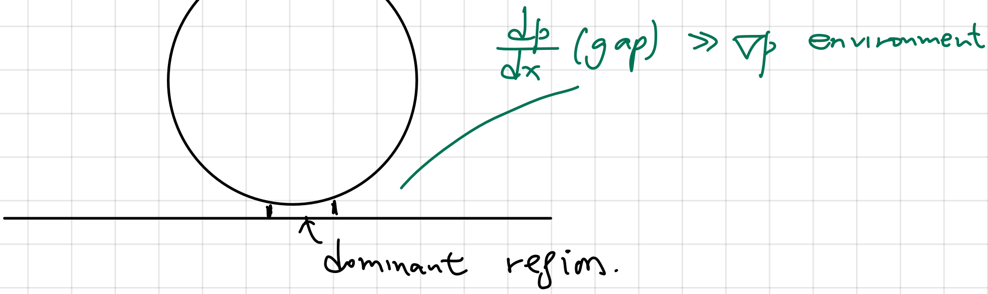
$$h-d = a(1-\cos\theta).$$

$$h \approx d + \frac{a}{2}\theta^2$$

$$\approx d\left(1+\frac{1}{2}\frac{x^2}{ad}\right)$$

since $\theta \simeq \sin\theta \simeq \frac{x}{a}$.

Relevant length scale is $L \sim \sqrt{ad}$. Consider $d \ll L \ll a$



b) quasi - parallel flow.

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\, y(h-y)$$

c) Mass- conservation

$$q = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}.$$

In general,

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\Rightarrow \frac{\partial q}{\partial x} = -\frac{\partial h}{\partial t} = V \quad \text{const.}$$

$$\Rightarrow q = Vx + \cancel{C}^{\nearrow 0} \text{ by symmetry}$$

$$\Rightarrow \frac{\partial p}{\partial x} = -\frac{12\mu V x}{d^3\left(1+\frac{1}{2}\frac{x^2}{ad}\right)^3}$$

$$\Rightarrow p = p_0 + \frac{6\mu a V}{d^2\left(1+\frac{1}{2}\frac{x^2}{ad}\right)^2}$$

Note that there is very rapid decay of pressure ($\propto \frac{1}{x^4}$) so can apply $p \to p_0$ as $x \to$ "∞".

d) Normal force on cylinder (per unit length into page)

$$F = \int_{-\text{``}\infty\text{''}}^{\text{``}\infty\text{''}} (p-p_0)\,dx.$$

$$= \frac{6\mu a V}{d^2}\sqrt{2ad}\int_0^\infty \frac{dt}{(1+t^2)^2}$$

$$= 6\pi\mu a V\sqrt{\frac{a}{2d^3}}$$

e) Motion of cylinder. (given weight $= F$)

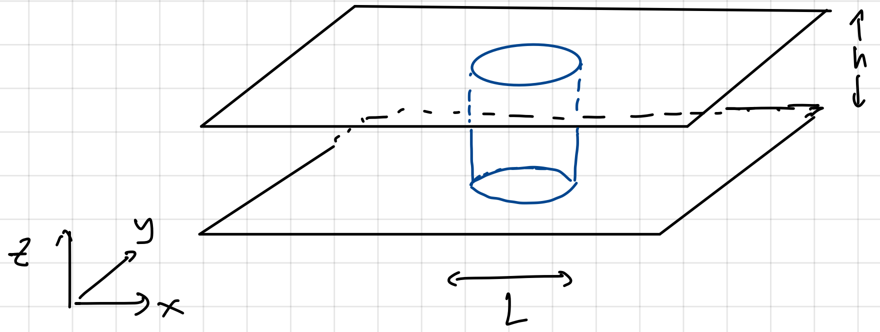
$$\frac{d}{dt} d = -V \propto -d^{3/2}.$$

$$\Rightarrow d^{-1/2} - d_0^{-1/2} \propto -t$$

$$\Rightarrow d = \left(d_0^{-1/2} + kt\right)^{-2}$$

$\Rightarrow d \to 0$ in infinite time.

5.5 Hele-Shaw Flow



If $h \ll L$, then thin film approx. valid.

If $\underline{u} = (u, v, w)$, then $w = O\left(\frac{h}{L}u, \frac{h}{L}v\right)$ by $\nabla \cdot \underline{u} = 0$, so approximately horizontal flow.

Inertia can be ignored if $\frac{h}{L} \cdot \frac{Uh}{\nu} \ll 1$ as before. Then

$$\mu \frac{\partial^2 \underline{u}_H}{\partial z^2} = \nabla_H \, p(x,y),$$

where $\underline{u}_H = (u, v)$

$$\Rightarrow \quad \underline{u}_H = -\frac{1}{2\mu} \nabla_H p \, z(h-z)$$

$$\Rightarrow \quad \underline{q}_H = \int_0^h \underline{u}_H \, dz = -\frac{h^3}{12\mu} \nabla_H p.$$

Continuity $\nabla \cdot \underline{u} = 0$

$$\Rightarrow \quad \nabla_H \cdot \underline{u}_H + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \quad \int_0^h \left(\nabla_H \cdot \underline{u}_H + \frac{\partial w}{\partial z}\right) dz = 0$$

$$\Rightarrow \quad \nabla_H \cdot \underline{q}_H + \cancel{[w]_0^h} = 0 \qquad \rightarrow 0 \ \because \text{no penetration}$$

So $\nabla_H \cdot \underline{q}_H = 0$.

$$\Rightarrow \quad \nabla_H^2 p = 0.$$

Note that p is a "velocity potential" for $\underline{q}_H$.

Define a Darcy velocity

$$\underline{u}_D(x,y) = \frac{1}{h}\underline{q}_H = \frac{1}{h}\int_0^h \underline{u}_H \, dz = \langle \underline{u} \rangle$$

Notes: · Avg (Darcy) velocity $\underline{u}_D$ is a 2D potential flow with potential

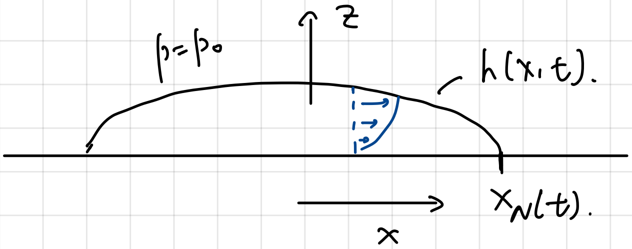
$$\varphi = -\frac{h^2}{12\mu} p.$$

· Flow through porous media (e.g. ground water flow) is often modelled using Darcy's eqn

$$\mu \underline{u} = -k \nabla p.$$

where the "permeability" k has dimensions length squared.

5.6 Viscous Gravity Current (on a horizontal surface)



Once $x_N \gg h$ (and $\frac{h}{x_N} \cdot \frac{U x_N}{\nu} \ll 1$), we use thin film approx.

Vertical: $$0 = -\frac{\partial p}{\partial z} - \rho g$$ with $p = p_0$ at $z = h(x,t)$.

$$\Rightarrow p = p_0 + \rho g (h - z)$$

Horizontal: $$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}.$$

$$\Rightarrow u = -\frac{g}{2\nu}\frac{\partial h}{\partial x} z(2h - z).$$

(satisfying no slip $u(z=0)=0$, no stress $\left.\frac{\partial u}{\partial z}\right|_{z=h} = 0$)

"Volume flux" is $q = \int_0^h u \, dz = -\frac{g}{3\nu} h^3 \frac{\partial h}{\partial x}$.

Local mass conservation

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 .$$

$$\Rightarrow \frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right). \qquad (1)$$

(1) is a non-linear diffusion eqn for h

Boundary conditions :

$$q = 0 \Rightarrow \frac{\partial h}{\partial x} = 0 \qquad \text{at } x = 0 . \qquad (2)$$

$$h = 0 \qquad \text{at } x = X_N(t). \qquad (3)$$

Global mass conservation

$$\text{Volume} = \int_0^{X_N(t)} h(x,t)\,dx = V \quad (\text{const}) \qquad (4).$$

Plus initial conditions.

Recall that diffusion equations forget about their initial conditions quite quickly.

<u>Similarity sol^n</u>

Scale x with L, h with H, t with T. Then

$$(1) \Rightarrow \quad \frac{H}{T} \sim \frac{g}{\nu}\,\frac{H^4}{L^2}$$

$$(4) \Rightarrow \quad HL \sim V$$

Only 2 scaling relationship for 3 scales, so $T = t$ and

$$\Rightarrow L \sim \left(\frac{V^3 g t}{\nu}\right)^{1/5}, \quad H \sim \left(\frac{\nu V^2}{g t}\right)^{1/5}$$

Write $h = \left(\frac{\nu V^2}{gt}\right)^{1/5} f(\eta)$, where $\eta = \frac{x}{L} = \left(\frac{\nu}{V^3 g t}\right)^{1/5} x$.

and
$$x_N(t) = \eta_N\left(\frac{V^3 g t}{\nu}\right)^{1/5}$$

Substitute into (1) to find

$$-\frac{1}{5}f - \frac{1}{5}\eta f' = \frac{1}{3}\left(f^3 f'\right)' \qquad (1)$$

$$-\frac{1}{3}f^3 f' = 0 \quad \text{at } \eta = 0 \qquad (2)$$

$$f = 0 \quad \text{at } \eta = \eta_N \qquad (3)$$

$$\int_0^{\eta_N} f \, d\eta = 1 \qquad (4)$$

Integrate (1) to find

$$-\frac{1}{5}\eta f = \frac{1}{3}f^3 f' + \underbrace{C}_{(2) \Rightarrow C=0}$$

$$\Rightarrow -\frac{1}{5}\eta = \frac{1}{3}f^2 f'$$

$$\Rightarrow -\frac{1}{10}\eta^2 = \frac{1}{9}f^3 + D$$

$$\Rightarrow f = \left(\frac{9}{10}\right)^{1/3}\left(\eta_N^2 - \eta^2\right)^{1/3}$$

From (4), with $y = \eta/\eta_N$.

$$\Rightarrow \left(\frac{9}{10}\right)^{1/3}\eta_N^{5/3}\int_0^1 (1-y^2)^{1/3}\, dy = 1$$

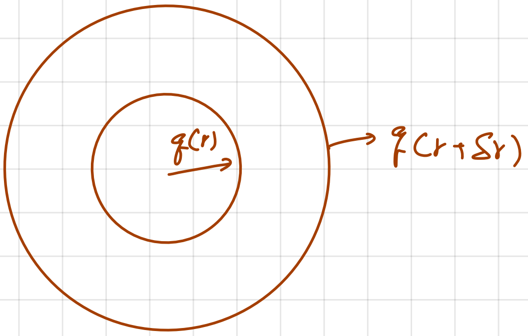
Note that $\int_0^1 (1-y^2)^{1/3}\, dy = \frac{1}{2}\,\mathrm{Beta}\left(\frac{1}{2}, \frac{4}{3}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{3}\right)}{2\,\Gamma\left(\frac{11}{6}\right)}$.

$$\Rightarrow \eta_N = \left[\left(\frac{9}{10}\right)^{1/3}\frac{\sqrt{\pi}}{5}\,\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}\right]^{-3/5} \approx 1.13.$$

Sheet 3 #4 (axisymmetric gravity current)

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rq) = 0.$$

(General: $\frac{\partial h}{\partial t} + \nabla \cdot q = 0$)



Global conservation of mass:

$$\int_0^{r_N} 2\pi r h(r,t)\, dr = V$$

6. Vorticity generation and confinement

6.1 Vorticity equation

Start with N-S with conservative body force $\underline{F} = -\nabla \chi$.

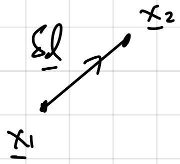
$$\rho \frac{D\underline{u}}{Dt} \equiv \rho\left(\underline{u}\cdot\nabla\underline{u} + \frac{\partial\underline{u}}{\partial t}\right) = -\nabla p + \mu\nabla^2\underline{u} - \nabla\chi .$$

Recall $\underline{u}\wedge(\nabla\wedge\underline{u}) = \nabla\left(\frac{1}{2}\underline{u}^2\right) - \underline{u}\cdot\nabla\underline{u}$. Take curl of N-S to get

$$\frac{\partial\underline{\omega}}{\partial t} + \underbrace{\underline{u}\cdot\nabla\underline{\omega}}_{\text{advection}} = \underbrace{\underline{\omega}\cdot\nabla\underline{u}}_{\text{stretching}} + \underbrace{\nu\,\nabla^2\underline{\omega}}_{\text{diffusion}}$$

6.2 Vortex Stretching

A material line element is advected according to



$$\frac{D}{Dt}(\delta\underline{l}) = \dot{\underline{x}}_2 - \dot{\underline{x}}_1$$

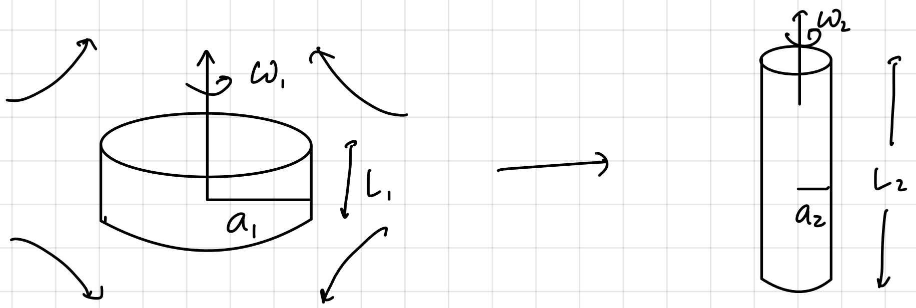
$$= \underline{u}_2 - \underline{u}_1$$

$$\approx (\underline{u}_1 + \delta\underline{l}\cdot\nabla\underline{u} + \dots) - \underline{u}_1$$

$$\Rightarrow \quad \frac{D}{Dt}(\delta\underline{l}) = \delta\underline{l}\cdot\nabla\underline{u} \quad \text{in the limit } \delta\underline{l}\to 0 .$$

Note $\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u}$ if no viscosity.

This is a consequence of conservation of angular momentum.



Mass conservation: $\rho a_1^2 L_1 = \rho a_2^2 L_2$.

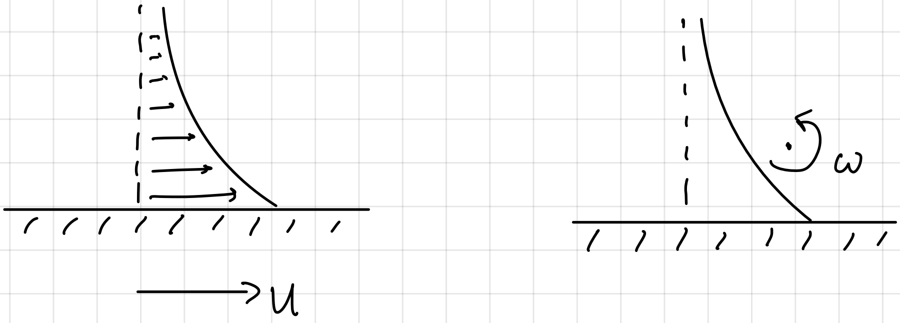
Average angular momentum:

$$\underbrace{\underbrace{(\rho a_1^2 L_2)}_{\text{mass}}\, a_1^2}_{\text{moment of inertia}} \omega_1 = (\rho a_2^2 L_2)\, a_2^2 \omega_2 .$$

$$\Rightarrow \qquad \omega_2 = \frac{L_2}{L_1}\omega_1$$

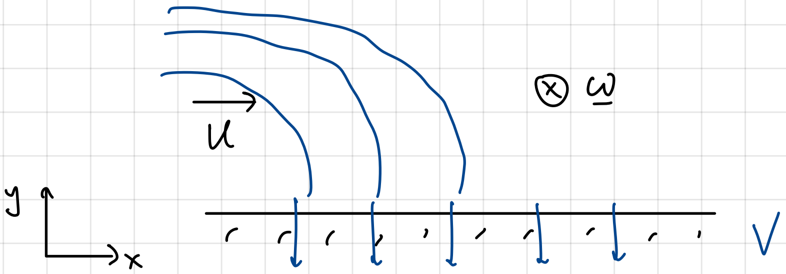
6.3 Diffusion of vorticity

See § 3, impulsively started plate.



Vorticity is generated at rigid boundaries by no-slip condition.

6.4 Confinement of vorticity



Consider flow past a rigid, porous plate through which fluid is sucked at fixed speed V.

This has sol^n

$$\underline{u} = (u(y), -V, 0).$$

Check $\nabla \cdot \underline{u} = 0$

x-momentum:

$$-V \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \quad u = U(1 - e^{-Vy/\nu})$$

$u = U(1 - e^{-y/\delta})$.

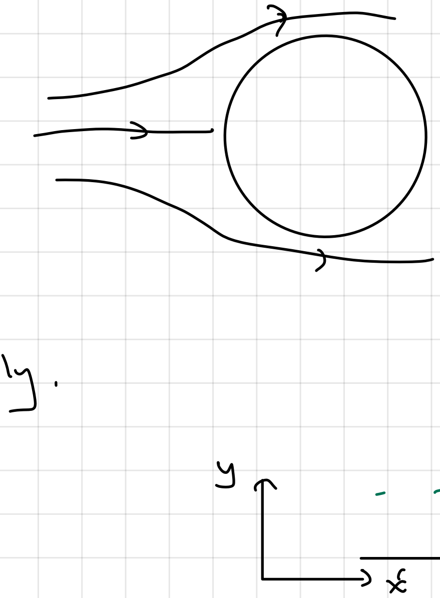
This is an exact sol^n of N-S.

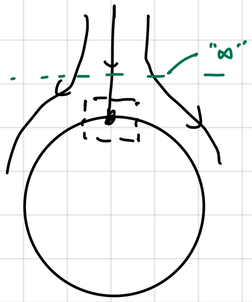
Note that steady state is achieved by balacing diffusion away from the wall, $\delta \sim \sqrt{\nu t}$, with advection $\delta \sim Vt$.

ie,

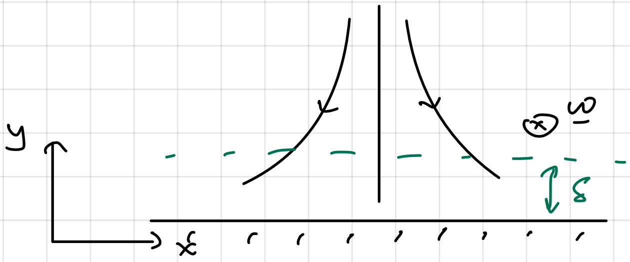
$$\sqrt{\nu t} \sim Vt$$
$$\Rightarrow t \sim \frac{\nu}{V^2}$$
$$\Rightarrow \delta \sim \frac{\nu}{V}.$$

6.4.2 Stagnation point on a rigid body





Locally,



"Far" from the wall,

$$\underline{u} \sim (Ex, -Ey, 0) = \underline{u}_\infty \text{ (say)}$$

No slip + no penetration

$$\Rightarrow \quad \underline{u} = 0 \quad \text{on} \quad y = 0.$$

Note: The outer flow satisfies no penetration, but not no slip on $y=0$.

Note $\nabla \wedge \underline{u}_\infty = 0$. Far field is irrotational. But, vorticity is generated at $y=0$ by no slip, diffuses away and is advected by flow.

Velocity scale U, then $E \sim U/L$.

$$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Estimate thickness of region of confined vorticity.

advection $\quad \rho v \frac{\partial u}{\partial y} \sim \rho (E\delta) \cdot \frac{Ex}{\delta} \sim \rho E^2 x.$

diffusion $\quad \mu \frac{\partial^2 u}{\partial y^2} \sim \mu \frac{Ex}{\delta^2}$

$$\Rightarrow \quad \delta \sim \sqrt{\nu / E}$$

Scale y with $\delta \sim \sqrt{\nu/E}$
u with Ex
v with $E\delta$.

Look for a sol^n with $u = Ex\, f(\eta)$, $v = -E\delta g(\eta)$, $\eta = y/\delta$.

$$\nabla \cdot \underline{u} = 0 \Rightarrow Ef(\eta) - E\delta\, g'(\eta) \frac{1}{\delta} = 0$$

$$\Rightarrow f = g'$$

and $\underline{u} = \left(Ex\, g'(\eta),\ -\sqrt{\nu E}\, g(\eta),\ 0\right)$

Substitute in N-S to find

$$E^2 x\left(g'^2 - gg''\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + E^2 x g''' \qquad (1)$$

$$E\sqrt{\nu E}\, gg' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - E\sqrt{\nu E}\, g'' \qquad (2)$$

Diff (2) w.r.t. x $\Rightarrow$ $p_{xy} = 0$

Diff (1) w.r.t. y $\Rightarrow$ $\left(g'^2 - gg''\right)' = g^{iv}$ (3)

Subject to b.c. $\underset{\substack{\uparrow\\ \text{no penetration}}}{g=0}$. $\underset{\substack{\uparrow\\ \text{no slip.}}}{g'=0}$, on $\eta = 0$.

$$g \sim \eta \ , \quad g' \to 1 \quad \text{as } \eta \to \text{"}\infty\text{"}$$

$\Rightarrow g'', g''' \to 0$

Integrate (3) once

$$\Rightarrow g''' = g'^2 - gg'' - \underset{\nwarrow}{1}$$

applying far-field b.c.

* Numerical Solution *

Write $y_1 = g$, $y_2 = g'$, $y_3 = g''$, then the DE can be written

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = y_2^2 - y_1 y_3 - 1$$

Write $\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, then

$$\underline{Y}' = \underline{F}(\underline{Y})$$

with I.C.

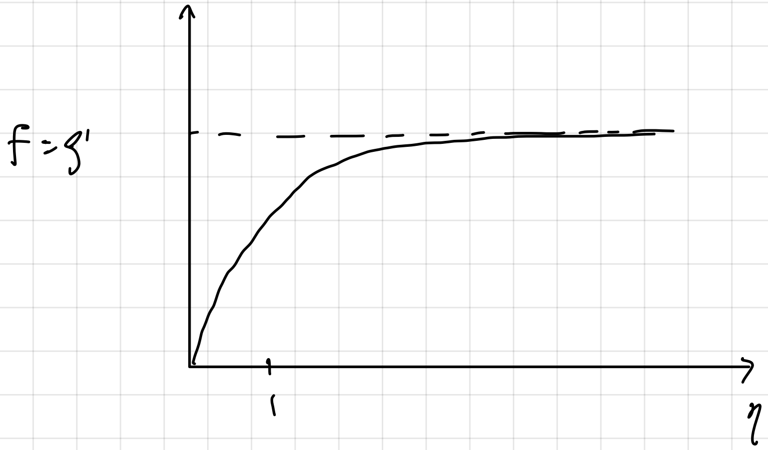
$$\underline{Y} = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}$$

where p is a guessed parameter. Integrate numerically from $\eta = 0$ to $\eta = L$ for the given value of p to determine the residual

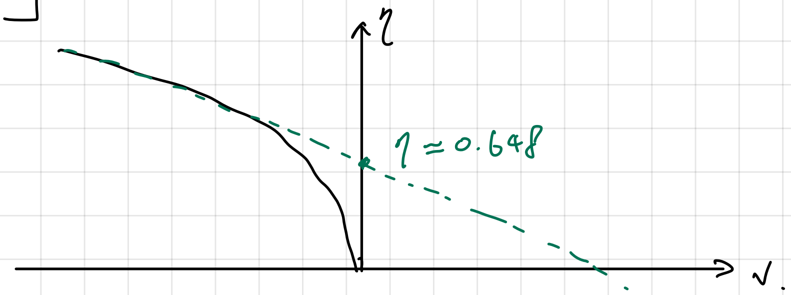
$$R(p, L) = y_2(L; p) - 1.$$

Note $y_2 \to 1$ as $\eta \to \infty$ when R suff. small, L suff. large.

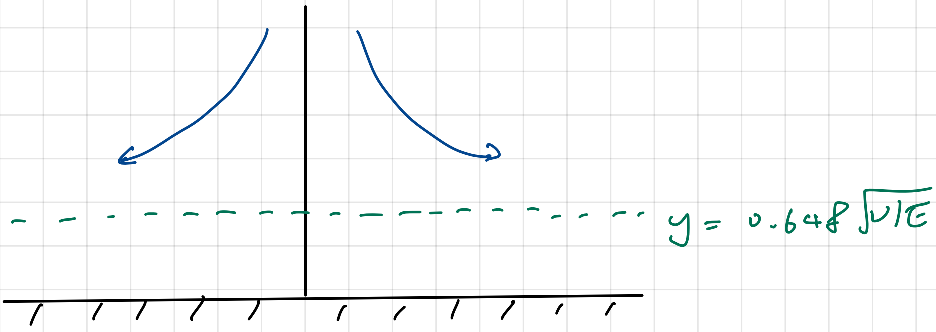
Find numerically that



Vertical velocity



So the "inviscid" outer flow looks like

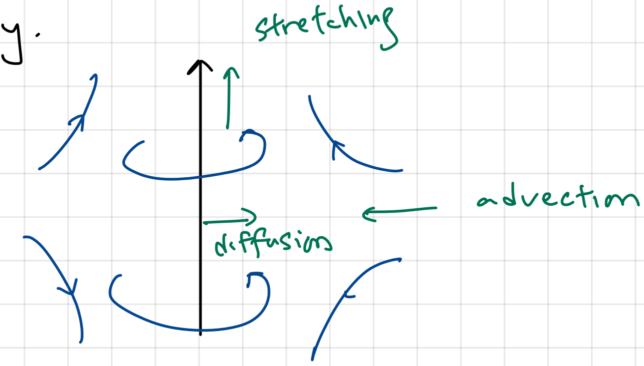


with slip and no penetration at a displaced wall at

$$y = 0.648\sqrt{\nu/E}$$ ← displacement thickness

6.5 Burger's Vortex

Steady balance between advection, stretching, and diffusion of vorticity.



Consider superposition of a vortical flow

$$\underline{u} = v(r)\ \underline{e}_\theta$$

and a straining flow

$$\underline{u} = -\alpha r\ \underline{e}_r + 2\alpha z\ \underline{e}_z$$

Check that $\nabla \cdot \underline{u} = 0$.

Vorticity is

$$\underline{\omega} = \nabla \times \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(rv)\ \underline{e}_z = \omega(r)\ \underline{e}_z$$

The steady vorticity eqn is

$$\underline{u} \cdot \nabla \underline{\omega} = \underline{\omega} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega}$$

gives

$$-\alpha r \frac{\partial \omega}{\partial r} = 2\alpha\omega + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right)$$

$$\Rightarrow -\alpha r^2 \frac{\partial \omega}{\partial r} - 2\alpha r \omega = \nu \frac{\partial}{\partial r}\left(r \frac{\partial \omega}{\partial r}\right)$$

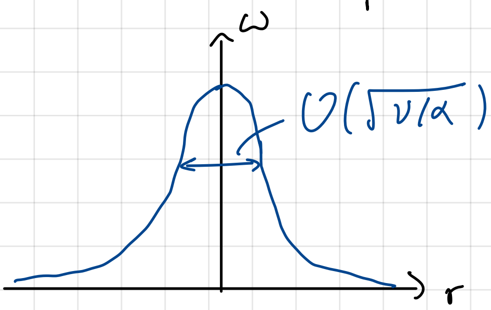
$$\Rightarrow -\alpha r^2 \omega = \nu r \frac{\partial \omega}{\partial r} + \cancel{C} = 0 \text{ at } r=0$$

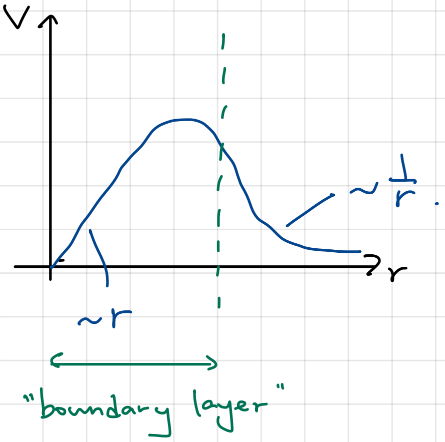
$$\Rightarrow \omega = \omega_0 e^{-\alpha r^2/2\nu}$$

$$\Rightarrow v = \frac{\omega_0 \nu}{\alpha} \frac{1 - e^{-\alpha r^2/2\nu}}{r}$$

Note circulation $2\pi r v \rightarrow \frac{2\pi\omega_0\nu}{\alpha}$ const. as $r \to \infty$.

So flow looks like (potential) flow around a line vortex.





As $r \to \infty$, v is the potential flow for a line vortex.

For $r \ll \sqrt{\nu/\alpha}$, the flow corresponds to a solid body rotation.

7. Boundary Layers ($Re \gg 1$).

7.1 Euler's equations

Scaling of N-S gives

$$Re \frac{D\underline{u}^*}{Dt^*} = -\left(\frac{PL}{\mu U}\right) \nabla^* p^* + \nabla^{*2} \underline{u}^*$$

This suggests (naively) when $Re \gg 1$ that

$$\left.\begin{aligned} \rho \frac{D\underline{u}}{Dt} &\approx -\nabla p \\ \nabla \cdot \underline{u} &= 0 \end{aligned}\right| \text{ Euler eqns}$$

i.e. inertial terms dominates viscous terms, and pressure scales with momentum flux.

$$\frac{PL}{\mu U} \sim Re = \frac{UL}{\nu} \Rightarrow P \sim \overset{\text{momentum flux}}{\rho U^2}$$

Euler eqns ⇒ irrotational flows remain irrotational

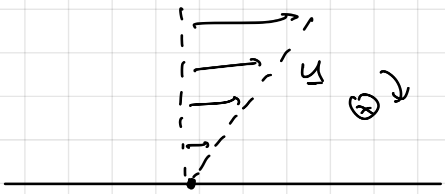
$$\nabla \wedge \underline{u} = 0 \Rightarrow \underline{u} = \nabla \phi$$

and $\underline{n} \cdot \underline{u} = 0$ on rigid boundaries.

Euler eqns are only 1st order in space, where N-S is second order. Can apply no penetration, but <u>not</u> no-slip.

7.2 Generation and confinement of vorticity

In real (viscous) flow, vorticity is generated at rigid boundaries by no-slip condition

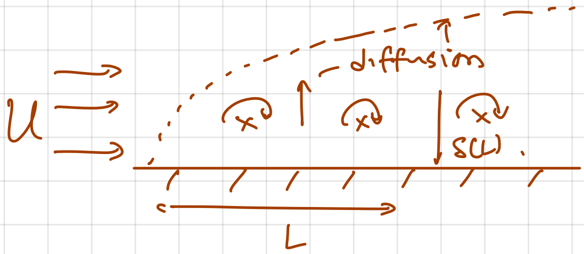


vorticity local to boundary

In many cases, advection keeps vorticity due to boundary, where it is generated

Example Flow past a rigid, flat plate

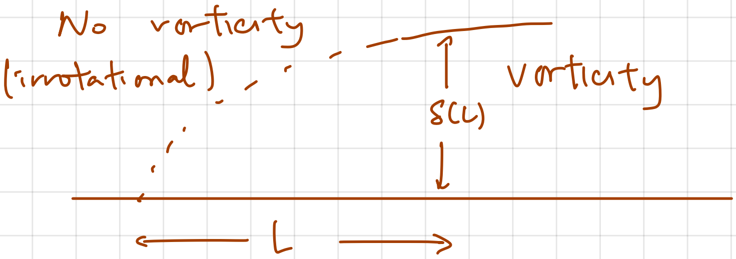
Vorticity generated at the boundary diffuses a distance $\delta \sim \sqrt{\nu t}$ in time t in y-direction



(away from the boundary), while being advected downstream a distance $L \sim Ut$ in x direction.

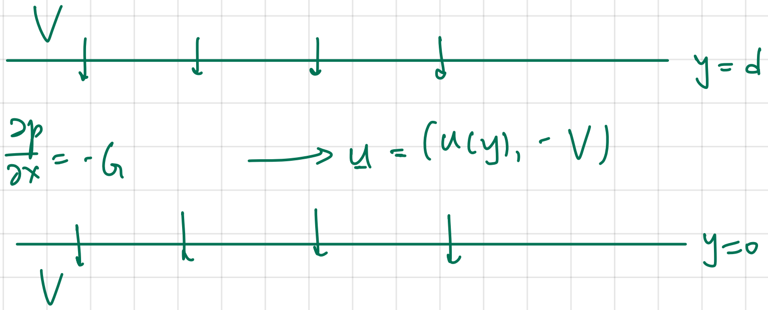
So diffusion balances advection (parameterised by t) at a distance (intrinsic lengthscale) δ, where

$$\delta \sim \sqrt{\nu t} \sim \sqrt{\nu L / U}.$$



Note that $\delta / L \sim \sqrt{\nu / UL} \sim Re^{-1/2} \ll 1$ when $Re \gg 1$.

Aside (cf Sheet 3 #6, §6.4.1)



$$-V \frac{\partial u}{\partial y} = \frac{1}{\rho} G + \nu \frac{\partial^2 u}{\partial y^2}$$

$u = 0$ on $y = 0, d$ (no slip)

Scale y with d, u with $Gd / \rho V$. (an inertial scaling) and write

$$y^* = y/d \qquad u^* = (\rho V / Gd) u.$$

Then $\quad -\frac{\partial u^*}{\partial y^*} = 1 + \varepsilon \frac{\partial^2 u^*}{\partial y^{*2}}$, where $\varepsilon = \frac{\nu}{Vd} = \frac{1}{Re}$.

Drop * :

$$-\frac{\partial u}{\partial y} = 1 + \varepsilon \frac{\partial^2 u}{\partial y^2}$$

$$u = 0 \quad \text{on} \quad y = 0, 1$$

Full sol^n. $\quad u = -y + \frac{1 - e^{-y/\varepsilon}}{1 - e^{-1/\varepsilon}}$. $\quad$ ε small

Consider $\varepsilon \ll 1$ and look for leading-order behaviour.

Naive approximation ("Euler sol^n")

$$\varepsilon u'' + u' = -1, \quad u = 0 \quad \text{on} \quad y = 0, 1.$$

$$\varepsilon \to 0 \quad \Rightarrow \quad u' = -1 \Rightarrow u = A - y.$$

Could try each b.c. in turn, or use physical understanding to apply b.c. at $y = 1$, where vorticity is not confined.

$$\Rightarrow \qquad A = 1$$

$$\Rightarrow u = U_{out} = 1 - y$$

Close to boundary where vorticity is confined, expect a new intrinsic lengthscale δ in which viscous and inertial term balance. (The local Reynolds number is $O(1)$)

$$\Rightarrow \quad \varepsilon \frac{u}{\delta^2} \sim \frac{u}{\delta}$$

$$\Rightarrow \quad \delta \sim \varepsilon$$

Introduce a new, scaled "inner" variable

$$Y = y/\delta$$

and substitute into full eqn to give

$$U_{YY} + U_Y = -\varepsilon$$

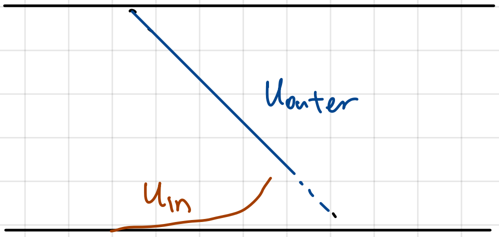
The leading order when $\varepsilon \to 0$ has

$$U_{YY} + U_Y = 0 , \qquad u = 0 \text{ on } Y = 0 .$$

$$\Rightarrow \quad u = u_{in} = B(1-e^{-Y})$$

The outer (Euler) flow has a slip velocity of the wall $y=0$

$$U_{out}(y=0) = 1$$

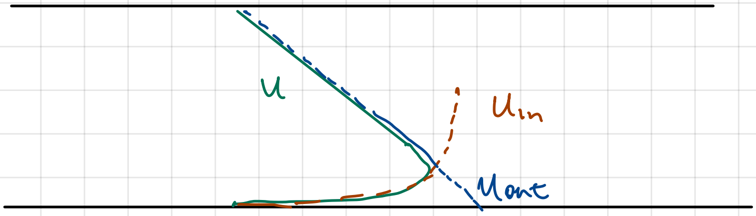


Choose B s.t. the inner boundary layer flow tends to the outer slip velocity as $Y \to \infty$. (ie. $\delta \to 0$ for any fixed y)

$$\lim_{Y \to \infty} U_{in}(Y) = \lim_{y \to 0} U_{out}(y)$$

$$\Rightarrow \quad B = 1$$

The outer and inner sol^n provide functional asymptotes to the full sol^n when $y \gg \varepsilon$ and $y = O(\varepsilon)$ resp.



Remark Composite sol^n

$$U_{comp} \sim U_{in} + U_{out} - \lim_{y \to 0} U_{out}(y)$$

$$\sim 1 - e^{-Y} + (1-y) - 1$$

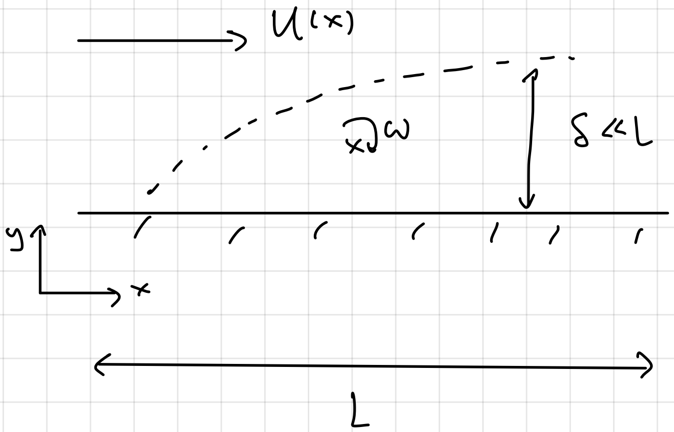
$$\sim 1 - y - e^{-y/\delta}$$

7.3 Viscous Flows of large Re

7.3.1 Solve outer, inviscous flow using Euler eqn / potential flow

See IB

7.3.2 Rescaling inside a viscous boundary layer



Outer Reynolds number $Re = \frac{UL}{\nu} \gg 1$.

Inside, inner, boundary layer, scale x with L, y with δ, t with L/U, u with U.

$$\nabla \cdot \underline{u} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

$$\Rightarrow \quad \frac{U}{L} \sim \frac{V}{\delta}$$

$$\Rightarrow \quad V \sim U \frac{\delta}{L}.$$

N-S momentum :

x-dir:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \cancel{\nu \frac{\partial^2 u}{\partial x^2}} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{U}{L/U} : U \frac{U}{L} : \frac{U\delta}{L} \frac{U}{\delta} : \frac{P}{\rho L} : \nu \frac{U}{L^2} : \nu \frac{U}{\delta^2}.$$

Expect $\delta \ll L$ (justify later). Therefore

$$\frac{U\delta^2}{\nu L} : \frac{P\delta^2}{\mu L U} : 1$$

Inside boundary layer, balance viscous term with inertial term by choosing $U\delta^2/\nu L \sim 1 \Rightarrow \delta \sim \sqrt{\nu L/U}$

$$\Rightarrow \delta/L \sim \sqrt{\nu/UL} = 1/\sqrt{Re} \ll 1 \text{ if } Re \gg 1.$$

Scale pressure variations to balance remaining terms

$$P \sim \frac{\mu L U}{\delta^2} \sim \rho U^2 \quad \text{(same as Euler scaling)}$$

y-dir:

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2}$$

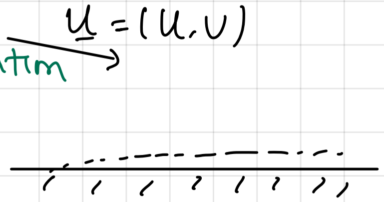
$$\frac{U\delta/L}{L/U} : \frac{1}{\rho}\cdot\frac{\rho U^2}{\delta} : \nu\frac{U\delta/L}{L^2} : \nu\frac{U\delta/L}{\delta^2}$$

$$\Rightarrow \frac{\delta^2}{L^2} : 1 : \nu\left(\frac{\delta}{L}\right)^4 : \nu\left(\frac{\delta}{L}\right)^2$$

So leading order eqn: $0 = \frac{\partial p}{\partial y}$

Pressure does not vary across the boundary layer. Pressure inside B.L. is equal to that of external Euler flow.

Just outside B.L.,



almost 0 ∵ no penetration

$$\underline{u} = \underline{U} = (U, \cancel{V})$$

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \underbrace{\cancel{V\frac{\partial U}{\partial y}}}_{\text{almost } 0}\right) = -\frac{\partial p}{\partial x} \quad \text{(Euler)}$$

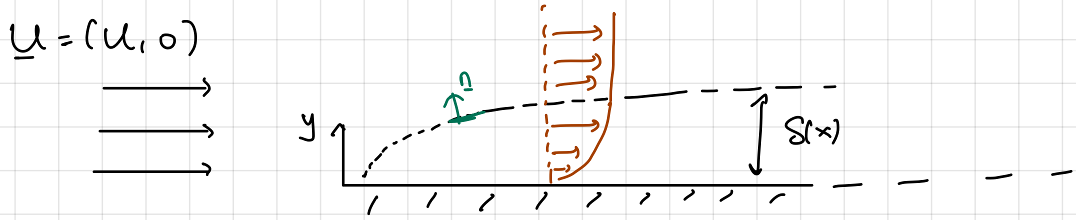
So the x-mom. eqn in the B.L. is

$$\rho\frac{Du}{Dt} = \rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right) + \mu\frac{\partial^2 u}{\partial y^2}$$

where $U(x,t)$ is the slip velocity of the outer Euler flow.

The inner velocity $\underline{u}(x,y,t)$ satisfies viscous boundary conditions (e.g. no slip, prescribed stress, ...) on boundary $y=0$, and tends to outer slip velocity as $y \to$ "∞".

7.4 Blassius boundary layer on a flat plate



Consider a uniform flow $\underline{u} = (U, 0)$ impinging on a semi-infinite flat plate at $y=0$, $x>0$.

No extrinsic lengthscale so $L \sim x$.

$$\Rightarrow \quad \delta \sim \sqrt{\frac{\nu x}{U}}$$

Introduce a stream f^n ψ s.t.

$$\underline{u} = (u, v) = \left(\frac{\partial \psi}{\partial y}, \; -\frac{\partial \psi}{\partial x}\right)$$

$$u \sim U \; . \Rightarrow \quad \psi \sim U\delta$$

So write $\psi = U\sqrt{\frac{\nu x}{U}}\, f(\eta)$. , where $\eta = \frac{y}{\delta} = \sqrt{\frac{U}{\nu x}}\, y$

Whernce

$$u = \frac{\partial \psi}{\partial y} = U f'(\eta).$$

$$V = -\frac{\partial \psi}{\partial x} = -\frac{U}{2}\sqrt{\frac{\nu}{Ux}}\, f - U\sqrt{\frac{\nu x}{U}}\, f'\left(-\frac{1}{2}\frac{1}{x}\sqrt{\frac{U}{\nu x}}\, y\right)$$

$$= \sqrt{\frac{\nu U}{x}}\left(-\frac{1}{2}f + \frac{1}{2}\eta f'\right)$$

Rmk In general, $\psi = U(x)\; \delta(x)\; f(y/\delta(x))$

Recall BL eqn's

$$\frac{Du}{Dt} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

For Blassius, $\underline{U} = (U, 0)$, so $\partial U/\partial t = 0$, $U\frac{\partial U}{\partial x} = 0$.

Steady flow:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

No penetration: $v = 0$ on $y = 0$

No slip: $u = 0$ on $y = 0$

Matching: $u \to U$ as $y \to$ "∞"

Then

$$u\frac{\partial u}{\partial x} = Uf'\left(-\frac{U}{2}\sqrt{\frac{U}{\nu x}}\,\frac{y}{x}f''\right)$$

$$= -\frac{1}{2}\frac{U^2}{x}\eta f'f''$$

$$v\frac{\partial u}{\partial y} = \sqrt{\frac{U\nu}{x}}\left(-\frac{1}{2}f + \frac{1}{2}\eta f'\right)U\sqrt{\frac{U}{\nu x}}\,f''$$

$$= -\frac{1}{2}\frac{U^2}{x}(f - \eta f')f''.$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \nu U\frac{U}{\nu x}f''' = \frac{U^2}{x}f'''.$$

So

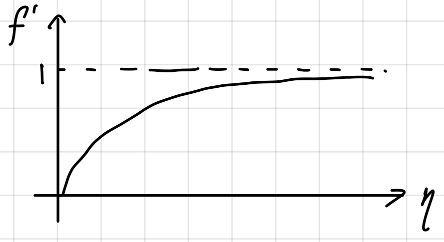
$$-\frac{1}{2}ff'' = f'''$$

$f' = 0$ on $\eta = 0$

$f = 0$ on $\eta = 0$

$f' \to 1$ as $\eta \to \infty$

Solve numerically using shooting with $f''(0) = p$ as a starting parameter. Adjust p until $f' \to 1$ as $\eta \to$ "∞".



Reaches 99% of far-field value when $\eta \approx 5.04$

$$\Rightarrow y \approx 5.04\sqrt{\frac{\nu x}{U}} .$$

From the numerical sol^n, we can also determine

$$V_\infty = \lim_{y\to\infty} v(y) = \sqrt{\frac{U\nu}{x}} \lim_{\eta\to\infty}\left(-\tfrac{1}{2}f + \tfrac{1}{2}\eta f'\right)$$

$$\approx 0.860\sqrt{\frac{U}{\nu x}}$$

Note $V_\infty > 0$ — detrainment velocity. At "edge" of BL,

$$\underline{u} \approx (U, V_\infty)$$

Say that the edge is at $y = \delta(x)$, then a normal to the edge is

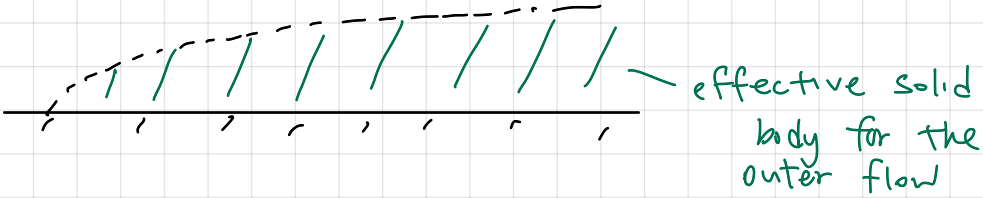
$$\underline{n} = (-\delta'(x), 1)$$

No penetration when $\underline{n}\cdot\underline{u} = 0$

$$\Rightarrow -U\delta'(x) + 0.860\sqrt{\frac{U}{\nu x}} = 0$$

$$\Rightarrow \delta = 1.72\sqrt{\frac{\nu x}{U}}$$

This is the displacement thickness.

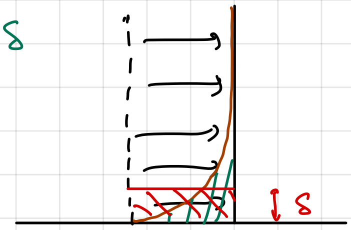


Get a higher-order correction to outer flow by solving for potential flow in $y > \delta(x)$.

Alternatively, compute the volume flux caused by the boundary layer.

defines δ

$$\int_0^\infty (U-u)\,dy = \delta U$$



$$\Rightarrow \delta = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

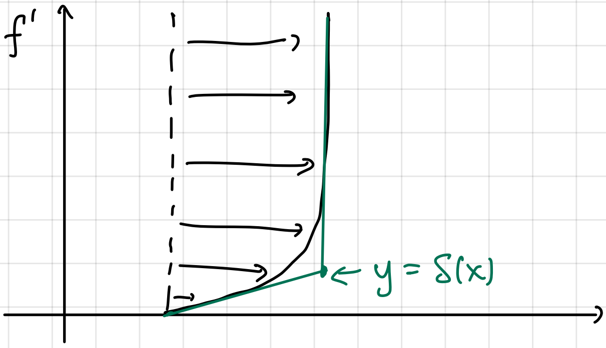
$$= \sqrt{\frac{\nu x}{U}} \int_0^\infty (1 - f') \, d\eta = 1.72 \sqrt{\frac{\nu x}{U}} \quad \text{as above.}$$

This gives us an alternative physical description of displacement thickness.

Skin friction

$$\tau = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu U \sqrt{\frac{U}{\nu x}} f''(0)$$

From numerical, $f''(0) \approx 0.332$

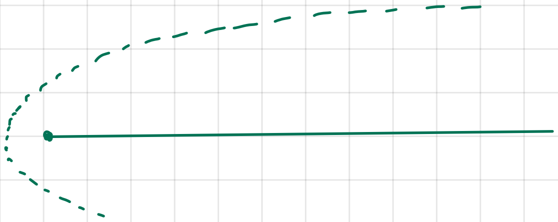


Define a sheer thickness by

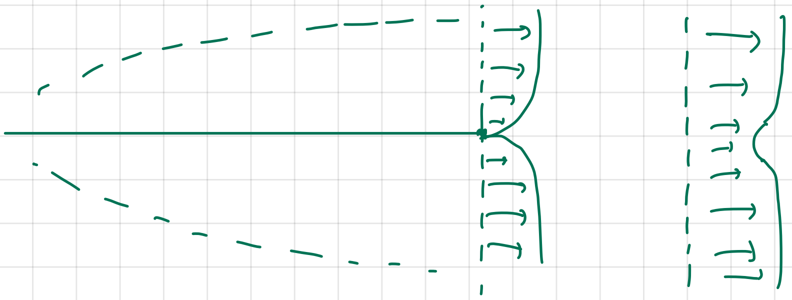
$$\tau = \mu \frac{U}{\delta_s}$$

$$\Rightarrow \delta_s \approx 3.01 \sqrt{\frac{\nu x}{U}}.$$

Aside Can develop higher-order corrections near leading and traiting edge.



The singularity of the similarity sol^n is displaced in the leading edge.



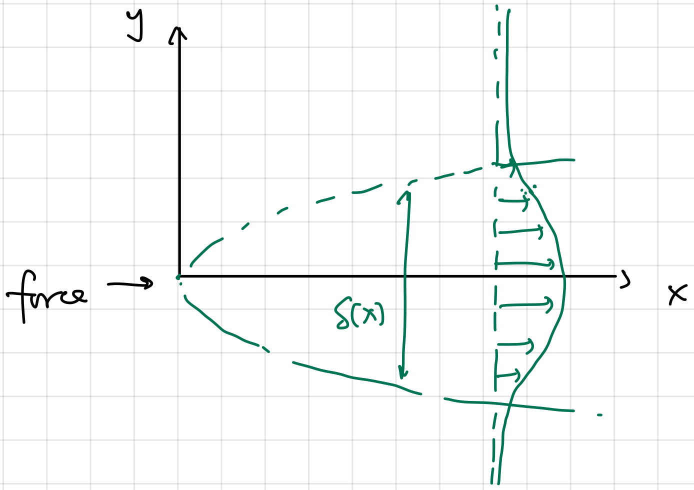
Large acceleration in "deficit wake" adjusts the pressure field, communicated back through the potential flow.

Uses "triple-deck theory".

7.5 2D Momentum jet

Consider a point force at $x=0, y=0$ in an otherwise still fluid.

$U=0$



With BL approx

x-mom:
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\cancel{\frac{\partial p}{\partial x}} + \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

0, since $p=$ const. in outer layer, and $\partial p/\partial y = 0$

Scaling:
$$\frac{U^2}{x} \sim \nu\frac{U}{\delta^2}$$

$$\Rightarrow \delta \sim \sqrt{\frac{\nu x}{U}}$$

(1) $\Rightarrow$
$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) \underbrace{- u\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial y}}_{=0 \text{ from } \nabla\cdot\underline{u}=0} = \nu\frac{\partial^2 u}{\partial y^2}.$$

Integrate in y.

$$\frac{d}{dx}\int_{-\infty}^{\infty} u^2\,dy + \underbrace{[uv]_{-\infty}^{\infty}}_{=0} = \underbrace{\left[\nu \frac{\partial u}{\partial y}\right]_{-\infty}^{\infty}}_{=0} .$$

$$\Rightarrow M = \underbrace{\int_{-\infty}^{\infty} \rho u^2\,dy}_{\text{momentum flux}} = \text{const.} \quad , \quad \rho \text{ const.}$$

Scaling : $\rho U^2 \delta \sim M$

Combining,

$$\delta(x) = \left(\frac{\rho \nu^2 x^2}{M}\right)^{1/3}, \qquad U(x) = \left(\frac{M^2}{\rho^2 \nu x}\right)^{1/3} .$$

Seek similarity sol^n of the form

$$\psi(x,y) = U(x)\,\delta(x)\,f(\eta), \qquad \eta = y/\delta(x) .$$

$$u = \frac{\partial \psi}{\partial y} = Uf' .$$

$$v = -\frac{\partial \psi}{\partial x} . = -\frac{\partial}{\partial x}(U\delta)\,f - U\delta f' \frac{\partial \eta}{\partial x} .$$

But $\frac{\partial}{\partial x}(U\delta) = \frac{U\delta}{3x}$, $\frac{\partial \eta}{\partial x} = \frac{-2\eta}{3x}$

$$\Rightarrow \quad v = \frac{U\delta}{x}\left(-\frac{f}{3} + \frac{2}{3}\eta f'\right)$$

$$(1) \Rightarrow \quad Uf'\left[-\frac{U}{3x}f' + Uf''\left(-\frac{2\eta}{3x}\right)\right] + \frac{U\delta}{x}\left(-\frac{f}{3} + \frac{2}{3}\eta f'\right)Uf''\cdot\frac{1}{\delta} = \frac{\nu U f'''}{\delta^2}$$

$$\Rightarrow \quad -\frac{1}{3}(f')^2 - \frac{1}{3}ff'' = f''' \qquad \text{using} \quad \delta^2 = \frac{xu}{U}$$

$$\Rightarrow \quad -\frac{1}{3}ff' = f'' + \underbrace{C}_{=0} \quad \because \lim_{y\to\infty} u, u_x = 0 \Rightarrow f, f' \xrightarrow{\eta\to\infty} 0$$

$$\Rightarrow \quad \frac{1}{6}(k^2 - f^2) = f' , \qquad k \text{ const.}$$

Solve separable ODE ,

$$f = k\tanh\left(\frac{k}{6}(\eta - \eta_0)\right)$$

$v=0$ at $y=0$ by symmetry $\Rightarrow f(0)=0 \Rightarrow \eta_0 = 0$

$$f' = \frac{k^2}{6}\operatorname{sech}^2\left(\frac{k\eta}{6}\right)$$

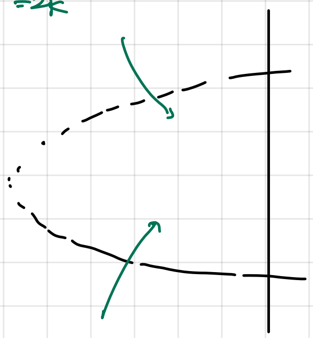
$$u(x,y) = U_c(x)\operatorname{sech}^2\left(\frac{k\eta}{6}\right).$$

where $U_c(x)$ is centreline velocity $(y=0)$.

Using $M = \int_{-\infty}^{\infty} \rho u^2 \, dy$, $\Rightarrow k = \left(\frac{9}{2}\right)^{1/3}$.

Mass flux: $$\int_{-\infty}^{\infty} \rho u \, dy = \int_{-\infty}^{\infty} \rho U \delta f' \, d\eta$$

$$= \rho U(x) \delta(x) \underbrace{\left[f\right]_{-\infty}^{\infty}}_{=2k} \propto x^{1/3}$$

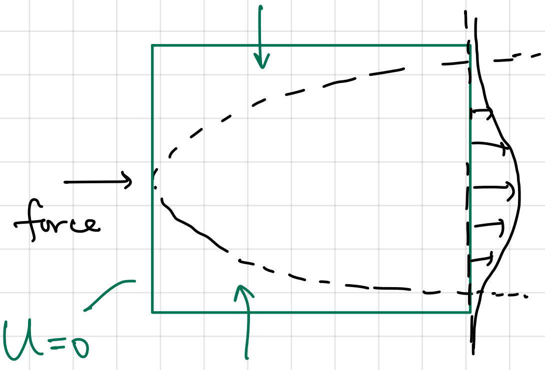


Increasing mass flux $\Rightarrow$ entrainment

Recall $v = \frac{U\delta}{x}\left(-\frac{f}{3} + \frac{2}{3}\eta f'\right)$

as $y\to\infty$, $f'=0$, $f\to k = \left(\frac{9}{2}\right)^{1/3}$

$\Rightarrow v \longrightarrow -\frac{U\delta}{x}\left(\frac{9}{2}\right)^{1/3} < 0$.



7.6 Accelerating / Decelerating external flows

Consider potential flow past a corner (c.f. §4.3) with sol^n of form

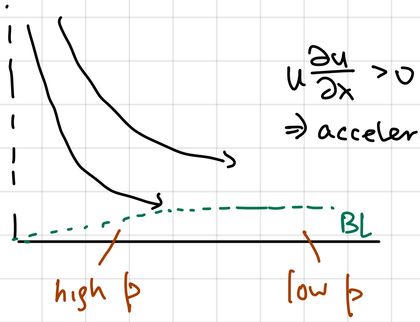
$$\phi = U_0 \frac{L}{m+1}\left(\frac{r}{L}\right)^{m+1}\cos((m+1)\theta)$$

$$U(x) = U_0(x)\left(\frac{x}{L}\right)^m$$

In outer layer, euler eqn :

x-mom : $$-\frac{\partial p}{\partial x} = \rho U \frac{\partial u}{\partial x} = \rho U^2 \frac{m}{x}.$$

m = 1 .

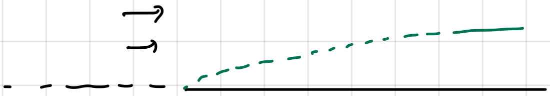


$u\frac{\partial u}{\partial x} > 0$

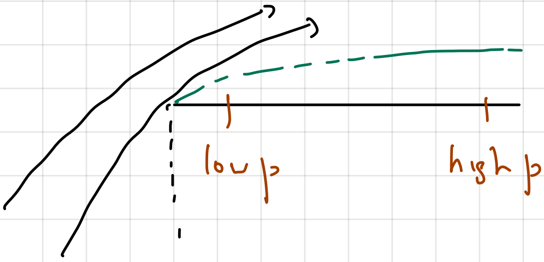
⇒ accelerating flow

m = 0 :

$u = U_0$



m = -1/3 :



Adverse pressure gradient

BL eqn becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{mU^2}{x} + \nu\frac{\partial^2 u}{\partial y^2}.$$

As before, scaling ⇒ $\delta \sim \sqrt{\nu x/U(x)}$.

Seek sol^n of the form $\psi = U(x)\,\delta(x)\,f(\eta),\ \eta = y/\delta(x) = \sqrt{\frac{U}{\nu x}}\,y$

$$\Rightarrow \boxed{m(f')^2 - \frac{m+1}{2} ff'' = m + f'''}$$

This is the Falkner-Skan eqn.

BL over flat plate (y=0). B.C. are

$u = v = 0$ on $y = 0$ ⇒ $f = f' = 0$ on $\eta = 0$.

$u \to U$ as $y \to \infty$ ⇒ $f' \to 1$ as $\eta \to \infty$

Solve numerically using shooting method. Recall

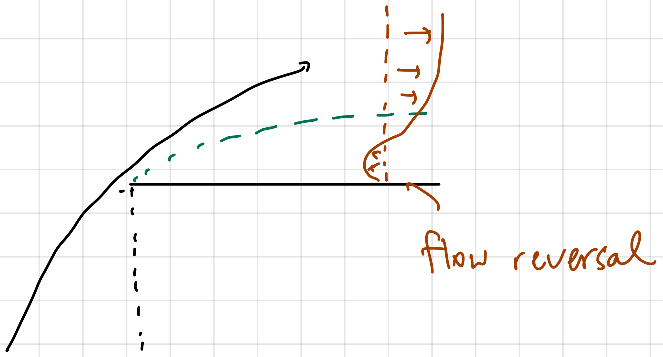
$$u = Uf' , \quad u_y = \frac{U}{\delta} f''$$

From numerical sol^n.

m	1	0	-0.09
$f''(0)$	1.3	0.3	0.003

$f''(0) < 0$ for smaller m ⇒ flow reversal.

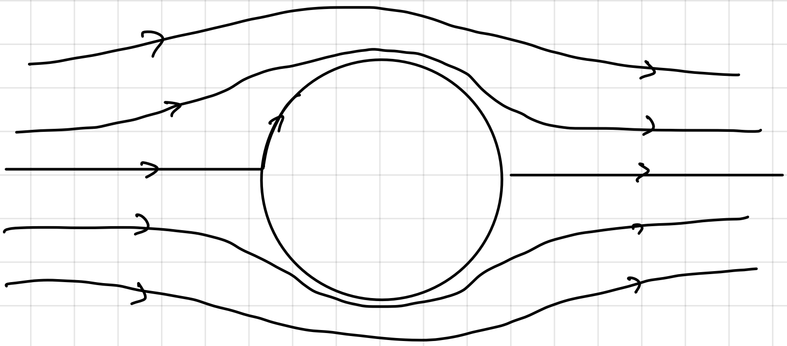
E.g. $m = -1/3$



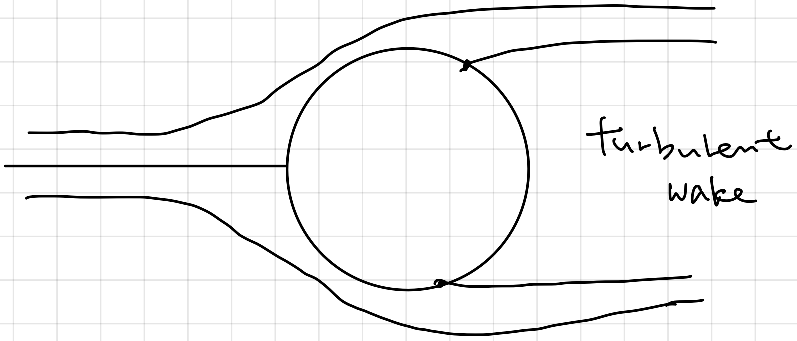
* Boundary layer separation *

For strongly decelerating flow with adverse pressure gradient, flow can separate from boundary, leading to turbulent wake.

E.g. Flow past sphere



Potential flow $U(x) = \frac{3}{2} U_0 \sin \frac{x}{a}$. Sol^n to BL eqn "blow up" at $\theta = \frac{x}{a} \approx 104.5^\circ$

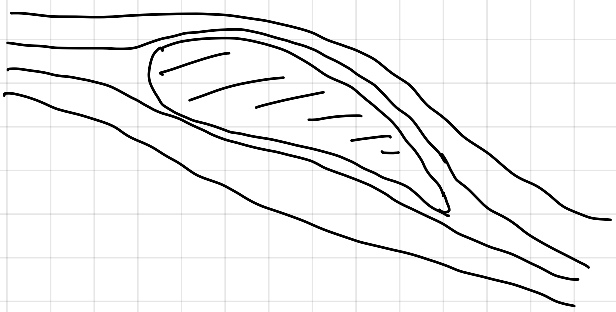


For flow past sphere, turbulance shifts BL separation to ~120°.

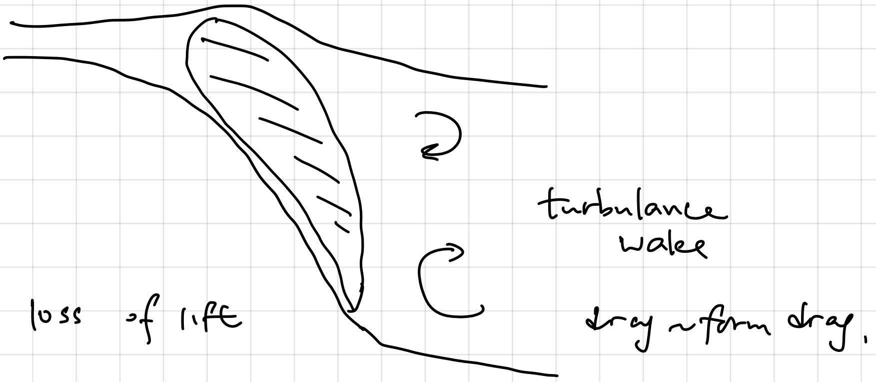
⇒ reduce drag.

E.g. Flow past aerofoil

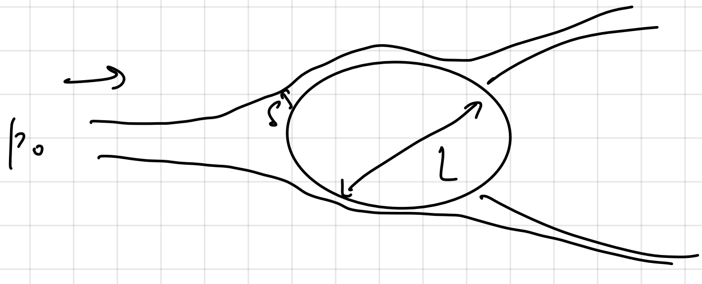
small angle

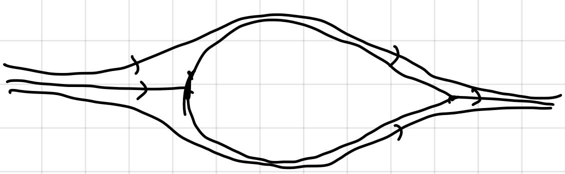


large angle



Drag on bulff and streamlined body





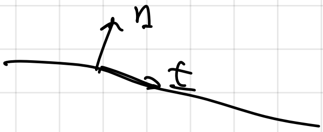
Skin friction $\sim \mu \frac{U}{\delta} L^2 \sim \mu L U Re^{1/2}$

Form drag $\sim \rho U^2 L^2$ for separated flow

$$\frac{\text{Form drag}}{\text{Skin friction}} \sim \frac{\rho U^2 L^2}{\mu L U Re^{1/2}} \sim Re^{1/2} \gg 1$$

7.7 Boundary layer at a free surface

At a free surface, the no-slip condition is replaced by a condition of zero tangential flow.



$$\underline{t} \cdot \underline{\underline{\sigma}} \cdot \underline{n} = 0$$

At $Re \gg 1$, the outer flow will typically not satisfy this condition. e.g. flow past a bubble

$$\varphi = -U_0 \cos\theta \left(r + \frac{a}{2r^2}\right)$$

$$\Rightarrow u_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = U_0 \sin\theta \left(1 + \frac{a}{2r^3}\right)$$

$$e_{r\theta} = \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\cancel{\frac{\partial u_r}{\partial \theta}}\right) \quad = 0 \ \because \text{no penetration}$$

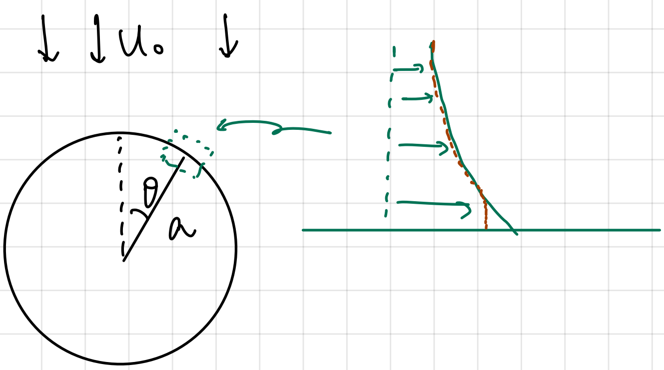
$$\sigma_{r\theta} = -\frac{3\mu U_0 \sin\theta}{a} \neq 0.$$

Very weak BL with velocity deficit

$$\Delta U \sim \delta \frac{U}{L}, \quad \delta \sim \sqrt{\nu L/U}$$

$$\Rightarrow \Delta U \sim U Re^{-1/2} \ll U$$

for $Re \gg 1$.



(Can linearise BL to leading order, so no separation, but often need second-order corrections for pactical purposes, so BL analysis is not straightforward.)

Dissipation

$$D = 2\mu \int_{BL} \underline{\underline{e}} : \underline{\underline{e}} \, dV + 2\mu \int_{Bulk} \underline{\underline{e}} : \underline{\underline{e}} \, dV$$

With no slip,

$$D \sim \mu \left(\frac{U}{\delta}\right)^2 L^2 \delta + \mu \left(\frac{U}{L}\right)^2 L^3$$

$$\sim \mu U^2 L \frac{L}{\delta} + \mu U^2 L .$$

So, in flows with rigid boundaries, dissipation is mainly in the viscous BL.

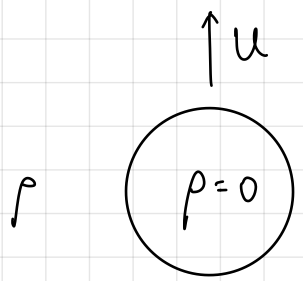
With no-stress,

$$D \sim \mu \left(\frac{\Delta U}{\delta}\right)^2 L^2 \delta + \mu \left(\frac{U}{L}\right)^2 L^3$$

$$\sim \mu U^2 L \left(\frac{\delta}{L}\right) + \mu U^2 L .$$

So in flows with free boundaries, dissipation is mainly in the bulk.

7.9 Rise velocity of a spherical bubble where $Re \gg 1$.



Buoyancy force

$$\underline{F} = -\frac{4}{3}\pi a^3 \rho \underline{g}$$

Work done by buoyancy is $\underline{F} \cdot \underline{U}$, equals dissipation

$$D = 2\mu \int_{r>a} \underline{\underline{e}} : \underline{\underline{e}} \, dV$$

Potential flow

$$\varphi = -\frac{a^3\,\underline{U}\cdot\underline{x}}{2r^3} \qquad \left(= A\ \underline{U}\cdot\nabla\left(\tfrac{1}{r}\right)\right)$$

$$\underline{u} = \nabla\varphi = -\frac{a^3}{2r^3}\underline{U} + \frac{3}{2}\,\frac{a^3(\underline{U}\cdot\underline{x})\ \underline{x}}{r^5}$$

$$\nabla\underline{u} = \frac{3}{2}\,\frac{a^3\,\underset{U_i x_j}{\underline{U}\,\underline{x}}}{r^5} + \frac{3}{2}a^3\left(\frac{\underline{U}\,\underline{x}}{r^5} + \frac{(\underline{U}\cdot\underline{x})}{r^5}\underline{\underline{I}} - \frac{5a^3(\underline{U}\cdot\underline{x})\,\underline{x}\,\underline{x}}{r^7}\right)$$

$$\underline{\underline{e}} = \frac{3}{2}a^3\left(\frac{\underline{U}\,\underline{x}+\underline{x}\,\underline{U}}{r^5} + \frac{\underline{U}\cdot\underline{x}}{r^5}\underline{\underline{I}} - \frac{5a^3(\underline{U}\cdot\underline{x})\,\underline{x}\,\underline{x}}{r^7}\right)$$

$$\underline{\underline{e}}:\underline{\underline{e}} = \frac{9}{4}\,\frac{a^6}{r^{10}}\left(2U^2r^2 + 4(\underline{U}\cdot\underline{x})^2\right) \qquad \text{(use suffix notation)}$$

$$\Rightarrow \quad D = 2\mu\int_{r>a}\underline{\underline{e}}:\underline{\underline{e}}\ dV$$

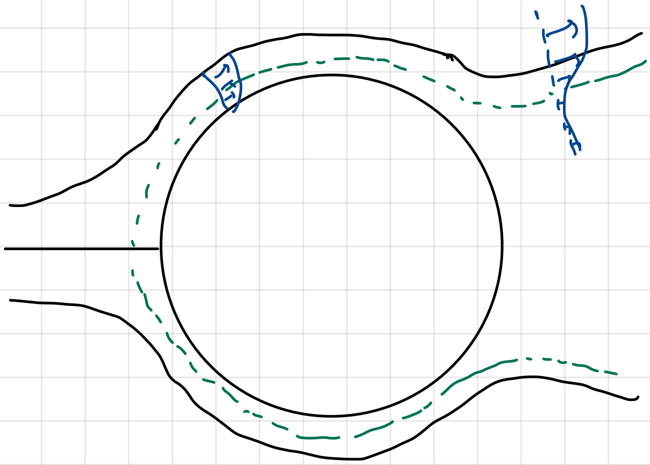
$$= 12\pi\mu a U^2 = \underline{F}\cdot\underline{U}$$

$$\Rightarrow \quad 12\pi\mu a U^2 = -\frac{4}{3}\pi a^3\rho\,\underline{g}\cdot\underline{U}$$

$$\Rightarrow \quad \underline{U} = -\frac{a^2}{9\nu}\,\underline{g}$$

using symmetry to deduce $\underline{U}$ anti-parallel to $\underline{g}$.

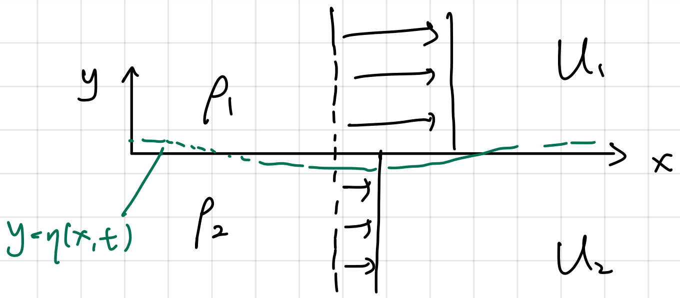
8. Stability



8.1 Kelvin-Helmholtz Instability

(c.f. water waves and Rayleigh-Taylor Instability)

Consider a shear layer between uniform streams.



Ignore the viscous boundary layer around $y=0$. Perturb this interface to $y=\eta(x,t)$ and consider limit of small (infinitesimal) perturbations ($|\eta_x| \ll 1$).

potential due to perturbation

Let $\underline{u} = (U_1, 0) + \nabla\varphi_1$ in $y>\eta$

$\underline{u} = (U_2, 0) + \nabla\varphi_2$ in $y<\eta$

For field $\varphi_1 \to 0$ as $y \to \infty$, $\varphi_2 \to 0$ as $y \to -\infty$

Infinitesimal perturbation $\Rightarrow$ $|\nabla\varphi| \ll U$.

Kinematic boundary conditions

Interface described by $y = \eta(x,t)$ for all time, so Lagrangian (material) derivative

$$\frac{D}{Dt}\left(y - \eta(x,t)\right) = 0.$$

$$\Rightarrow -\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} - v = 0$$

$$\Rightarrow -\frac{\partial \eta}{\partial t} - (U + \underbrace{\frac{\partial \varphi}{\partial x})\frac{\partial \eta}{\partial x}}_{\substack{\text{quadratically}\\ \text{small}\\ \downarrow \\ \text{ignore}}} + \frac{\partial \varphi}{\partial y} = 0.$$

Linearised KBC:

$$\left.\begin{aligned}\frac{\partial \eta}{\partial t} + U_1\frac{\partial \eta}{\partial x} &= \frac{\partial \varphi_1}{\partial y}\\ \frac{\partial \eta}{\partial t} + U_2\frac{\partial \eta}{\partial x} &= \frac{\partial \varphi_2}{\partial y}\end{aligned}\right|$$

expand about $y=0$, look at leading order.

Note $v_1 \neq v_2$ at $y=0$.

Dynamic boundary conditions

Continuous pressure $p_1 = p_2$. Use expression for pressure in irrotational flows

$$\rho\frac{\partial \varphi}{\partial t} + \frac{1}{2}\rho|\underline{u}|^2 + p + \Phi = f(t)$$

indpt of $\underline{x}$.

$$\rho\frac{\partial \varphi}{\partial t} + \frac{1}{2}\rho\left(U^2 + 2\underline{U}\cdot\nabla\varphi + \overset{\text{quadratic}}{\cancel{|\nabla\varphi|^2}}\right) + p + \Phi = f(t)$$

So eqns for perturbations are

$$\rho_1\frac{\partial \varphi_1}{\partial t} + \rho_1 U_1\frac{\partial \varphi_1}{\partial x} + p_1 + \rho_1 g\eta = f_1(t).$$

$$\rho_2\frac{\partial \varphi_2}{\partial t} + \rho_2 U_2\frac{\partial \varphi_2}{\partial x} + p_2 + \rho_2 g\eta = f_2(t)$$

If $p_1 = p_2$,

$$\rho_1\left(\frac{\partial \varphi_1}{\partial t} + U_1\frac{\partial \varphi_1}{\partial x} + g\eta\right) = \rho_2\left(\frac{\partial \varphi_2}{\partial t} + U_2\frac{\partial \varphi_2}{\partial x} + g\eta\right) + f(t).$$

Having linearised the perturbation eqn, can use Fourier Transforms to consider each wave number independently.

Set $\eta = A e^{ikx + \sigma t}$.

$$\nabla^2 \varphi = 0 \quad \Rightarrow \quad \varphi_1 = B_1 e^{-ky} e^{ikx+\sigma t}$$

$$\varphi_2 = B_2 e^{ky} e^{ikx+\sigma t}$$

Apply B.C.s on $y=0$.

KBC: $$\sigma A + ikU_1 A = -kB_1 \qquad (1)$$

$$\sigma A + ikU_2 A = kB_2 \qquad (2)$$

Consider the case $\rho_1 = \rho_2$, then

DBC: $$\sigma B_1 + ikU_1 B_1 = \sigma B_2 + ikU_2 B_2 \quad (f=0) \qquad (3)$$

Substitute for B_1, B_2 using (1), (2).

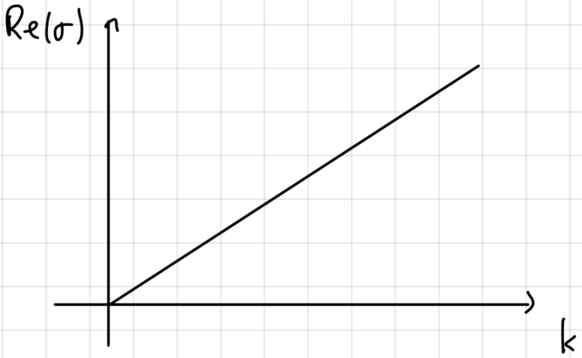
$$-(\sigma + ikU_1)^2 = (\sigma + ikU_2)^2$$

$$\Rightarrow \sigma = -ik\frac{U_1+U_2}{2} \pm k\frac{U_2-U_1}{2}$$

$$\Rightarrow \eta = \underbrace{A\exp\left(ik\left(x - \frac{U_1+U_2}{2}t\right)\right)}_{\text{disturbance wave travelling with mean velocity}} \underbrace{\exp\left(\pm\frac{1}{2}(U_2-U_1)kt\right)}_{\text{one growing and one decaying sol}^n}$$

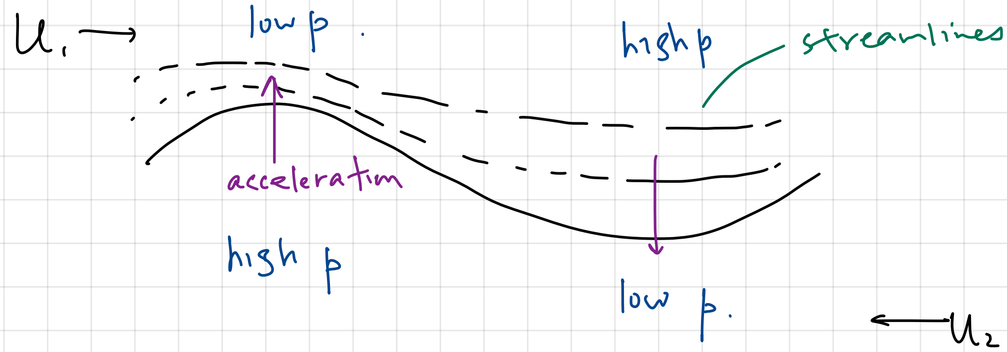
There is always one growing sol^n for arbitrary initial condition.

So flow unstable if $U_1 \neq U_2$.



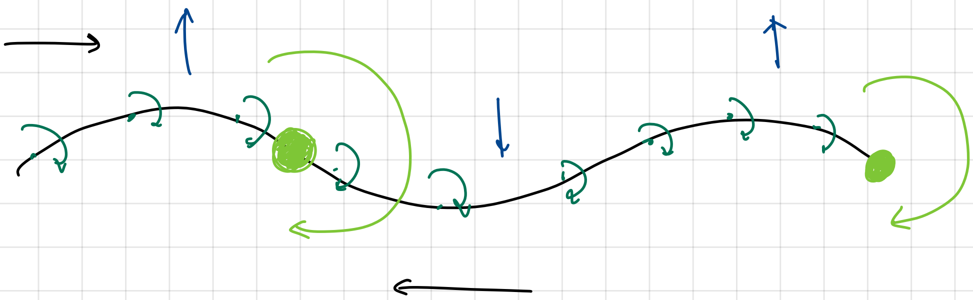
Modes with largest wavenumber k, smallest wavelength $\lambda = \frac{2\pi}{k}$, grow the fastest.

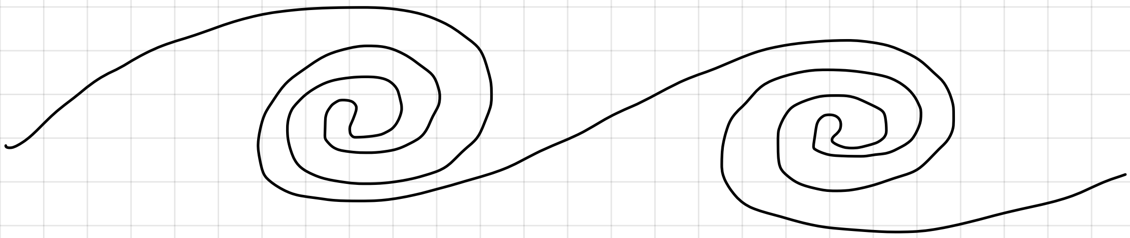
Physical mechanism of instability.



Bernoulli indicates low pressure over peaks, high pressure over troughs, which gives accelerations as indicated, tending to amplify the disturbance.

Alternatively, consider advection of vorticity.





Accumulation of vorticity tends to push crests up and trough down ⇒ instability.

Get rolled up into Kelvin - Helmholtz billows.

Stabilisation of long waves by gravity

Suppose $\rho_2 > \rho_1$. so light fluid over heavy.

DBC becomes

$$\rho_1 (\sigma + ikU_1) B_1 + \rho_1 g A = \rho_2 (\sigma + ikU_2) B_2 + \rho_2 g A.$$

Find B_1 and B_2 in terms of A from KBC

$$\Rightarrow \rho_1 (\sigma + ikU_1)^2 + \rho_2 (\sigma + ikU_2)^2 + (\rho_2 - \rho_1) gk = 0.$$

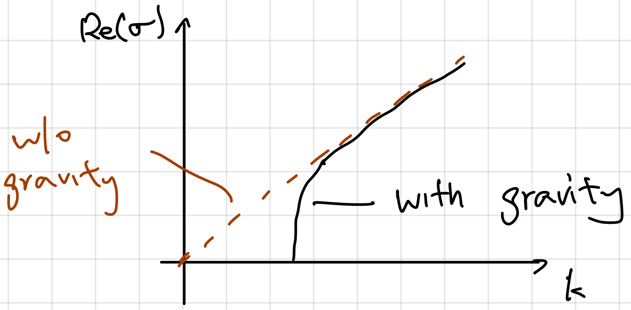
$$\Rightarrow \quad \sigma = \frac{1}{\rho_1 + \rho_2} \left[-ik(\rho_1 U_1 + \rho_2 U_2) \pm \sqrt{\Delta} \right]$$

where

$$\Delta = \rho_1 \rho_2 (U_1 - U_2)^2 k^2 - (\rho_2^2 - \rho_1^2) gk$$

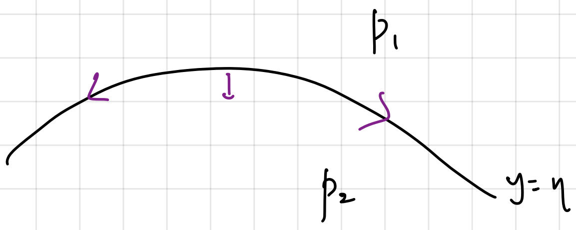
There is instability iff $\Delta > 0$, so that σ has a positive real part. So instability if

$$k > \frac{\rho_2^2 - \rho_1^2}{\rho_1 \rho_2} \frac{g}{(U_1 - U_2)^2}$$



Stabilisation of short waves by surface tension

If fluid are immiscible, then there is a jump in pressure across the interface equal to the surface tension γ times the curvature.



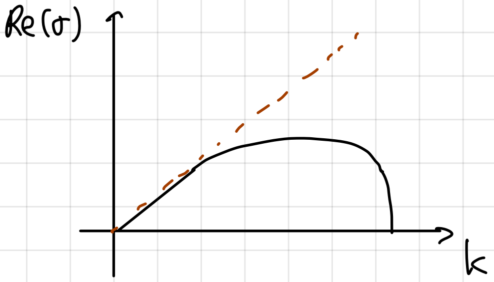
$$p_2 - p_1 \approx -\gamma\,\eta_{xx}$$

Consider $\rho_1 = \rho_2$, $\gamma > 0$.

DBC: $$(\sigma + ikU_1)\,B_1 - (\sigma + ikU_2)\,B_2 - \frac{\gamma}{\rho}k^2 A = 0$$

$$\Rightarrow (\sigma + ikU_1)^2 + (\sigma + ikU_2)^2 + \frac{\gamma}{\rho}k^3 = 0$$

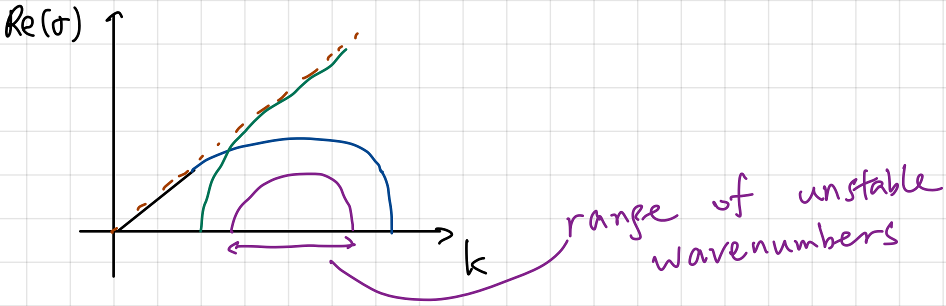
$$\Rightarrow \Delta = -4k^2(U_1+U_2)^2 + 8(U_1^2+U_2^2)k^2 - 8\frac{\gamma}{\rho}k^3$$



Unstable ($\Delta > 0$) if $k < \rho(U_1 - U_2)^2/2\gamma$.

Altogether, if $\rho_2 > \rho_1$, $\gamma > 0$, get

$$\frac{\Delta}{4k^2} = \rho_1\rho_2(U_1 - U_2)^2 - (\rho_1 + \rho_2)\left[\frac{(\rho_2 - \rho_1)g}{k} + \gamma k\right]$$



$$\max_k \left(\frac{\Delta}{4k^2} \right) = \rho_1 \rho_2 (U_1 - U_2)^2 - (\rho_1 + \rho_2) 2 \sqrt{(\rho_2 - \rho_1) g \gamma}$$

achieved at $k = \sqrt{\frac{(\rho_2 - \rho_1) g}{\gamma}}$.

So stable at all wavenumbers if

$$(U_1 - U_2)^2 < 2 \frac{(\rho_1 + \rho_2)}{\rho_1 \rho_2} \sqrt{(\rho_2 - \rho_1) g \gamma}$$