Further Complex Methods

§1 Complex variable

1.1 Revision

Def^n A neighbourhood of a point $z \in \mathbb{C}$ is an open set which contains z.

Def^n The extended complex plane is $\mathbb{C} \cup \{\infty\}$, notation $\bar{\mathbb{C}}$, $\mathbb{C}^*$.

All directions to the point ∞ are equivalent.

Def^n A f^n $f(z)$ is differentiable at z if

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

exists independently of the direction of δz. We say f is analytic (regular / holomorphic) at z if $\exists$ nbd of z s.t. $f'(z)$ exists everywhere in the nbd. We say f is analytic in domain D if f' exists throughout D.

Cauchy-Riemann relations: if $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$, $u, v \in \mathbb{R}$ is diff. at z, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Conversely, if C-R holds and u, v have cts partial derivatives, then f is diff.

Cor Write $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$ and regard $z, \bar{z}$ as independent variables, using chain rule

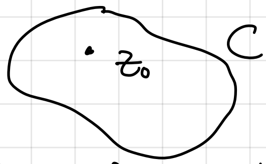
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then $$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv)$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0.$$

ie analytic f^ns depend on z, but not $\bar{z}$.

<u>Thm (Cauchy's thm)</u> If $f(z)$ analytic within and a closed curve C (i.e. in simply connected domain), then
$$\oint_C f(z) = 0.$$
implying integral between two points is path-independent.

Cauchy integral formula: if f analytic, for z_0 within C, then
$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0). \qquad (\dagger)$$



Thus knowledge of $f(z)$ on ∂D $(= C)$ determines $f(z)$ within D.

$\left(\frac{d}{dz_0}\right)^n (\dagger)$ gives
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

All derivatives of f exists at z_0. (An analytic f^n is infinitely diff.).

Def^n A complex f^n is <u>entire</u> if analytic throughout $\mathbb{C}$.

<u>Thm</u> (Liouville's thm) If f entire and bounded on $\mathbb{C} \cup \{\infty\}$, then f is const.

(An interesting f^n has to have singularities)

Pf: Consider a circular contour $|z - z_0| = R$, with $|f(z)| \le M$.

Apply Cauchy integral formula with $n=1$,

$$f'(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} \, dz .$$

$$\Rightarrow |f'(z_0)| \leq \frac{1}{2\pi} \left| \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} \, dz \right|$$

$$\leq \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R} .$$

Let $R \to \infty \Rightarrow |f'(z_0)| = 0$, hence $f'(z_0) = 0$, and f const. $\square$

Laurent expansion : Suppose $f(z)$ has an <u>isolated</u> sing. at z_0.
(i.e. f analytic in a nbd of z_0, except at z_0 itself), then
we can write

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n ,$$

contour enclosing nbd of z_0.

where

$$C_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} \, dz .$$

We classify isolated sing. of f as follows :

- if series truncate $f(z) = \sum_{n=N}^{\infty} C_n (z-z_0)^n$.
  - $N \geq 0$: regular point
  - $-\infty < N < 0$: z_0 is a <u>pole</u> of order $-N$. ($N=-1$: simple pole)
  - $N = -\infty$: z_0 is <u>essential</u> sing. (e.g. $e^{-1/z}$ at $z=0$).

<u>Def^n</u> <u>Residue</u> of f at z_0 is the coeff. of $(z-z_0)^{-1}$, i.e

$$\operatorname*{Res}_{z=z_0} f(z) = C_{-1} .$$

If z_0 is a pole of order N, then

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \frac{1}{(N-1)!} \left[\left(\frac{d}{dz} \right)^{N-1} (z-z_0)^N f(z) \right].$$

Thm (Residue thm) If $f(z)$ analytic in a simply connected domain except at a finite or countable number of isolated sing. $z_1, z_2, \dots$, then

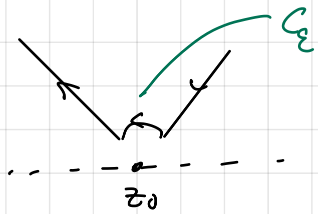
$$\underset{\text{anticlockwise}}{\oint_C} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

where sum is over set of sing. enclosed in C.

The Indentation lemma

Useful in evaluating integrals of f's with a simple pole.

Contour C as depicted, part of a circle of radius ε, centre at z_0. Increase in $\arg(z-z_0)$ around the circle is $\beta - \alpha$.



Lem Let $f(z)$ have a simple pole at z_0, Contour C_ε defined by $\{z : z = z_0 + \varepsilon e^{i\theta},\ \alpha < \theta < \beta\}$. then

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\,dz = i(\beta - \alpha)\ \mathrm{res}(f(z) : z_0).$$

Pf: f has simple pole at z_0, hence a Laurent expansion around z_0, and $\exists$ analytic f^n $g(z)$ s.t. $f(z) = \dfrac{\mathrm{res}(f(z):z_0)}{z-z_0} + g(z)$, with $|g(z)| < M$ in $|z - z_0| < r$ for some r.

Then for $0 < \varepsilon < r$,

$$\int_{C_\varepsilon} f(z)\,dz = \mathrm{res}(f(z):z_0) \int_{C_\varepsilon} \frac{dz}{z-z_0} + \int_{C_\varepsilon} g(z)\,dz.$$

$$= \mathrm{res}(f(z):z_0) \int_\alpha^\beta i\,d\theta + \underbrace{\int_{C_\varepsilon} g(z)\,dz}_{|\ \ | < M \cdot 2\pi\varepsilon}$$

$$\to i(\beta - \alpha)\ \mathrm{res}(f(z) : z_0) \quad \text{as} \quad \varepsilon \to 0. \qquad \square$$

Cauchy's Argument Principle

We say $f(z)$ is meromorphic in D if $f(z)$ analytic throughout D except at isolated singularities which are poles.

Cauchy's AP: let $f(z)$ is meromorphic inside a simple closed curve γ with no sing. or zeros on γ. then

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\,(N-P),$$

where N is no. of zeros within γ.
P poles

} take into account multiplicity.

Pf: $f(z)=0$ at $z_1, z_2, \ldots$ with mult. $k_1, k_2, \ldots$

$f(z)$ has poles at $\tilde{z}_1, \tilde{z}_2, \ldots$ with orders $l_1, l_2, \ldots$

Then $f(z) = \dfrac{(z-z_1)^{k_1}(z-z_2)^{k_2}\cdots}{(z-\tilde{z}_1)^{l_1}(z-\tilde{z}_2)^{l_2}\cdots}\, g(z)$, where $g(z)$ analytic and non-zero in γ.

$$\frac{f'(z)}{f(z)} = \left(\frac{k_1}{z-z_1} + \frac{k_2}{z-z_2} + \cdots\right) - \left(\frac{l_1}{z-\tilde{z}_1} + \frac{l_2}{z-\tilde{z}_2} + \cdots\right) + \frac{g'(z)}{g(z)}.$$

Apply residue thm then done. $\square$

1.2 Analyticity of function defined by an integral

Consider f^ns of form

$$F(z) = \int_C f(z,t)\,dt,$$

where C is a path in complex plane (not necessarily closed).

For what values of z is $F(z)$ defined and analytic?

Conditions for analyticity:

(i) f cts jointly in z and t

(ii) Integral converges uniformly in each compact subset of domain of z.

(iii) $f(z,t)$ is analytic in z for each t.

Recall the integral $\int_0^\infty f(z,t)\,dt$ is uniformly converging for $z \in \mathcal{U}$ if given ε, $\exists B_0$ s.t. $B_0 < B_1 < B_2$,

$$\left| \int_{B_1}^{B_2} f(z,t)\,dt \right| < \varepsilon$$

$\forall z \in \mathcal{U}$.

<u>Example</u> $F_1(z) = \int_{-\infty}^{\infty} e^{-zt^2}\,dt$

(a) Existence

For what values of z is the integral defined? (if not defined can't be analytic).

Integral converges if $Re(z) > 0$, diverges if $Re(z) < 0$.

If $Re(z) = 0$, integral exists — convergence relies on oscillation, but not absolute.

(b) Analyticity

(i), (iii) satisfied.

For (ii), convergence not uniform in $Re(z) > 0$ but is uniform in $Re(z) \geq \varepsilon > 0$ for any $\varepsilon > 0$.

We know that $F_1(z) = \left(\frac{\pi}{z}\right)^{1/2}$, indeed analytic in $Re(z) > 0$.

Example $F_2(z) = \int_0^\infty \frac{t^{z-1}}{1+t}\,dt$ (Note that $t^{z-1} = \exp((z-1)\log t)$.

(a) Existence

Check limits 0 and ∞:

- Near $t=0$, integrand $\sim t^{z-1}$. Consider

$$\int_0 t^{z-1}\,dt = \left[\frac{t^z}{z}\right]_0$$

well defined if $Re(z)>0$.

- $t\to\infty$, integrand $\sim t^{z-2}$

$$\int^\infty t^{z-2}\,dt = \left[\frac{t^{z-1}}{z-1}\right]^\infty$$

well defined if $Re(z)<1$

So integral exists if $0<Re(z)<1$.

(b) analyticity

(i), (iii) satisfied.

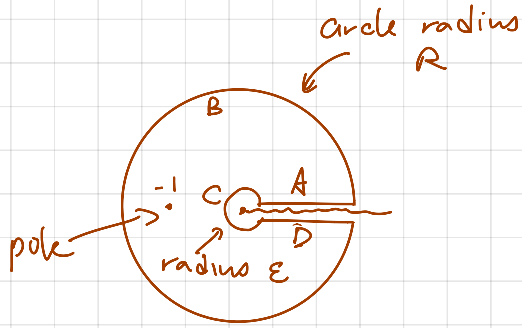
Can show integral converges uniformly if $0<a<Re(z)<b<1$

$\forall\ 0<a,b<1$, so (ii) satisfied

$\Rightarrow$ analyticity for $0<Re(z)<1$.

Evaluate integral

$$\oint_C \frac{t^{z-1}}{1+t}\,dt$$



Parameterise:

A	$t=u$	$\varepsilon\le u\le R$
B	$t=Re^{i\theta}$	$0<\theta<2\pi$
C	$t=\varepsilon e^{i\theta}$	$2\pi>\theta>0$
D	$t=u^{2\pi i}$	$R>u>\varepsilon$.

As $\varepsilon \to 0$, $R \to \infty$, $\quad I_A = \int_0^\infty \frac{u^{z-1}}{1+u}\,du$, $\quad I_B = O(R^{z-1}) \to 0$

$$I_B = -\int_0^\infty \frac{u^{z-1}}{1+z}\,du\, e^{2\pi i z} \qquad I_D = O(\varepsilon^z) \to 0$$

$$(2\pi i)\,\mathrm{res}\left(\frac{t^{z-1}}{1+t} : t=-1\right) = (1-e^{2\pi i z}) \int_0^\infty \frac{u^{z-1}}{1+u}\,du$$

$$\Rightarrow \int_0^\infty \frac{u^{z-1}}{1+u}\,du = \pi \csc(\pi z).$$

This is analytic f^n of z for $0 < Re(z) < 1$.

1.3 Analytic (or meromorphic) continuation

Thm (Identity Thm) Let $g_1(z)$, $g_2(z)$ be analytic (holomorphic) in connected open set $D \subseteq \mathbb{C}$, with $g_1 = g_2$ in $\tilde{D}$ a non-empty subset of D, then $g_1 = g_2$ on D.

(extension - result holds if $\tilde{D}$ is a curve or if $\tilde{D}$ is countable set with a communication point)

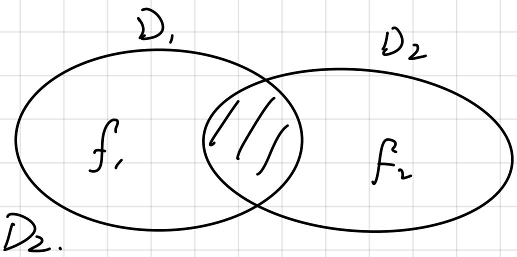
Pf (Sketch): Expand $g_1 - g_2$ as a Taylor series in $\tilde{D}$ - the "zero" Taylor series. Expanding about z_0 - can choose so that radius of convergence expand outside $\tilde{D}$, therefore $g_1 = g_2$ on point of D that is larger than $\tilde{D}$. Now carry on this procedure. $\square$

Analytic Continuation

Let D_1, D_2 open sets in $\mathbb{C}$, $D_1 \cap D_2 \neq \phi$.

f_1 analytic on D_1, f_2 analytic on D_2,

and $f_1 = f_2$ on $D_1 \cap D_2$.



f_2 is the analytic continuation of f_1 in D_2.

Prop The continuation is unique.

Pf: Assume $\exists$ 2 continuation of f_1 into D_2, e.g. $f_2, \tilde{f}_2$, then $f_2 = \tilde{f}_2$ on $D_1 \cap D_2$, but $f_2 \neq \tilde{f}_2$ on $D_2 \setminus (D_1 \cap D_2)$.

In notation of identity thm.

$$g_1 = f_1 \text{ on } D_1, \quad f_2 \text{ on } D_2 \setminus (D_1 \cap D_2)$$

$$g_2 = f_1 \text{ on } D_1, \quad \tilde{f}_2 \text{ on } D_2 \setminus (D_1 \cap D_2)$$

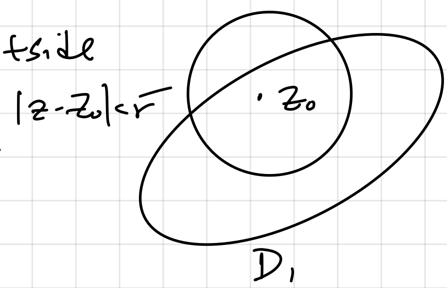
Since $g_1 = g_2$ on $D_1 \cap D_2 \Rightarrow g_1 = g_2$ on $D_2 \setminus (D_1 \cap D_2)$

$$\Rightarrow f_2 = \tilde{f}_2 \text{ on } D_2 \setminus (D_1 \cap D_2) \qquad \square$$

Methods of analytic continuation.

(1) Taylor Series

Suppose f analytic in D. Pick z_0 close to boundary of D_1, construct Taylor series expansion about z_0, which cgs in circle that extends outside D_1. Hence defines an analytic ction of f_1 into region larger than D_1.



Example Continuation of $\frac{1}{1-z}$.

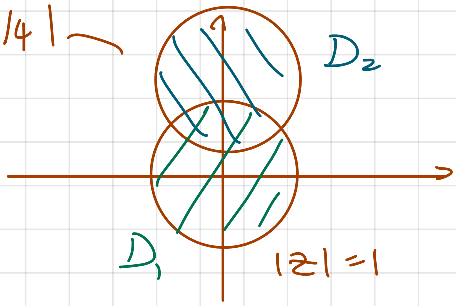
Note $$\frac{1}{1-z} = \frac{1}{1-z_0} \cdot \frac{1}{1-\frac{z-z_0}{1-z_0}} = \frac{1}{1-z_0} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(1-z_0)^n}$$

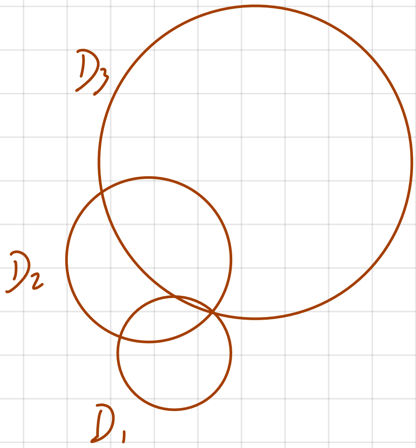
converges in disk $|z-z_0| < |1-z_0|$.

Let $f_1(z) = \sum_{n=0}^{\infty} z^n$. defined in $|z|<1$, (D_1)

Choose $z_0 = 3i/4$, then $f_2(z) = \frac{1}{1-3i/4} \sum_{n=0}^{\infty} \frac{(z-3i/4)^n}{(1-3i/4)^n}$ cgs in $|z-3i/4| < |1-3i/4|$ (D_2). In particular, $f_1(z) = f_2(z)$ on $D_1 \cap D_2$. Hence $f_2(z)$ defines an analytic continuation of f_1 into D_2.

$|z - 3i/4| < |1 - 3i/4|$





repeating gives an extension of $f_1(z)$ into $\mathbb{C} \setminus (1)$.

$z = 1$ excluded. — this is a meromorphic continuation.

prop $\Rightarrow$ the continuation is unique.

What can go wrong? e.g. singularities of $f_1(z)$ are so numerous on boundary of D_1 that they prevent continuation.

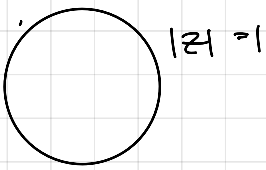
'Natural boundary' example: Define $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ cgs for $|z| < 1$. (e.g. ratio test). Consider

$$f(z^2) = \sum_{n=0}^{\infty} z^{2^{n+1}} = \sum_{n=1}^{\infty} z^{2^n} = f(z) - z$$

$$f(z^4) = f(z^2) - z^2 = f(z) - z - z^2$$

$$f(z^{2^n}) = f(z) - \sum_{m=0}^{n-1} z^{2^m}$$

but $f(1) = \infty$. Hence, $f(\omega) = \infty$ $\forall \omega$ s.t. $\omega^{2^n} = 1$ for any finite n.

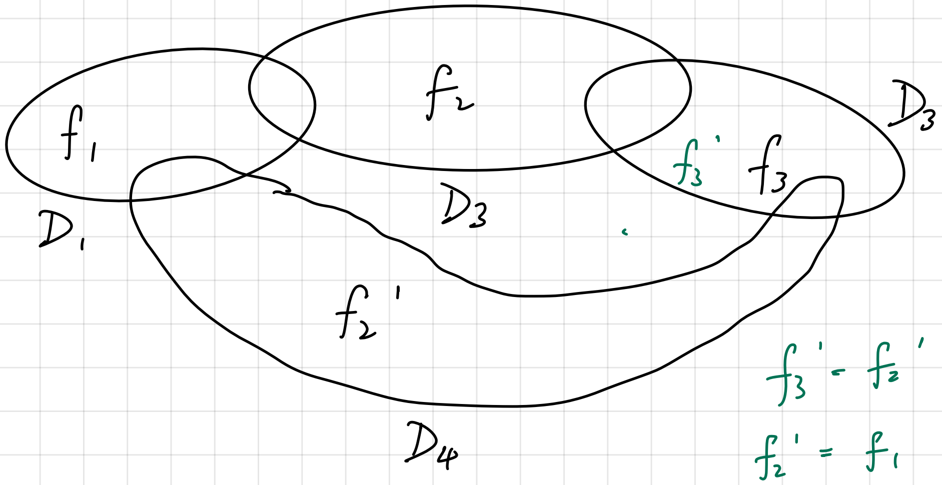


ω's are dense on $|z| = 1$.

— can't extend by finding Taylor series that converges in circle extending outside $|z| = 1$.

$|z|=1$ is 'natural barrier' or 'natural boundary' for f.

<u>Potential non-uniqueness</u>



$f_3' = f_2'$ in $D_3 \cap D_2'$

$f_2' = f_1$ in $D_2' \cap D_1$

No overlap between D_2' and D_2 — f_3' not guaranteed to be the same as f_3.

Relavant to multivalued f^ns with branch points.

<u>(2) Contour deformation</u>

Let

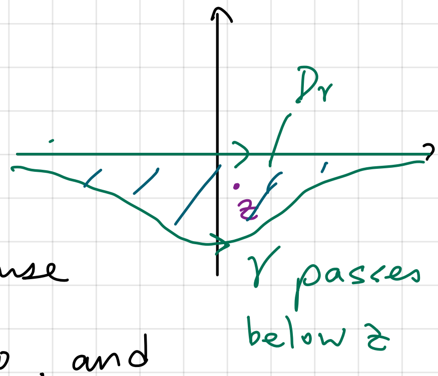
$$G(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z}\, dt$$

applies for $\mathrm{Im}(z) > 0$ — defines analytic f^n for $\mathrm{Im}(z) > 0$.

We seek to continue $G(z)$ to $\mathrm{Im}(z) < 0$.

For $\mathrm{Im}(z) < 0$.

$$G_a(z) = \int_\gamma \frac{e^{it}}{t-z}\, dt$$



If $\mathrm{Im}(z) > 0$ — then $G_a(z) = G(z)$ because integral around closed contour is zero, and furthermore $G_a(z)$ is an analytic f^n, hence $G(z) = G_a(z)$ in $\mathrm{Im}(z) > 0$ and $G_a(z)$ is analytic for z lying above γ.

So $G_a(z)$ provides continuation of $G(z)$ into shaded region.

Another possibility?

$$G_b(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z}\, dt$$

for $Im(z) \neq 0$.

But for $z \in D_\gamma$, then

$$G_a(z) - G_b(z) = 2\pi i \,\mathrm{res}\left(\frac{e^{it}}{t-z}, z\right) = 2\pi i e^{iz}$$

$$G_b(z) = \begin{cases} G_a(z) & \text{if } Im(z) > 0. \\ G_a(z) - 2\pi i e^{iz} & \text{if } Im(z) < 0. \end{cases}$$

So $G_b(z)$ jumps by value $2\pi i e^{iz}$ as we cross real axis.

$G_b(z)$ is <u>not</u> analytic — not analytic continuation.

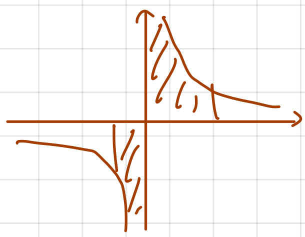
<u>Example</u> $F(z) = \int_\gamma f(z,t)\, dt$ defined and analytic for $z \in \tilde{D}$.

$\exists$ unique continuation to $z \in D$ with $D \supset \tilde{D}$ — needs care.

1.4 Cauchy Principle Value

Motivation — can we assign a finite value to an unbounded integral?

<u>Example</u> $\int_{-1}^{2} \frac{dx}{x} = \log 2 - \log|-1| = \log 2$?



If $f(x)$ is not integrable at $x = c$ and $a < c < b$, define the Cauchy Principle Value (CPV) by

$$P\left(\int_a^b f(x)\, dx\right) = \lim_{\varepsilon \to 0}\left(\int_a^{c-\varepsilon} f(x)\, dx + \int_{c+\varepsilon}^{b} f(x)\, dx\right)$$

if limit exists. (Can extend to many non-integrable points)

WLOG, set $c=0$. Consider $f(x)=\frac{g(x)}{x}$, where $g(x)$ well-behaved at $x=0$. Set $f(x)=C_{-1}x^{-1}+C_0+C_1x+\dots$, then

$$P\left(\int_a^b f(x)\,dx\right)=\lim_{\varepsilon\to 0}\left[C_{-1}\log\left|\frac{\varepsilon}{a}\right|+C_{-1}\log\left|\frac{b}{\varepsilon}\right|+C_0\left(b-a+\mathcal{O}(\varepsilon)\right)+\dots\right]$$

$$=C_{-1}\log\frac{|b|}{|a|}+C_0(b-a)+\dots$$

May also use CPV to deal with infinite limits.

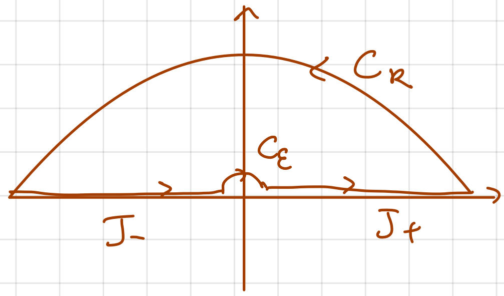
$$P\left(\int_{-\infty}^{\infty}f(x)\,dx\right)=\lim_{R\to\infty}\int_{-R}^{R}f(x)\,dx$$

if limit exists

Example $I=P\left(\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx\right)$, where $f(z)$ analytic in $\mathrm{Im}(z)\geq 0$ and $|f(z)|\to 0$ as $z\to\infty$.

Need CPV because pole on path of integration.

$$\oint_C\frac{f(z)}{z}\,dz=\int_{J_+}+\int_{J_-}+\int_{C_\varepsilon}+\int_{C_R}=0$$



because no enclosed sing.

$J_-=\{x:-R\leq x\leq-\varepsilon\}$

$J_+=\{x:\varepsilon\leq x\leq R\}$,

$C_\varepsilon=\{z:z=\varepsilon e^{i\theta},\ \pi\geq\theta\geq 0\}$, $C_R=\{z:z=Re^{i\theta},\ 0\leq\theta\leq\pi\}$.

limit $\varepsilon\to 0$, $R\to\infty$, then $\int_{C_R}\to 0$ $\because$ $|f(z)|\to 0$. $\int_{J_-}+\int_{J_+}\to I$

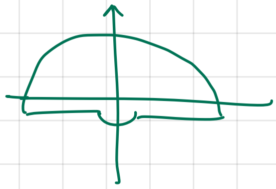
$$\int_{C_\varepsilon}=\lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{f(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\,i\varepsilon e^{i\theta}\,d\theta\to i\pi f(0).$$

Hence

$$I=i\pi f(0).$$

Remarks

- Could have closed contour below $z=0$ rather than above.
- If f real on real axis, then result imply



$$f(0)=0$$

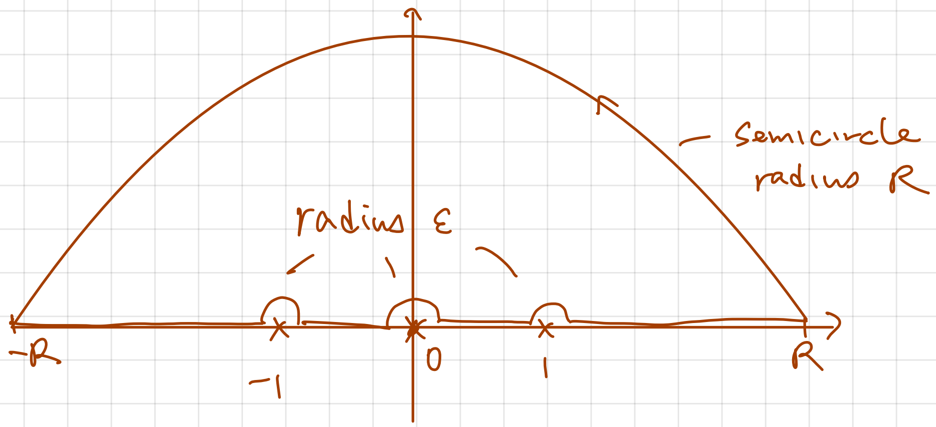
(consequence of analyticity).

Example $P\left(\int_0^\infty \frac{\sin x}{x(x^2-1)}\,dx\right)$.

Note that this is non-integrable (sing. at $x=1$)

$$P\left(\int_0^\infty \frac{\sin x}{x(x^2-1)}\,dx\right) = \frac{1}{2}P\left(\int_{-\infty}^\infty \frac{\sin x}{x(x^2-1)}\,dx\right)$$

$$= \frac{1}{2}\,\mathrm{Im}\,P\left(\int_{-\infty}^\infty \frac{e^{ix}}{x(x^2-1)}\,dx\right)$$



Close contour

$$\int_{-R}^{-1-\varepsilon} \frac{e^{ix}}{x(x^2-1)}\,dx - \pi i\,(\text{res at } -1) + O(\varepsilon)$$

$$+ \int_{-1+\varepsilon}^{-\varepsilon} \frac{e^{ix}}{x(x^2-1)}\,dx - \pi i\,(\text{res at } 0) + O(\varepsilon)$$

$$+ \int_{\varepsilon}^{1-\varepsilon} \frac{e^{ix}}{x(x^2-1)}\,dx - \pi i\,(\text{res at } 1) + O(\varepsilon) + \int_{1+\varepsilon}^{R} \frac{e^{ix}}{x(x^2-1)}\,dx = 0$$

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} P\int_{-\infty}^\infty \frac{e^{ix}}{x(x^2-1)}\,dx = \pi i\left(\frac{e^{-i}}{(-1)(-2)} + \frac{1}{(-1)} + \frac{e^{i}}{(1)(2)}\right)$$

$$= \pi i\,(\cos 1 - 1)$$

$$\Rightarrow \quad P\left(\int_0^\infty \frac{\sin x}{x(x^2-1)}\,dx\right) = \frac{\pi}{2}(\cos 1 - 1)$$

Hilbert Transform

Defn Hilbert transform of f^n $f(x)$ defined by

$$\mathcal{H}(f)(y) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(x)}{x-y} dx$$

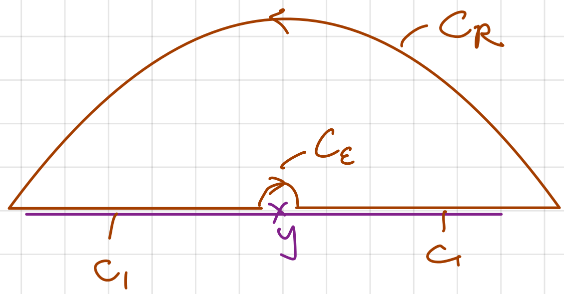
- $\mathcal{H}$ is linear operator
- If assume $f(x)$ has Fourier Transform, can deduce $\mathcal{H}(f)$ by linearity if we know $\mathcal{H}(e^{i\omega x})$.

We will show that

$$\mathcal{H}(e^{i\omega x})(y) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x-y} dx = \begin{cases} ie^{i\omega y} & \omega > 0 \\ -ie^{i\omega y} & \omega < 0 \end{cases}$$

Pf: Good choice for $\omega > 0$.

$$\oint_{e} \frac{e^{i\omega z}}{z-y} dz = 0. = \int_{C_R} + \int_{C_\varepsilon} + \int_{C_1}$$



$\lim_{R\to\infty} \int_{C_R} \to 0$ using Jordan's lemma.

$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} = -\pi i \cdot (\text{residue at } y)$

Hence, $\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x-y} dx = \frac{\pi i}{\pi} e^{i\omega y} = ie^{i\omega y} \quad (\omega > 0)$.

If $\omega < 0$, close contour below. $\Rightarrow$ $= -ie^{i\omega y}$, $(\omega < 0)$ $\square$

Note $\mathcal{H}(\mathcal{H}(e^{i\omega x}))(y) = e^{-i\omega y}$ $\Rightarrow$ $\mathcal{H}^2(f) = -f$, so $\mathcal{H}$ is anti self-inverse ($\mathcal{H}^2 = -I$, I identity).

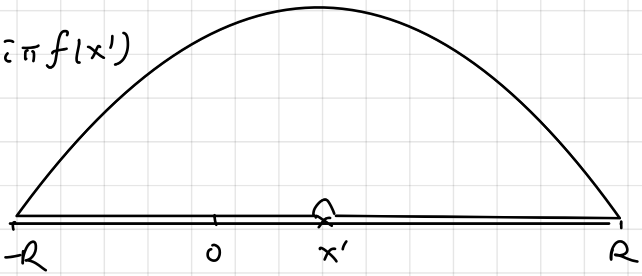
More generally, if $g = \mathcal{H}(f)$, then $\mathcal{H}(g) = -f$.

Kramer - Kronig relations

Let $f(z)$ analytic in $\mathrm{Im}(z) > 0$, with $f \to 0$ as $|z| \to \infty$ in $\mathrm{Im}(z) > 0$.

Let $x' \in \mathbb{R}$ and consider integral around contour as drawn

$$\oint \frac{f(z)}{z - x'} dz = P\left(\int_{-\infty}^{\infty} \frac{f(x)}{x - x'} dx \right) - i\pi f(x')$$
$$= 0$$



(follows from Cauchy's thm and follows from analyticity in $\mathrm{Im}(z) > 0$).

Write $f(z) = u(x, y) + i v(x, y)$

For $z \in \mathbb{R}$, $x = z$, and $f(z) = u(x, 0) + i v(x, 0) = u_0(x) + i v_0(x)$.

Result above:

$$P \int_{-\infty}^{\infty} \frac{u_0(x)}{x - x'} dx = -\pi v_0(x') \qquad \mathcal{H}(u_0) = -v_0$$

or

$$P \int_{-\infty}^{\infty} \frac{v_0(x)}{x - x'} dx = \pi u_0(x') \qquad \mathcal{H}(v_0) = u_0.$$

This is the Kramer-Kronig relation.

Application: consider simple linear system

$$\frac{dy}{dt} = -\alpha y + F(t). \qquad (\alpha > 0)$$

Can be solved by FT.

$$\rightarrow (-i\omega + \alpha)\, \hat{y} = \hat{F}(\omega)$$

(using $\hat{F}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt$)

$$\Rightarrow \hat{y}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t} \hat{F}(\omega)}{-i\omega + \alpha} d\omega$$

i.e. $\hat{F}(\omega)$ multiplied by f^n which is analytic in upper half plane.

pole in lower half plane

More general linear system

$$\hat{y}(\omega) = \hat{F}(\omega)\hat{s}(\omega),$$

(expect $\hat{s}(\omega)$ analytic in lower half plane if system as usual)

Write $\hat{S}(\omega) = \hat{S}_r(\omega) + i\hat{S}_i(\omega)$ for ω real, then

$$\pi S_r(\omega') = P\int_{-\infty}^{\infty} \frac{S_i(\omega)}{\omega - \omega'}\,d\omega \,, \quad \pi S_i(\omega') = -P\int_{-\infty}^{\infty} \frac{S_r(\omega)}{\omega-\omega'}\,d\omega \,,$$

for ω' real.

$S_r(\omega)$ determines sol^n in phase with forcing, e.g. dissipation of sound.

$S_i(\omega)$ determines sol^n out of phase

1.5 Multivalued functions

A multivalued f^n $F(z)$ admits more than one value for given z.

Example. $z^{1/2}$ single valued at 0 (and ∞), but elsewhere in $\mathbb{C}$ it may take

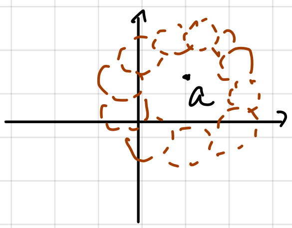
$$z = re^{i\theta} \Rightarrow z^{1/2} = r^{1/2} e^{i(\frac{1}{2}\theta + n\pi)} \,, \quad n \in \mathbb{Z}$$
$$= \pm r^{1/2} e^{i\theta/2} \,, \quad \theta \in [0, 2\pi).$$

Example $\log z = \log r + i\theta$ (for $z \neq 0$), may also take values $\log r + i\theta + 2\pi i n$, $n \in \mathbb{Z}$.

Branch points (BP)

$z = a$ is a BP of $f(z)$ if $f(a + re^{2\pi i}) \neq f(a+r)$ for small r where $f(a+re^{2\pi i})$ by analytic continuation.

e.g. analytic continuation by Taylor series.



A **branch** of $f(z)$ is single valued f^n obtained by taking one value for each z. This f^n will have curves of discontinuity called **branch cuts**. (Branch cuts join branch points, but not all branch points can be joined by branch cuts)

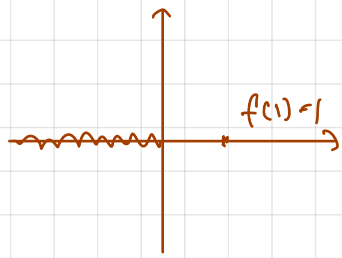
A branch is defined by (i) $f(z)$, (ii) set of branch cuts, (iii) a value of f at one point., then all other values can be obtained by analytic continuation.

Example $f(z) = z^{1/2}$ has two BPs : $z=0, \infty$.

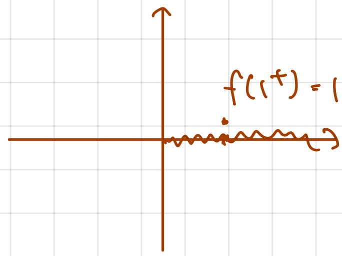
$z=0$: BP follows from writing $z = re^{i\varphi}$.

$z=\infty$: write $t = 1/z$, $t^{-1/2}$ has BP at $t=0$.

Branch cut : choose $(-\infty, 0)$ and specify $f(1) = 1$ gives a unique def^n on $\varphi = \arg(z) \in (-\pi, \pi)$



Alternative : $(0, \infty)$ as BC, $f(1^+) = 1$ (just above the real axis)

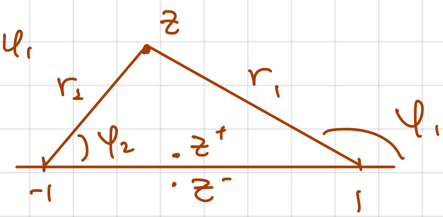


The def^n is unique on $\arg(z) \in (0, 2\pi)$

Example let $f(z) = (1-z^2)^{1/2} = \left((1+z)(1-z)\right)^{1/2}$

Claim : $f(z)$ has BP at $z = \pm 1$.

Parameterise $z = -1 + r_2 e^{i\varphi_2} = 1 + r_1 e^{i\varphi_1}$



For BP at $z=-1$. consider

$$z^+ = -1 + r_2 e^{i0^+} = 1 + r_1 e^{i\pi^-}$$

$$z^- = -1 + r_2 e^{i2\pi^-} = 1 + r_1 e^{i\pi^+}$$

$(r_1 + r_2 \approx 2)$

Then $f(z^+) = f(-1+r_2) = \left(r_2 e^{i0^+}(-r_1 e^{i\pi^-})\right)^{1/2} = \pm(r_1r_2)^{1/2} e^{i0/2}$

$$f(z^-) = f(-1+r_2e^{2\pi i}) = \left(r_2 e^{i2\pi^-}(-r_1 e^{i\pi^+})\right)^{1/2} = \pm(r_1r_2)^{1/2}e^{i\pi}$$
$$= \mp(r_1r_2)^{1/2}.$$

Similar for BP at $z=1$. So

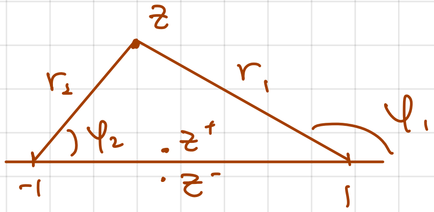
$$f(\pm1+r) = -f(\pm1+re^{2\pi i})$$

f is multivalued.

Introduction of a branch cut gives single value of f and discty

We now define a branch of f

(i) $f(z) = (1-z^2)^{1/2} = \left((1+z)(1-z)\right)^{1/2}$



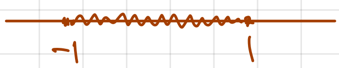
(ii) Branch cut

(iii) Value of f at one point

Parameterise $z = 1 + r_1 e^{i\varphi_1} = -1 + r_2 e^{i\varphi_2}$

$$f(z) = \left(r_2 e^{i\varphi_2}(\underbrace{-1}_{e^{i\pi}}\, r_1 e^{i\varphi_1})\right)^{1/2} = \pm(r_1r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2 + i\pi/2}$$
$$= \pm i(r_1r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$$

Two natural choices of BCs



$[-1,1]$

e.g. choose $f(0^+)=1$.

or

$(-\infty,-1)\cup(1,\infty)$

e.g. choose $f(0)=1$

Choose $[-1,1]$, $f(0^+)=1$

$$f(0^+) = 1 \overset{!}{=} \pm i\, 1^{1/2} e^{i(\pi+0)/2} = \pm i\cdot i = \pm(-1)$$

So choose "$-$", and

$$f(z) = -i(r_1r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2} = -i(1-z^2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}.$$

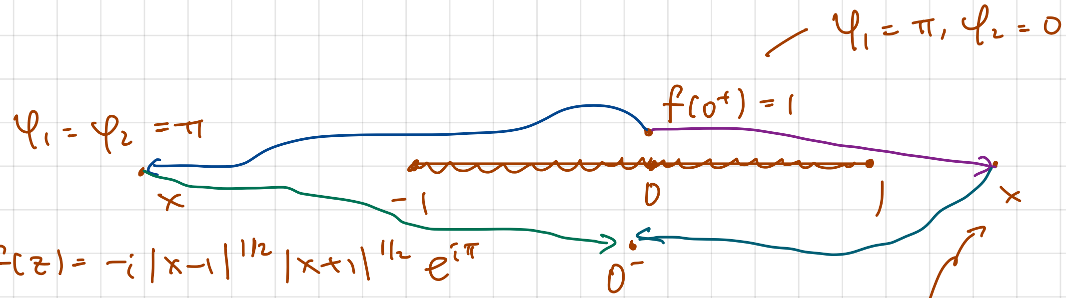
Now evaluate $f(z)$ at other points by considering evolution of φ_1 and φ_2 along paths that do not cross the branch cut.

$$f(z) = -i(r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$$

$$= -i\,|1-z^2|^{1/2} e^{i(\varphi_1+\varphi_2)/2}$$

Just above real axis, $\varphi_1 = \pi$, $\varphi_2 = 0$

$$f(x) = -i\,|x+1|^{1/2}|x-1|^{1/2} e^{i\pi/2} = (1-x^2)^{1/2}$$



$$f(z) = -i\,|x-1|^{1/2}|x+1|^{1/2} e^{i\pi}$$

$$= i\,|x-1|^{1/2}|x+1|^{1/2}$$

$$\simeq -ix \quad \text{for large } x$$

$\varphi_1 = \varphi_2 = 0$

$$f(z) = -i\,|x-1|^{1/2}|x+1|^{1/2}$$

$$\simeq -ix \quad \text{for large } x.$$

$\varphi_1 = \pi$, $\varphi_2 = 2\pi$

$$f(z) = -i\,|1+x|^{1/2}|1-x|^{1/2} e^{i3\pi/2} = -(1-x^2)^{1/2}.$$

or $\varphi_1 = -\pi$, $\varphi_2 = 0$

$$f(z) = -i\,|1+x|^{1/2}|1-x|^{1/2} e^{-i\pi/2} = -(1-x^2)^{1/2}$$

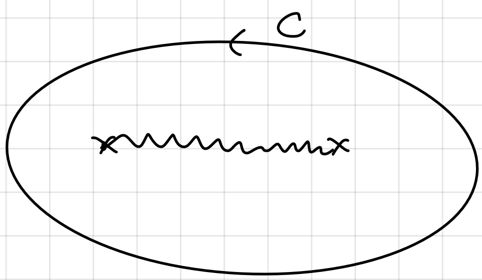
So $f(0^-) = -1$

Branch cut algorithm

Aim: evaluate $I = \int_{-1}^{1} (1-x^2)^{1/2}\, dx$.

Let $f(z)$ be branch of $(1-z^2)^{1/2}$. $BC = [-1,1]$, $f(0^+) = 1$.

Consider $\oint_C f(z)\, dz$, C "big".



large z: $\varphi_1 \approx \varphi_2$, $z \approx r_1 e^{i\varphi_1} \approx r_2 e^{i\varphi_2}$

$$f(z) = (1-z^2)^{1/2} = -i\,|1-z^2|^{1/2} e^{i(\varphi_1+\varphi_2)/2}$$

$$\approx -i\,|z|\,e^{i\varphi_1} = -iz$$

Lorentz expansion : $(1-z^2)^{1/2} = \left(-z^2\left(1-\frac{1}{z^2}\right)\right)^{1/2}$

$$= \pm iz\left(1-\frac{1}{z^2}\right)^{1/2}$$

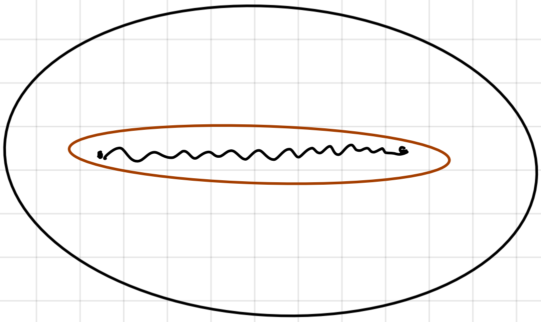
$$= \underbrace{\pm}_{\text{take "}-\text{"}} iz\left(1-\frac{1}{2z^2}+\dots\right)$$

Hence.

$$\oint_C f(z)\,dz = -\oint_C iz\,dz + i\oint_C \frac{dz}{2z} + \dots$$

$$= \frac{i}{2}\cdot 2\pi i = -\pi$$

But also can collapse integral onto branch cut



$$\int_1^{-1}(1-x^2)^{1/2}\,dx + \int_{-1}^{1} -(1-x^2)^{1/2}\,dx$$

$$= -2\int_{-1}^{1}(1-x^2)^{1/2}\,dx = -2I$$

Hence $I = \frac{\pi}{2}$.

arcsin defined as an integral

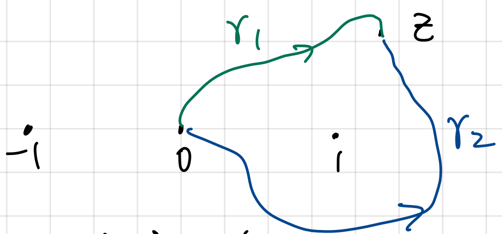
(a) we know that $\frac{d}{dz}\arcsin(z) = (1-z^2)^{-1/2}$, $z \neq \pm 1$.

Can we define $\arcsin(z) = \int_0^z \frac{1}{(1-t^2)^{1/2}}\,dt$?

Need to fix branch of $(1-t^2)^{1/2}$ at $t=0$, then define by continuation as we move along contour of integration.

Key point : $(1-t^2)^{-1/2}$ has BP at $t=\pm 1$. value of integral will depend on the path.

$$\int_{\gamma_1}\frac{1}{(1-t^2)^{1/2}}\,dt \neq \int_{\gamma_2}\frac{1}{(1-t^2)^{1/2}}\,dt$$



Hence analytic continuation of $\arcsin(z)$ is not unique.

(b) We define a single valued f^n Arcsin(z).

$$\text{Arcsin}(z) = \int_0^z \frac{dt}{(1-t^2)^{1/2}}$$

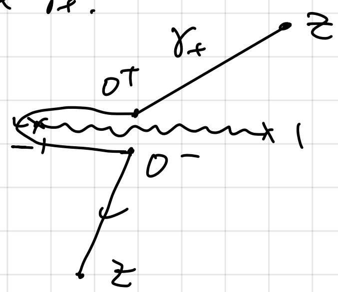
with branch as defined previously $=1$ at $t=0$.

Take branch cut of $(1-t^2)^{-1/2}$ to be $[-1,1]$ — intepreting $t=0$ as $t=0^+$ lying just above Branch cut.

The path of integration is not allowed to cross BC.

Define as follows. If $\text{Arg}\, z \in (0,\pi)$, use γ_+.

If $\text{Arg}\, z \in [\pi, 2\pi)$, use γ_-, i.e. integrate around BC to 0^-, then straight to z.



Where is Arcsin(z) analytic?

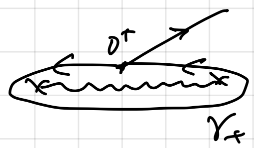
For $0 < \text{Arg}\, z < 2\pi$, Arcsin(z) analytic except on BC.

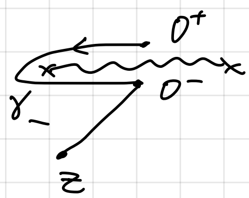
Discty in Arcsin(z) across BC $[-1,1]$

Also a discty across $[1,\infty)$, hence BC of Arcsin(z) req'd from $-1 \to \infty$.

Now define a new f^n $\displaystyle\int_0^z \frac{1}{(1-t^2)^{1/2}}\,dt = \underset{(\pi,3\pi)}{\text{Arcsin}}\, z$

with $\text{Arg}(z) \in (2\pi, 3\pi)$ — γ_+. , with $\text{Arg}(z) \in (\pi, 2\pi)$ — γ_-

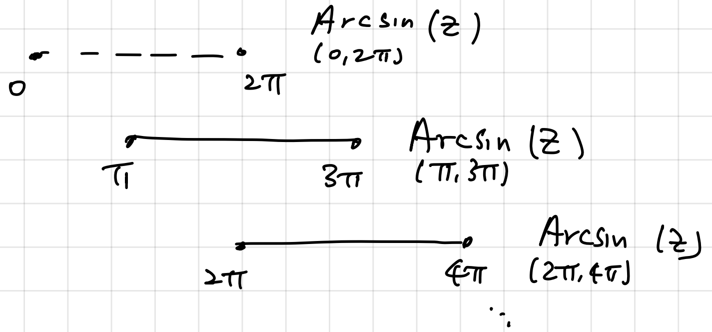




With $\underset{(\pi,3\pi)}{\text{Arcsin}(z)} = \underset{(0,2\pi)}{\text{Arcsin}}$ if $\pi < \text{Arg}\, z < 2\pi$.

Both f^{n}s are analytic, hence $\underset{(\pi,3\pi)}{\text{Arcsin}(z)}$ is analytic continuation of $\underset{(0,2\pi)}{\text{Arcsin}(z)}$ into $2\pi < \text{Arg}\, z < 3\pi$.

Repeat process



Hence have analytic multivalued f^n arcsin(z) with discty across $[-1,1]$.

Dependence on Arg(z):

· If $0 < \operatorname{Arg} z < 2\pi$. $\arcsin(ze^{2\pi i}) = \underset{(0,2\pi)}{\operatorname{Arcsin}}(z) + \oint_{BC} \frac{1}{(1-t^2)^{1/2}}\,dt.$

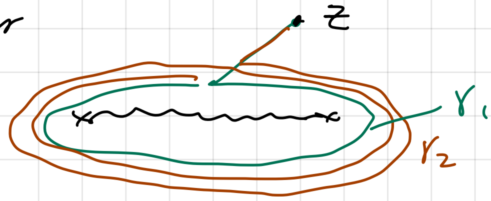
$$= \underset{(0,2\pi)}{\operatorname{Arcsin}} - 2\pi$$

In general, all possible values of arcsin(z) are $\underset{(0,2\pi)}{\operatorname{Arcsin}}(z) + 2n\pi,\ n \in \mathbb{Z}$.

(Could resolve analyticity at $z=0$ by choosing a different BC).

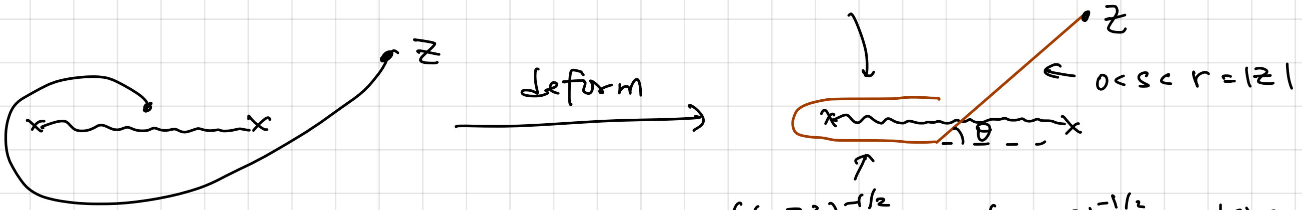
(c) Alternative to above procedure is to acknowledge path dependence.

Each of $\gamma_0, \gamma_1, \gamma_2, \ldots$ give answers differ by 2π.



What about paths that encloses $[-1,1]$?

$(1-z^2)^{-1/2} = (1-x^2)^{-1/2},\quad 0 > x > -1$



$(1-z^2)^{-1/2} = -(1-x^2)^{-1/2},\quad -1 < x < 0.$

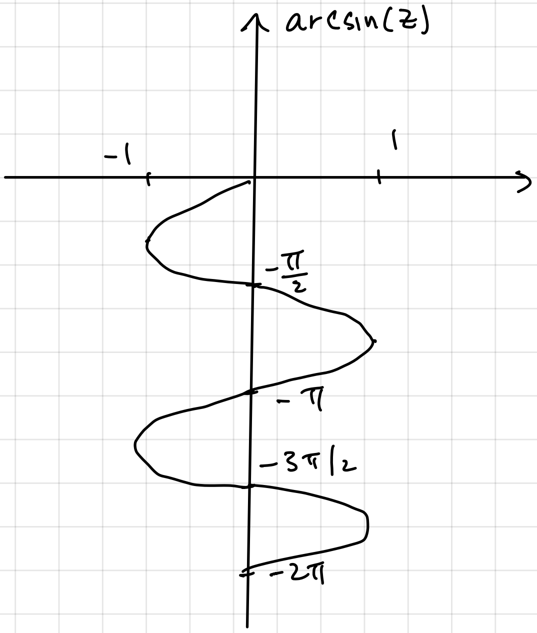
$$\underset{\tilde{\gamma}}{\arcsin}(z) = \int_0^{-1} (1-x^2)^{-1/2}\,dx + \int_{-1}^{0} -(1-x^2)^{-1/2}\,dx + \int_0^r -(1-s^2e^{2i\theta})^{-1/2} s e^{i\theta}\,d\theta$$

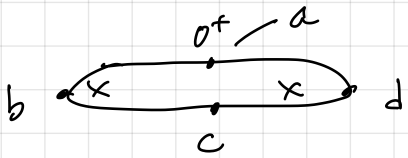
$$= -\pi - \underset{(0,2\pi)}{\operatorname{Arcsin}}(z).$$

Another set of possible values for $\arcsin(z)$ is

$$-\pi + 2n\pi - \underset{(0,2\pi)}{\mathrm{Arcsin}}\,(z) \quad , \quad n \in \mathbb{Z}.$$

(d) Inverse of $\arcsin(z)$.





Consider the inverse of arcsin — call it "sin"

$$W = \underset{(0,2\pi)}{\mathrm{Arcsin}}\,(z), \qquad \sin w = z.$$

$$\arcsin(z) = \underset{(0,2\pi)}{\mathrm{Arcsin}}\,(z) + 2\pi n \Rightarrow \sin(\arcsin(z)) = z - \sin w = \sin(w + 2\pi).$$

$$\arcsin(z) = (2n-1)\pi - \underset{(0,2\pi)}{\mathrm{Arcsin}}\,(z) \Rightarrow \sin w = \sin((2n-1)\pi - w)$$

Inverse of our multivalued f^ns is a <u>periodic</u> f^n.

Complication of multivalued f^n is avoided by taking inverse.

Note $\int_0^z \frac{1}{(1-t^2)^{1/2}}\,dt \sim \log z$ as z becomes large, consistent with behaviour of $\sin f^n$

$$\sin W = \frac{1}{2i}\left(e^{iw} - e^{-iw}\right)$$

together with the fact that only singularity of sin is essential at ∞.

1.6 Elliptic functions

Def^n Periodic f^n: $f(z+\alpha) = f(z)$ for some $\alpha \in \mathbb{C}$.

Def^n Doubly periodic f^n (DPF), have $\omega_1, \omega_2 \in \mathbb{C}$, $\omega_1/\omega_2 \notin \mathbb{R}$, i.e. $\arg(\omega_1) \neq \arg(\omega_2)$, s.t.

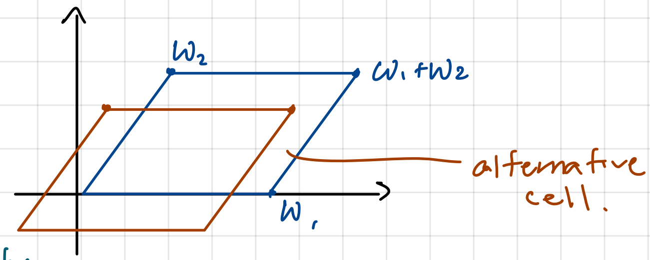
$$f(z+\omega_1) = f(z+\omega_2) = f(z) \quad \forall z \in \mathbb{C}.$$

More generally, $f(z+m\omega_1+n\omega_2) = f(z)$ $\forall z \in \mathbb{C}$ $\forall m,n \in \mathbb{Z}$.

Def^n Elliptic f^n is DPF which is meromorphic in $\mathbb{C}$.

Consider the parallelogram defined by $\omega_1, \omega_2 \in \mathbb{C}$. (WLOG assume $\omega_1 \in \mathbb{R}$.)

All zeros / poles of elliptic f^n are copies of zeros / poles in one cell.



0, ω_1, ω_2, $\omega_1+\omega_2$ are lattice points

Properties of elliptic f^n:

(1) Number of zeros and poles in a cell is finite.

(and no. of poles = no. of zeros taking account of multiplicity)

(2) An elliptic f^n with no poles in cell is const. (follows from Liouville's thm)

Weierstrass $\wp$ f^n — a non-const elliptic f^n.

Set $\omega_{m,n} = m\omega_1 + n\omega_2$, $m,n \in \mathbb{Z}$ (labels lattice points)

Define

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0,0\}} \left(\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2} \right).$$

$\wp(z)$ meromorphic with poles of order 2 at each lattice point.

Exercise Show that expression for $\wp(z)$ cgt and $\wp(z)$ is DPF.

Prop $\wp(z)$ satisfies

$$\wp'(z) = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where $g_2 = 60\sum \omega_{m,n}^{-4}$, $g_3 = 140 \sum \omega_{m,n}^{-6}$, sum over $\{m,n\} \in \mathbb{Z}^2 \setminus \{0,0\}$.

Pf (Sketch):

Consider $\wp(z) = \frac{1}{z^2} + Q(z)$. $Q(z)$ analytic in cell containing $z=0$.

Check that $Q(z) = O(z^2)$.

$$\wp'(z)^2 = \left(-\frac{2}{z^3} + Q'(z)\right)^2 = \frac{4}{z^6} - 4\underbrace{\frac{Q'(z)}{z^3}}_{O(z^{-2})} + \underbrace{Q'(z)^2}_{O(z^2)}$$

$$4\wp(z)^3 = 4\left(\frac{1}{z^2} + Q(z)\right)^3 = \frac{4}{z^6} + \underbrace{\frac{12Q(z)}{z^4}}_{O(z^{-2})} + \underbrace{\frac{12Q(z)^2}{z^2}}_{O(z^2)} + \underbrace{4Q(z)^3}_{O(z^6)}.$$

$$\Rightarrow \qquad \wp'(z)^2 - 4\wp(z)^3 = \underbrace{O(z^{-2})}_{\exists g_2, g_3,\ g_2\wp(z)+g_3} + O(z^2).$$

Hence $\wp'(z) - 4\wp(z)^3 - g_2\wp(z) - g_3 = O(z^2) = 0$

LHS analytic DPF,

analytic DPF
$=0$ at $z=0$,
so const. $=0$

Consider detail of $O(z^{-2})$ and $O(1)$ terms to deduce g_2, g_3. $\square$

Following reasoning for arcsin etc,

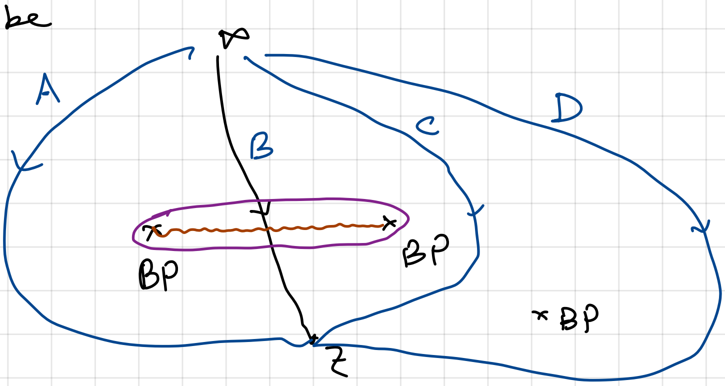
$$\wp^{-1}(z) = -\int_z^\infty \frac{ds}{(4s^3 - g_2 s - g_3)^{1/2}}$$

choice of upper limit $\Rightarrow \wp^{-1}(\infty) = 0$. (one possible value)

Integrand has 4 BPs (including one at ∞)

Periodicity of $\wp$ follows from multivaluedness of $\wp^{-1}$ which follows from multivaluedness of integrand

Difference between C and A will be integral around branch cut.



Note Branch cuts are not essential

As $z \to \infty$, $\wp^{-1}(z) \approx C + O(z^{-1/2})$

i.e. $(\wp^{-1}(z) - C)^{-2} \sim z$ as $z \to \infty$

$\Rightarrow$ $\wp(\omega) \sim (\omega - C)^{-2}$ as $\omega \to C$.

Elliptic f^ns important in many areas of mathematics

- e.g. Elliptic curves $y^2 = 4x^3 - g_2 x - g_3$ in $\mathbb{C}^2$.

  parameterized by $\wp(z)$: $z \longmapsto (x(z), y(z)) = (\wp(z), \wp'(z))$.

Originally arose in mechanics. – Simple pendulum

$$l\ddot{\theta} = -g\sin\theta$$

Energy: $\frac{1}{2}\dot{\theta}^2 = \omega^2\cos\theta + E = \omega^2(\cos\theta - \cos\theta_0)$ ($\dot{\theta} = 0$ at $\theta = \theta_0$).

$$\Rightarrow \dot{\theta} = \sqrt{2}\,\omega\,(\cos\theta - \cos\theta_0)^{1/2}$$

transformation $x = \dfrac{\sin\theta/2}{\sin\theta_0/2}$, then $\dot{x} = \omega(1-x^2)^{1/2}(1 - x^2\sin^2\theta_0/2)^{1/2}$.

Time requires evaluation of

$$\int_0^x \frac{du}{(1-u^2)^{1/2}(1-u^2\sin^2\theta_0/2)^{1/2}} = G^{-1}(x)$$

defines multivalued f^n in complex plane

$\Rightarrow$ can transform to Weierstrass elliptic f^n.

§2 Special Functions

Exploit machinery of analytic continuation to extend elementary definiation of well known f^n's.

2.1 Gamma Function

Extend def^n of factorial f^n $n! = 1\times 2\times \dots \times n$. from integers to complex plane.

Let
$$I(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad \text{(Euler's integral)}$$

Integral converges and is analytic for $Re(z) > 0$.

Note $I(z+1) = \int_0^\infty t^{z} e^{-t}\, dt = z I(z)$. , hence
$$I(n+1) = n!\, I(1) = n!$$

Now define $\Gamma(z) = I(z)$ for $Re(z) > 0$. – continue to larger part of $\mathbb{C}$.

$\frac{I(z+1)}{z}$ analytic for $Re(z) > -1$, $z \neq 0$. $\Rightarrow$ Use as continuation of $I(z)$ into $Re(z) > -1$, $z \neq 0$. Similarly,
$$\frac{I(z+n+1)}{z(z+1)\dots(z+n)}$$
analytic for $Re(z) > -n-1$ and $z \neq 0, 1, 2, \dots, -n$. Increase n, then provides meromorphic continuation of $I(z)$ in $Re(z) \leq 0$, $z \neq 0, -1, \dots$

Euler product formula for $\Gamma(z)$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\dots(z+n)} \qquad \forall n \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

Recall $e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n$
$$\Gamma(z) = I(z) = \lim_{n\to\infty} \int_0^n (1 - t/n)^n\, t^{z-1}\, dt$$

$$= \lim_{n\to\infty} n^z \int_0^1 (1-\tau)^n \tau^{z-1}\, d\tau \qquad \tau = t/n$$

$$= \lim_{n\to\infty} n^z \left(\left[\frac{\tau^z (1-\tau)^n}{z} \right]_0^1 + \int_0^1 \frac{n}{z} \tau^z (1-\tau)^{n-1} d\tau \right)$$

$$= \lim_{n\to\infty} \frac{n^z \; n(n-1)\ldots 1}{z(z+1)\ldots(z+n-1)} \int_0^1 \tau^{z+n-1} d\tau$$

$$= \lim_{n\to\infty} \frac{n^z \; n!}{z(z+1)\ldots(z+n)}$$

Provides explicit formula for $\Gamma(z)$ valid in all $\mathbb{C} \setminus \{0,-1,\ldots\}$.

Gauss Product formula

Claim $\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1+1/n)^z}{(1+z/n)}$

Pf: $\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\ldots(z+n)}$

$$= \lim_{n\to\infty} \frac{1}{z} \cdot \frac{n^z}{(1+z)(1+\frac{z}{2})\ldots(1+\frac{z}{n})}$$

$$= \lim_{n\to\infty} \frac{1}{z} \frac{\left(\frac{n+1}{n}\right)^z \left(\frac{n}{n-1}\right)^z \ldots \left(\frac{2}{1}\right)^z \left(\frac{n}{n+1}\right)^z}{(1+z)(1+\frac{z}{2}) \ldots (1+\frac{z}{n})}$$

$$= \lim_{n\to\infty} \frac{1}{z} \cdot \frac{(1+1/n)^z}{(1+z/n)} \frac{(1+\frac{1}{n-1})^z}{(1+\frac{z}{n-1})} \ldots \underbrace{\left(\frac{n}{n+1}\right)^z}_{\to 1 \text{ as } n\to\infty}$$

$$= \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1+1/n)^z}{1+z/n}. \qquad \square$$

e.g. $\Gamma(z)$ has no zeros. $\Rightarrow$ $1/\Gamma(z)$ entire.

Weierstrass Canonical Product

Claim: $\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}$.

γ is the Euler – Mascheroni const, where

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right) = 0.577\ldots$$

Use Gauss Product Formula.

Pf: $$\frac{1}{\Gamma(z)} = z\prod_{n=1}^{\infty}\frac{(1+z/n)}{(1+1/n)^{z}} = z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\frac{n}{(n+1)^{z}}$$

$$= z\lim_{n\to\infty}\left(\frac{1}{(n+1)^{z}}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right).$$

$$(n+1)^{-z} = \exp(-z\log n)\,\exp\left(-z\,O\left(\tfrac{1}{n}\right)\right)$$

$$= \exp\left(-z\left(\sum_{k=1}^{n}\frac{1}{k}-\gamma\right)\right)\exp\left(-z\,O\left(\tfrac{1}{n}\right)\right)$$

$$= \left(\prod_{k=1}^{\infty}e^{-z/k}\right)e^{\gamma z}\exp\left(-z\,O\left(\tfrac{1}{n}\right)\right)$$

$$\Rightarrow \frac{1}{\Gamma(z)} = ze^{\gamma z}\lim_{n\to\infty}\underbrace{\prod_{k=1}^{\infty}e^{-z/k}\left(1+\frac{z}{k}\right)}_{1+O(z^{2})\text{ as }z\to 0.} \qquad \square$$

e.g. $$\frac{1}{z\Gamma(z)} \sim 1+\gamma z+O(z^{2}) \quad \text{as } z\to 0$$

$$\Gamma'(1) = -\gamma \quad \text{etc.}$$

<u>Reflection formula</u>

$$\Gamma(z)\cdot\Gamma(1-z) = \pi\csc(\pi z) \quad \text{for } z\notin\mathbb{Z}.$$

First consider $0<\mathrm{Re}(z)<1$. — can use integral formula.

$$\Gamma(z)\,\Gamma(1-z) = \left(\int_0^{\infty}t^{z-1}e^{-t}\,dt\right)\left(\int_0^{\infty}s^{-z}e^{-s}\,ds\right)$$

let $t = r\sin^2\theta$, $s = r\cos^2\theta$, $0\le r<\infty$, $0\le\theta\le\pi/2$.

$$\frac{\partial(s,t)}{\partial(r,\theta)} = 2r\cos\theta\sin\theta.$$

$$= 2\int_0^{\infty}re^{-r^{2}}dr\int_0^{\pi/2}(\sin\theta)^{2z-2}(\cos\theta)^{-2z}\cos\theta\sin\theta\,d\theta$$

$$= 2\int_0^{\pi/2}(\tan\theta)^{2z-1}\,d\theta$$

$\tan\theta = u^{1/2}$

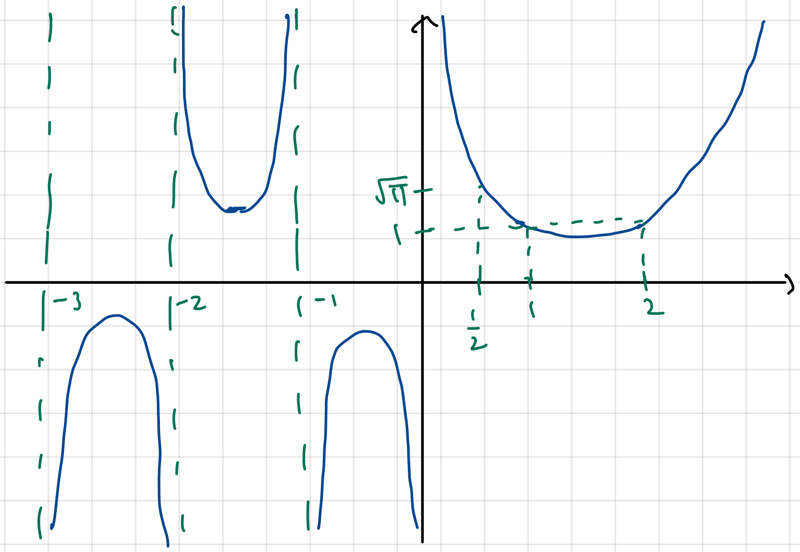
$$= \int_0^{\infty}\frac{u^{z-1}}{1+u}\,du = \pi\csc(\pi z).$$

$$\Rightarrow \Gamma(z)\,\Gamma(1-z) = \pi\csc\pi z \quad \text{for } 0<\mathrm{Re}(z)<1.$$

Hence equality for all $z \in \mathbb{C} \setminus \mathbb{Z}$ by the identity thm.

Hence $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ etc..

Graph of $\Gamma(x)$ for $x \in \mathbb{R}$.



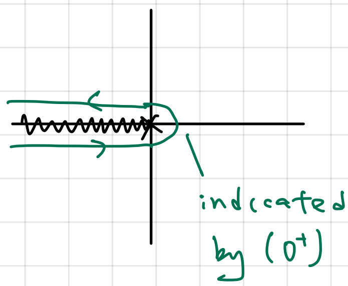
- $\Gamma(n+1) = n!$ $\forall n \in \mathbb{N}$
- Residue at $z=-n$ are $-\frac{1}{(n-1)!}$
- $\Gamma(z) \neq 0$ $\forall$ finite z from Gauss formula.

Hankel Representation of $\Gamma(z)$

$$\Gamma(z) = \frac{1}{2i\sin\pi z} \int_{-\infty}^{(0^+)} e^{-t} t^{z-1} \, dt, \quad z \neq 0, -1, -2, \dots \quad (+)$$

$-\pi < \arg t < \pi$

The contour is Hankel contour.



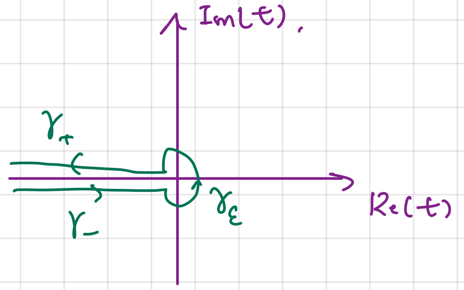
Integral defines analytic f^n $\forall z$.

Convergence as $t \to -\infty$ is assured by e^t factor

and avoid potential sing. at $z=0$.

Claim $(+)$ provide an analytic continuation of $I(z) = \int_0^\infty t^{z-1} e^{-t} dt$ $(\operatorname{Re} z > 0)$ to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$

Pf: Divide contour into $\gamma_-, \gamma_+, \gamma_\varepsilon$.



let
$$J(z) = \int_{-\infty}^{(0^+)} e^{t}\, t^{z-1}\, dt$$

Write $J(z) = \int_{\gamma_-} + \int_{\gamma_\varepsilon} + \int_{\gamma_+}$

On γ_- : $t = x e^{-i\pi}$, $\infty > x \geq \varepsilon$, $dt = e^{-i\pi} dx$

γ_ε : $t = \varepsilon e^{i\theta}$, $-\pi < \theta \leq \pi$, $dt = i\varepsilon e^{i\theta} d\theta$

γ_+ : $t = x e^{i\pi}$, $\varepsilon \leq x < \infty$, $dt = e^{i\pi} dx$.

if $Re(z) > 0$, $O(\varepsilon^z)$

$$J(z) = -\int_{\infty}^{\varepsilon} e^{-t} dt\, t^{z-1}\left(e^{-i\pi(z-1)} - e^{i\pi(z-1)}\right) + \cancel{\int_{\theta=-\pi}^{\pi} (\varepsilon e^{i\theta})^z (1 + O(\varepsilon))\, d\theta}$$

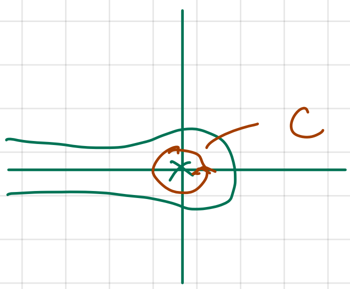
$$= \int_0^\infty e^{-t}\, t^{z-1}\, 2i \sin \pi z\, dt = 2i \sin \pi z\, I(z).$$

Hence for $Re(z) > 0$, $J(z)/2i\sin\pi z = I(z)$, and analytic for $z \notin \mathbb{Z}$, so provides analytic continuation to $\mathbb{C}\setminus\mathbb{Z}$, in particular into $Re(z) \leq 0$. □

Notes: (1) Apparent simple poles $\forall z \in \mathbb{Z}$, but sing. are removable when z pos. integer — no BC, Hankel integral is 0.

(2) Residues at $z = 0, -1, -2, \ldots$ For these z, there is pole in integral at $t=0$ — Hankel integral corresponds to contribution from residue at $t=0$.

$$J(-m) = \oint_C e^{-t}\, t^{-m+1}\, dt$$
$$= 2\pi i \operatorname{res}\left(e^{-t} t^{-(m+1)}, t=0\right)$$



$$e^{-t} t^{-(m+1)} = \sum_{k=0}^{\infty} \frac{t^k\, t^{-(m+1)}}{k!} \Rightarrow \text{Residue} = \frac{1}{m!}$$

Hence $J(-m) = 2\pi i / m!$. Now recall $\Gamma(z) = \dfrac{J(z)}{2\pi i \sin(\pi z)}$. $\Gamma(z)$ has simple poles at $z = -m$, $m = 0, 1, 2, \ldots$ with residue

$$\frac{2\pi i}{2\pi i \sin(\pi z)} \cdot \frac{1}{m!} = \frac{(-1)^m}{m!}$$

(consistent with previous approaches).

Uniqueness of $\Gamma(z)$. Is $\Gamma(z)$ as we constructed a unique extension of $(\,)!$ to $\mathbb{C}$? [Question is not uniqueness of analytic continuation of $I(z)$.]

Counterexample: Consider entire $g(z)$ s.t. $g(n)=1 \ \forall n \in \mathbb{N}$, and $g(z+1)=g(z)$.

Now write

$$\tilde{\Gamma}(z) = g(z)\,\Gamma(z).$$

$$\tilde{\Gamma}(z+1) = \Gamma(z+1)\,g(z+1) = \Gamma(z)\, z\, g(z) = z\,\tilde{\Gamma}(z).$$

Hence $\tilde{\Gamma}(n+1) = n!$ — $\tilde{\Gamma}$ is another possible extension of $(\,)!$

Does a suitable $g(z)$ exist? Yes, e.g. $g(z) = 1 + \sin(2\pi z)$.

However, $\Gamma(z)$ is the only extension of $(\,)!$ that is bounded on $\mathrm{Re}(z) \in [1,2]$ and analytic.

Thm (Wielandt's thm) (A uniqueness for $\Gamma(z)$) If $F(z)$ satisfies

(i) $F(z)$ analytic for $\mathrm{Re}(z) > 0$

(ii) $F(z+1) = zF(z)$

(iii) $F(z)$ bounded in $\mathrm{Re}(z) \in [1,2]$

(iv) $F(1) = 1$

Then $F(z) = \Gamma(z)$ defined by $I(z)$ and its analytic continuation.

Lem Let $F(z)$ analytic f^n satisfy (i) – (iv). Define $f(z) = F(z) - \Gamma(z)$, then $f(z)$ is entire.

Pf: (i), (ii) $\Rightarrow$ $F(z)$ can be analytically (meromorphically) continued to $\mathrm{Re}(z) \le 0$ (poles at $z = 0, -1, \ldots$) – as for $\Gamma(z)$.

$$F(z) = \frac{F(z+N)}{z(z+1)\ldots(z+N-1)}$$

$$\Rightarrow \mathrm{Res}\,(F(z) : z = -n) = F(1)\,\frac{(-1)^n}{n!} = \frac{(-1)^n}{n!}.$$

i.e. same as $\Gamma(z)$.

$F(z)$, $\Gamma(z)$ meromorphic in $\mathbb{C}$ with poles coinciding and same residues at each pole $\Rightarrow F - \Gamma$ has no sing in $\mathbb{C}$ (e.g. using Laurent expansion). Hence f entire. $\square$

Lem $f(z) = F(z) - \Gamma(z)$ bounded in strip $0 \le \mathrm{Re}(z) \le 1$.

Pf: First establish that f bounded on $1 \le \mathrm{Re}(z) \le 2$, then use recurrence relation. $F(z)$ assumed bounded on $1 \le \mathrm{Re} \le 2$, hence only need to show boundedness of $\Gamma(z)$.

$$|\Gamma(z)| = \left| \int_0^\infty e^{-t}\, t^{z-1}\, dt \right| \le \int_0^\infty e^{-t}\, t^{\mathrm{Re}(z)-1}\, dt = \Gamma(\mathrm{Re}(z)) \le 1$$

for $1 \le \mathrm{Re}(z) \le 2$.

Assertion follows from

$$\Gamma''(x) = \int_0^\infty e^{-t} (\log t)^2\, t^{x-1}\, dt > 0 \quad \forall x > 0$$

plus $\Gamma(1) = \Gamma(2) = 1$. Hence $f(z)$ bounded in $1 \le \mathrm{Re}(z) \le 2$.

In $0 \le \mathrm{Re}(z) \le 1$,

$$f(z) = F(z) - \Gamma(z) = \frac{F(z+1) - \Gamma(z+1)}{z} = \frac{f(z+1)}{z}.$$

$F(z+1)$, $\Gamma(z+1)$ both analytic, $F(1) = \Gamma(1)$, hence $f(z)$ finite as $z \to 0$.

Hence from above $f(z)$ bounded. $\square$

Combine the two lemma.

Pf of Thm : Let $s(z) = f(z) f(1-z)$. Then $s(z)$ entire by first lem, and bounded in $0 \le Re(z) \le 2$ by lem 2.

Now consider $s(z+1) = f(z+1) f(-z)$

$$= z f(z) \cdot \frac{f(1-z)}{-z} = -s(z).$$

Hence $s(z+2) = s(z)$. and since $s(z)$ bounded in $0 \le Re(z) \le 2$, $s(z)$ is bounded everywhere, hence $s(z)$ const. by Liouville thm

But $s(1) = f(1) f(0) = 0$ ($\because F(1) = \Gamma(1)$).

Hence, $s(z) = f(z) f(1-z) = 0 \quad \forall z \in \mathbb{C}. \Rightarrow F(z) = \Gamma(z) \quad \forall z \in \mathbb{C}$.

(in particular for $Re(z) \ge 0$). $\square$

There are other uniqueness on Γ, but these tend to focus on argument being a real variable. (so approaches other than analyticity must be used.)

2.2 The Beta Function

Define

$$B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} \, dt$$

for $Re\, p > 0$, $Re\, q > 0$.

By analytic continuation, for more general p, q.

Setting $t = \sin^2\theta$ gives

$$B(p,q) = 2 \int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta$$

Properties:

(i) $B(p,q) = B(q,p)$

(ii) $B(1,q) = \int_0^1 (1-t)^{q-1}\,dt = \frac{1}{q}$ $(\mathrm{Re}\, q > 0)$

(iii) $B(p,q+1) = \frac{q}{p+q} B(p,q)$

Pf: $B(p,q+1) = \int_0^1 t^{p-1}(1-t)^q\,dt$

$$= \int_0^1 t^{p-1}(1-t)^{q-1}\,dt - \int_0^1 t^p(1-t)^{q-1}\,dt$$

$$= B(p,q) - \left[\frac{t^p(1-t)^q}{q}\right]_0^1 - \frac{p}{q}\int_0^1 t^{p-1}(1-t)^q\,dt$$

$$\Rightarrow \left(1+\frac{p}{q}\right) B(p,q+1) = B(p,q) \qquad \square$$

(iv) $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. Hence $B(n,m) = \frac{(n-1)!\,(m-1)!}{(m+n-1)!}$ for $m,n \in \mathbb{N}$

Pf: Consider

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-s}s^{p-1}\,ds \int_0^\infty e^{-t}t^{q-1}\,dt$$

Set $s = r\sin^2\theta$, $t = r\cos^2\theta$

$$= \int_0^\infty dr \int_0^{2\pi} d\theta\; e^{-r} r^{p+q-2} \sin^{2p-2}\theta \cos^{2q-2}\theta\,(2r\sin\theta\cos\theta)$$

$$= 2\int_0^\infty e^{-r} r^{p+q-1}\,dr \cdot \int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2p-1}\theta\,d\theta$$

$$= \Gamma(p+q)\,B(p,q) \qquad \square$$

Now consider analytic continuation. Fix p with $\mathrm{Re}(p) > 0$. Write $q = z$.

$$\text{(iii)} \Rightarrow B(p,z) = \frac{p+z}{z} B(p,z+1).$$

gives analytic continuation into $-1 \leq \mathrm{Re}(z) \leq 0$ (except for $z=0$ where there is a pole).

Just as $\Gamma(z)$, similarly

$$B(p,z) = \frac{p+z}{z} \cdot \frac{p+z+1}{z+1} B(p,z).$$

continues into $-2 \leq \text{Re} \leq 0$ apart from $z=0, -1$, etc.

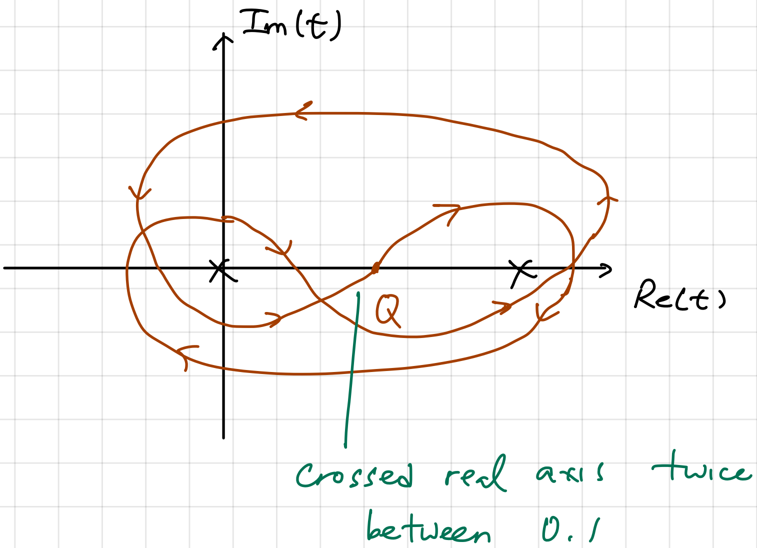
Hence analytic continuation $\forall z \in \mathbb{C}$ except poles at $z = 0, -1, -2, \dots$

Could also use (iv) based on continuation of $\Gamma(z)$.

Pochhammer representation of $B(p,q)$

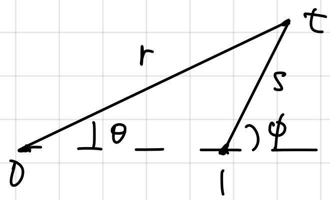
Let $J(p,q) = \oint_P f(t)\, dt$, P is Pochhammer contour,

with $f(t) = t^{p-1}(1-t)^{q-1}$ with branches determined by analytic continuation along path.



$Q \in \mathbb{R}$, $0 < Q < 1$.

We finish on same branch of $t^{p-1}(1-t)^{q-1}$ as we started with. Parameterise by local angles.



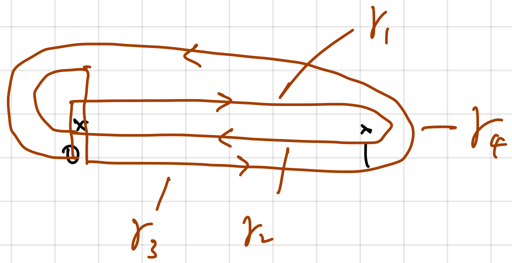
$$t = 1 + se^{i\phi} = re^{i\theta}.$$

$$f(t) = t^{p-1}(1-t)^{q-1}$$

$$= (re^{i\theta})^{p-1}(-se^{i\phi})^{q-1} = r^{p-1}s^{q-1}e^{i(p-1)\theta}e^{i(q-1)(\phi-\pi)}.$$

For $Re(p) > 0$, $Re(q) > 0$..

ensures that contributions from small circles can be ignored semicircles



On γ_1: $\theta = 0$, $\phi = \pi$, $f(t) = x^{p-1}(1-x)^{q-1}$, $x = Re(t)$.

γ_2: $\theta = 0$, $\phi = -\pi$. $f(t) = x^{p-1}(1-x)^{q-1} e^{-2\pi i q}$

γ_3: $\theta = -2\pi$, $\phi = -\pi$, $f(t) = x^{p-1}(1-x)^{q-1} e^{-2\pi i(p+q)}$.

γ_4: $\theta = -2\pi$, $\phi = \pi$, $f(t) = x^{p-1}(1-x)^{q-1} e^{-2\pi i p}$, $0 \leq x \leq 1$.

Hence,

$$J(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \left(1 - e^{-2\pi i q} + e^{2\pi i(p+q)} - e^{-2\pi i p}\right)$$

$$= B(p,q)\, e^{\pi i(p+q)} \cdot -4 \sin p\pi \sin q\pi.$$

Hence,

$$B(p,q) = -\frac{1}{4} e^{\pi i(p+q)} \csc p\pi \csc q\pi \underbrace{J(p,q)}_{=0 \text{ for } p,q \text{ integers}}$$

RHS is analytic for all p, q except for $p = 0, -1, -2, \ldots$, $q = 0, -1, -2, \ldots$

sing for $p = 1, 2, \ldots$, $q = 1, 2, \ldots$ are removable.

2.3 Riemann zeta function

Define

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

for $Re(z) > 1$. By analytic continuation o/w.

$\zeta(z)$ analytic in $Re(z) > 1$. Series converge.

$$\zeta'(z) = \sum_{n=1}^{\infty} (\log n)\, n^{-z}$$

also converges for $Re(z) > 1$.

Named after Riemann but previously studied by Euler who showed

$$\zeta(2) = \pi^2/6.$$

Integral representation of $\zeta(z)$.

Prop For $\mathrm{Re}(z)>1$,
$$\zeta(z)=\frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt. \qquad (\dagger)$$

Pf: Recall that $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt = n^z\int_0^\infty s^{z-1}e^{-ns}\,ds.$ $(t=ns)$.

$$\zeta(z)\Gamma(z)=\sum_{n=1}^\infty n^{-z}n^z\int_0^\infty s^{z-1}e^{-ns}\,ds$$
$$=\int_0^\infty s^{z-1}\sum_{n=1}^\infty e^{-ns}\,ds.$$
$$=\int_0^\infty \frac{s^{z-1}}{1-e^{-s}}\cdot e^{-s}\,ds$$
$$=\int_0^\infty \frac{s^{z-1}}{e^s-1}\,ds.$$

Integrable singly at $s=0$, $\mathrm{Re}(z)>1$. $\square$

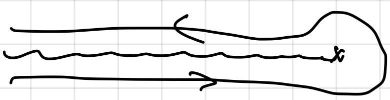
Now use as basis for more general integral representation.

Hankel representation of $\zeta(z)$

$$\zeta(z)=\frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{(0^+)}\frac{t^{z-1}}{e^{-t}-1}\,dt.$$

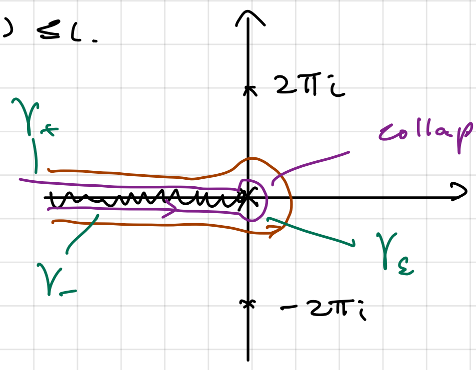
We need to show that

$$\frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{(0^+)}\frac{t^{z-1}}{e^{-t}-1}\,dt=\frac{1}{\Gamma(z)}\int_0^\infty\frac{t^{z-1}}{e^t-1}\,dt$$



for $\mathrm{Re}(z)>1$.

If true then since integrand is entire in z and smooth on t, on Hankel contour, then LHS gives analytic continuation of $\zeta(z)$ into $\mathrm{Re}(z)\le 1$.



collapsing Hankel contour onto real axis, - useful if $\mathrm{Re}(z)>1$.

Parameterisation:

$$\gamma_- : \quad t = xe^{-i\pi}, \qquad \infty > x \geq \varepsilon$$

$$\gamma_\varepsilon : \quad t = \varepsilon e^{i\theta}, \qquad -\pi \leq \theta \leq \pi$$

$$\gamma_+ : \quad t = xe^{i\pi}, \qquad \varepsilon \leq x < \infty.$$

$$\int_{-\infty}^{(0^+)} = J_- + J_\varepsilon + J_+, \quad \text{with}$$

$$J_\pm = \pm \int_\varepsilon^\infty \frac{(xe^{\pm i\pi})^{z-1}}{e^{-(xe^{\pm i\pi})}} e^{i\pi} dx = \mp e^{\pm i\pi z} \int_\varepsilon^\infty \frac{x^{z-1}}{e^x - 1} dx,$$

$$J_\varepsilon = O(\varepsilon^{Re(z)-1})$$

$$\lim_{\varepsilon \to 0} J_- + J_\varepsilon + J_+ \longrightarrow \int_0^\infty \frac{x^{z-1}}{e^x - 1} dx \, (e^{i\pi z} - e^{-i\pi z})$$

$$= 2i \sin(\pi z) \, \Gamma(z) \, \zeta(z) \quad \text{by } (\dagger).$$

$$\Rightarrow \frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{(0^+)} \frac{t^{z-1}}{e^{-t}-1} dt = \underbrace{\frac{\Gamma(1-z)}{2\pi i} 2i \sin(\pi z) \, \Gamma(z)}_{=1.} \, \zeta(z)$$

$$= \zeta(z)$$

Note residues of $\frac{t^{z-1}}{e^t - 1}$ at poles on imaginary axis

$$Res\left(\frac{t^{z-1}}{e^{-t}-1}, 2\pi i n\right) = -(2\pi i n)^{z-1}.$$

Thm $\zeta(z)$ may be continued into a meromorphic f^n of z on $\mathbb{C}$, with only sing. a pole at $z=1$ with residue 1.

Pf: Hankel representation implies sing. only possible when $z = 1, 2, 3, \ldots$ corresponding to poles of $\Gamma(1-z)$. We know $\zeta(z)$ analytic in $Re(z) > 1$, so only possibility is $z=1$.

Consider $$\lim_{z \to 1} \left(\frac{(z-1)\,\Gamma(1-z)}{2\pi i} \int_{-\infty}^{(0^+)} \frac{t^{z-1}}{e^{-t}-1} dt \right)$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{(0^+)} \frac{1}{e^{-t}-1} dt = -\frac{1}{2\pi i} \cdot 2\pi i \, Res\left(\frac{1}{e^{-t}-1}, 0\right) = 1. \quad \square$$

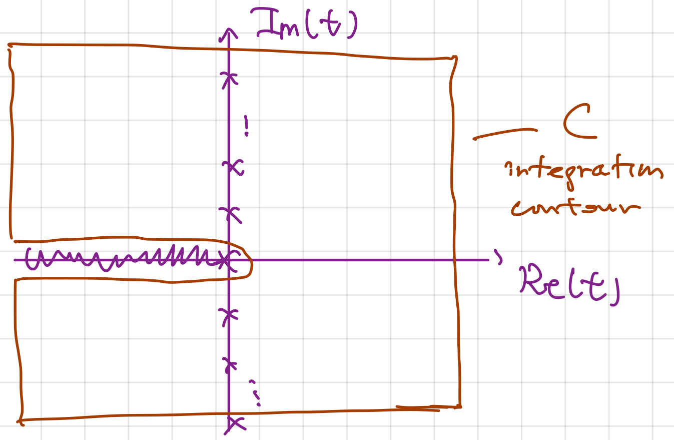
Functional equation for $\zeta(z)$.

Provides an explicit analytic continuation of $\zeta(z)$ from $\mathrm{Re}(z)>1$ to $\mathrm{Re}(z)\le 1$

Prop $\quad \zeta(z) = 2^z \pi^{z-1} \sin\frac{\pi z}{2}\, \Gamma(1-z)\,\zeta(1-z) \quad \forall z\in\mathbb{C}. \qquad (*)$

Pf: Derive $(*)$ for $\mathrm{Re}(z)<0$.

Start with integral around Hankel contour – close with a large rectangle, vertices at $\pm R \pm (2N+1)\pi i$, $R\gg 1$, $N \gg 1$, integer.



Take $R=\mu N$, μ fixed.

Consider

$$J(z) = \oint_C \frac{t^{z-1}}{e^{-t}-1}\,dt .$$

Integrand has BP at 0, BC along $-$ve real axis, neither is inside contour.

Hence $\quad J(z) = 2\pi i \cdot$ (sum of residues from poles along Im axis).

$$= 2\pi i \sum_{n=-N}^{N} -(2\pi i n)^{z-1} - (-2\pi i n)^{z-1}$$

Take $N\to\infty$.

$$J(z) = -4\pi i\,(2\pi)^{z-1}\cos\frac{\pi(z-1)}{2}\sum_{n=1}^{\infty} n^{z-1}$$

We can show that contributions from rectangular sides $\to 0$ as $N\to\infty$, $R\to\infty$

$$\left|\int_\gamma \frac{t^{z-1}}{e^{-t}-1}\,dt\right| \le \frac{\max_\gamma |t^{z-1}|}{\min_\gamma |e^{-t}-1|}\, L_\gamma = M_\gamma \quad \text{(upper bound on integral)}$$

L_γ: length of γ

$x = R\ (= \mu N)$: $\quad |e^{-t}| \le e^{-R} \Rightarrow |e^{-t} - 1| \ge 1 - e^{-R}$.

$$\Rightarrow \max_{\gamma} |t|^{z-1} \le (\sqrt{2}R)^{Re(z)-1}, \quad \text{and} \quad L_\gamma = (4N+2)\pi$$

Hence,

$$M_\gamma = \frac{(4N+2)\pi}{1-e^{-\mu N}} (\sqrt{2}\mu N)^{Re(z)-1} \to 0 \text{ as } N \to \infty$$

$x = -R\ (= -\mu N)$: $\quad |e^{-t}| \ge e^{R} \Rightarrow |e^{-t} - 1| \ge e^{R} - 1$

$$\max_{\gamma} |t^{z-1}| = (\sqrt{2}R)^{Re(z)-1}, \quad L_\gamma = (4N+2)\pi.$$

Hence

$$M_\gamma = \frac{(4N+2)\pi}{e^{\mu N}-1} (\sqrt{2}\mu N)^{Re(z)-1} \to 0 \text{ as } N \to \infty$$

$y = (2N+1)\pi$: $\quad |e^{-t} - 1| = |e^{x} + 1| \ge 1$

Similarly for $|t^{z-1}|$, $L_\gamma = 2R = 2\mu N$.

$$\Rightarrow M_\gamma = 2\mu N \left(\sqrt{2}(2N+1)\pi\right)^{Re(z)-1} \to 0 \text{ as } N \to \infty.$$

Contribution from rectangular sides $\to 0$ as $N, R \to \infty$. So

$$\int_{-\infty}^{(0^+)} \frac{t^{z-1}}{e^{-t}-1}\, dt = \frac{2\pi i\, \zeta(z)}{\Gamma(1-z)} = 4\pi i (2\pi)^{z-1} \sin\frac{\pi z}{2} \zeta(1-z).$$

Hence,

$$\zeta(z) = 2^{z} \pi^{z-1} \sin\frac{\pi z}{2} \Gamma(1-z) \zeta(1-z) \qquad (Re\, z < 0).$$

This is analytic $\forall z$ except pole at $z=1$, hence equality holds for all z by analytic continuation of RHS. $\square$

Can use to obtain $\zeta(2n)$.

$$\zeta(1-2n) = 2^{1-2n} \pi^{-2n} \underbrace{\sin\frac{\pi}{2}(1-2n)}_{(-1)^n} \Gamma(2n) \zeta(2n)$$

$$\| \qquad\qquad\qquad\qquad$$

$\dfrac{(-1)^{2n-1} B_{2n}}{2n}$, where B_n are Bernoulli number s.t.

$$\frac{1}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m t^{m-1}}{m!}$$ — $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, ...

Hence,
$$\zeta(2n) = \frac{(-1)^{n-1} \pi^{2n}}{2n!} 2^{2n-1} B_{2n}$$

(Recall $\zeta(2) = \pi^2/6$)

Zeros of $\zeta(z)$ and the Riemann hypothesis

$$\zeta(s) = \sum_{s=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} \quad . \quad Re(s) > 1 \qquad (**).$$

Riemann showed that

$$\sum_{\substack{p \text{ prime} \\ m \geq 1, p^m \leq x}} (\log p) = x - \sum_{\rho : \zeta(\rho) = 0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}.$$

weighted measure of #primes $\leq x$

zeros of ζ

$(**) \Rightarrow \zeta(s) \neq 0$ for $Re(s) > 1$.

So functional relation gives information on zeros of ζ in $Re(s) < 0$.

Connection between $\zeta(s)$ and primes

Chebyshev psi f^n

$$\psi(x) = \sum_{p^m \leq x} \log p = \sum_{n \leq x} \Lambda(n),$$

where $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \\ 0 & \text{o/w} \end{cases}$.

e.g. $\psi(10) = 3 \log 2 + 2 \log 3 + \log 5 + \log 7$.

Consider $\sum_{n=2}^{\infty} \Lambda(n) n^{-s}$ $(Re(s) > 1)$ generating f^n for Λ. Can show

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \Lambda(n) n^{-s} \qquad (Re(s) > 1)$$

Pf:

$$-\frac{d}{ds}\log(\zeta(s)) = \frac{d}{ds}\sum_p \log(1-p^{-s})$$
$$= \sum_p \frac{s\log p}{(1-p^{-s})} p^{-s}$$
$$= \sum_p s\log p\left(\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right) = \sum_{n=2}^{\infty}\Lambda(n)n^{-s} \qquad \square$$

Now consider

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)x^s\frac{ds}{s} \circledast \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\sum_{n=2}^{\infty}\Lambda(n)\left(\frac{x}{n}\right)^s\frac{ds}{s}$$

Choose $a>1$ so $Re(s)>1$,

$$\circledast \frac{1}{2\pi i}\sum_{n=2}^{\infty}\Lambda(n)\int_{a-i\infty}^{a+i\infty}\left(\frac{x}{n}\right)^s\frac{ds}{s}$$

- $x<n$, close contour to the right $\Rightarrow 0$
- $x>n$, close contour to the left $\Rightarrow$ integral $=1$ from residue at $s=0$.

$$\Rightarrow RHS = \sum_{n<x}\Lambda(n) = \psi(x)$$

Hence

$$\psi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{\overset{a=1}{a+i\infty}}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}ds = x - \underbrace{\frac{\zeta'(0)}{\zeta(0)}}_{=\log(2\pi)} - \sum_{\rho_i}\frac{x^{\rho_i}}{\rho_i}$$

series of increasing order of $|Im(\rho_i)|$

$\psi(x)$ for large x? Euler product formula $\Rightarrow$ no ρ_i with $Re(\rho_i)>1$.

Key result: $Re(\rho_i)<1 \Rightarrow \psi(x)\sim x$ as $x\to\infty$.

Consequence: $\pi(x) = (\#\text{ primes } <x) \sim \int_2^x \frac{1}{\log t}dt$ as $x\to\infty$.

Locations of zeros $\rho_1,\rho_2,\dots$ control error in above approximation.

$$\zeta(z) = 2^z\pi^{z-1}\sin\frac{\pi z}{2}\,\Gamma(1-z)\,\zeta(1-z).$$

gives information on $Re(z)<0$.

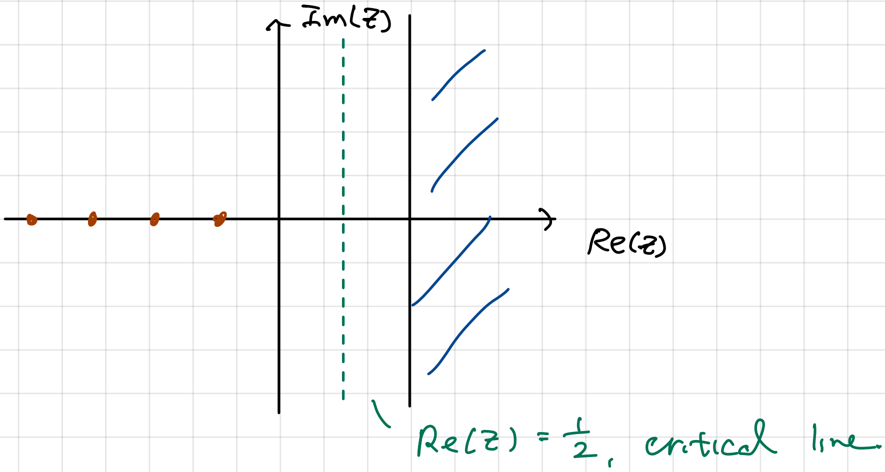
$$\Rightarrow \zeta(-2n) = 0, \quad n = 1, 2, 3, \ldots$$

($\because$ $\sin n\pi = 0$, $\Gamma(1+2n)$ finite)

$\zeta(0) \neq 0$ since zero of $\sin\frac{\pi z}{2}$ coincides with simple pole of $\Gamma(1-z)$

$z = -2n$ are trivial zeros of ζ. Are there non-trivial ones?

<u>Riemann hypothesis</u>: All non-trivial zeros of ζ lie on $\mathrm{Re}(z) = 1/2$.



Consider $$F(z) = \pi^{-z/2}\,\Gamma(z/2)\,\zeta(z).$$
$$= 2^z \pi^{z/2-1} \sin\frac{\pi z}{2}\,\Gamma(z/2)\,\Gamma(1-z)\,\zeta(1-z)$$
$$= \pi^{z/2-1}\,\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) = F(1-z)$$

Hence $$F\left(\tfrac{1}{2}+iy\right) = F\left(\tfrac{1}{2}-iy\right) = F\left(\overline{\tfrac{1}{2}+iy}\right)$$
$$\Rightarrow F\left(\tfrac{1}{2}+iy\right) \in \mathbb{R}.$$

Requires $F(\bar{z}) = \overline{F(z)}$ — follows from properties of $\pi^{-z/2}$, $\Gamma(z/2)$, $\zeta(z)$, etc.

It is known that:

- Hardy (1915): $\exists$ infinitely many zeros on critical line
- Hadamard: $\nexists$ no zeros on boundary of critical strip.

(i) Calculations of zeros on critical lines — look for sign changes in $F(\frac{1}{2}+iy)$.

(ii) Calculate Cauchy Formula:

$$\frac{1}{2\pi i}\oint_C \frac{\zeta'(z)}{\zeta(z)}\,dz = \#\text{zeros} - \#\text{poles}$$

Apply to C containing the strip $0 < Re(z) < 1$.

If no. of zeros from (i) matches no. from (ii), then all zeros lie on critical line.

Platt and Trudgian (2021): all zeros $\beta + i\gamma$ with $0 \le |\gamma| \le 3\times 10^{12}$ have $\beta = \frac{1}{2}$.

Consequence of Riemann hypothesis:

- $\pi(x) = \int_2^x \frac{dt}{\log t} + O(\sqrt{x}\log x)$.

Zeta f^n — many generalisation, e.g. $\sum \lambda^{-s}$, & eval. of e.g. elliptic operator — applications in theoretical physics.

3. Solution of DEs by transform methods

3.1 Sol^n to ODE by Integral Representation

Seek general sol^n to linear ODE as integral in $\mathbb{C}$.

Example Airy's eq^n

$$W''(x) + x\,W(x) = 0 \qquad (1)$$

Applications: wave propagation, e.g. optics, as an approximation to

$$W''(x) + f(x)\,W(x) = 0.$$

with $f(x)$ changing in sign.

We could try to solve using FT.

$$\tilde{W}(k) = \int_{-\infty}^{\infty} e^{-ikx}W(x)\,dx, \qquad W(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{W}(k)\,dk$$

Deduce $\tilde{w}(k)$ and invert

$$w(x) = \frac{A}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{ik^3}{3} + ikx\right) dk.$$

This is one sol^n of (1). but (1) is 2nd order. So $\exists$ 2 LI sol^n.

But second sol^n is not bounded for large x so does not have FT.

Instead, consider (1) in complex plane, treating w as $\mathbb{C} \to \mathbb{C}$ f^n

– use z in place of x for indpt. variable.

Write $w(z)$ as

$$w(z) = \int_\gamma e^{zt} f(t)\, dt,$$

with $f(t)$ and γ to be determined

It will turn out that $f(t)$ is determined up to a const.,

but here will be multiple choice of γ, allowing 2 LI sol^{n}s.

Substitute representation of $w(z)$ into (1):

$$w''(z) = \frac{d^2}{dz^2} \int_\gamma e^{zt} f(t)\, dt = \int_\gamma t^2 e^{zt} f(t)\, dt$$

Hence $w'' + zw = 0$

$$\Rightarrow \int_\gamma (t^2 + z^2)\, e^{zt} f(t)\, dt = 0$$

Want $f(t)$ be z-indpt. IBP and get

$$z \int_\gamma e^{zt} f(t)\, dt = \int_\gamma \frac{d}{dt}\left(e^{zt} f(t)\right) - e^{zt} f'(t)\, dt$$

$$= \left[e^{zt} f(t)\right]_\gamma - \int_\gamma e^{zt} f'(t)\, dt$$

Hence,

$$\left[e^{zt} f(t)\right]_\gamma + \int_\gamma \left(t^2 f(t) - f'(t)\right) e^{zt}\, dt = 0.$$

We find $f(t)$, γ s.t. 1 and 2 are individually 0.

Note $[e^{zt} f(t)]_\gamma = 0$ if $f(t)$ analytic within γ and γ is closed.

(2) : $t^2 f(t) - f'(t) = 0 \Rightarrow f(t) = A e^{t^3/3}$.

(1) : Finding γ s.t. $[e^{zt} e^{t^3/3}]_\gamma = 0$.

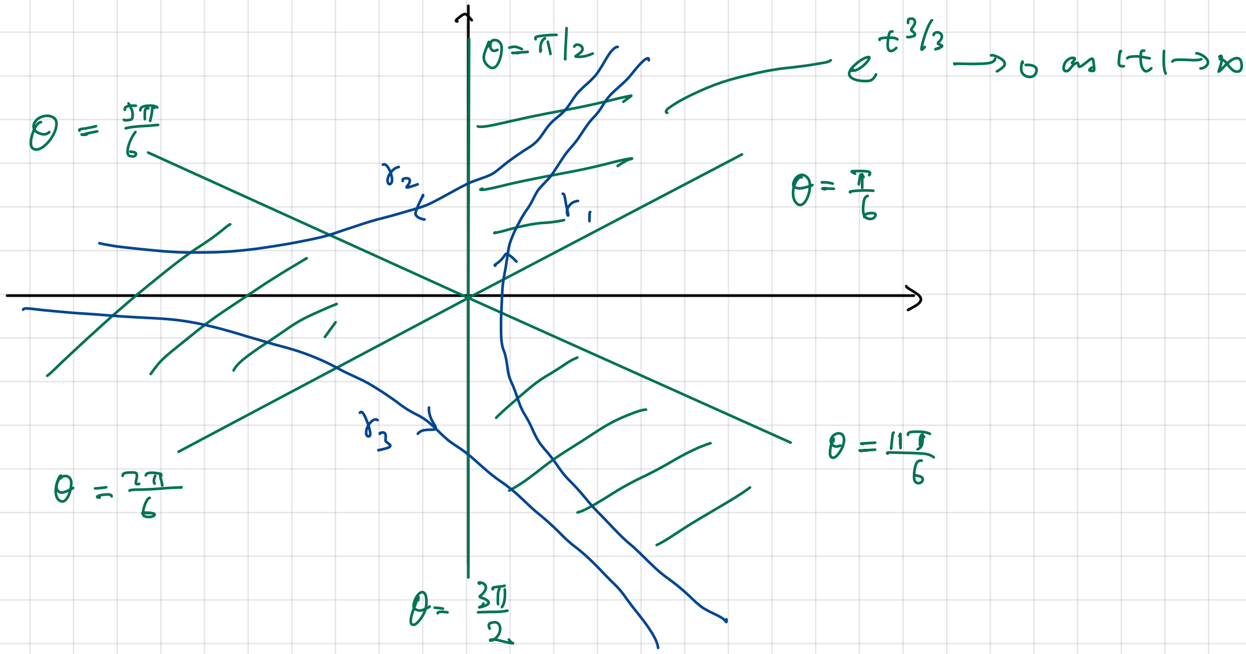
Note if γ closed, then

$$\oint_\gamma e^{zt} f(t)\, dt = \oint_\gamma e^{zt + t^3/3}\, dt = 0$$

not useful. Instead, seek $e^{zt + t^3/3} \to 0$.

Put $t = |t| e^{i\theta}$. For large $|t|$, have $|t^3/3| \gg |zt|$

$$e^{t^3/3} = e^{|t|^3/3\, e^{3i\theta/3}} \to 0 \quad \text{if} \quad \cos 3\theta < 0.$$



Any of $\gamma_1, \gamma_2, \gamma_3$ give required property that

$$[e^{t^3/3} e^{zt}]_\gamma = 0$$

Any two gives LI sol^n (note $\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} = 0$)

γ_1 can be determined to coincide with the imaginary axis.

$$t = iy, \quad w_1(z) = iA \int_{-\infty}^{\infty} e^{izy - iy^3/3}\, dy$$

Same as FT representation.

$$\Rightarrow \quad W_1(z) = A\left(i \int_{-\infty}^{\infty} \cos(y^3/3 - zy)\,dy + \underbrace{\cancel{\int_{-\infty}^{\infty} \sin(y^3/3 - zy)\,dy}}_{\text{odd integrand}} \right)$$

Hence,

$$W_1(z) = A \int_{-\infty}^{\infty} \cos(y^3/3 - zy)\,dy$$

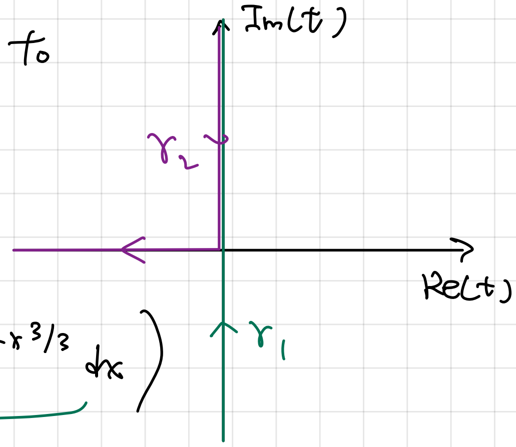
is a sol^n of (1), with $A = 1/2\pi$.

$$W_1(z) = Ai(z) \qquad (\text{Airy } f^n \text{ of 1st kind.})$$

For second sol^n, choose γ_2 and deform to

$t = iy, \quad \infty > y \geq 0$

$t = -x, \quad 0 \leq x < \infty.$



$$W_2(z) = A\left(i \int_{\infty}^{0} e^{izy - \frac{1}{3} i y^3}\,dy - \underbrace{\int_0^{\infty} e^{-zx - x^3/3}\,dx}_{\text{unbounded as } z \to \infty \;\Rightarrow\; \text{no FT.}} \right)$$

$$\Rightarrow \quad W_2(z) = -A \int_0^{\infty} \sin\left(\frac{x^3}{3} - xz\right) + \exp(-zx - x^3/3)\,dx.$$

Set $A = \frac{1}{\pi}$, $W_2(z) = Bi(z)$, Airy f^n of 2^{nd} kind.

Integral representations can be used to extract useful info.

$$Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(y^3/3 - zy)\,dy$$

Hence

$$Ai(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(y^3/3)\,dy = 3^{-2/3} \frac{\sqrt{3}}{2} \Gamma\left(\tfrac{1}{3}\right).$$

$$Ai'(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y \sin(y^3/3 - zy)\,dy$$

$$\Rightarrow \quad Ai'(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y \sin(y^3/3)\,dy = \frac{\sqrt{3}}{2}\, 3^{-1/3}\, \Gamma\left(\tfrac{2}{3}\right).$$

Corresponding result for $Bi(z)$. Note that LI can be established from Wronskian $Ai(0)\,Bi'(0) - Ai'(0)\,Bi(0)$.

Example $zw'' - (1+z)w' + 2(1-z)w = 0.$

Seek sol^n of $w(z) = \int_\gamma e^{zt} f(t)\,dt.$

$$w'(z) = \int_\gamma t e^{zt} f(t)\,dt \qquad w''(t) = \int_\gamma t^2 e^{zt} f(t)\,dt$$

Hence $\int_\gamma \left(z(t^2 - t - 2) + (2-t) \right) e^{zt} f(t)\,dt = 0$

IBP $\Rightarrow$ $\left[(t^2 - t - 2) e^{zt} f(t) \right]_\gamma + \int_\gamma \left[(2-t) f(t) - \frac{d}{dt}\left((t^2 - t - 2) f(t) \right) \right] e^{zt}\,dt = 0$

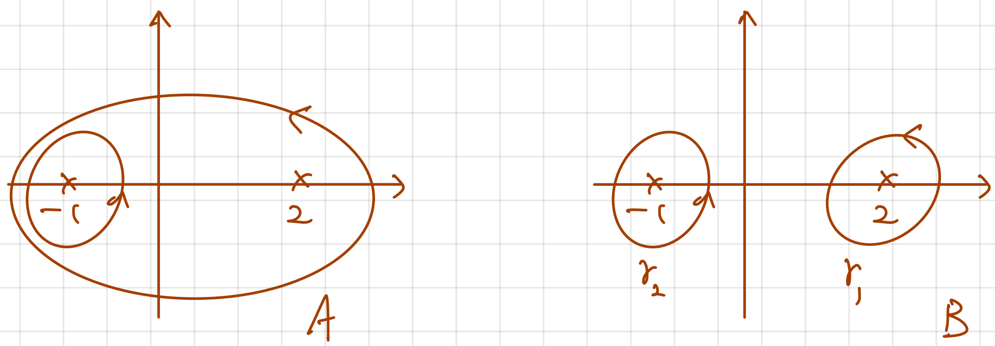
$$\Rightarrow \quad (2-t) f(t) - \frac{d}{dt}\left((t^2 - t - 2) f(t) \right) = 0$$

$$\Rightarrow \quad f = \frac{1}{(t+1)^2 (t-2)} \quad \text{up to a const.}$$

Now require $\left[\frac{(t^2 - t - 2) e^{zt}}{(t+1)^2 (t-2)} \right]_\gamma = \left[\frac{e^{zt}}{t+1} \right]_\gamma = 0$

Find γ s.t. above holds with $\int_\gamma \frac{e^{zt}}{(t+1)^2 (t-2)}\,dt \neq 0.$

Choose γ closed and encircling one or more poles.



Choose B.

$$\int_{\gamma_1} \frac{e^{zt}}{(t+1)^2 (t-2)}\,dt = 2\pi i \operatorname{Res}\left(\frac{e^{zt}}{(t+1)^2 (t-2)},\ t = 2 \right)$$
$$= \frac{2\pi i}{9} \cdot e^{2z}$$

$$\int_{\gamma_2} \frac{e^{zt}}{(t+1)^2 (t-2)}\,dt = 2\pi i \operatorname{Res}\left(\frac{e^{zt}}{(t+1)^2 (t-2)},\ t = -1 \right)$$
$$= -\frac{2\pi i}{9} e^{-z} (1 + 3z).$$

Omit the factors.

$$w_1(z) = e^{2z} \quad . \quad w_2(z) = (1+3z)e^{-z}$$

(Check LI using Wronskian)

Solving ODE using integral representations — general method for eqn of form

$$a(z)\, w'' + b(z)\, w' + c(z)\, w = 0$$

with a, b, c low-order poly. in z, the general approach is

$$w(z) = \int_\gamma k(z,t)\, f(t)\, dt$$

Substitute into eqn to determine $f(t)$ etc.

Various options for Kernel $k(z,t)$.

- e^{zt} Laplace
- $(z-t)^\sigma$ Euler
- t^z Mellen

Example (Euler Kernel)

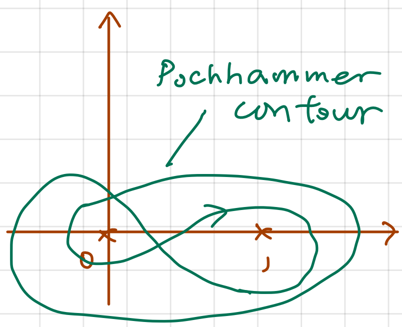
$$w(z) = \int_\gamma t^{a-c}(1-t)^{c-b-1}(t-z)^{-a}\, dt,$$

a, b, c const. is a sol^n of

$$z(1-z)w'' + (c-(a+b+1)z)w' - ab\,w = 0$$

(hypergeometric eqn) provided that

$$\left[t^{a-c+1}(1-t)^{c-b}(t-z)^{1-a}\right]_\gamma = 0.$$



Sometimes choice of contour depends on parameters in eqn, e.g. Ex 2 #6 — also Hermite eqn

$$w'' - 2zw' + 2\nu w = 0$$ choice depend on ν

3.2 Solving PDEs by integral transform.

Laplace transform: extension of FT. to f^ns that are non-zero only on semi-interval, e.g. $f(x)\begin{cases} =0 & x<0 \\ \neq 0 & x>0 \end{cases}$

Def^n (Laplace transform)

$$\hat{f}(p) = \int_0^\infty e^{-px} f(x)\, dx$$

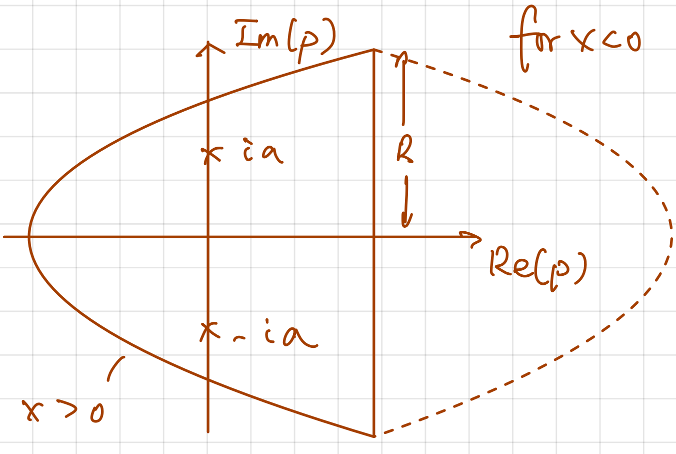
$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px} \hat{f}(p)\, dp.$$

where c fixed real const. Contour $c-i\infty$ to $c+i\infty$ is the Bromwich contour. Choose c s.t. contour is to the right of all sing. of $\hat{f}(p)$.

Rmk If $x<0$, can close contour to the right (recall Jordan's lemma). Obtain $f(x)=0$.

Example Inverse LT of $\hat{f}(p) = \frac{1}{p^2+a^2}$, $a \in \mathbb{R}$ const.

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{p^2+a^2} e^{px}\, dp.$$



$x<0$: close to right $\Rightarrow f(x)=0$

$x>0$: $f(x) = 2\pi i\left(\mathrm{Res}\left(\frac{e^{px}}{p^2+a^2}, p=ia\right) + \mathrm{Res}\left(\frac{e^{px}}{p^2+a^2}, p=-ia\right)\right)$

$$= \frac{e^{iax}}{2ia} + \frac{e^{-iax}}{-2ia} = \frac{\sin ax}{a}$$

$$\Rightarrow f(x) = \begin{cases} \frac{\sin ax}{a} & x>0 \\ 0 & x<0 \end{cases}$$

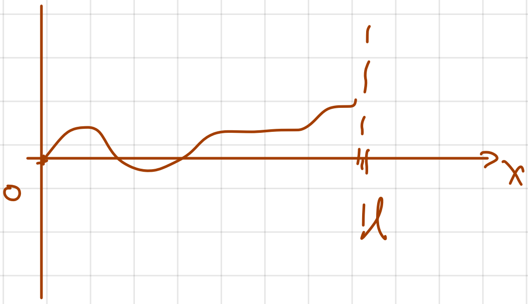
Exercise Verify $\int_0^\infty \frac{\sin ax}{a} e^{-px}\, dx = \frac{1}{p^2+a^2}$

Solving PDEs using LT.

Example (Waves on finite string)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 \le x \le l$$

$$\left.\begin{aligned} y(0,t) &= 0 \\ y(l,t) &= y_0(t) \end{aligned}\right\} t \ge 0$$



$$y(x,0) = y_t(x,0) = 0, \quad 0 \le x \le l.$$

Several different approaches available (e.g. Fourier Series).

IVP suggests LT as good option.

Take LT of eqn:

$$\hat{y}(x,p) = \int_0^\infty e^{-pt} y(x,t)\, dx$$

$$y(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\hat{y}(x,p)\, dp.$$

$$p^2 \hat{y}(x,p) \underbrace{- py(x,0) - y_t(x,0)}_{=0} = c^2 \frac{\partial^2 \hat{y}}{\partial x^2}(x,p)$$

$$\hat{y}(0,p) = 0, \quad \hat{y}(l,p) = \int_0^\infty e^{-pt} y_0(t)\, dt$$

Simplify $y_0(t) = \begin{cases} 0 & t \le 0 \\ y_0 t & t > 0 \end{cases}$, y_0 const.

$$\Rightarrow \hat{y}(0,p) = 0, \quad \hat{y}(l,p) = y_0/p.$$

Thus $\hat{y}(x,p) = A(p)\cosh\left(\frac{px}{c}\right) + B(p)\sinh\left(\frac{px}{c}\right).$

$$\hat{y}(0,p) = 0 \rightarrow A(p) = 0$$

Hence,

$$\hat{y}(x,p) = \frac{y_0}{p}\,\frac{\sinh(px/c)}{\sinh(pl/c)}.$$

Method 1 : explicit inversion

$$y(x,t) = \frac{y_0}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \underbrace{\frac{\sinh(px/c)}{\sinh(pl/c)}\,\frac{e^{pt}}{p}}_{F(x,t\,;p)}\,dp$$

$F(x,t\,;p)$ has simple poles at $p = \frac{\pi i c m}{L}$, $m \in \mathbb{Z}$.

For $\mathrm{Re}(p)$ large >0,
$$\frac{\sinh(px/c)}{\sinh(pl/c)} \sim e^{(x-l)p/c}.$$

Hence
$$F(x,t\,;p) \sim e^{p(t + \frac{x-l}{c})}.$$

Can close to right if $t + \frac{x-L}{c} < 0$. Can certainly close to right if $t<0$ (justifies taking $\alpha>0$).

$$\Rightarrow y(x,t) = 0 \quad \text{for } t<0.$$

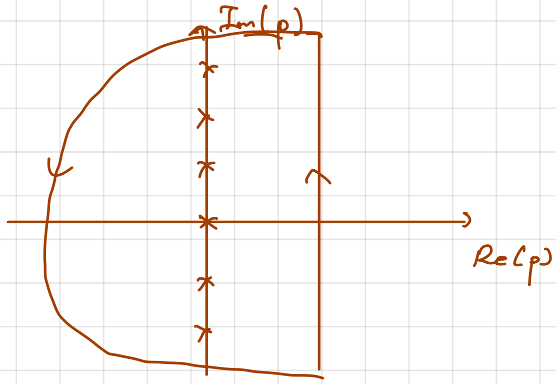
But can also close to right for $0<t<\frac{l-x}{c}$, i.e.

$$y(x,t) = 0 \quad \text{for } ct < L - x.$$

Sol^n 'switches on' for $ct > L-x$ (consistent with wave propagation).

For $\mathrm{Re}(p)$ large and <0, $\frac{\sinh(px/c)}{\sinh(pl/c)}e^{pt} \sim e^{(\frac{L}{c} - \frac{x}{c} - t)p}$, small as $\mathrm{Re}(p) \to -\infty$ if $L/c - x/c + t > 0$, i.e. $t > \frac{x-L}{c}$.

Then $y(x,t) = 2\pi i \cdot \sum$ residues.



residues at $\frac{\pi i m c}{l}$, $(m \neq 0)$,

$$\frac{c}{L}\,\frac{\sinh\left(\frac{\pi i m x}{l}\right)}{\cosh(\pi i m)}\, e^{i\pi m c t/l}\cdot\frac{l}{\pi i m c}$$

At $m=0$, residue $= x/l$.

Hence
$$y(x,t) = y_0\left(\frac{x}{l} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sin\frac{\pi m x}{l}\,\frac{e^{i\pi m c t/L}}{\pi m c}(-1)^m\right)$$
$$= y_0\left(\frac{x}{l} + 2\sum_{m=1}^{\infty} \sin\frac{\pi m x}{l}\,\cos\frac{m\pi c t}{l}\cdot\frac{(-1)^m}{\pi m c}\right)$$

Two forms of sol^n are consistent — FS cancels $\frac{x}{L}$ if t small.

<u>Method 2</u> :

$$\hat{y}(x,p) = \frac{y_0}{p}\,\frac{e^{xp/c} - e^{-xp/c}}{e^{lp/c} - e^{-lp/c}}$$

$$= \frac{y_0}{p}\,\frac{e^{xp/c} - e^{-xp/c}}{e^{pl/c}}\left(1 - e^{-2pl/c}\right)^{-1}$$

$$= \frac{y_0}{p}\,\frac{e^{xp/c} - e^{-xp/c}}{e^{pl/c}}\left(1 + e^{-2pl/c} + e^{-4pl/c} + \dots\right)$$

Bromwich contours in $\mathrm{Re}(p) > 0$ justifies expansion.

Each term is of the form $\frac{ke^{\alpha p}}{p}$, LT of $kH(t+\alpha)$ by shifting thm. Hence

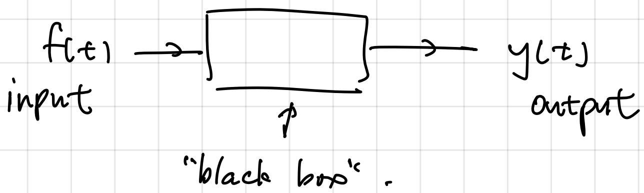
$$y(x,t) = y_0 \sum_{n=0}^{\infty} \underset{\longleftarrow}{H(ct + x - (2n+1)l)} - \underset{\longrightarrow}{H(ct - x - (2n+1)l)}.$$

Train of leftward and rightward propagating waves — appearing in $0 < x < l$ as repeated from different ends of string.

3.3 Causality and Stability

Linear input-output system, e.g. a linear ODE.

$y(t) = \mathcal{L} f(t)$,



Expected properties of $\mathcal{L}$

(1) <u>Causality</u> : If $f(t) = 0$ for $t < 0$, then $y(t) = 0$ for $t < 0$.

Output can't precede input.

(2) <u>Stability</u> : In rough terms, if $\exists R$ s.t. $f(t) = 0$ for $|t| \geq R$

i.e. $f(t)$ non-zero only for finite time, then $\mathcal{L}f(t) \to 0$ as $t \to \infty$.

Example Consider linear operator representing damped harmonic oscillator.

$$y'' + 2by' + (a^2+b^2)y = f(t).$$

$b > 0$

Specify $f(t) = 0$ for $t < 0$, $y(0) = y'(0) = 0$.

Take FT $\quad \tilde{y}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} y(t)\, dt.$

$$(i\omega)^2 \tilde{y} + 2i\omega b \tilde{y} + (a^2+b^2)\tilde{y} = \tilde{f}$$

Hence

$$\tilde{y} = \tilde{R}(\omega)\tilde{f}, \quad \tilde{R} = \frac{1}{(i\omega)^2 + 2i\omega b + (a^2+b^2)}$$

transfer f^n

$$\tilde{R}(\omega) = \frac{1}{2ia}\left(\frac{1}{i\omega + b - ia} - \frac{1}{i\omega + b + ia}\right).$$

$$R(t) = \begin{cases} \frac{1}{2ia}\left(e^{(-b+ia)t} - e^{(-b-ia)t}\right), & t > 0, \\ 0 & t < 0 \end{cases}$$

Hence, $\quad y(t) = \int_0^t R(t-\tau) f(\tau)$

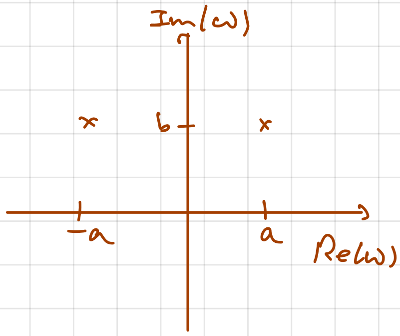
$$= \frac{1}{a}\int_0^t e^{-b(t-\tau)} \sin(a(t-\tau)) f(\tau)\, d\tau.$$

Hence, system is both causal and stable ($\because b > 0$).

Consider location of poles of $\tilde{R}(\omega)$.

$$\tilde{R}(\omega) = \frac{1}{(i\omega + b + ia)(i\omega + b - ia)}$$

$b > 0$, then poles of $\tilde{R}(\omega)$ are confined to upper half ω plane.



General property: causal / stable if no poles of $\tilde{R}(\omega)$ in lower half plane.

Compare with LT approach.

$$\hat{y}(s) = \int_0^\infty e^{-st} y(t)\, dt$$

ODE $\Rightarrow$ $s^2\hat{y} + 2bs\hat{y} + (a^2+b^2)\hat{y} = \hat{f}(s) + y'(0) + (s+2b)y(0).$

$$\Rightarrow \hat{y}(s) = \hat{R}(s)\hat{f}(s) = \frac{\hat{f}(s)}{s^2+2bs+a^2+b^2}$$

$$= \frac{1}{2ia}\left(\frac{1}{s+b-ia} - \frac{1}{s+b+ia}\right)\hat{f}(s).$$

Poles at $-b \pm ia$ in left half plane if $b>0$.

Stable and causal if no poles in right-hand plane.

Extend to more general systems:

- FT casuality / stability if $\tilde{R}(\omega)$ has no poles in lower half plane
- LT 〃 $\hat{R}(s)$ 〃 right.

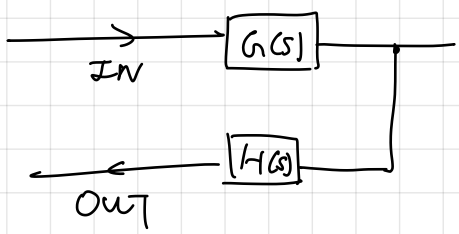
3.4 Nyquist Stability Criterion

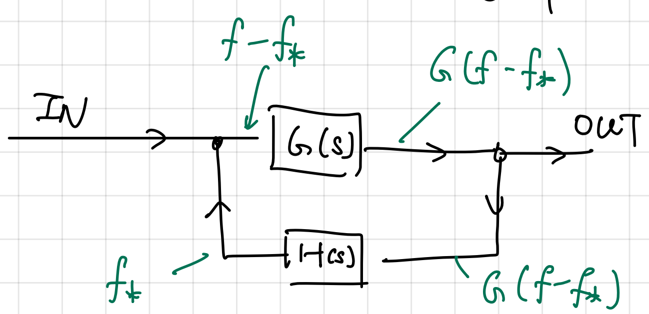
Nyquist stability criterion is a graphical technique for determining stability, e.g. used in control theory (mechanical / electrical systems)

Consider linear system – characterise by transfer f^n $G(s)$ (omit ^) with negative feedback process with transfer f^n $H(s)$.

Open loop transfer f^n $G(s)H(s)$.

If $G(s)H(s)$ has poles in RHP, then unstable.





(negative) feedback

$$\Rightarrow f_* = HG(f - f_*)$$

$$\Rightarrow f_* = \frac{HG}{1+HG}.$$

$$\Rightarrow f - f_* = \frac{1}{1+HG}, \qquad G(f-f_*) = \frac{G}{1+HG}.$$

Overall stability determined by poles of G and zeros of $1+HG$.

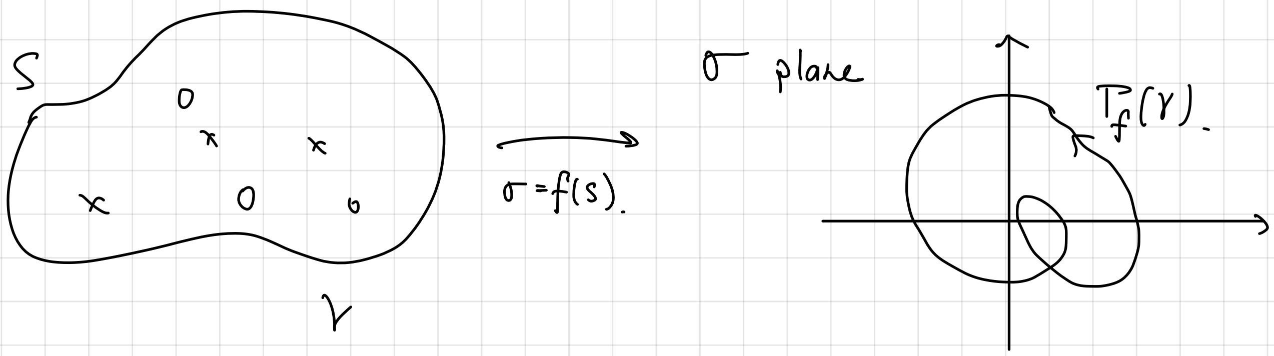
Zeros of $1+H(s)G(s)$ in RHP $\Rightarrow$ poles of $\frac{G}{1+HG}$ in RHP $\Rightarrow$ instab.

Finding zeros of $1+GH$ can be difficult.

Nyquist criteria determines no. of zeros in RHP. Uses Cauchy argument principle, recall: $f(s)$ meromorphic inside simple curve γ with no zeros/poles on γ, then

$$\frac{1}{2\pi i}\oint \frac{f'(s)}{f(s)}\,ds = Z - P \qquad \text{(anticlockwise)}$$

no. of zeros   no. of poles.



$$\frac{1}{2\pi i}\int_{\Gamma_f} \frac{d\sigma}{\sigma} = \#\text{ times that } \Gamma_f \text{ encircles origin}$$

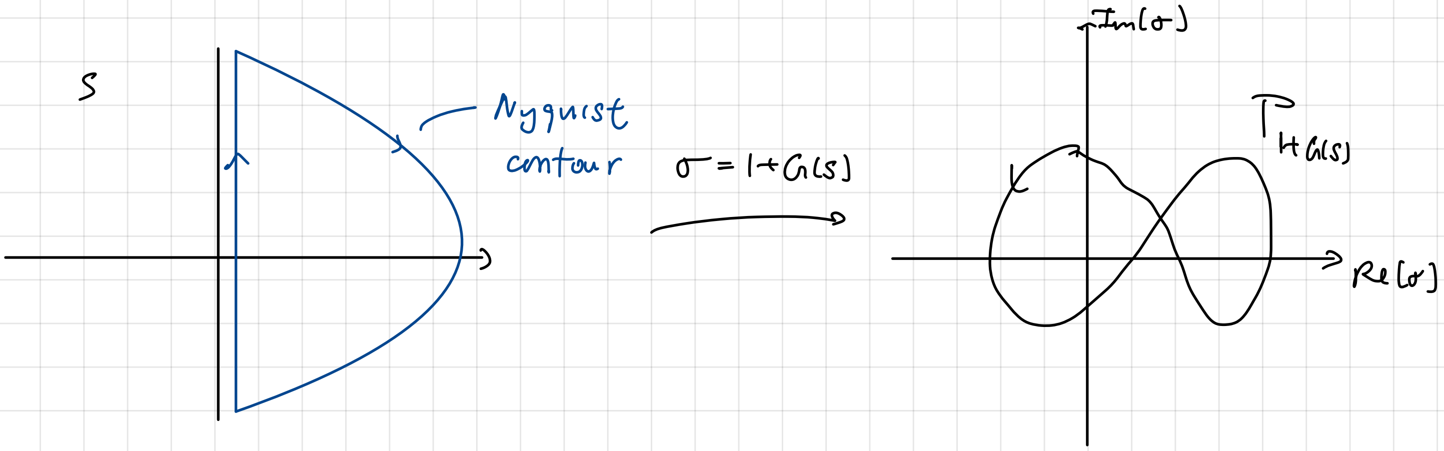
Hence, no. of times that Γ_f encircles origin $= N = Z - P$ (anticlockwise)

$= P - Z$ (clockwise).

For simplicity, consider unity feedback sys. $H(s) = 1$.

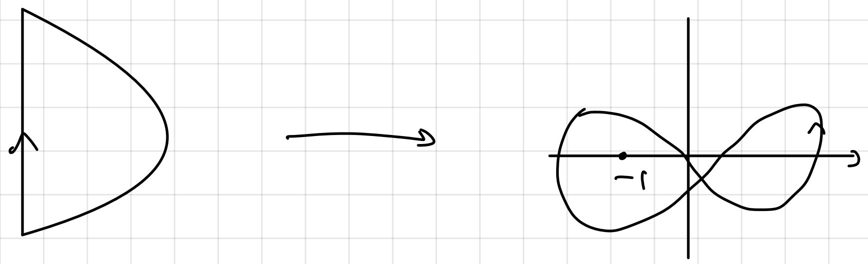
Closed loop transfer f^n $\frac{G(s)}{1+G(s)}$.

Nyquist contour



No. of encirclement of origin by $\Gamma_{1+G} = N = P - Z$ for $1+G(s)$ in RHP.

We can obtain the same info. by mapping according to G rather than $1+G$. Plot shifted to the left by -1, quantity of interest is no. of increments of -1.



No. of anticlockwise encirclements of -1 by $\Gamma_G = P - Z$ for $1+G(s)$ in RHP. Measure this quantity by measuring $G(i\omega)$. i.e. oscillatory input for varying frequency.

Example $G(s) = \frac{\alpha(s+3)}{s^2-4}$, $\alpha > 0$

Open loop system is unstable (pole at $s=2$).

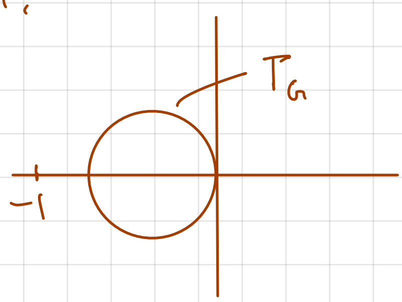
Closed loop transfer f^n is

$$\frac{\alpha(s+3)/s^2-4}{1-\alpha(s+3)/s^2-4}$$

Poles where $s^2-4+\alpha(s+3)=0$, hence $s=-\frac{\alpha}{2}\pm\sqrt{4-3\alpha+\frac{\alpha^2}{4}}$.

Confined to LHP if $\alpha > 4/3$.

For $\alpha=1$,

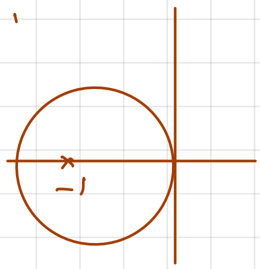


Counting $P-Z \Rightarrow P-Z=0$

$\Rightarrow \underbrace{\#\text{ poles in RHP}}_{=1} = \#\text{ zeros in RHP}$.

$\Rightarrow \#\text{ zeros} = 1 \Rightarrow$ unstab.

For $\alpha=2$,



$P-Z=1$

$\Rightarrow \#\text{ zeros} = 0$ in RHP $\Rightarrow$ Stable.

Exercise Show that image of imaginary axis under mapping $G(s)$ is a circle.

4. Second Order ODEs in the complex plane

Start with IA DE applied to complex variables.

$$W''(z)+p(z)W'(z)+q(z)W(z)=0 \qquad (\dagger)$$

with $p(z), q(z)$ meromorphic in $\mathbb{C}$.

4.1 Classification of singular points

(a) $z=z_0$ is an ordinary point at $(\dagger)$ if p, q both analytic at $z=z_0$. O/W, $z=z_0$ is a singular point.

(b) If $z=z_0$ sing., but $(z-z_0)p(z)$ and $(z-z_0)^2 q(z)$ are analytic, then z_0 is a regular sing. point.

(c) o/w, z_0 is irregular sing point.

For linear ODEs, sing. of sol^n are indpt of IC and determined by $p(z)$, $q(z)$.

Note not true for NL eqn., e.g. $w' + w^2 = 0$ $\Rightarrow w(z) = \dfrac{w(0)}{1+w(0)z}$, sing determined by IC.

Def^n of (a), (b) can be extended to $z=\infty$ by writing $z = 1/t$.

(+) becomes

$$\frac{d^2w}{dt^2} + \left(\frac{2}{t} - \frac{p(t^{-1})}{t^2}\right)\frac{dw}{dt} + \frac{q(t^{-1})}{t^4}\, w = 0.$$

Hence $z=\infty$ $(t=0)$ if OP if $\frac{2}{t} - \frac{p(t^{-1})}{t^2}$ and $\frac{q(t^{-1})}{t^4}$ analytic at $t=0$.

If $\frac{2}{t} - \frac{p(t^{-1})}{t^2} = -f(t)$, hence $p(z) = \frac{2}{z} + \frac{1}{z^2} f\left(\frac{1}{z}\right)$ with f analytic as $z\to\infty$, and $\frac{q^{-1}(t)}{t^4} = g(t)$, hence $q(z) = \frac{g(1/z)}{z^4}$ with $g(1/z)$ analytic as $z\to\infty$.

Correspondingly, $z=\infty$ $(t=0)$ is RSP if

$$t\left(\frac{2}{t} - \frac{p(t^{-1})}{t^2}\right) = 2 - p(z)z.$$

is analytic as $z\to\infty$, i.e. $p(z) = \frac{2}{z} + \frac{1}{z} f(1/z)$. with f analytic as $z\to\infty$, and

$$\frac{q(t^{-1})}{t^2} = z^2 q(z)$$

analytic. i.e. $q(z) = \frac{1}{z^2} g(1/z)$ with g analytic as $z\to\infty$.

Example $$\frac{d^2w}{dz^2} - \frac{2z}{1-z^2}\frac{dw}{dz} + \frac{n(n+1)w}{1-z^2} = 0.$$

$z = \pm 1$ RSP. (verify).

$z = \infty$: $\quad q(z) = \dfrac{n(n+1)}{z^2} + O\left(\dfrac{1}{z^4}\right)$, $\quad p(z) = -\dfrac{2z}{1-z^2} = \dfrac{2}{z}\left(1 + O\left(\dfrac{1}{z^2}\right)\right)$.

Hence ∞ is RSP.

For large z, $$\frac{d^2w}{dz^2} + \frac{2}{z}\frac{dw}{dz} - \frac{n(n+1)}{z^2}w \simeq 0$$

Compatable with $w \simeq z^\sigma$ for large σ.

(indicates RSP but more precise conditions will be used later).

4.2 Indicial Equation

Aim: to find LI sol^{n}s to $(\dagger)$. Consider RSP at $z=0$. Seek power series sol^n.

$$w(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n.$$

in nbd of $z=0$.

Using criteria for RSP:

$$p(z) = \sum_{m=-1}^{\infty} p_m z^m, \quad q(z) = \sum_{m=-2}^{\infty} q_m z^m.$$

Sub. into $(\dagger)$.

$$\sum_{n=0}^{\infty}\Big(z^{n+\sigma-2} a_n (n+\sigma)(n+\sigma-1) + \Big(\sum_{m=-1}^{\infty} p_m z^m\Big)(n+\sigma)a_n z^{n+\sigma-1}$$
$$+ \Big(\sum_{m=-2}^{\infty} q_m z^m\Big) a_n z^{n+\sigma}\Big) = 0.$$

Coeff. of lowest power $(\sigma - 2)$:

$$a_0\left(\sigma(\sigma-1) + \sigma p_{-1} + q_{-2}\right) = 0 \qquad \text{(indicial eqn).}$$

Roots σ_1, σ_2 and hence generate two sol^n.

$$w_1(z) = z^{\sigma_1}(a_0 + a_1 z + a_2 z^2 + \dots), \quad w_2(z) = z^{\sigma_2}(b_0 + b_1 z + \dots)$$

σ_1, σ_2 are exponents at $z=0$. (sol^n LI if $\sigma_1 \neq \sigma_2$)

Now consider RSP at ∞. Define $t = 1/z$.

Write
$$W(z) = t^{\sigma} \sum_{n=0}^{\infty} a_n t^n = z^{-\sigma} \sum_{n=0}^{\infty} a_n z^{-n}.$$

Note sign convention on σ.

For RSP at ∞, require $p(z) = \frac{2}{z} + \frac{1}{z} \underbrace{f(1/z)}$, $q(z) = \frac{1}{z^2} \underbrace{g(1/z)}$,

analytic as argument $\to 0$

ie.
$$p = \sum_{m=1}^{\infty} p_m z^{-m}, \quad q = \sum_{m=2}^{\infty} q_m z^{-m}.$$

Sub into (+)

$$z^{-\sigma} \sum_{n=0}^{\infty} a_n (n+\sigma)(n+\sigma+1) z^{-n-2}$$
$$- z^{-\sigma} \sum_{n=0}^{\infty} a_n (n+\sigma) z^{-n-1} \left(\sum_{m=1}^{\infty} p_m z^{-m} \right) + z^{-\sigma} \sum_{n=0}^{\infty} a_n z^{-n} \left(\sum_{m=2}^{\infty} q_m z^{-m} \right) = 0.$$

Identify terms with highest power of z, ie. $z^{-\sigma-2}$.

$$a_0 \sigma(\sigma+1) - a_0 \sigma p_1 + a_0 q_2 = 0$$

with roots σ_1, σ_2.

Summary: RSP at $z=z_0$ if $p(z) = \sum_{n=-1}^{\infty} p_n (z-z_0)^n$, $q(z) = \sum_{n=-2}^{\infty} q_n (z-z_0)^2$,

with
$$W(z) = (z-z_0)^{\sigma} \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

with σ root of
$$\sigma^2 + (p_{-1} - 1)\sigma + q_{-2} = 0.$$

RSP at $z=\infty$ if $p(z) = \sum_{m=1}^{\infty} p_m z^{-m}$, $q(z) = \sum_{m=2}^{\infty} q_m z^{-m}$, with
$$w(z) = z^{-\sigma} \sum_{n=0}^{\infty} a_n z^{-n}$$

with σ root of
$$\sigma^2 + (1-p_1)\sigma + q_2 = 0.$$

4.3 Sol^n of second order ODEs near OPs at RSPs

Example $\quad W'' + \underbrace{\frac{4}{z-1}}_{p(z)} W' + \underbrace{\frac{2}{(z-1)^2}}_{q(z)} W = 0.$

$p(z)$, $q(z)$ analytic in $|z|<1$.

Near $z=0$ (OP), two series sol^n

$$W_1(z) = \sum_{n=0}^{\infty} z^n, \quad W_2(z) = \sum_{n\geq 0}^{\infty} n z^n.$$

Note $W_1(0) \neq 0$, $W_2(0) = 0$, both analytic in $|z|<1$.

Near $z=1$ (RSP), write

$$W(z) = (w-1)^{\sigma} \sum_{n=0}^{\infty} a_n (z-1)^n$$

$\sigma(\sigma-1) + 4\sigma + 2 = 0 \implies \sigma_1 = -1, \ \sigma_2 = -2.$

Hence, 2 LI sol^n

$$W_3(z) = \frac{1}{1-z}, \quad W_4(z) = \frac{1}{(1-z)^2}.$$

Using analytic continuation, can't integrate through sing. point, but we can integrate around and use analyticity. May have to accept multivaluedness.

Note $\sigma_1 - \sigma_2 \in \mathbb{Z}$ – to be discussed later.

$z \to \infty$: Write $z-1 \approx z$, $\quad W'' + \frac{4}{z} W + \frac{2}{z^2} W = 0$

Indicial eqn: $\quad \sigma(\sigma+1) - 4\sigma + 2 = 0 \implies \sigma = -2, -1$

$$W_5(z) = z^{-1} \sum_{n=0}^{\infty} z^{-n}, \quad W_6(z) = z^{-2} \sum_{n=0}^{\infty} (n+1) z^{-n}$$

$\exists$ relations between (W_1, W_2), (W_3, W_4), (W_5, W_6).

General properties of sol^n near OP and RSP of

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0.$$

with p, q meromorphic on $\mathbb{C}$.

Sol^n near OP:

Thm If $p(z), q(z)$ analytic in disc $|z|<R$, then $\exists$ two LI sol^n $w_1(z), w_2(z)$ s.t.

(1) $w_1(z)$, $w_2(z)$ analytic on $|z|<R$. (may be analytic in part of $|z|\geq R$)

(2) $w_1(0)\neq 0$, $w_2(0)=0$., $w_2'(0)=0$. (roots of indicial eqn are $\sigma=0$, $\sigma=1$)

Pf: Sub Taylor series for $p(z), q(z)$. Write down series for $w(z)=\sum_{n=0}^{\infty} a_n z^n$, equate coeff. of z^n and check cgs. $\square$

Example sol^n is analytic in region that includes sing. of $p(z), q(z)$.

$$w'' - \frac{2}{z-1}w' + \frac{2}{(z-1)^2}w = 0.$$

has sol^n $w = A(z-1) + B(z-1)^2$. (entire).

(2) is eqv. to saying that $\exists$ sol^n satisfying $w(0)$, $w'(0)$ given.

Sol^n near RSP: indicial eqn has roots σ_1, σ_2 with $w(z)\sim z^{\sigma_1}, z^{\sigma_2}$. LI if $\sigma_1\neq\sigma_2$.

If $\sigma_1=\sigma_2$, then $w(z)\sim z^{\sigma_1}$ and $w(z)\sim(\log z)z^{\sigma_1}$ LI.

Thm If $z=0$ is RSP, $\exists$ 2 LI sol^n $w_1(z), w_2(z)$ with

(1) $\sigma_1-\sigma_2\notin\mathbb{Z}$: $w_1(z) = z^{\sigma_1}u_1(z)$, $w_2(z) = z^{\sigma_2}u_2(z)$, with u_1, u_2 analytic in $|z|<R$ with $u_1(0), u_2(0)\neq 0$

(2) $\sigma_1 = \sigma_2$: $\quad w_1(z) = z^{\sigma_1} u_1(z), \quad w_2(z) = w_1(z)\log z + z^{\sigma_1} u_2(z),$

with u_1, u_2 analytic in $|z|<R$, $u_1(0), u_2(0) \neq 0$.

(3) $\sigma_1 \neq \sigma_2$, $\sigma_1 - \sigma_2 \in \mathbb{Z}$. (say $\sigma_1 > \sigma_2$).

$$w_1(z) = z^{\sigma_1} u_1(z), \qquad w_2(z) = C w_1(z) \log z + z^{\sigma_2} u_2(z),$$

with u_1, u_2 analytic in $|z|<R$ and $u_1(0), u_2(0) \neq 0$.

Pf: (1) Pose power sol^n with $w(z) = z^{\sigma} \sum_{n=0}^{\infty} a_n z^n$. Sub. into (†).

$$\sum_{n=0}^{\infty} a_n (n+\sigma)(n+\sigma-1) z^{n-2+\sigma} + \sum_{n=0}^{\infty} a_n (n+\sigma) z^{n-1+\sigma} \left(\sum_{m=-1}^{\infty} p_m z^m\right)$$

$$+ \sum_{n=0}^{\infty} a_n z^{n+\sigma} \left(\sum_{m=-2}^{\infty} q_m z^m\right) = 0$$

Then coeff. of $z^{\sigma-2}$:

$$a_0 \sigma(\sigma-1) + a_0 \sigma p_{-1} + a_0 q_{-2} = 0 = a_0 F(\sigma)$$

$F(\sigma) = 0$ determines σ_1, σ_2, then higher power : $z^{\sigma+r-2}$,

$$a_r F(\sigma+r) = -\sum_{m=0}^{r-1} a_m \left((m+\sigma) p_{r-m-1} + q_{r-m-2}\right).$$

Recurrence relation for a_r in terms of a_m $(m<n)$ (Straightforward unless $\sigma_1 - \sigma_2 \in \mathbb{Z}$)

(2), (3) include $\log z$ term., This will generate 2 LI sol^n. □

RSP at ∞ : sol^n will be of the form

$$w_1(z) = z^{-\sigma_1} u_1(1/z) + z^{-\sigma_2} u_2(1/z).$$

(Case 1 – similar, similar generalisation for (2), (3)).

Example (Bessel's eqn) $w'' + \frac{1}{z}w' + \left(1 - \frac{\gamma^2}{z^2}\right)w = 0$.

$z=0$ RSP. Indicial eqn:

$$\sigma(\sigma-1) + \sigma - \gamma^2 = \sigma^2 - \gamma^2 = 0$$

Hence, $w = z^\sigma \sum_{m=0}^{\infty} a_n z^n$

$$\Rightarrow z^\sigma \left(\sum_{n=0}^{\infty} a_n(n+\sigma)(n+\sigma-1) z^{n-2} + a_n (n+\sigma) z^{n-2} - \gamma^2 a_n z^{n-2} + a_n z^n \right) = 0.$$

$\sigma = \pm\gamma$ from $n=0$

$n=1$: $\quad a_1\left((\sigma+1)^2 - \gamma^2\right) = 0$

$n=r,\ r\geq 2$ $\quad a_r\left((r+\sigma)^2 - \gamma^2\right) = -a_{r-2}$

e.g. $\gamma=0$ $(\sigma_1 = \sigma_2)$ $\quad a_n = -\frac{a_{n-2}}{n^2} \Rightarrow a_{2n} = \frac{(-1)^n}{2^{2n}(n!)^2}$.

$$w_1(z) = 1 - \frac{1}{4}z^2 + \frac{1}{64}z^4 + \dots = J_0(z) \quad \text{Bessel } f^n \text{ of 1st kind.}$$

Second sol^n is $w_2(z) = J_0(z)\log z + \sum_{n=0}^{\infty} b_n z^n$.

- $\gamma \neq$ integer / half integer — two power series
- $\gamma =$ " ~ Need log term.

4.4 Fuchsian equations

An ODE of form

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0 \quad (\dagger)$$

is called Fuchsian if it has at most 3 RSPs.

0 RSP

Claim $\nexists$ ODE $(\dagger)$ with no RSP in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Pf: For such an eqn $p(z), q(z)$ analytic $\forall z \in \mathbb{C}$, but $z=\infty$ is an

OP, hence $p(z) = \frac{2}{z} + O\left(\frac{1}{z^2}\right)$ as $z \to \infty$, hence $p(z) \to 0$ as $z \to \infty$
$\Rightarrow$ p bounded. hence by Liouville thm, p const., hence $p(z) \equiv 0$
(so can't go as $2/z$ as $z \to \infty$) $\cancel{\#}$ $\square$

1 RSP

WLOG, $z=0$ is RSP. Then all other points including $z=\infty$ are ordinary points

$\Rightarrow z\,p(z) = P(z)$, $z^2 = q(z) = Q(z)$. Both P, Q analytic in $\mathbb{C}$.

$z=\infty$ OP $\Rightarrow$ $p(z) = \frac{2}{z} + \frac{1}{z^2} f(1/z)$, $q(x) = \frac{1}{z^4}\, g(1/z)$ with f, g analytic at origin.

Hence, $P(z) = 2 + \frac{1}{z} f(1/z)$, $Q(z) = \frac{1}{z^2}\, g(1/z)$.

Hence P, Q bounded and hence const. $\Rightarrow$ $P(z) = 2$, $Q(z) = 0$.

Hence, the only possible form is

$$w'' + \frac{2}{z} w' = 0$$

with sol^n $w(z) = \alpha + \beta/z$, α, β const. Exponents: $0, -1$.

Before moving to 2, 3 RSPs, note that a Mobius transformation $\mathcal{M}: z \to t$ will send $(\dagger)$ to another eqn of the same general form (with different p, q).

RSPs are moved and according to $z_0 \mapsto \mathcal{M} z_0$, with exponents unchanged.

Illustration : Apply to 1 RSP case with RSP at $z=z_0$.

$$w'' + \frac{2}{z-z_0} w' = 0$$

Solve : $w = \alpha + \beta/(z-z_0)$. Now apply Mobius trans^n $u = \frac{1}{z-z_0}$, $\frac{d}{dz} = -u^2 \frac{d}{du}$

$$\Rightarrow u^2 \frac{d}{du}\left(u^2 \frac{dw}{du}\right) - 2u^3 \frac{dw}{du} = \frac{d^2 w}{du^2} = 0$$

Sol^n: $W = A + Bu = Au^{-0} + B(u^{-1})^{-1} = A + B/(z-z_0)$.

Preservation of exponents follows from local behaviour near RSP, won't be affected by presence of other RSP.

Exercise Simple example with single RSP works $\forall \sigma_1, \sigma_2$.

Also, note that Mobius trans^n $\mathcal{M}: z \to t$, then $(\dagger) \to$ another 2nd ODE of form $(\dagger)$ with different $p(z), q(z)$.

2 RSP

WLOG assume RSP at 0, 1 and all other points including ∞ OPs.

Write $p(z) = \frac{1+A}{z} + \frac{1+B}{z-1} + P(z)$, with $P(z)$ entire, A, B const.

$z=\infty$ OP $\Rightarrow p(z) = \frac{2}{z} + O(\frac{1}{z^2})$. hence $P(z) = 0$. $A+B = 0$

Correspondingly, $q = \frac{Cz+D}{z^2} + \frac{Ez+F}{(z-1)^2} + Q_2(z)$, $Q_2(z)$ entire

$$= \frac{Q_3(z)}{z^2(z-1)^2} + Q_2(z), \quad Q_3 \text{ poly, deg} \leq 3.$$

$z=\infty$ OP $\Rightarrow z^4 q(z)$ bounded as $z \to \infty \Rightarrow Q_2(z) = 0$, $Q_3(z) = Q$ const.

Thus DE has form

$$w'' + \left(\frac{1+A}{z} + \frac{1-A}{z-1}\right) w' + \frac{Qw}{z^2(z-1)^2} = 0$$

Near $z=0$, $w'' + \frac{1+A}{z} w' + \frac{Qw}{z^2} = 0$, hence indicial eqn

$$\sigma^2 + A\sigma + Q = 0 \qquad (I)$$

Near $z=1$, $w'' + \frac{1-A}{z-1} w' + \frac{Qw}{(z-1)^2} = 0$, hence ind. eqn.

$$\Rightarrow \sigma^2 - A\sigma + Q = 0 \qquad (II)$$

Now illustrate effect of Mobius trans^n, e.g. $t = 1/z$: $0 \to \infty$, $1 \to 1$ RSP at $t=1$, $t=\infty$. Compare exponents.

3 RSP

The Papperitz eqn (Riemann P eqn), have 8 parameters.

(i) 3 pos. of RSPs

(ii) 6 exponents (2 at each RSP)

(iii) 1 constraint on exponents — exponents must sum to 1.

Suppose that RSPs are at $z = a, b, c \in \mathbb{C}$, distinct, and all other points, including ∞, are OPs.

$p(z)$ must have form $p(z) = \frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} + \underset{\text{entire}}{P(z)}$

($\alpha + \alpha'$, etc. defined, and α, α' not defined individually)

$z = \infty$ is OP, then $p(z) = \frac{2}{z} + O(\frac{1}{z^2}) \Rightarrow P(z) = 0$ (Liouville).

$z\,p(z) \to 2$ as $z \to \infty$ $\Rightarrow$ $1-\alpha-\alpha' + 1-\beta-\beta' + 1-\gamma-\gamma' = 2$

$$\Rightarrow \alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

$q(z)$ has form $q(z) = \frac{K_a z + L_a}{(z-a)^2} + \frac{K_b z + L_b}{(z-b)^2} + \frac{K_c z + L_c}{(z-c)^2} + \underset{\text{entire}}{Q_2(z)}$

$$= \frac{Q_1(z)}{(z-a)^2(z-b)^2(z-c)^2} + Q_2(z).$$

with $Q_1(z)$ poly, deg ≤ 5.

$z=\infty$ is OP, $q(z) = \frac{1}{z^4} g(1/z)$ with $g(t)$ analytic as $t \to 0$.

$z^4 q(z)$ bounded as $z \to \infty$, hence $Q_2(z) = 0$, and $Q_1(z)$ is at most quadratic.

Hence, expressing using partial fractions,

$$q(z) = \frac{1}{(z-a)(z-b)(z-c)} \left[\frac{q_a}{z-a} + \frac{q_b}{z-b} + \frac{q_c}{z-c} \right]$$

Now write (WLOG a, b, c distinct).

$$q_a = \alpha\alpha'(a-b)(a-c), \quad q_b = \beta\beta'(b-a)(b-c), \quad q_c = \gamma\gamma'(c-a)(c-b).$$

Hence,

$$w'' + \left(\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} \right) w'$$

$$- \frac{(b-c)(c-a)(a-b)}{(z-a)(z-b)(z-c)} \left[\frac{\alpha\alpha'}{(z-a)(b-c)} + \frac{\beta\beta'}{(z-b)(c-a)} + \frac{\gamma\gamma'}{(z-c)(a-b)} \right] w = 0 \quad (\dagger\dagger)$$

with $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$. This is the Papperitz eqn.

Near $z=a$, $(\dagger\dagger)$ reduces to

$$w'' + \frac{1-\alpha-\alpha'}{z-a} w' + \frac{\alpha\alpha'}{(z-a)^2} w \simeq 0.$$

Hence ind. eqn:

$$\sigma^2 - (\alpha+\alpha')\sigma + \alpha\alpha' = 0$$

$\Rightarrow$ exponents $\sigma = \alpha, \alpha'$ are exponents at $z=a$.

Similarly, β, β' exponents at $z=b$, γ, γ' exponents at $z=c$.

Form of $(\dagger\dagger)$ preserved by Mobius trans^n as argued previously.

Exercise $t = 1/z$, $a \mapsto 1/a$, $b \mapsto 1/b$, $c \mapsto 1/c$, $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ preserved.

Papperitz eqn (††) can be written using Papperitz symbol.

$$w(z) = P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

for "$w(z)$ is a sol^n of (††)".

4.5 Hypergeometric eqn

Mobius maps shifts RSPs of linear ODE w/o changing exponents.

If $M: (a, b, c, z) \mapsto (a', b', c', z')$

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a' & b' & c' & \\ \alpha & \beta & \gamma & z' \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

Consider multiplying sol^n of P eqn by $\left(\frac{z-a}{z-b}\right)^k$

$$W(z) = \left(\frac{z-a}{z-b}\right)^k w(z).$$

Certainly $W(z)$ is sol^n of ODE of standard form (†).

Recall that OPs and RSPs distinguished by local behaviour of sol^n.

At any OP $z=z_0$, 2 LI sol^n $w_1(z)$, $w_2(z)$ with $w_1(0) \neq 0$,

$w_2(0)=0$ but $w_2'(z) \neq 0$. w_1, w_2 analytic in nbd of z_0

So $\left(\frac{c-a}{z-b}\right)^k$ non-zero and analytic at $z=z_0$. Property unchanged by multiplication $\Rightarrow$ OP $\to$ OP.

Near RSP at $z=a$: $w(z) = A(z-a)^{\alpha} u_1(z) + B(z-a)^{\alpha'} u_2(z)$ with u_1, u_2 analytic in nbd of $z=a$ and $u_1(a) \neq 0$, $u_2(a) \neq 0$.

Hence, $W(z) = A(z-a)^{\alpha+k} U_1(z) + B(z-a)^{\alpha'+k} U_2(z)$, with U_1, U_2 analytic in nbd of $z=a$, $U_1(a) \neq 0$, $U_2(a) \neq 0$.

Hence eqn for $W(z)$ has RSP at $z=a$ w/ exponents $\alpha+k$ and $\alpha'+k$.
Similarly, eqn for $W(z)$ has RSP at $z=b$ with exponents $\beta-k$, $\beta'-k$, and has RSP at $z=c$ with exponents unchanged.

$$W = \left(\frac{z-a}{z-b}\right)^k P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a & b & c & \\ \alpha+k & \beta-k & \gamma & z \\ \alpha'+k & \beta'-k & \gamma' & \end{Bmatrix}$$

Note sum of exponents remains 1.

Consider RSP at ∞, say $b=\infty$, and

$$w = P\begin{Bmatrix} a & \infty & c & \\ \alpha & \beta & \gamma, & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

Consider $W = (z-a)^k w$.

As $z\to\infty$, $w = Az^{-\beta}u_1(1/z) + Bz^{-\beta'}u_2(1/z)$, u_1, u_2 analytic and non-zero as $z\to\infty$. Hence,

$$W = (z-a)^k w = Az^{k-\beta}U_1(1/z) + Bz^{k-\beta'}U_2(1/z).$$

With $U_1(1/z)$ and $U_2(1/z)$ analytic as $z\to\infty$. Hence,

$$(z-a)^k P\begin{Bmatrix} a & \infty & c & \\ \alpha & \beta & \gamma, & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a & \infty & c & \\ \alpha+k & \beta-k & \gamma & z \\ \alpha'+k & \beta'-k & \gamma' & \end{Bmatrix}.$$

Can apply following procedure to general P eqn

(1) move RSPs to $(0,1,\infty)$ by Mobius transⁿ.

(2) Shift exponents so that $\alpha=\beta=0$.

giving

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \gamma+\alpha+\beta & z \\ \alpha'-\alpha & \beta'-\beta & \gamma'+\alpha+\beta & \end{Bmatrix}$$

Now relabel $\gamma+\alpha+\beta = A$, $\gamma'+\alpha+\beta = B$, $\beta'-\beta = C-A-B$. Hence

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & A & z \\ 1-C & C-A-B & B & \end{matrix}\right\}.$$

6 param $\to$ 3 param., representing sol^n of

$$w'' + \left(\frac{C}{z} + \frac{1+A+B-C}{z-1}\right) w' + \frac{AB}{z(z-1)} = 0 \qquad (*)$$

the <u>hypergeometric eqn.</u>

One of exponents at $z=0$ is 0, hence there is a sol^n $w(z)$ which is analytic and non-zero at $z=0$. WLOG $w(0)=1$.

This sol^n is the <u>hypergeometric f^n</u> $F(A,B;C,z)$.

Properties:

- $F(A,B;C,0)=1$
- $F(A,B;C,z) = F(B,A;C,z)$

<u>Claim</u>: $F(A,B;C,z) = \sum_{n\geq 0}^{\infty} \frac{(A)_n (B)_n}{(C)_n\, n!} z^n$,

where $(X)_n := X(X+1)\cdots(X+n-1)$ is the Pochhammer symbol. Series converges for $|z|<1$. (note RSP at $z=1$)

<u>Pf</u>: Let $F(A,B;C,z) = \sum_{n\geq 0} a_n z^n$. Can show

$$a_{n+1} = \frac{(A+n)(B+n)}{(C+n)(n+1)} a_n$$

by comparing coeff. $\square$

<u>Note</u> Potential problem if C is negative integer.

First few terms: $F(A,B;C,z) = 1 + \frac{AB}{C} z + \frac{A(A+1)B(B+1)}{2C(C+1)} z^2 + \cdots$

Examples • $(1-z)^n = F(-n, 1; 1, z)$,

• $\log(1-z) = z\, F(1, 1; 2, z)$

• $e^z = \lim_{B \to 0} F(1, B; 1, z/B)$.

Integral representation of HGF:

$$F(A, B; C, z) = \frac{\Gamma(C)}{\Gamma(A)\Gamma(C-B)} \int_0^1 t^{B-1}(1-t)^{C-B-1}(1-tz)^{-A}\,dt$$

for $Re(C) > Re(B) > 0$ and $|z| < 1$ (needed for cgs of integral)

2nd sol^n near $z=0$ $(\sigma_2 = 1-C)$ is $w_2(z) = z^{1-C} g(z)$, with $g(z)$ analytic and $g(0)=1$. Aim to express $w_2(z)$ in F.

$w_2(z)$ is a sol^n to $(*)$, hence by exponential shifting, $w_2(z)$ is sol^n

$$z^{C-1} P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & A & z \\ 1-C & C-A-B & B & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ C-1 & 0 & A-C+1 & z \\ 0 & C-A-B & B-C+1 & \end{Bmatrix}$$

$$= P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & A-C+1 & z \\ C-1 & C-A-B & B-C+1 & \end{Bmatrix}$$

$$= P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & A' & z \\ 1-C' & C'-A'-B' & B' & \end{Bmatrix}$$

where $C' = 2-C$, $A' = A-C+1$, $B' = B-C+1$. Hence

$$z^{C-1} w_2(z) = F(A', B'; C', z)$$

$$\Rightarrow w_2(z) = z^{1-C} F(A', B'; C', z)$$

$$= z^{1-C} F(A-C-1, B-C+1; 2-C, z)$$

(potential problem if $C \in \mathbb{Z}_{>0}$, so avoid $C \in \mathbb{Z}$)

Sol^n near $z=\infty$

Aim to write the sol^n in terms of HGF.

Sol^n of form $W_a(z) = z^{-a} U_a(z)$, $W_b(z) = z^{-b} U_b(z)$, with $U_a(1/z)$, $U_b(1/z)$ analytic as $1/z \to 0$.

W_a is sol^n of $P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\}$, then $U_a(z) = z^a W_a(z)$ is sol^n of

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ a & 0 & 0 & z \\ 1-c+a & c-a-b & b-a & \end{matrix}\right\} = P\left\{\begin{matrix} \infty & 1 & 0 & \\ a & 0 & 0 & \frac{1}{z} \\ 1-c+a & c-a-b & b-a & \end{matrix}\right\}$$

$$= P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & \frac{1}{z} \\ b-a & c-a-b & 1-c+a & \end{matrix}\right\}$$

$$= P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a' & \frac{1}{z} \\ 1-c' & c'-a'-b' & b' & \end{matrix}\right\}$$

Where $a' = a$, $b' = 1-c+a$, $c' = 1+a-b$.

$U_a(z)$ analytic at $t = 1/z = 0$, so if $U_a(\infty) = 1$, then

$$W_a(z) = z^{-a} F(a, 1-c+a; 1+a-b, z^{-1})$$

Similarly,

$$W_b(z) = z^{-b} F(b, 1-c+b; 1+b-a, z^{-1})$$

Note possibility of multivalued f^n – above statement apply to principle branch

Furthermore, since $W_a(z)$, $W_b(z)$ LI, we can write

$$F(a,b;c,z) = A z^{-a} F(a, 1-c+a; 1+a-b, z^{-1}) + B z^{-b} F(b, 1-c+b; 1+b-a, z^{-1})$$

Where A, B can be determined e.g. from integral representations.

Example (Legendre eqn)

$$(1-z^2)w'' - 2zw' + \left[n(n+1) - \frac{m^2}{1-z^2}\right] w = 0$$

(e.g. arises in sol^n of Laplace eqn in sph. polars with ϕ variation $\propto e^{im\phi}$ and $z = \cos\theta$.)

In standard notation,

$$p(z) = -\frac{2z}{1-z^2}, \qquad q(z) = \frac{n(n+1)}{1-z^2} - \frac{m^2}{(1-z^2)^2}.$$

$z = \pm 1$, $z = \infty$ RSPs, all other points are OPs. Hence Papperitz eqn and we can determine sol^n as linear combination of HGFs.

Near $z=1$: $\quad w'' + \frac{1}{z-1} w' - \frac{m^2}{4(z-1)^2} w \approx 0$. Ind. eqn:

$$\sigma^2 - \frac{m^2}{4} = 0 \Rightarrow \sigma_{1,2} = \pm m/2$$

Near $z=-1$: as $z=1$ by symmetry

$z=\infty$: $\quad w'' + \frac{2}{z} w' - \frac{n(n+1)}{z^2} w = 0$. Ind. eqn:

$$\sigma(\sigma+1) - 2\sigma - n(n+1) = 0 \Rightarrow \sigma = -n, n+1.$$

Hence P symbol for Legendre eqn is $P\begin{Bmatrix} 1 & -1 & \infty & \\ m/2 & -m/2 & -n & z \\ -m/2 & m/2 & n+1 & \end{Bmatrix}$.

Now $\quad P\begin{Bmatrix} 1 & -1 & \infty & \\ m/2 & -m/2 & -n & z \\ -m/2 & m/2 & n+1 & \end{Bmatrix} \Leftrightarrow \frac{(1+z)^{m/2}}{(1-z)^{m/2}} P\begin{Bmatrix} 1 & -1 & \infty & \\ 0 & 0 & -n & z \\ m & -m & n+1 & \end{Bmatrix}$.

and mobius trans^n $t = \frac{1-z}{2}$ has $-1 \mapsto 1$, $1 \mapsto 0$, $\infty \mapsto \infty$. Hence

$$\Leftrightarrow \frac{(1+z)^{m/2}}{(1-z)^{m/2}} \underbrace{P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & -n & \frac{1-z}{2} \\ m & -m & n+1 & \end{Bmatrix}}_{a=-n,\ b=n+1,\ c=1-m}$$

So (if $m \notin \mathbb{Z}$), 2 LI sol^n written as

$$w_1(z) = \left(\frac{1+z}{1-z}\right)^{m/2} F\left(-n, n+1 ; 1-m, \frac{1-z}{2}\right),$$

$$w_2(z) = (1-z^2)^{m/2} F\left(m-n, n+m-1 ; 1+m, \frac{1-z}{2}\right).$$

Case of $m \in \mathbb{Z}$

Restriction above to $m \notin \mathbb{Z}$ because of need for $\log z$ term.

WLOG take $m \leq 0$, then $w_1(z) = (1-z)^{-m/2}(1+z)^{m/2} F(\ldots)$ remains as sol^n.

$w_2(z) = (1-z)^{m/2}(1+z)^{m/2} F(\ldots)$ may need to be replaced by a sol^n with log form.

If in addition, $n \in \mathbb{Z}$, then the series for

$$F\left(m-n, n+m-1 ; 1+m, \frac{1+z}{2}\right)$$

will terminate, giving a poly. If $-n+m > m$ or $n+m+1 < n$, (i.e. unless $n=0$ or -1), series terminates before need for $\log z$

– no replacement of $w_2(z)$ req'd.

4.6 Monodromy

Monodromy = "running around sing.", how mathematical objects behave as they are followed around a sing.

Consider eqn $w'' + pw' + qw = 0$ $(*)$ in nbd of RSP at $z=0$, with no other SP in $|z| < R$. Ind. eqn has roots σ_1, σ_2.

If $\sigma_1 - \sigma_2 \notin \mathbb{Z}$, then have sol^n

$$W_1(z) = z^{\sigma_1} u_1(z), \qquad W_2(z) = z^{\sigma_2} u_2(z),$$

u_1, u_2 analytic in $|z|<R$.

If $\sigma_1 - \sigma_2 \in \mathbb{Z}$, sol^n

$$W_1(z) = z^{\sigma_1} u_1(z), \quad W_2(z) = \log z \cdot W_1(z) + z^{\sigma_2} u_2(z),$$

u_1, u_2 analytic in $|z|<R$.

Define $\mathcal{D} = \{z : 0 < |z| < R\}$, the largest punctured disc about $z=0$, that does not contain other sing. Let $z_0 \in \mathcal{D}$ be an OP, hence $\exists$ open disc $\mathcal{D}_0$ centred at z_0, in which $(*)$ has analytic sol^n which forms a vec. space of dim 2.

LI sol^n: $W_1(z), W_2(z)$. We can analytically continue the f^n W_1, W_2 along the curve C via a seq. of discs back to $\mathcal{D}_0$. Hence, $\hat{W}_1(z) = W_1(e^{2\pi i} z)$ and $\hat{W}_2(z) = W_2(e^{2\pi i} z)$, where $\hat{W}_1, \hat{W}_2$ sol^n to $(*)$. So $\exists$ non-sing. matrix M s.t.

$$\begin{pmatrix} \hat{W}_1(z) \\ \hat{W}_2(z) \end{pmatrix} = M \begin{pmatrix} W_1(z) \\ W_2(z) \end{pmatrix}$$

M is the <u>monodromy matrix</u>. Note if 0 were OP, then $M = I$.

M can be brought into JNF

$$\text{(i)} \begin{pmatrix} e^{2\pi i \sigma_1} & 0 \\ 0 & e^{2\pi i \sigma_2} \end{pmatrix} \quad \text{or} \quad \text{(ii)} \begin{pmatrix} e^{2\pi i \sigma_1} & 0 \\ 1 & e^{2\pi i \sigma_1} \end{pmatrix}$$

For both cases $W_1(e^{2\pi i} z) = e^{2\pi i \sigma_1} W_1(z)$. Write $g(z) = z^{-\sigma_1} W_1(z)$, then $g(e^{2\pi i} z) = (e^{2\pi i} z)^{-\sigma_1} W_1(e^{2\pi i} z) = z^{-\sigma_1} W_1(z)$. Hence $g(z)$ is single valued and can be written as Taylor Series,

$$w_1(z) = z^{\sigma_1} \sum_{n=0}^{\infty} a_n z^n.$$

For (i), same argument applies to $w_2(z) = z^{\sigma_2} \sum_{n=0}^{\infty} b_n z^n$.

Consider (ii):

$$w_2(e^{2\pi i} z) = w_1(z) + e^{2\pi i \sigma_1} w_2(z)$$

Let $f(z) := z^{-\sigma_1} w_2(z) - \frac{e^{-2\pi i \sigma_1}}{2\pi i} z^{-\sigma_1} \log z \; w_1(z)$. Then

$$\begin{aligned} f(e^{2\pi i} z) &= e^{-2\pi i \sigma_1} z^{-\sigma_1} w_2(e^{2\pi i} z) \\ &\quad - \frac{e^{-2\pi i \sigma_1}}{2\pi i} e^{-2\pi i \sigma_1} z^{-\sigma_1} (\log z + 2\pi i) \, w_1(e^{2\pi i} z) \\ &= e^{-2\pi i \sigma_1} z^{-\sigma_1} \left(w_1(z) + e^{2\pi i \sigma_1} w_2(z) \right) \\ &\quad - \frac{e^{-2\pi i \sigma_1}}{2\pi i} e^{-2\pi i \sigma_1} z^{-\sigma_1} (\log z + 2\pi i) \, e^{2\pi i \sigma_1} w_1(z). \\ &= z^{-\sigma_1} w_2(z) - \frac{e^{-2\pi i \sigma_1}}{2\pi i} z^{-\sigma_1} \log z \; w_1(z) = f(z). \end{aligned}$$

Hence f single valued and hence

$$w_2(z) = w_1 \log z + z^{\sigma_1} \sum_{n=0}^{\infty} b_n z^n.$$

i.e. the existence/need for the log. term in the sol^n is associated with non-diag. of M. More generally, M encodes important info. about the structure of ODE and its sol^n.