Electrodynamics

1. Relativistic Formulation

Maxwell equations for fields,

$$\underline{E}(\underline{x},t)\ ,\quad \underline{B}(\underline{x},t)\quad ,\ \underline{x}\in\mathbb{R}^3,\ t\in\mathbb{R}.$$

In presence of sources, $\rho(\underline{x},t)$: charge density,
$\underline{J}(\underline{x},t)$: current density

$$\nabla\cdot\underline{E} = \rho/\varepsilon_0 \qquad (M1) \qquad\qquad \nabla\cdot\underline{B} = 0 \qquad (M3)$$

$$\nabla\times\underline{E} = -\frac{\partial\underline{B}}{\partial t} \qquad (M2) \qquad\qquad \nabla\times\underline{B} = \mu_0\underline{J} + \mu_0\varepsilon_0\frac{\partial\underline{E}}{\partial t} \qquad (M4)$$

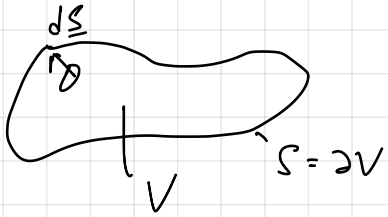
where $c = (\mu_0\varepsilon_0)^{-1/2}$. (1)

(M1), (M4) ⇒ local charge conservation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\underline{J} = 0 \qquad (2)$$

The total charge in region $V\subset\mathbb{R}^3$ is

$$Q(t) := \int_V d^3\underline{x}\ \rho(\underline{x},t)$$



$$\Rightarrow \frac{dQ}{dt} = \frac{d}{dt}\int_V d^3x\ \rho(\underline{x},t)$$

$$= \int_V d^3\underline{x}\ \frac{\partial\rho}{\partial t}$$

$$= -\int_V d^3\underline{x}\ \nabla\cdot\underline{J} = -\int_S d\underline{S}\cdot\underline{J}\ , \qquad (3)$$

i.e. the rate of change of charge in V = total flux through ∂V.

Convinent to express $\underline{E}$ and $\underline{B}$ in terms of potentials.

$\phi(\underline{x},t)$: scaler potential, $\underline{A}(\underline{x},t)$: vector potential

as

$$\underline{E} = -\frac{\partial\underline{A}}{\partial t} - \nabla\phi\ ,\quad \underline{B} = \nabla\times\underline{A} \qquad (4).$$

We see that $\underline{E}$ and $\underline{B}$ are invariant under gauge transformations

$$\phi \mapsto \phi - \frac{\partial \chi}{\partial t}, \quad \underline{A} \mapsto \underline{A} + \nabla \chi \qquad (5)$$

where $\chi = \chi(\underline{x}, t)$.

Relativistic Electrodynamics

Work in Minkowski spacetime $\mathbb{R}^{3,1}$ with metric tensor

$$\eta_{\mu\nu} := \text{diag}(-1, +1, +1, +1),$$

Spacetime coordinate $x^\mu = (ct, \underline{x})$.

Metric is invariant under Lorentz transformation (LT).

$$x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\Lambda^\mu{}_\alpha \eta^{\alpha\beta} \Lambda^\nu{}_\beta = \eta^{\mu\nu}$$

Potential form 4-vector field

$$A^\mu(x) = A^\mu(\underline{x}, t) := \left(\phi(\underline{x}, t)/c, \underline{A}(\underline{x}, t)\right)$$

$$A^\mu \xrightarrow{\Lambda} A'^\mu := \Lambda^\mu{}_\nu A^\nu$$

The gauge transformation in (5) becomes

$$A^\mu \longrightarrow A^\mu + \partial^\mu \chi$$

Remark $\partial^\mu := \eta^{\mu\nu} \partial_\nu$, $\partial_\nu := \partial / \partial x^\nu$

Sources also form a 4-vector field

$$J^\mu(x) = J^\mu(\underline{x}, t) := \left(c\rho(\underline{x}, t), \underline{J}(\underline{x}, t)\right)$$

$$J^\mu \xrightarrow{\Lambda} J'^\mu := \Lambda^\mu{}_\nu J^\nu$$

Charge conservation: $\partial_\mu J^\mu = 0$ (2')

Electric and magnetic field live in Maxwell Field Strength tensor

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu, \qquad (11)$$

which is a second rank antisymmetric tensor (2,0).

Antisymmetry : $F^{\mu\nu} = - F^{\nu\mu}$

tensor : $F^{\mu\nu} \xrightarrow{\Lambda} F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$ (12)

$$F^{\mu\nu} \overset{(4),(9),(11)}{=} \left(\begin{array}{c|cccc} \mu\backslash\nu & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & E_x/c & E_y/c & E_z/c \\ 1 & -E_x/c & 0 & B_z & -B_y \\ 2 & -E_y/c & -B_z & 0 & B_x \\ 3 & -E_z/c & B_y & -B_x & 0 \end{array} \right)$$

We have $A^\mu \to A^\mu + \partial^\mu \chi$, so

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \longrightarrow F^{\mu\nu} + (\underbrace{\partial^\mu \partial^\nu - \partial^\nu \partial^\mu}_{=0})\chi$$

sym of partial derivatives

Lorentz group has two invariant tensors :

- metric tensor: $\eta_{\mu\nu}$
- alternating tensor : $\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} \pm 1 & (\mu,\nu,\rho,\sigma) = \omega(1,2,3,4), \ \omega \in S_4, \ \mathrm{sgn}(\omega) = \pm 1 \\ 0 & \text{o/w} \end{cases}$

We have

$$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = |\underline{B}|^2 - |\underline{E}|^2/c^2 .$$

$${}^*F^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Lorentz invariant scalers

and

$$-\frac{1}{4} F_{\mu\nu} {}^*F^{\mu\nu} = \frac{1}{c} \underline{E} \cdot \underline{B}$$

We can rewrite Maxwell equations.

$$(M1),\ (M4) \Rightarrow \qquad \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \qquad (\mu 1)$$

$$(M2),\ (M3) \Rightarrow \qquad \partial_\mu {}^*F^{\mu\nu} = 0 \qquad (\mu 2)$$

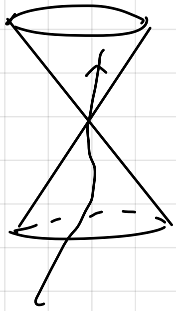
Check that $(\mu 2)$ satisfied if we have $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$,

$$\partial_\mu {}^*F^{\mu\nu} = \frac{1}{2}\partial_\mu \varepsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\underbrace{\partial_\mu\partial_\rho A_\sigma}_{=0} - \underbrace{\partial_\mu\partial_\sigma A_\rho}_{=0}) = 0.$$

Relativistic Dynamics

Particle in 3D Minkowski space $\mathbb{R}^{3,1}$ follows worldline parameterised by proper time τ.



$$x^\mu = x^\mu(\tau) = (ct(\tau),\ \underline{x}(\tau))$$

In sine inertial frame,

$$3\text{-velocity}: \quad \underline{v} = \frac{d\underline{x}}{dt}.$$

Coordinate time t related to proper time

$$\frac{dt}{d\tau} = \gamma(v) = \frac{1}{\sqrt{1-v^2/c^2}}, \qquad v = |\underline{v}|.$$

Covariant description involves 4-velocity.

$$U^\mu := \frac{dx^\mu}{d\tau} = \left(c\frac{dt}{d\tau},\ \frac{dt}{d\tau}\frac{d\underline{x}}{dt}\right) = (\gamma c,\ \gamma\underline{v}).$$

Then $U_\mu U^\mu = -c^2$.

Conserved 4-momentum

$$P^\mu = mU^\mu = (E,\ \underline{p})$$

relativistic energy

relativistic momentum

where $E = m\gamma c^2$, $\underline{p} = m\gamma\underline{v}$.

Lorentz force law

Relativistic particle of rest mass m, charge q.

$$\underset{\text{4-force}}{F^{\mu}} := \frac{dp^{\mu}}{d\tau} = q F^{\mu\nu} U_{\nu} \qquad (14)$$

- spatial components:

$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B}) . \qquad (15)$$

- time component:

$$\underset{\text{rate of change of energy}}{\frac{dE}{dt}} = q\, \underline{E} \cdot \underset{\text{rate of work done due to } \underline{E}}{\underline{v}} \qquad (16)$$

Action Principle

Non-relativistic mechanics

Particle of mass m moving in $\mathbb{R}^3$ $\underline{x} = \underline{x}(t) \in \mathbb{R}^3$ in potential field $V(\underline{x})$.

$$m\ddot{\underline{x}} = -\nabla V .$$

E.O.M. can be derived from principle of least action.

Def^n (Lagrangian) $L(t) = L(\underline{x}(t), \dot{\underline{x}}(t)) = \frac{1}{2} m \dot{\underline{x}}^2 - V(\underline{x}) .$ $(*)$

Def^n (Action) $S_{t_i, t_f}[\underline{x}(t)] := \int_{t_i}^{t_f} L(\underline{x}, \dot{\underline{x}}, t)\, dt$

We vary the path $\underline{x}(t) \mapsto \underline{x}(t) + \delta\underline{x}(t)$. with fixed endpoints, $\delta\underline{x}(t_i) = \delta\underline{x}(t_f) = 0$. The path of the particle is determined by requiring that

$$\delta S_{t_i, t_f} = 0 .$$

This is the principle of least action (Hamilton).

Calculus of variations $\Rightarrow$ Euler-Lagrange eqn

$$\frac{\partial L}{\partial \underline{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}}\right) = 0$$

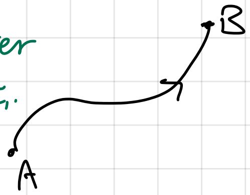
$$\overset{(*)}{\Rightarrow} \qquad m\,\ddot{\underline{x}} = -\nabla V$$

Action of Relativistic Particle

Particle, rest mass m, follows worldline in $\mathbb{R}^{3,1}$

$$x^\mu = x^\mu(\lambda).$$

← worldline parameter increases with τ. $\frac{d\lambda}{d\tau} > 0$.



from event A to event B.

Natural candidate for action is length of path.

$$S_{A,B} = \mathcal{N}\int_A^B (-ds^2)^{1/2} \quad , \quad ds^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu .$$

Then

$$S_{A,B} = \mathcal{N}\int_{\lambda(A)}^{\lambda(B)} \left(-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda$$

special choice: $\lambda = \tau$, then $ds^2 = -c^2 d\tau^2$; and

$$S_{A,B} = \mathcal{N}c\int_{\tau(A)}^{\tau(B)} d\tau = \mathcal{N}c\,\left(\tau(B) - \tau(A)\right).$$

Introduce coords $x^\mu(\tau) = (ct(\tau), \underline{x}(\tau))$, with

$$\frac{dt}{d\tau} = \gamma(v) = \frac{1}{\sqrt{1-|\dot{\underline{x}}|^2/c^2}} .$$

and

$$S_{A,B} = \mathcal{N}c\int_{\tau(A)}^{\tau(B)} d\tau = \mathcal{N}c\int_{t(A)}^{t(B)} \frac{1}{\gamma(v)}\,dt$$

$$= \mathcal{N}c\int_{t(A)}^{t(B)} \sqrt{1-|\dot{\underline{x}}|^2/c^2}\,dt .$$

$$= \int_{t(A)}^{t(B)} L(t)\,dt ,$$

where $L(t) = \mathcal{N}c\,\sqrt{1-|\dot{\underline{x}}|^2/c^2}$.

Taylor expanding,

$$L(t) = \mathcal{N}c\left(1 - \frac{1}{2}\frac{|\dot{x}|^2}{c^2} + \mathcal{O}\left(\frac{|\dot{x}|^4}{c^4}\right)\right)$$

Compare to $L_{NR} = KE - PE = \frac{1}{2}m|\dot{x}|^2$, choose $\mathcal{N} = -mc$, then

$$L(t) \approx -mc^2 + \frac{1}{2}m|\dot{x}|^2 + \mathcal{O}(|\dot{x}|^4/c^4)$$

Hence, in general,

$$S_{A,B} = -mc\int_{\lambda(A)}^{\lambda(B)}\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2}d\lambda \qquad (17)$$

Couply EM must be

- Lorentz invariant
- Gauge invariant
- Recover correct eqn of motion

The total action is

$$S_T = S_{A,B} + \tilde{S}_{A,B},$$

where $\tilde{S}_{A,B} = q\int_{\lambda(A)}^{\lambda(B)} A_\mu(x(\lambda))\frac{dx^\mu}{d\lambda}\,d\lambda.$

Consider gauge transformation (5') $A_\mu(\underset{\sim}{x}) \mapsto A_\mu(x) + \partial_\mu\chi(x)$.

$\forall\,\chi(x)$, $\chi(x_A) = \chi(x_B) = 0$, then $\tilde{S}_{A,B} \longrightarrow \tilde{S}_{A,B} + \delta\tilde{S}_{A,B}$,

$$\delta\tilde{S} = q\int_{\lambda_A}^{\lambda_B}\partial_\mu\chi\frac{dx^\mu}{d\lambda}\,d\lambda.$$

$$= q\left[\chi(x_A) - \chi(x_B)\right] = 0.$$

So it is guage invariant.

For $$L_T = -mc\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} + qA_\mu(x(\lambda))\frac{dx^\mu}{d\lambda} \qquad (19)$$

E-L:

$$\frac{\partial L_T}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial L_T}{\partial(dx^\mu/d\lambda)}\right) = 0$$

by varying $$S_T = \int_{\lambda(A)}^{\lambda(B)} L_T[x^\mu, \frac{dx^\mu}{d\lambda}]\, d\lambda.$$

Note also $$\frac{\partial L_T}{\partial x^\mu} = q\, \partial_\mu A_\nu(x) \frac{dx^\mu}{d\lambda}$$

$$\frac{\partial L_T}{\partial(dx^\mu/d\lambda)} = \frac{mc\,\eta_{\mu\nu} \frac{dx^\nu}{d\lambda}}{\left(-\eta_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda}\right)^{1/2}} + q A_\mu(x)$$

and consider $\frac{dx^\rho}{d\lambda} = \frac{dx^\rho}{d\tau} \cdot \frac{d\tau}{d\lambda} = u^\rho \frac{d\tau}{d\lambda}$, we have

$$\left(-\eta_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda}\right)^{1/2} = c \frac{d\tau}{d\lambda}$$

using $u_\rho u^\rho = -c^2$. Then

$$\frac{\partial L_T}{\partial(dx^\mu/d\lambda)} = m\eta_{\mu\nu} \frac{dx^\nu}{d\tau} + q A_\mu(x(\lambda)),$$

which gives E-L:

$$m\eta_{\mu\nu} \frac{d}{d\lambda}\left(\frac{dx^\nu}{d\tau}\right) = q\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\frac{dx^\nu}{d\lambda} = q F_{\mu\nu} \frac{dx^\nu}{d\lambda}.$$

Specialising to use $\lambda = \tau$ (proper time).

$$\Rightarrow \quad m\eta_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} = q F_{\mu\nu} \frac{dx^\nu}{d\tau}$$

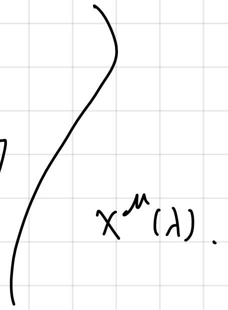
$$\Rightarrow \quad \boxed{\frac{dp_\mu}{d\tau} = q F_{\mu\nu} u^\nu.}$$

which is the Lorentz force law.

Action of EM fields and Sources

Action $S[A, J]$. Require Lorentz invariant, gauge invariant and reproduces Maxwell eqns.

We will write S in terms of Lagrange density



$$\mathcal{L}(x) = \mathcal{L}(\underset{A_\mu(x)}{A(x)}, \overset{J^\mu(x)}{J(x)}, \dots).$$

$$S = \int_{\mathbb{R}^{3,1}} \overset{d^4x = c\,dt \cdot d^3x.}{d^4x}\ \mathcal{L}(x).$$

We write

$$\mathcal{L} = -\frac{1}{4\mu_0 c} F_{\mu\nu} F^{\mu\nu} + \frac{1}{c} A_\mu J^\mu \qquad (20).$$

Lorentz invariant is automatic. For gauge invariant, consider $A_\mu \to A_\mu + \partial_\mu \chi$ for any $\chi(x) \to 0$ as $|\underline{x}| \to \infty$ or $t \to \pm\infty$.

Varying $\mathcal{L} + \delta\mathcal{L}$, where

$$\delta\mathcal{L} = \frac{1}{c}\partial_\mu \chi \cdot J^\mu = \frac{1}{c}\partial_\mu(\chi J^\mu) - \underbrace{\frac{1}{c}\chi\,\partial_\mu J^\mu}_{=0\ \because\ \partial_\mu J^\mu = 0.}$$

Then

$$\delta S = \frac{1}{c}\int_{\mathbb{R}^{3,1}} d^4x\ \partial_\mu \chi \cdot J^\mu$$

$$= \frac{1}{c}\int_{\mathbb{R}^{3,1}} d^4x\ \partial_\mu(\chi J^\mu)$$

$$= \frac{1}{c}\int_{\partial(\mathbb{R}^{3,1})} dS_\mu\ \underbrace{\chi}_{\chi = 0 \text{ on } \partial(\mathbb{R}^{3,1})} J^\mu . = 0$$

Principle of least action applied on $\mathcal{L}$

$$\Rightarrow \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \qquad (21)$$

(See Sheet 1 #1.2).

Coupled System

A relativistic particle rest mass m, charge q, moving along worldline $y^\mu(\lambda) = (ct(\lambda), \underline{y}(\lambda))$. This corresponds to a soure $J^\mu(x) = (c\rho(\underline{x},t), \underline{J}(\underline{x},t))$ with

$$\rho(\underline{x},t) = q\,\delta^{(3)}(\underline{x} - \underline{y}(\lambda(t)))$$

Defining 3-veloeity $\underline{v} = d\underline{y}/dt$, then

$$\underline{J}(\underline{x},t) = q\underline{v}(t)\,\delta^{(3)}(\underline{x} - \underline{y}(\lambda(t)))$$

The interaction term is

$$S_{int} = \frac{1}{c}\int d^4x\, J_\mu A^\mu$$

$$= \frac{q}{c}\int c\,dt \int d^3x\; \delta^{(3)}(\underline{x} - \underline{y}(\lambda(t)))\; A_\mu(X)\, w^\mu$$

where $w^\mu = (c, \underline{v}) = U^\mu/\gamma(v) = \frac{dy^\mu}{d\tau}\cdot\frac{d\tau}{dt}$. So

$$S_{int} = q\int dt\; A_\mu(t, \underline{y}(\lambda(t)))\cdot\frac{dy^\mu}{d\tau}\cdot\frac{d\tau}{dt}$$

$$= q\int d\lambda\;\frac{dy^\mu}{d\lambda}\,A_\mu(y(\lambda)).$$

The combined action is then

$$S_{total} = -\frac{1}{4\mu_0 c}\int F_{\mu\nu}F^{\mu\nu}\,d^4x - mc\int\left(-\eta_{\mu\nu}\frac{dy^\mu}{d\lambda}\frac{dy^\nu}{d\lambda}\right)^{1/2} d\lambda + S_{int}.$$

Variate $\delta A_\mu \longrightarrow$ Maxwell eqn.

$\delta y^\mu \rightsquigarrow$ Lorentz force law.

Motion in Constant Field

$$\frac{dU^\mu}{d\tau} = \frac{q}{m}F^\mu{}_\nu U^\nu$$

Interested in $\partial_\rho F^\mu{}_\nu = 0$, ie. constant homogenous field.

Integrate to get

$$U^\mu(\tau) = \exp\left(\frac{q\tau}{m}F\right)^\mu{}_\nu \, U^\nu(0).$$

where

$$F^\mu{}_\nu = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$\nearrow$ − sign induced in time coord. by lowering indices

and

$$\exp\left(\frac{q\tau}{m}F\right)^\mu{}_\nu = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{q\tau}{m}\right)^n (F^n)^\mu{}_\nu.$$

$$= \delta^\mu{}_\nu + \frac{q\tau}{m}F^\mu{}_\nu + \frac{1}{2}\left(\frac{q\tau}{m}\right)^2 F^\mu{}_\rho F^\rho{}_\nu. + O(F^3)$$

Different cases depending on Lorentz invariant,

$$-\frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu} = \frac{1}{c}\,\underline{E}\cdot\underline{B}$$

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = |\underline{B}|^2 - \frac{1}{c^2}|\underline{E}|^2.$$

If $\underline{E}\cdot\underline{B} \neq 0$, then perform a LT to a frame where

$$\underline{E} = (E, 0, 0) \quad , \quad \underline{B} = (B, 0, 0).$$

Then

$$F^\mu{}_\nu = \begin{pmatrix} E/c\,\mathbb{L} & 0 \\ 0 & \beta\,\mathbb{M} \end{pmatrix},$$

$\mathbb{L}^2 = I_2 \rightarrow$

where $\mathbb{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbb{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\alpha = \dfrac{qE}{mc}$, $\beta = \dfrac{qB}{mc}$, and

$\mathbb{M}^2 = -I_2 \nearrow$

$$\exp\left(\frac{q\tau}{m}F\right)^\mu{}_\nu = \begin{pmatrix} \exp(\alpha\tau\,\mathbb{L}) & 0 \\ 0 & \exp(\beta\tau\,\mathbb{M}) \end{pmatrix}$$

with

$$\exp(\alpha\tau\,\mathbb{L}) = \sum_{n=0}^{\infty} \frac{(\alpha\tau)^n}{n!}\mathbb{L}^n$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha\tau)^{2m}}{(2m)!} I_2 + \sum_{m=0}^{\infty} \frac{(\alpha\tau)^{2m+1}}{(2m+1)!}\mathbb{L}.$$

$$= \cosh(\alpha\tau)\, I_2 + \sinh(\alpha\tau)\, \mathbb{L} .$$

Similarly, $\exp(\beta\tau \mathbb{M}) = \cos(\beta\tau)\, I_2 + \sin(\beta\tau)\, \mathbb{M}$.

Integrate $u^\mu = \frac{dx^\mu}{d\tau}$ w.r.t. τ, we get

$$x^\mu(\tau) - c^\mu = \begin{pmatrix} \mathbb{R}(\tau) & 0 \\ 0 & \mathbb{S}(\tau) \end{pmatrix}^\mu{}_\nu \, u^\nu(0) .$$

where $\mathbb{R}(\tau) = \frac{1}{\alpha} \sin(\alpha\tau)\, I_2 + \frac{1}{\alpha} \cosh(\alpha\tau)\, \mathbb{L}$,

$\mathbb{S}(\tau) = \frac{1}{\beta} \sin(\beta\tau)\, I_2 - \frac{1}{\beta} \cos(\beta\tau)\, \mathbb{M}$.

Energy and Momentum in EM (see IB)

EM field carries energy density

$$\mathcal{E} = \frac{\varepsilon_0}{2} |\underline{E}|^2 + \frac{1}{2\mu_0} |\underline{B}|^2$$

- EM field does work on point charge q at a rate

$$W = \frac{d}{dt}\left(\mathcal{E}_{mechanical} \right) = q\, \underline{E} \cdot \underline{v}$$

Current-charge distribution is

$$\mathcal{W} = \underline{J} \cdot \underline{E}$$

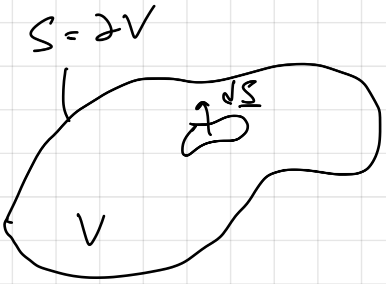
work per unit volume per unit time

Define Poynting vector $\underline{N} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$. then Maxwell eqns gives

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \underline{N} = - \underline{J} \cdot \underline{E}} \qquad (31).$$

If we interpret $\underline{N} \cdot d\underline{S}$ as energy flux through $d\underline{S}$, then (31) express conservation of energy.

Integrate over any region $V \subset \mathbb{R}^3$



$$\frac{d}{dt}\int_V \mathcal{E}\, dV + \int_V \underline{J}\cdot\underline{E}\, dV = -\int_V \nabla\cdot\underline{N}\, dV$$

$$= -\int_{\partial V} \underline{N}\cdot d\underline{S}$$

$\Rightarrow$ $\frac{d}{dt}$ (EM energy in V) + $\frac{d}{dt}$ (total work done in V)

= − (Energy flux through ∂V).

Momentum conservation

Point particle charge q, in an EM field.

$$\underline{F} = \frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v}\times\underline{B})$$

For a charge distribution,

$$\underline{f} = \rho\underline{E} + \underline{J}\times\underline{B} \qquad (32)$$

force density = r.o.c. of mechanical momentum per unit volume

Eliminate the sources using Maxwell eqn.

$$\rho = \varepsilon_0 \nabla\cdot\underline{E} \qquad (M1)$$

$$\underline{J} = \frac{1}{\mu_0}\nabla\times\underline{B} - \varepsilon_0\frac{\partial\underline{E}}{\partial t}. \qquad (M4).$$

Then

$$\underline{f} = \varepsilon_0(\nabla\cdot\underline{E})\underline{E} + \frac{1}{\mu_0}(\nabla\times\underline{B})\times\underline{B} - \varepsilon_0\frac{\partial\underline{E}}{\partial t}\times\underline{B} \qquad (33)$$

Use identities

$$(\nabla\times\underline{B})\times\underline{B} = (\underline{B}\cdot\nabla)\underline{B} - \frac{1}{2}\nabla|\underline{B}|^2. \qquad (34)$$

Then

$$\frac{\partial\underline{E}}{\partial t}\times\underline{B} = \frac{\partial}{\partial t}(\underline{E}\times\underline{B}) - \underline{E}\times\frac{\partial\underline{B}}{\partial t}$$

$$= \frac{\partial}{\partial t}(\underline{E}\times\underline{B}) + \underline{E}\times(\nabla\times\underline{E})$$

$$= \frac{\partial}{\partial t}(\underline{E}\times\underline{B}) - (\underline{E}\cdot\nabla)\underline{E} + \frac{1}{2}\nabla|\underline{E}|^2 \qquad (35)$$

Combining,

$$\underline{f} = \varepsilon_0 \underline{E}(\nabla\cdot\underline{E}) + \frac{1}{\mu_0}\left[(\underline{B}\cdot\nabla)\underline{B} - \frac{1}{2}\nabla|\underline{B}|^2\right] - \varepsilon_0\left[\frac{\partial}{\partial t}(\underline{E}\times\underline{B}) + \frac{1}{2}\nabla|\underline{E}|^2 - (\underline{E}\cdot\nabla)\underline{E}\right]$$

Convenient to add $\frac{1}{\mu_0}\underline{B}(\nabla\cdot\underline{B}) = 0$. So

$$\underline{f} = \varepsilon_0\left[(\nabla\cdot\underline{E})\underline{E} + (\underline{E}\cdot\nabla)\underline{E} - \frac{1}{2}\nabla|\underline{E}|^2\right] + \frac{1}{\mu_0}\left[(\underline{B}\cdot\nabla)\underline{B} + \underline{B}(\nabla\cdot\underline{B}) - \frac{1}{2}\nabla|\underline{B}|^2\right] - \varepsilon_0\frac{\partial}{\partial t}(\underline{E}\times\underline{B}) \tag{36}$$

Terms in brackets are total derivatives

$$\left[\underline{E}(\nabla\cdot\underline{E}) + (\underline{E}\cdot\nabla)\underline{E} - \frac{1}{2}\nabla|\underline{E}|^2\right]_j = \frac{\partial}{\partial x^i}\left(E^iE^j - \frac{1}{2}|\underline{E}|^2\delta_{ij}\right). \tag{37}$$

Define momentum density

$$\underline{g} = \varepsilon_0\,\underline{E}\times\underline{B} \tag{38}$$

and Maxwell stress tensor

$$\sigma^{ij} = -\varepsilon_0\left(E^iE^j - \frac{1}{2}|\underline{E}|^2\delta^{ij}\right) - \frac{1}{\mu_0}\left(B^iB^j - \frac{1}{2}|\underline{B}|^2\delta^{ij}\right) \tag{39}$$

Rewrite (36) as conservation eqn

$$\boxed{\frac{\partial g^i}{\partial t} + \frac{\partial\sigma^{ij}}{\partial x^j} = -(\rho\underline{E} + \underline{J}\times\underline{B})_i} \tag{40}$$

Integrating over some region $V\subset\mathbb{R}^3$

$$\frac{d}{dt}\int_V g^i\,dV + \int_V(\rho\underline{E} + \underline{J}\times\underline{B})^i\,dV = -\int_V\frac{\partial\sigma^{ij}}{\partial x^j}\,dV = -\int_{\partial V}\sigma^{ij}\,dS^j.$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}\text{momentum carried}\\ \text{by field in } V\end{pmatrix} + \frac{d}{dt}\begin{pmatrix}\text{total mechanical}\\ \text{momentum in } V\end{pmatrix} = -\begin{pmatrix}\text{flux of momentum}\\ \text{through } V\end{pmatrix}$$

Stress - Energy Tensor

In relativistic theory, the components

$$\mathcal{E}, \quad N_i, \quad g_j, \quad \sigma_{ij}$$

These combine to form a (2,0) Lorentz tensor

$$T^{\mu\nu} = \left(\begin{array}{c|cc} \mu \backslash \nu & 0 & i \\ \hline 0 & \mathcal{E} & cg_j \\ i & N_i/c & \sigma_{ij} \end{array}\right)$$

Matter dist. of particles, rest mass m, number density n_0.

at rest in some inertial frame S.

$$\mathcal{E} = mc^2 n_0 \quad . \quad \underline{N}, = \underline{g} = \sigma = 0$$

$$\boxed{T^{\mu}{}_{\mu} = -\mathcal{E} + \sigma_{ii} = -\mathcal{E}} \qquad (41)$$

Perform a boost with speed v ∥ x-dim.

$$t' = \gamma(v)\left(t - \frac{v}{c^2}x\right) \qquad y' = y$$

$$x' = \gamma(v)(x - vt) \qquad z' = z$$

in S'. Particle move with 3-velocity $\underline{v} = (-v, 0, 0)$

Energy per particle : $m\gamma(v)c^2$

momentum per particle : $mv\gamma(v)$

Number density in S' increased by Lorentz contraction

$$n_0' = \gamma(v)\, n_0 .$$

Then in S',

Energy density : $\mathcal{E}' = \gamma(v)\, mc^2 n_0' = \gamma^2 mc^2 n_0 = \gamma^2(v)\,\mathcal{E}$

Energy flux : $N_x' = -\mathcal{E}' v = -\gamma^2(v)\, v\,\mathcal{E}$

Momentum density : $g_x' = -\gamma(v)\, m v\, n_0' = -\gamma^2(v)\, m v n_0$

$$= -\gamma^2(v)\, \frac{v}{c^2}\, \mathcal{E}$$

Momentum flux : $\sigma'_{xx} = -g_x' v = \gamma^2(v)\, \frac{v^2}{c^2}\, \mathcal{E}.$

So

$$x^\mu \longrightarrow x^{\mu'} = \Lambda^\mu{}_\nu x^\nu , \quad \Lambda = \left(\begin{array}{cc|c} 1 & -v/c & \\ -v/c & 1 & 0 \\ \hline & 0 & \mathbb{1} \end{array}\right)$$

and

$$T'^{\mu\nu} = \begin{pmatrix} \gamma^2 \mathcal{E} & -\gamma^2(v)\, v\mathcal{E}/c & 0 \\ -\gamma^2(v)\, v\mathcal{E}/c & \gamma^2(v)\, v^2\mathcal{E}/c^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Can now check that $T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$

Properties of stress-energy tensor:

- Symmetric $T^{\mu\nu} = T^{\nu\mu}$, so energy flux $\propto$ momentum density.
- Trace $T^\mu{}_\mu = -\mathcal{E} = -mc^2 n_0$

EM Stress-energy Tensor

We have

$$\mathcal{E} = \frac{\varepsilon_0}{2} |\underline{E}|^2 + \frac{1}{2\mu_0} |\underline{B}|^2 .$$

$$\underline{N} = \frac{1}{\mu_0}\, \underline{E} \times \underline{B} .$$

$$\underline{g} = \varepsilon_0\, \underline{E} \times \underline{B} = \underline{N}/c^2 .$$

$$\sigma_{ij} = -\varepsilon_0 \left(E_i E_j - \frac{1}{2} |\underline{E}|^2 \delta_{ij} \right) - \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} |\underline{B}|^2 \delta_{ij} \right)$$

Then

$$T^\mu{}_\mu = -\mathcal{E} + \sum_{i=1}^{3} \sigma_{ii}$$

$$= \varepsilon_0 |\underline{E}|^2 \left(-\frac{1}{2} - 1 + \frac{3}{2} \right) + \frac{1}{\mu_0} |\underline{B}|^2 \left(-\frac{1}{2} - 1 + \frac{3}{2} \right) = 0$$

- $T^{\mu\nu}$ is $(2,0)$ symmetric, traceless, gauge-invariant, quadratic in fields. so

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}\right) \tag{43}$$

Conservation Law becomes

$$\boxed{\partial_\mu T^{\mu\nu} = - F^{\nu}{}_{\rho} J^{\rho}} \tag{44}$$

Radiation

Maxwell's eqn admit wave-like sol^n (in vacuum)

$$\underline{E} = \underline{E_0} \sin(\underline{k}\cdot\underline{x} - \omega t) \qquad \underline{B} = \underline{B_0} \sin(\underline{k}\cdot\underline{x} - \omega t).$$

Maxwell gives

$$\omega = c|\underline{k}| \qquad , \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

Waves propagate at speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

$$\nabla \cdot \underline{E} = \nabla\cdot\underline{B} = 0. \Rightarrow \quad \underline{k}\cdot\underline{E_0} = \underline{k}\cdot\underline{B_0}$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad \Rightarrow \quad \underline{k}\times\underline{E_0} = \omega \underline{B_0} \quad \Rightarrow \quad c|\underline{B_0}| = |\underline{E_0}|$$

we introduce the wave 4-vector

$$k^\mu = (\omega/c, \underline{k})$$

- $w = c|\underline{k}| \Rightarrow \eta_{\mu\nu} k^\mu k^\nu = 0$
- $\underline{k}\cdot\underline{x} - \omega t = k_\mu x^\mu.$

Then for field-strength tensor, noting that

$$\underline{k} = (k, 0, 0), \quad \underline{E_0} = (0, cB_0, 0), \quad \underline{B_0} = (0, 0, B_0),$$

we have

$$F^{\mu\nu} = B_0 \sin(k_\mu x^\mu) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The energy-stress tensor is

$$T^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right)$$

Noting

$$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = |\underline{B}|^2 - \frac{1}{c^2} |\underline{E}|^2 = 0,$$

So

$$T^{\mu\nu} = \frac{1}{\mu_0} F^{\mu\alpha} F^{\nu}{}_{\alpha} = \frac{B_0^2}{\mu_0} \sin^2(k_\mu x^\mu) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{B_0^2}{\mu_0} \sin^2(k_\mu x^\mu) \left(\begin{array}{cc|c} 1 & 1 & \multirow{2}{*}{0} \\ 1 & 1 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right)$$

Hence,

$$\mathcal{E} = \frac{B_0^2}{\mu_0} \sin^2(k_\mu x^\mu)$$

And

$$\sigma_{xx} = \frac{N_x}{c} = c g_x = \mathcal{E}$$

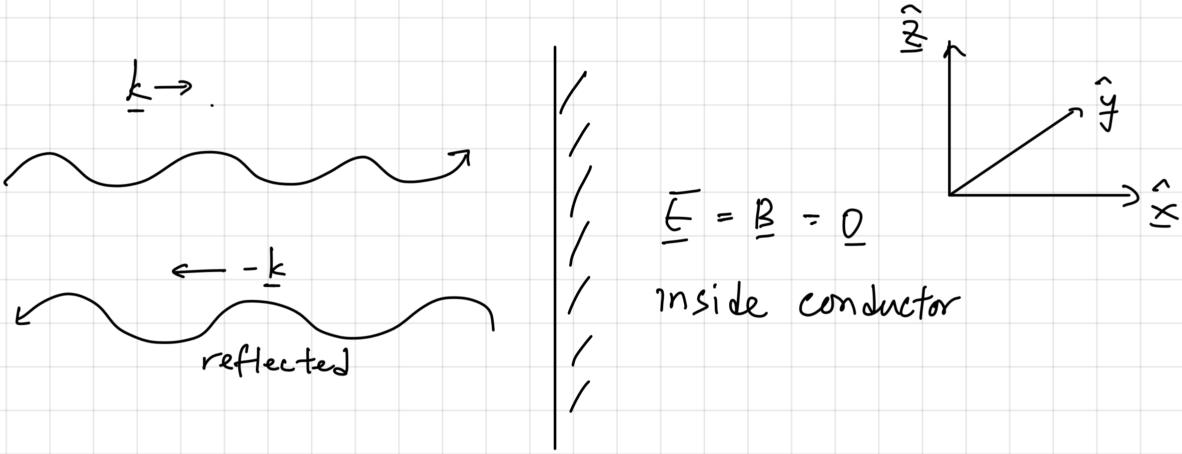
So we can treat EM fields as a gas of photons

$$E_\gamma = p_\gamma c.,$$

with $p_\gamma = |\underline{p}_\gamma|$

Radiation Pressure

Consider EM wave incident an perfect conductor



$\underline{E} = \underline{B} = \underline{0}$

inside conductor

$$\underline{k} = k\underline{x}, \quad \underline{E} \propto \hat{y}, \quad \underline{B} \propto \hat{z}.$$

Gauss's law :

$$E_+ - E_- = \sigma_s n/\varepsilon_0$$

$$\nabla\cdot\underline{E} = \rho/\varepsilon_0 \Rightarrow \underline{E} \text{ cts } (\parallel \text{ to conductor}).$$

$$\Rightarrow \underline{E}(\underline{x}=0, t) = \underline{0}$$

$$\Rightarrow E_{tot} = B_0 c\left[\sin(kx-\omega t) - \sin(-kx-\omega t)\right]\hat{y}.$$

Similarly, from Maxwell.

$$\underline{B}_{tot} = B_0\left[\sin(kx-\omega t) + \sin(-kx-\omega t)\right]\hat{z}$$

$(\underline{k}\times\underline{E}_0 = \underline{B}_0)$ So $\underline{B}$ discontinuous at $x=0$ ⇒ surface current.

$$\sigma_{xx} = \frac{1}{2\mu_0}\left[\frac{|\underline{E}|^2}{c^2} + |\underline{B}|^2 - \underbrace{\frac{E_x^2}{c^2}}_{=0} - \underbrace{B_x^2}_{=0}\right]$$

$$= \frac{B_0^2}{\mu_0}\left(\sin^2(kx-\omega t) + \sin^2(-kx-\omega t)\right)$$

$$\Rightarrow \boxed{\sigma_{xx}\big|_{x=0} = \frac{2B_0^2}{\mu_0}\sin^2(-\omega t).}$$

This is the "force exerted on a conductor per unit area", or "radiation pressure".

2. Radiation

4-vector potential $A_\mu(x)$ generated by an arbitrary source $J_\mu(x)$

(M1) $\Rightarrow$ $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, then we have

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = -\mu_0 J^\nu \qquad (1)$$

where $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$.

Perform gauge-transformation $A^\mu \longmapsto \tilde{A}^\mu = A^\mu + \partial^\mu \chi$, choosing

$$\Box \chi = -\partial_\mu A^\mu. \qquad (2)$$

Then

$$\partial_\mu \tilde{A}^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \chi$$
$$= 0$$

This is the Lorentz gauge. And in Lorentz gauge, (1) becomes

$$\Box A^\mu = -\mu_0 J^\mu \qquad (3)$$

drop tilde

Taking Fourier Transform w.r.t. time,

$$A^\mu(t,\underline{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{A}^\mu(\omega,\underline{x})\, e^{i\omega t}\, d\omega$$

$$J^\mu(t,\underline{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{J}^\mu(\omega,\underline{x})\, e^{i\omega t}\, d\omega$$

Then $\Box A^\mu = -\mu_0 J^\mu$ becomes

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A^\mu = -\mu_0 J^\mu$$

$$\Rightarrow \quad (\nabla^2 + k^2)\,\tilde{A}^\mu = -\mu_0 \tilde{J}^\mu. \qquad (4)$$

where $k := \omega/c$ is the wavenumber.

Solve by finding Green's f^n $G(\underline{x};\underline{x}')$ for (∇^2+k^2) obeying

$$(\nabla_x^2+k^2)\,G(\underline{x};\underline{x}') = \delta^{(3)}(\underline{x}-\underline{x}') \qquad (5)$$

So that

$$\tilde{A}^\mu(\omega,\underline{x}) = -\mu_0\int d^3x'\; G(\underline{x};\underline{x}')\,\tilde{J}^\mu(\omega,\underline{x}) \qquad (6)$$

Standard analysis gives

$$G_\pm(\underline{x};\underline{x}') = -\frac{1}{4\pi}\,\frac{e^{\pm ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}$$

Taking "−" sol^n, (6) gives

$$\tilde{A}^\mu(\omega,\underline{x}) = \frac{\mu_0}{4\pi}\int d^3x'\;\frac{e^{-ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}\,J^\mu(\omega,\underline{x}'),$$

and thus

$$A^\mu(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{A}^\mu(\omega,\underline{x})\,e^{i\omega t}\,d\omega$$

$$= \frac{\mu_0}{4\pi}\int d^3x'\int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}\;\frac{e^{i\omega t-i\omega|\underline{x}-\underline{x}'|/c}}{|\underline{x}-\underline{x}'|}\;\tilde{J}^\mu(\omega,\underline{x}')$$

$$= \frac{\mu_0}{4\pi}\int d^3x'\;\frac{J^\mu(t_{ret},\underline{x}')}{|\underline{x}-\underline{x}'|} \qquad (7)$$

where $t_{ret} := t - \underset{\uparrow}{\frac{|\underline{x}-\underline{x}'|}{c}}$ is the retarded time.

time taken for signal to pass from $\underline{x}$ to $\underline{x}'$.

Time component of (7)

$$\phi(\underline{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\;\frac{\rho(t_{ret},\underline{x}')}{|\underline{x}-\underline{x}'|}$$

Remark G_+ yields similar formula but $t_{ret}\longrightarrow t_{adv} = t+\frac{|\underline{x}-\underline{x}'|}{c}$, but violates casuality.

Relativistic invariance

For any event $x^\mu=(ct,\underline{x})$, define *past light-cone* to be

$$\omega_x^- = \{y\in\mathbb{R}^{3,1} : \eta_{\mu\nu}(x^\mu-y^\mu)(x^\nu-y^\nu)=0\,,\ y^0<x^0\}.$$

(7) can be written as

$$A^\mu(x) = \frac{\mu_0}{2\pi}\int_{W_x^-} d^3y \;\; J^\mu(y)$$

$$= \frac{\mu_0}{2\pi}\int_{\mathbb{R}^{3,1}} d^4y \;\; \delta(\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)) \cdot \textcircled{H}(x^0 - y^0)\, J^\mu(y) \qquad (8)$$

This is manifestly invariant

Pf: Recall f^n $f(y)$ with isolated zeros $y = y_i^*$, $i = 1, \dots, l$

$$\int_{-\infty}^{\infty} dy \;\; \delta(f(y))\, g(y) = \sum_{i=1}^{l} \frac{g(y_i^*)}{|f'(y_i^*)|} .$$

we have

$$\delta(f(y)) = \sum_{i=1}^{l} \frac{\delta(y - y_i^*)}{|f'(y_i^*)|}$$

Apply this to

$$f(y^0) = \eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)$$

$$= |\underset{\sim}{x} - \underset{\sim}{y}|^2 - (x^0 - y^0)^2$$

$$= \left(|\underset{\sim}{x} - \underset{\sim}{y}| - x^0 + y^0\right)\left(|\underset{\sim}{x} - \underset{\sim}{y}| + x^0 - y^0\right)$$

vanishes at $y^0 = y^0_\pm = x^0 \pm |\underset{\sim}{x} - \underset{\sim}{y}|$, and

$$f'(y^0)\Big|_{y^0 = y^0_\pm} = \pm 2|\underset{\sim}{x} - \underset{\sim}{y}|$$

Then

$$\delta(f(y^0)) = \frac{\delta(y^0 - y^0_+)}{2|\underset{\sim}{x} - \underset{\sim}{y}|} + \frac{\delta(y^0 - y^0_-)}{2|\underset{\sim}{x} - \underset{\sim}{y}|}$$

$$\Rightarrow \quad \delta(f(y^0))\, \textcircled{H}(x^0 - y^0) = \frac{\delta(y^0 - y^0_+)}{2|\underset{\sim}{x} - \underset{\sim}{y}|} \qquad (*)$$

Hence,

$$\frac{\mu_0}{2\pi}\int d^4y \;\; \delta(\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu))\, \textcircled{H}(x^0 - y^0)\, J^\mu(y)$$

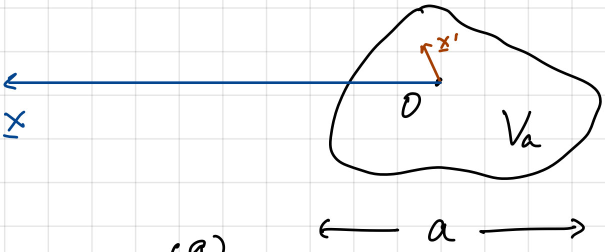
$$= \frac{\mu_0}{4\pi}\int d^4y \;\; \frac{\delta(y^0 - x^0 + |\underset{\sim}{x} - \underset{\sim}{y}|)}{|\underset{\sim}{x} - \underset{\sim}{y}|}\, J^\mu(y)$$

$$= \frac{\mu_0}{4\pi}\int d^3y \;\; \frac{J^\mu(y^0 = ct_{ret}, \underset{\sim}{y})}{|\underset{\sim}{x} - \underset{\sim}{y}|} = A^\mu(\underset{\sim}{x}, t)$$

$\square$

Radiation in dipole approximation

Consider radiation due to a source J^μ supported in some region $V_a \subset \mathbb{R}^3$ of size a. and WLOG contain origin



For $\underline{x}' \in V_a$, $|\underline{x}'| \ll |\underline{x}|$

$$|\underline{x}-\underline{x}'| = [(\underline{x}-\underline{x}')\cdot(\underline{x}-\underline{x}')]^{1/2}$$

$$\approx r - \hat{\underline{x}}\cdot\underline{x}' + \mathcal{O}(a/r) \qquad (9)$$

where $r = |\underline{x}|$, $\hat{\underline{x}} = \underline{x}/r$, for $r \gg a$.

$$(7) \Rightarrow \qquad \underline{A}(t,\underline{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\underline{J}(t-\frac{|\underline{x}-\underline{x}'|}{c}, \underline{x}')}{|\underline{x}-\underline{x}'|}$$

Using (9),

$$\underline{J}(t-\frac{|\underline{x}-\underline{x}'|}{c}, \underline{x}') \approx \underline{J}(t-\frac{r}{c}, \underline{x}') + \frac{\hat{\underline{x}}\cdot\underline{x}'}{c}\,\dot{\underline{J}}(t-\frac{r}{c}, \underline{x}') + \dots$$

$\underline{J}$ oscillates at characteristic frequency ω (e.g. $\underline{J} = \underline{J}_0(\underline{x})\sin(\omega t)$)

For $r \gg a$ (α) and $\omega \ll c/a$ (β), then

$$\underline{A}(t,\underline{x}) \approx \frac{\mu_0}{4\pi r}\int \underline{J}(t-\frac{r}{c}, \underline{x}')\, d^3x'. \qquad (10)$$

Recall point charge q at position $\underline{x}$, the <u>electric dipole moment</u> is

$$\underline{p} = q\underline{x}$$

Similarly, for charge distribution $\rho(t,\underline{x})$,

$$\underline{p}(t) = \int \rho(t,\underline{x})\,\underline{x}\, d^3x$$

Charge conservation:
$$\frac{\partial \rho}{\partial t} + \nabla\cdot\underline{J} = 0$$

Then $$\dot{\underline{p}} = \frac{d\underline{p}}{dt} = \int \frac{\partial \rho}{\partial t}\, \underline{x}\; d^3x$$

$$= -\int (\nabla \cdot \underline{J})\, \underline{x}\; d^3x$$

$$\overset{\text{div. thm.}}{=} \int (\underline{J} \cdot \nabla)\, \underline{x}\; d^3x$$

$$= \int \underline{J}\,(\underline{t}, \underline{x})\; d^3x$$

(10) becomes

$$\underline{A}(t, \underline{x}) \approx \frac{\mu_0}{4\pi r}\, \dot{\underline{p}}\left(t - \frac{r}{c}\right). \qquad (11)$$

Now calculate fields.

$$\underline{B} = \nabla \times \underline{A} = \nabla \times \left(\frac{\mu_0}{4\pi r}\, \dot{\underline{p}}\left(t - \frac{r}{c}\right)\right)$$

$$= -\frac{\mu_0}{4\pi r^2}\left[\overset{\nabla r = \hat{\underline{x}}}{\hat{\underline{x}}} \times \dot{\underline{p}}(t - r/c) + \frac{r}{c}\, \underline{x} \times \ddot{\underline{p}}\left(t - \frac{r}{c}\right)\right]$$

Assume the dipole oscillates with freq. ω. Second term will scale like $\frac{r\omega}{c}$ relative to the first, will dominate if

$r \gg \frac{c}{\omega}$ (γ)

In this regime $((\alpha), (\beta), (\gamma))$,

$$\underline{B}(t, \underline{x}) = -\frac{\mu_0}{4\pi r c}\, \hat{\underline{x}} \times \ddot{\underline{p}}\left(t - \frac{r}{c}\right). \qquad (12)$$

Find $\underline{E}(t, \underline{x})$ by solving (M4).

$$\frac{\partial \underline{E}}{\partial t} = c^2\, \nabla \times \underline{B} = -\frac{\mu_0 c}{4\pi}\, \nabla \times \left(\frac{\hat{\underline{x}}}{r} \times \ddot{\underline{p}}\left(t - \frac{r}{c}\right)\right)$$

With identical arguments (γ), leading term comes from action of ∇ on $t - \frac{r}{c}$.

$$\frac{\partial \underline{E}}{\partial t} \approx \frac{\mu_0}{4\pi r} \left(\hat{\underline{x}} \times (\hat{\underline{x}} \times \ddot{\underline{p}}(t-\tfrac{r}{c})) \right)$$

Integrating w.r.t. t,

$$\underline{E}(t,\underline{x}) = \underline{E}_{\text{static}}(\underline{x}). + \frac{\mu_0}{4\pi r} \left(\hat{\underline{x}} \times (\hat{\underline{x}} \times \ddot{\underline{p}}(t-\tfrac{r}{c})) \right) \qquad (13).$$

Solving (M1),

$$\underline{E}_{\text{static}}(\underline{x}) \overset{r\to\infty}{\sim} \frac{\hat{\underline{x}}}{4\pi r^2} Q .$$

where $Q = \int_{V_a} \rho(t,\underline{x})\, d^3x.$

So the sol^n is

$$\underline{E}(t,\underline{x}) \approx \underline{E}_{\text{static}}(\underline{x}) + \underline{E}_{\text{rad}}(t,\underline{x})$$

$$\underline{B}(t,\underline{x}) \approx \underline{B}_{\text{rad}}(t,\underline{x})$$

with
$$\underline{B}_{\text{rad}} = -\frac{\mu_0}{4\pi r c} \hat{\underline{x}} \times \ddot{\underline{p}}(t-r/c)$$

$$\underline{E}_{\text{rad}} = -c\, \hat{\underline{x}} \times \underline{B}_{\text{rad}}(t,\underline{x})$$

We assumed

$$r \gg a \quad (\alpha), \qquad w \ll \frac{c}{a} \quad (\beta). \qquad r \gg \frac{c}{\omega} \quad (\gamma)$$

ω is the characteristic angular freq. of source,

$$\underline{p}(t) = \underline{p}_0 \sin(wt).$$

have

$$\ddot{\underline{p}} = -\omega^2 \underline{p}(t).$$

Set $\underline{k} := k\hat{\underline{x}}$, with $k := w/c.$, $\hat{\underline{k}} = \hat{\underline{x}}$, then

$$\ddot{\underline{p}}(t-r/c) = -w^2 \underline{p}_0 \sin(\omega t - \underline{k}\cdot\underline{x})$$

$$\Rightarrow \quad \underline{B}_{\text{rad}} = \underline{B}_0(r) \sin(\omega t - \underline{k}\cdot\underline{x})$$

$$\underline{E}_{\text{rad}} = \underline{E}_0(r) \sin(\omega t - \underline{k}\cdot\underline{x})$$

with $\underline{B}_0(r) = \dfrac{\mu_0 \omega^2}{4\pi r c} \underline{k} \times \underline{p}_0$, $\underline{E}_0(r) = -c\, \underline{k} \times \underline{B}_0(r).$

$$\Rightarrow \quad \underline{k} \cdot \underline{B}_0(r) = \underline{k} \cdot \underline{E}_0(r) = 0$$

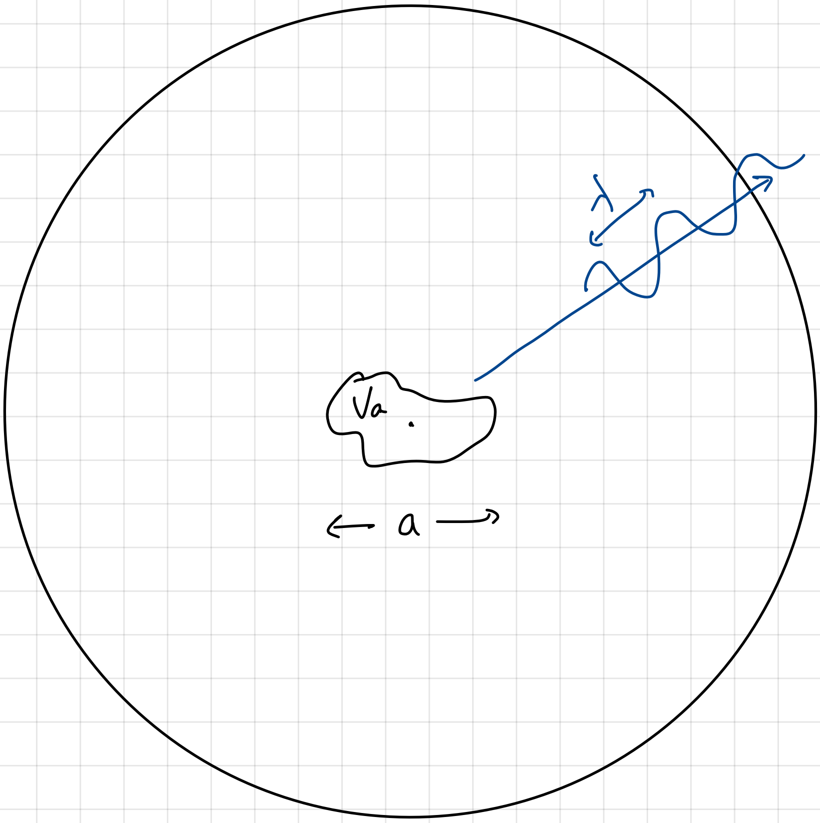
and

$$\underline{k} \times \underline{E}_0(r) = \omega \, \underline{B}_0(r)$$

These are outgoing spherical waves, with wave vector

$$\underline{k} = \left(\frac{\omega}{c}\right) \hat{\underline{x}}$$

and wavelength $\lambda = 2\pi / k = \frac{2\pi}{\omega} c$.



Assumptions $(\alpha), (\beta), (\gamma)$ is equivalent to

$$r \gg \lambda \gg a$$

"far-field region".

Power radiated

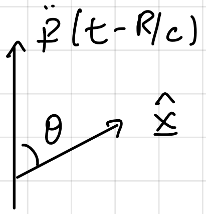
Flux of energy is

$$\underline{N} = \frac{1}{\mu_0} \underline{E}_{rad} \times \underline{B}_{rad}$$

$$= -\frac{c}{\mu_0} \left(\hat{\underline{x}} \times \underline{B}_{rad} \right) \times \left(\underline{B}_{rad} \right)$$

$$= \frac{c}{\mu_0} |\underline{B}_{rad}|^2 \, \hat{\underline{x}} \qquad (\text{since } \hat{\underline{x}} \cdot \underline{B}_{rad} = 0)$$

Work at fixed time t, introduce sph. polar with z-axis ∥ to $\ddot{\underline{p}}(t-r/c)$. For some field under R, $R \gg \lambda \gg a$,



$$|\underline{B}_{rad}|^2 = \left(\frac{\mu_0}{4\pi Rc}\right)^2 |\hat{\underline{x}} \times \ddot{\underline{p}}(t-\tfrac{R}{c})|^2 .$$

$$= \left(\frac{\mu_0}{4\pi Rc}\right)^2 \sin^2\theta \; |\ddot{\underline{p}}(t-R/c)|^2 .$$

and

$$P = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \; R^2 \underline{N}\cdot d\underline{S}\Big|_{r=R}$$

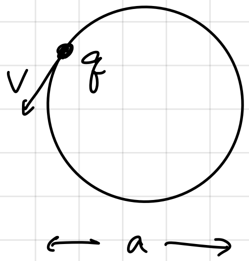
$$= \frac{\mu_0}{(4\pi)^2 c} |\ddot{\underline{p}}(t-R/c)|^2 \int_0^{2\pi} d\phi \int_{-1}^{1} \sin^2\theta \, d(\cos\theta)$$

$$= \frac{\mu_0}{6\pi c} |\ddot{\underline{p}}(t-R/c)|^2$$

Larmar formula

For a point particle following trajectory $\underline{x}(t)$.

We have dipole



$\omega \sim v/a$

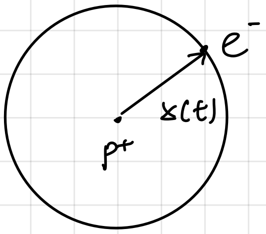
$$\underline{p}(t) = q\,\underline{x}(t)$$

not momentum

$$P(t) = \frac{\mu_0}{6\pi c} |\ddot{\underline{p}}(t-R/c)|^2 = \frac{\mu_0 q^2}{6\pi c} |\underline{a}(t-R/c)|^2$$

Dipole approximation requires $w \ll c/a$, valid if $v = |\dot{\underline{x}}| \ll c$

Application: instability of classical H-atom



e^- moves in Coulomb field of p^+.

$$m_e \ddot{\underline{x}} = -\frac{e^2}{4\pi\varepsilon_0} \frac{\hat{\underline{x}}}{r^2}$$

Dipole moment

$$\underline{p}(t) = e\,\underline{x}(t) .$$

$$\Rightarrow \ddot{\underline{p}}(t) = e\,\ddot{\underline{x}}(t) = -\frac{e^3}{4\pi\varepsilon_0 m_e r^2} \hat{\underline{x}} .$$

The emitted power is

$$P = \frac{\mu_0}{6\pi c} |\underline{\ddot{p}}|^2 = \frac{\mu_0}{6\pi c} \cdot \left(\frac{e^3}{4\pi\varepsilon_0 m_e r^2} \right)^2$$

Simplication: assume orbit is circular.

$$E = \frac{1}{2} m_e |\underline{\dot{x}}|^2 - \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r} = - \frac{e^2}{4\pi\varepsilon_0} \frac{1}{2r} = E(r).$$

$$\Rightarrow \frac{dE}{dt} = \frac{e^2}{8\pi\varepsilon_0} \frac{\dot{r}}{r^2} = -P = -\frac{\mu_0}{6\pi c} \left(\frac{e^3}{4\pi\varepsilon_0 m_e r^2} \right)^2$$

$$\Rightarrow \quad \dot{r} = - \frac{\mu_0 e^4}{12\pi^2 c \varepsilon_0 m_e^2} \cdot \frac{1}{r^2}$$

Starting from $r = r_0$ at time $= t_0$, e^- reaches the origin $r=0$ at time

$$T = \int_0^T dt = \int_{r_0}^0 \frac{1}{\dot{r}} \, dr = \frac{4\pi^2 c \varepsilon_0 m_e^2 r_0^3}{\mu_0 e^4}$$

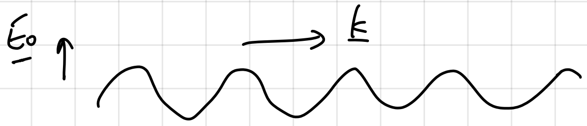
Setting $r_0 = 5 \times 10^{-11}$ m (Bohr radius),

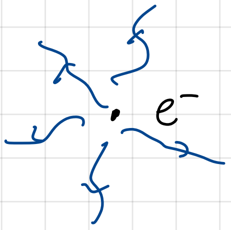
$$T \approx 10^{-11} \text{ s}$$

Classical atoms are unstable (need QM).

Scattering

Consider incidence EM radiation on free electron





$$\underline{E} = \underline{E_0} \sin(\underline{k} \cdot \underline{x} - \omega t)$$

e^- experiences a force, so

$$m_e \underline{\ddot{x}} = -e \underline{E}$$

e^- undergoes oscillation of amplitude $A \propto |\underline{E_0}|$

If $A \ll \lambda = \frac{2\pi}{|\underline{k}|}$, then can evaluate $\underline{E}(\underline{x},t)$ at avg position $\langle \underline{x} \rangle = 0$

$$\Rightarrow \quad \ddot{\underline{x}} \approx -\frac{e}{m_e} \underline{E_0} \sin(-\omega t)$$

$$\Rightarrow \quad \underline{x}(t) = \frac{e}{m_e} \underline{E_0} \frac{1}{\omega^2} \sin(-\omega t)$$

Check: $\quad A = \frac{e|\underline{E_0}|}{\omega^2 m_e}$. If $v = A\omega \ll c \Rightarrow A \ll \lambda = \frac{2\pi}{\omega} c$.

The oscillating e^- has a dipole moment,

$$\underline{p} = -e\,\underline{x}(t) = \frac{e^2}{m_e} \underline{E_0} \frac{\sin(-\omega t)}{\omega^2}.$$

Hence e^- emits radiation with power

$$P(t) = \frac{\mu_0}{6\pi c} \left| \ddot{\underline{p}}(t - R/c) \right|^2$$

$$= \frac{\mu_0}{6\pi c} \cdot \frac{e^4}{m_e^2} |\underline{E_0}|^2 \sin^2(\omega t - kR).$$

Averaged power is

$$\langle P \rangle := \frac{1}{T} \int_0^T P(t)\, dt \qquad T = \frac{2\pi}{\omega}$$

$$= \frac{\mu_0}{12\pi c} \frac{e^4}{m_e^2} |\underline{E_0}|^2$$

This is Thomson scattering of incident radiation by e^-.

The incident energy flux is

$$|\underline{N}| = \frac{1}{\mu_0} |\underline{E} \times \underline{B}| = \frac{|\underline{E_0}|^2}{\mu_0 c} \sin^2 \omega t$$

and average

$$\langle |\underline{N}| \rangle = \frac{|\underline{E_0}|^2}{2\mu_0 c}$$

The "amount of scattering" is characterised by cross-section

$$\sigma = \frac{\langle P \rangle}{\langle |\underline{N}| \rangle} = \frac{\mu_0^2 e^4}{6\pi m_e^2}$$

Rayleigh Scattering

Similar process, EM radiation scatters of e^- in neutral atoms

External electric field induces a dipole moment.

$$\underline{p}(t) = \underset{\uparrow}{\alpha}\, \underline{E}(t).$$

α polarisability, medium dependent

Incident wave $\underline{E} = \underline{E}_0 \sin(\underline{k}\cdot\underline{x} - \omega t)$

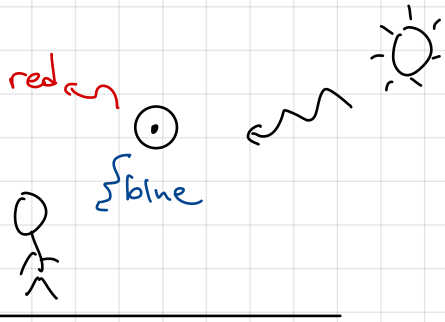
$$\ddot{\underline{p}} = -\alpha\omega^2 \underline{E}.$$

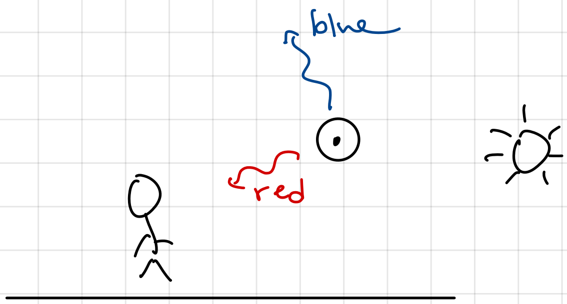
Dipole approx

$$\Rightarrow \quad \langle P \rangle = \frac{\mu_0}{6\pi c} \cdot \frac{1}{2} \alpha^2 \omega^4 |\underline{E}_0|^2$$

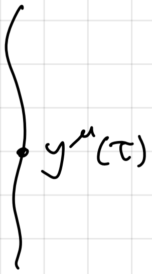
$$\Rightarrow \quad \sigma = \frac{\langle P \rangle}{\langle |\underline{N}| \rangle} = \frac{\mu_0^2}{6\pi} \alpha^2 \omega^4 \sim \frac{1}{\lambda^4}.$$

Enhanced scattering with small λ ⇒ blue sky and red sunset





Radiation from point charge



Relativistic point charge. World line $y^\mu = y^\mu(\tau)$

$$A^\mu(x) = \frac{\mu_0}{4\pi} \int d^4 z \, \frac{\delta(x^0 - z^0 + |\underline{x} - \underline{z}| \; J^\mu(z)}{|\underline{x} - \underline{z}|}$$

$J^\mu(x) = (c\rho, \underline{J})$, $\rho(\underline{x},t) = q\,\delta^{(3)}(\underline{x} - \underline{y}(\tau))$, $\underline{J}(\underline{x},t) = q\,\underline{v}\,\delta^{(3)}(\underline{x} - \underline{y}(\tau))$

$$\underline{v} = \frac{d\underline{y}}{dt}.$$

So can write

$$J^\mu(x) = qc \int \delta^{(4)}(x - y(\tau))\, \dot{y}^\mu(\tau)\, d\tau$$

$$\Rightarrow A^\mu(x) = \frac{\mu_0 qc}{4\pi}\int d\tau \int d^4z\, \delta^{(4)}(z - y(t))\, \dot{y}^\mu(\tau)\, \frac{\delta(x^0 - z^0 - |\underline{x}-\underline{z}|)}{|\underline{x}-\underline{z}|}$$

$$= \frac{\mu_0 qc}{4\pi}\int d\tau\, \dot{y}^\mu(\tau)\, \frac{\delta(x^0 - y^0 - |\underline{x}-\underline{y}|)}{|\underline{x}-y(\tau)|}$$

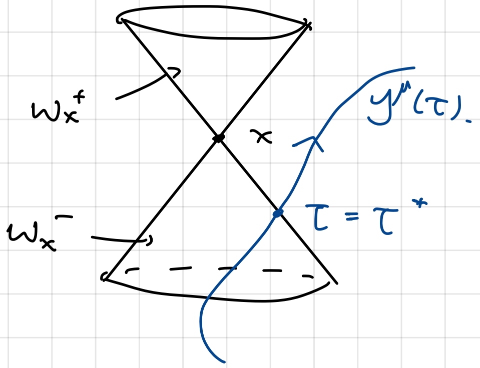
Recall we can replace

$$\frac{\delta(x^0 - y^0 - |\underline{x}-\underline{y}|)}{|\underline{x}-y(\tau)|} = 2\,\Theta(R^0(\tau))\,\delta(\eta_{\mu\nu}R^\mu(\tau)R^\nu(\tau))$$

with $R^\mu(\tau) = x^\mu - y^\mu(\tau)$.

δ-f^n has support at unique point

$\tau = \tau^*$ on the worldline.



Then

$$\dot{y}^\mu = \frac{dy^\mu}{d\tau}$$

$$A^\mu(x) = \frac{\mu_0 qc}{4\pi}\int d\tau\, \dot{y}(\tau)\,\Theta(R^0(t))\,\delta(\eta_{\mu\nu}R^\mu(\tau)R^\nu(\tau))$$

Note that

$$\frac{d}{d\tau}(\eta_{\mu\nu}R^\mu(\tau)R^\nu(\tau)) = 2\eta_{\mu\nu}R^\mu(\tau)\frac{d}{d\tau}(x^\nu - y^\nu(\tau))$$

$$= -2\eta_{\mu\nu}R^\mu\,\dot{y}(\tau).$$

So

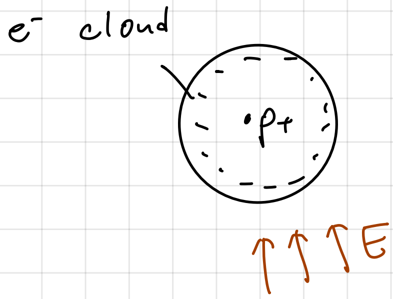
$$A^\mu(x) = \frac{\mu_0 qc}{4\pi}\int d\tau\, \dot{y}(\tau)\, \frac{\delta(\tau - \tau^*)}{|R^\nu(\tau_*)\,\dot{y}_\nu(\tau_*)|}$$

$$= \frac{\mu_0 qc}{4\pi}\,\frac{\dot{y}^\mu(\tau_*)}{|R^\nu(\tau_*)\,\dot{y}_\nu(\tau_*)|}$$

3. Electromagnetism in Media

Dielectric materials

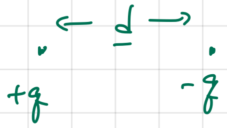
Properties :
- no mobile charges
- consist of neutral atoms



Polarisation

External $\underline{E}$ induces electric dipole moment

Revision: Simple dipole



$$\underline{p} := q\underline{d}$$

↖ electric dipole moment.

Electrostatic potential

$$\phi(\underline{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\underline{x}|} - \frac{q}{|\underline{x}+\underline{d}|}\right)$$

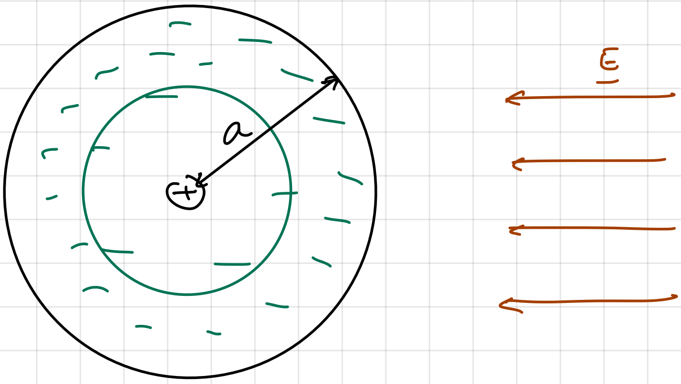
$$\overset{|\underline{x}| \gg |\underline{d}|}{\approx} \frac{1}{4\pi\varepsilon_0}\frac{\underline{p}\cdot\underline{x}}{|\underline{x}|^3}$$

The corresponding $\underline{E}$ is

$$\underline{E}(\underline{x}) \overset{|\underline{x}|\gg|\underline{d}|}{\approx} \frac{1}{4\pi\varepsilon_0}\frac{3(\underline{p}\cdot\hat{\underline{x}})\hat{\underline{x}} - \underline{p}}{|\underline{x}|^3} \qquad (1)$$

Simple model for atomic polarisation

Nucleus at $r = |\underline{x}| = 0$



Electric field due to cloud at radius $r<a$

$$\underline{E}_{\text{cloud}} = -\frac{1}{4\pi\varepsilon_0} q \left(\frac{r}{a}\right)^3 \frac{\hat{\underline{x}}}{r^3} = -\frac{1}{4\pi\varepsilon_0} \frac{qr}{a^3} \hat{\underline{x}}$$

Apply external field $\underline{E}$. Nucleus is displaced to position $\underline{d}$ where

$$\underline{E} + \underline{E}_{\text{cloud}} = 0 \Rightarrow \underline{d} = \frac{4\pi\varepsilon_0 a^3}{q} \underline{E}$$

Resulting dipole is

$$\underline{p} = q\underline{d} = (4\pi\varepsilon_0 a^3)\, \underline{E}.$$

To good approximation, many materials exhibit linear polarisation

$$\underline{p} = \alpha \underline{E} \qquad (2)$$

atomic dipole moment

material dependent const.

Focus on average electric dipole moment per unit volume.

$$\underline{P} \approx n \langle \underline{p} \rangle$$

density

Average lengthscale l <u>large</u> compared to atomic scale a, but <u>small</u> compared to size L of sample, i.e. $L \gg l \gg a$.

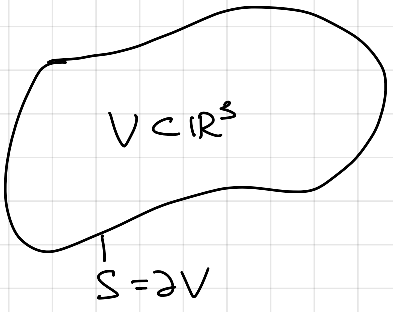
Per unit volume, $\underline{P} = \underline{P}(\underline{x}, t)$ varies in space and time.

macroscopic polarisation

Bound charge

Contribution to electrostatic potential at point $\underline{x}$ from atomic dipole in volume content $d^3\underline{x}'$

$$d\phi(\underline{x}) \overset{(1)}{\approx} \frac{1}{4\pi\varepsilon_0} \frac{\underline{P}(\underline{x}') \cdot (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} d^3\underline{x}'$$



Integrate over sample,

$$\phi(\underline{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\underline{x}' \; \frac{\underline{P}(\underline{x}')\cdot(\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3\underline{x}' \; \underline{P}(\underline{x}')\cdot\nabla_{\underline{x}'}\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)$$

$$\overset{(IBP)}{=} \frac{1}{4\pi\varepsilon_0}\int_S \frac{d\underline{S}\cdot\underline{P}(\underline{x}')}{|\underline{x}-\underline{x}'|} - \frac{1}{4\pi\varepsilon_0}\int_V d^3\underline{x}' \; \frac{\nabla\cdot\underline{P}(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

Compare 2^{nd} term with potential due to charge distribution $\rho(\underline{x})$,

$$\phi(\underline{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\underline{x}' \; \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

Identify bulk density of bound charge

$$\rho_{bound}(\underline{x},t) = -\nabla\cdot\underline{P} \qquad (3)$$

$$\Rightarrow \nabla\cdot\underline{E} = \frac{1}{\varepsilon_0}\rho_{bound}.$$

Similarly, 1^{st} term corresponds to surface density of bound charge on S

$$\sigma = \underline{P}(\underline{x})\cdot\underline{n} \qquad (4)$$

Interpretation: spacial variation of polarisation $\underline{P}(\underline{x})$ lead to a build up of net charge in some regions of sample.

Contribution to total charge is

$$\delta Q = \int_V dV \, \rho_{bound} = -\int_V \nabla\cdot\underline{P}\, dV = -\int_S \underline{P}\cdot d\underline{S}$$

cancelled by surface charge

$$\delta Q_{surface} = \int_S d^2y \; \sigma_{bound} = \int_S \underline{P}\cdot d\underline{S} = -\delta Q$$

Electric displacement

Introduce additional "free charges" which can move (conduction bound electrons) with ρ_{free}

$$\nabla\cdot\underline{E} = \frac{1}{\varepsilon_0}\left(\rho_{free} + \rho_{bound}\right)$$

$$= \frac{1}{\varepsilon_0}\left(\rho_{free} - \nabla\cdot\underline{P}\right)$$

Absorb effect of bound charges by defining electric displacement

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P} \qquad (5)$$

which diverges

$$\boxed{\nabla\cdot\underline{D} = \rho_{free}} \qquad (6)$$

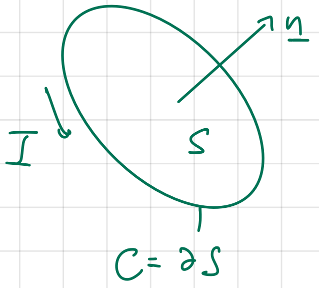
For linear materials

$$\underline{P} = n\underline{p} = n\alpha\underline{E}$$

$$\Rightarrow \quad \underline{D} = \varepsilon\underline{E} \quad , \qquad \varepsilon = \varepsilon_0 + n\alpha$$

Magnetic fields in matter

Revision Magnetic dipole: consider steady current I flowing around a planar loop $C = \partial S$ with normal vector $\hat{\underline{n}}$, area A.



Vector potential

$$\underline{A}(\underline{x}) = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\underline{x}'}{|\underline{x}-\underline{x}'|}$$

Leading large distance behaviour ($|\underline{x}| \gg \sqrt{A}$),

$$\underline{A}(\underline{x}) \approx \frac{\mu_0}{4\pi}\frac{\underline{m}\times\underline{x}}{|\underline{x}|^3}$$

where $\underline{m} = IA\underline{n}$ is the magnetic dipole.

Field ($\underline{x} \gg \sqrt{A}$)

$$\underline{B} = \nabla\times\underline{A}$$

$$\approx \frac{\mu_0}{4\pi}\left(\frac{3(\underline{m}\cdot\hat{\underline{x}})\hat{\underline{x}} - \underline{m}}{|\underline{x}|^3}\right) \qquad (7)$$

(c.f. electric dipole (1))

Magnetic dipoles / current loops exist within materials, e.g. electrons orbiting atom, electron spin.

Define $\underline{M}(\underline{x},t)$ as average dipole moment per unit volume

$$\underline{M}(\underline{x},t) = \underbrace{n}_{\text{density}} \overset{\text{atomic magnetic dipole.}}{\langle \underline{m} \rangle} . \qquad (8)$$

Except in ferromagnets, dipoles point in random directions

$$\Rightarrow \quad \underline{M} = 0 .$$

In ferromagnets, like to (anti-) align with external B-field.

Magnetisation well approximated by linear response

$$\underline{M} = \frac{1}{\mu_0} \frac{\chi_m}{1+\chi_m} \underline{B} .$$

where χ_m is the <u>magnetic susceptability</u>.

- Dimagnetisation : $-1 < \chi_m < 0$.
- Paramagnetisation : $\chi_m > 0$

Some materials are non-linear. In particular, can have <u>ferromagnetism</u> $\underline{M} \neq 0$ for $\underline{B} = 0$.

Bound Currents

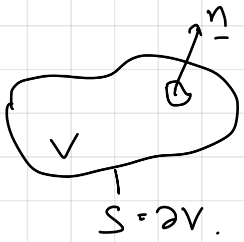
Contribution to vector potential from volume element d^3x'.

$$d\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \frac{\underline{M}(\underline{x}') \times (\underline{x} - \underline{x}')}{|\underline{x}-\underline{x}'|^3} d^3x' .$$

Integrate over a region $V \subset \mathbb{R}^3$ with $S = \partial V$

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3\underline{x}' \frac{\underline{M}(\underline{x}') \times (\underline{x} - \underline{x}')}{|\underline{x}-\underline{x}'|^3}$$

$$= \frac{\mu_0}{4\pi} \int_V d^3\underline{x}' \; \underline{M}(\underline{x}') \times \nabla' \left(\frac{1}{|\underline{x}-\underline{x}'|} \right)$$



$$\overset{\text{IBP}}{=} -\frac{\mu_0}{4\pi}\int_S \frac{d\underline{S}\times\underline{M}(\underline{x}')}{|\underline{x}-\underline{x}'|} + \frac{\mu_0}{4\pi}\int_V d^3\underline{x}' \frac{\nabla'\times\underline{M}(\underline{x}')}{|\underline{x}-\underline{x}'|} \tag{9}$$

Comparing to the general expansion to vector potential due to steady current, define <u>bound current density</u>

$$\underline{J}_{\text{bound}} = \nabla\times\underline{M} \tag{10}$$

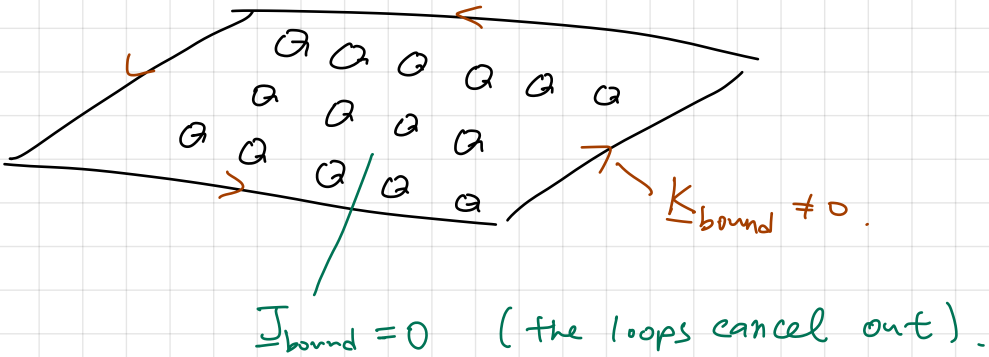
Have $\nabla\cdot\underline{J}_{\text{bound}} = 0 \Rightarrow$ steady current.

Similarly, 1st term in (9) corresponds to surface current density on S

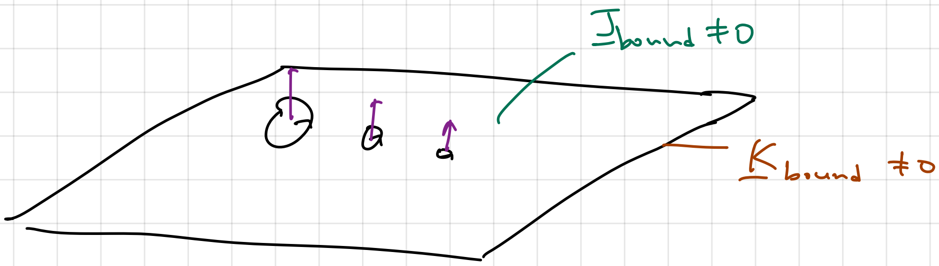
$$\underline{K}_{\text{bound}} = \underline{M}\times\underline{n} \tag{11}$$

- Constant $\underline{M} \Rightarrow \underline{J}_{\text{bound}} = \nabla\times\underline{M} = 0$

  $\Rightarrow$ uniform distribution of current loop.



- Varying $\underline{M} \Rightarrow \underline{J}_{\text{bound}} \neq 0$



<u>Ampere's Law</u> (Steady current

Introduce "free" current density $\underline{J}_{\text{free}}$.

$$\nabla\times\underline{B} = \mu_0(\underline{J}_{\text{free}} + \underline{J}_{\text{bound}})$$
$$= \mu_0(\underline{J}_{\text{free}} + \nabla\times\underline{M})$$

Absorb dependence on the bound current by defining magnetising field $\underline{H}$ by

$$\underline{H} = \frac{1}{\mu_0}\underline{B} - \underline{M} \qquad (12)$$

$$\Rightarrow \quad \nabla \times \underline{H} = \underline{J}_{\text{free}} \qquad (13)$$

Linear magnetisation $\quad \underline{M} = \frac{1}{\mu_0}\frac{\chi_m}{1+\chi_m}\underline{B}$

permeability

$$\Rightarrow \quad \underline{B} = \mu \underline{H}, \quad \text{with} \quad \mu = \mu_0(1+\chi_m)$$

As in vacuum case, Ampere's law needs modification for time-dependence field

$$\nabla \cdot \underline{J}_{\text{bound}} + \frac{\partial \rho_{\text{bound}}}{\partial t} = 0 \qquad (*)$$

where

$$\rho_{\text{bound}} = -\nabla \cdot \underline{P}$$

Need new definition

$$\underline{J}_{\text{bound}} = \nabla \times \underline{M} + \frac{\partial \underline{P}}{\partial t}.$$

Turn $(*)$ into

$$\nabla \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J}_{\text{free}} + \mu_0 \underline{J}_{\text{bound}}$$

$$= \mu_0 \underline{J}_{\text{free}} + \mu_0 \nabla \times \underline{M} + \mu_0 \frac{\partial \underline{P}}{\partial t}.$$

Rewrite in terms of $\underline{D}$ and $\underline{H}$

electric displacement

magnetising field

$$\Rightarrow \quad \nabla \times \underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{J}_{\text{free}}$$

For linear materials, $\underline{D} = \varepsilon \underline{E}$, $\underline{B} = \mu \underline{H}$.

Macroscopic Maxwell's equations

$$\nabla \cdot \underline{D} = \rho_{free} \quad (\mu 1) \qquad \nabla \times \underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{J}_{free} \quad (\mu 2)$$

$$\nabla \cdot \underline{B} = 0 \quad (\mu 3) \qquad \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad (\mu 4).$$

EM waves in matter (no "free" charges)

$$\nabla \cdot \underline{D} = 0 \qquad \nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t}$$

$$\nabla \cdot \underline{B} = 0 \qquad \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

where $\underline{D} = \varepsilon \underline{E}$, $\underline{B} = \mu \underline{H}$.

$$\frac{\partial^2 \underline{E}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{1}{\varepsilon}\frac{\partial \underline{D}}{\partial t}\right) = \frac{1}{\varepsilon}\frac{\partial}{\partial t}(\nabla \times \underline{H})$$

$$= \frac{1}{\mu\varepsilon} \nabla \times \frac{\partial \underline{B}}{\partial t}$$

$$= -\frac{1}{\mu\varepsilon} \nabla \times (\nabla \times \underline{E})$$

$$= \frac{1}{\mu\varepsilon} \nabla^2 \underline{E} \qquad (\text{using } \nabla \cdot \underline{E} = 0)$$

Here $$\frac{1}{v^2}\frac{\partial^2 \underline{E}}{\partial t^2} - \nabla^2 \underline{E} = 0$$

Similarly $$\frac{1}{v^2}\frac{\partial^2 \underline{H}}{\partial t^2} - \nabla^2 \underline{H} = 0 ,$$

with $v^2 = \frac{1}{\mu\varepsilon} < c^2$.

$$\Rightarrow \quad \begin{aligned} \underline{E} &= \underline{E}_0 \sin(\underline{k} \cdot \underline{x} - \omega t) \\ \underline{B} &= \mu \underline{H} = \underline{B}_0 \sin(\underline{k} \cdot \underline{x} - \omega t). \end{aligned}$$

where $\omega = v\,|\underline{k}|$,

$$\underline{k} \cdot \underline{E}_0 = \underline{k} \cdot \underline{B}_0 = 0$$

$$\underline{k} \times \underline{E}_0 = \omega \underline{B}_0$$

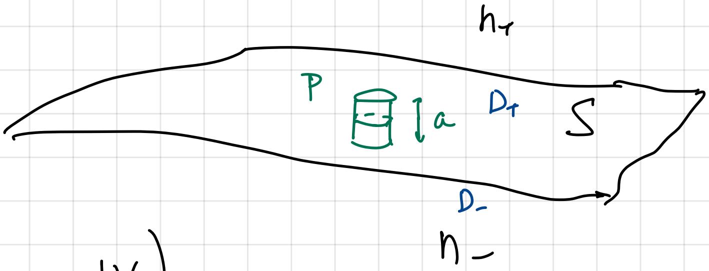
So have wave propagating with speed $v \leq c$.

Define index of refraction

$$n = c/v \geqslant 1.$$

Non-trivial effects occur when n varies: two regions $n = n_+$, $n = n_-$ separated by a surface S.

Integrate ($\mu 1$) inside P.



$$\nabla \cdot \underline{D} = \rho_{free}$$

$$\lim_{a \to 0}\left(\int_P \nabla \cdot \underline{D} \, dV\right) = \lim_{a \to 0}\left(\int_P \rho_{free} \, dV\right)$$

$$\Rightarrow \quad \underline{n} \cdot (\underline{D}_+ - \underline{D}_-) = \sigma \leftarrow \text{surface charge density}$$

Similarly ($\mu 3$) $\Rightarrow$ $\underline{n} \cdot (\underline{B}_+ - \underline{B}_-) = 0.$

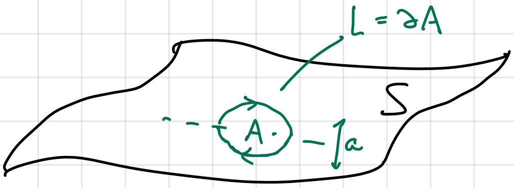
If no surface charge $\sigma = 0 \Rightarrow$ normal components of $\underline{D}$ and $\underline{B}$ are continuous at S. So normal components of

$$\underline{E} = \frac{1}{\varepsilon} \underline{D}, \qquad \underline{H} = \frac{1}{\mu_0} \underline{B}$$

are discontinuous.

Tangential components: integrate ($\mu 2$), ($\mu 4$) over a loop L.

Integrate ($\mu 4$)



$$\lim_{a \to 0}\left(\int_L \underline{E} \cdot d\underline{l}\right) = \lim_{a \to 0}\left(\int_A \nabla \times \underline{E} \cdot d\underline{S}\right)$$

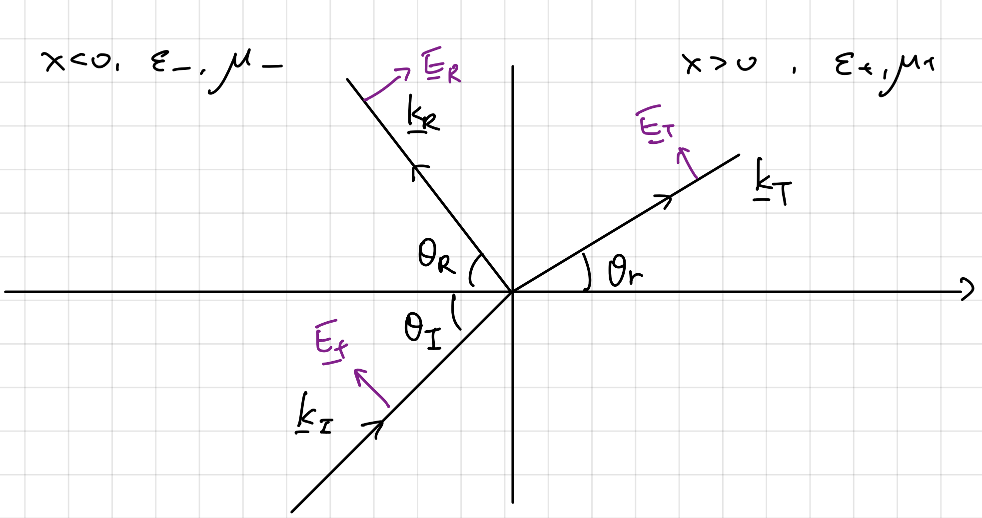
$$= \lim_{a \to 0}\left(\int_A -\frac{\partial \underline{B}}{\partial t} \cdot d\underline{S}\right) = 0$$

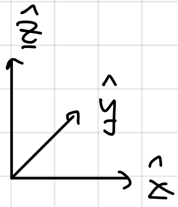
$$\Rightarrow \quad \underline{n} \times (\underline{E}_+ - \underline{E}_-) = 0$$

Similarly on ($\mu 2$), $\quad \underline{n} \times (\underline{H}_+ - \underline{H}_-) = \underline{K} \leftarrow$ surface current

In absence of surface currents, tangential components of $\underline{E}$ and $\underline{H}$ are cts ($\Rightarrow$ $\underline{D}$, $\underline{B}$ discts)

Reflection and Refraction





$$\underline{E}(\underline{x},t) = \begin{cases} \underline{E}_{inc} + \underline{E}_{Ref} & x<0 \\ \underline{E}_{trans} & x>0 \end{cases}$$

Incident wave
$$\underline{E}_{inc} = \underline{E}_I \sin(\underline{k}_I\cdot\underline{x} - \omega_I t)$$
$$\underline{k}_I = k_I\cos\theta_I\,\hat{\underline{x}} + k_I\sin\theta_I\,\hat{\underline{z}}.$$

Reflected wave
$$\underline{E}_{Ref} = \underline{E}_R \sin(\underline{k}_R\cdot\underline{x} - \omega_R t)$$
$$\underline{k}_R = -k_R\cos\theta_R\,\hat{\underline{x}} + k_R\sin\theta_R\,\hat{\underline{z}}.$$

Transmitted wave
$$\underline{E}_{trans} = \underline{E}_T \sin(\underline{k}_T\cdot\underline{x} - \omega_T t)$$
$$\underline{k}_T = k_T\cos\theta_T\,\hat{\underline{x}} + k_T\sin\theta_T\,\hat{\underline{z}}.$$

$$\hat{\underline{x}}\cdot(\underline{D}_+ - \underline{D}_-) = \hat{\underline{x}}\times(\underline{E}_+ - \underline{E}_-) = 0 \qquad (\rho)$$

where $\underline{D}_\pm(y,z,t) = \lim\limits_{\varepsilon\to 0}\left(\underline{D}(x=\pm\varepsilon, y, z, t)\right).$

These require the phase factors $\underline{k}\cdot\underline{x} - \omega t$ to match at $x=0$ $\forall\, y, z, t$.

$$\omega_I = \omega_R = \omega_T \tag{κ1}$$

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \tag{κ2}$$

In each region, freq and wavevector are related by

$$\omega_I = V_- k_I \quad . \omega_R = V_- k_R , \tag{η1}$$

$$\omega_T = V_+ k_T \tag{η2}$$

where $V_\pm = 1/\sqrt{\mu_\pm \varepsilon_\pm}$.

(κ1), (η1) :

$$\omega_I = \omega_R \Rightarrow k_I = k_R$$

$$\overset{(κ2)}{\Rightarrow} \quad \sin\theta_I = \sin\theta_R$$

$$\Rightarrow \quad \boxed{\theta_I = \theta_R} \qquad \text{(law of reflection)} \qquad (\mathrm{I})$$

(κ1), (η2) :

$$\omega_T = \omega_I = V_- R_I$$

$$\Rightarrow k_I = \frac{V_+}{V_-} k_T = \frac{n_-}{n_+} k_T \quad , \quad n_\pm = c/V_\pm$$

$$\overset{(κ2)}{\Rightarrow} \boxed{n_- \sin\theta_I = n_+ \sin\theta_T} \qquad \text{(Snell's law of refraction)} \qquad (\mathrm{II})$$

More information from polarisation vector...

Special case: Parallel polarisation ($\underline{E_I}, \underline{E_R}, \underline{E_T}$ lie in incident plane)

$$\underline{k_I} = k_I \cos\theta_I \, \underline{\hat{x}} + k_I \sin\theta_I \, \underline{\hat{z}}$$

$$\Rightarrow \underline{E_I} = -E_I \sin\theta_I \, \underline{\hat{x}} + E_I \cos\theta_I \, \underline{\hat{z}}$$

Magnetic field:

$$\underline{B}_{inc} = \underline{B_I} \sin(\underline{k_I} \cdot \underline{x} - \omega_I t)$$

$$\underline{B_I} = (\underline{\hat{k}_I} \times \underline{E_I})/v_- = B_I \, \underline{\hat{y}} \quad . \quad B_I = E_I/v_-$$

As $\theta_R = \theta_I$ and $k_R = k_I$,

$$\underline{k_R} = -k_I \cos\theta_I \, \underline{\hat{x}} + k_I \sin\theta_I \, \underline{\hat{z}}$$

$$\underline{E_R} = E_R \sin\theta_I \, \underline{\hat{x}} + E_R \cos\theta_I \, \underline{\hat{z}}$$

$$\underline{B}_{ref} = \underline{B}_R \sin(\underline{k}_R \cdot \underline{x} - \omega_R t)$$

$$\underline{B}_R = (\hat{\underline{k}}_R \times \underline{E}_R)/v_- = -B_R \hat{y}, \quad B_R = E_R/v_- \qquad (\nu 1)$$

$$\underline{k}_T = k_T \cos\theta_T \hat{\underline{x}} + k_T \sin\theta_T \hat{\underline{z}}$$

$$\underline{E}_T = -E_T \sin\theta_T \hat{\underline{x}} + E_T \cos\theta_T \hat{\underline{z}}$$

$$\underline{B}_{Trans} = \underline{B}_T \sin(\underline{k}_T \cdot \underline{x} - \omega_T t)$$

$$\underline{B}_T = (\hat{\underline{k}}_T \times \underline{E}_T)/V_+ = B_T \hat{y}, \quad B_T = E_T/V_+ \qquad (\nu 2)$$

Boundary conditions:

$$\hat{\underline{x}} \times (\underline{E}_+ - \underline{E}_-) = 0 \Rightarrow (E_I + E_R)\cos\theta_I = E_T \cos\theta_T \qquad (\psi 1)$$

$$\hat{\underline{x}} \cdot (\underline{B}_+ - \underline{B}_-) = 0 \Rightarrow B_I - B_R = B_T$$

$$\Rightarrow \frac{E_I - E_R}{V_-} = \frac{E_T}{V_+} \qquad (\psi 2)$$

Solve $(\psi 1), (\psi 2)$ to get Fnernel eqn

$$\frac{E_R}{E_I} = \frac{n_- \cos\theta_T - n_+ \cos\theta_I}{n_- \cos\theta_T + n_+ \cos\theta_I}$$

$$\frac{E_T}{E_I} = \frac{2n_- \cos\theta_I}{n_- \cos\theta_+ + n_+ \cos\theta_I}$$

Where $n_\pm = c/V_\pm$.

Combine w/ snell's law

$$\frac{n_-}{n_+} = \frac{\sin\theta_T}{\sin\theta_I}$$

We find, angle of incidence $\theta_I = \theta_B$ Brewster's angle s.t. $E_R = 0$ (no reflection)

$$E_- = 0 \Rightarrow n_- \cos\theta_T = n_+ \cos\theta_B \qquad (\chi 1)$$

$$\frac{n_-}{\sin\theta_T} = \frac{n_+}{\sin\theta_B} \qquad (\chi 2)$$

$(\chi 1)/(\chi 2) \Rightarrow \qquad \sin(2\theta_T) = \sin(2\theta_B)$

$\theta_T \neq \theta_B$, so must have $\theta_T = \frac{\pi}{2} - \theta_B$. Sub into $(\chi 1)$,

$$\boxed{\tan\theta_B = \frac{n_+}{n_-}}$$

Total internal reflection

If $n_- > n_+$,

$$\sin\theta_T = \left(\frac{n_-}{n_+}\right)\sin\theta_I$$

This has no real sol^n as an eq^n for θ_T when

$$\theta_I > \theta_{cr} = \sin^{-1}\left(\frac{n_+}{n_-}\right)$$

$$\underline{E}_{trans} = \underline{E}_T \sin(\underline{k}_T\cdot\underline{x} - \omega_T t)$$

$$= \mathrm{Re}\,(-i\,\underline{E}_T \exp(i(\underline{k}_T\cdot\underline{x} - \omega_T t))$$

$(\kappa 1) \Rightarrow \quad \omega_T = \omega_I$

$$\underline{k}_T = \hat{\underline{z}}(\underline{k}_T\cdot\hat{\underline{z}}) + \hat{\underline{x}}(\underline{k}_T\cdot\underline{x})$$

$(\kappa 2) \Rightarrow \quad \underline{k}_T\cdot\hat{\underline{z}} = \underline{k}_I\cdot\hat{\underline{z}} = \frac{\omega_I}{V_-}\sin\theta_I$

$$|\underline{k}_T|^2 = \frac{\omega_T^2}{V_+^2} = \frac{\omega_I^2}{V_+^2}$$

$$\underline{k}_T\cdot\hat{\underline{x}} = \pm\sqrt{|\underline{k}_T|^2 - (\underline{k}_T\cdot\hat{\underline{z}})^2}$$

$$= \pm\sqrt{\omega_I^2/V_+^2 - \omega_I^2\sin^2\theta_I/V_-^2}$$

$$= \pm\frac{\omega_I}{V_+}\sqrt{1 - \left(\frac{n_-}{n_+}\right)^2\sin^2\theta_I}$$

$$= \pm\frac{i\omega_I}{V_+}\alpha$$

$\alpha \in \mathbb{R}$ for $\theta_I > \sin^{-1}\left(\frac{n_+}{n_-}\right) = \theta_{cR}$.

Then

$$\underline{E}_{trans} = \mathrm{Re}\left(-i\,\underline{E}_T \exp\left(i\left(k_T \cdot \hat{\underline{z}} - \omega_I t\right) \exp\left(\mp \frac{\omega_I}{v_+} \alpha x\right)\right)\right)$$

Pick "$-$" sign for physical sol^n which decays exponentially for $x > 0$ (<u>evanescent wave</u>)

<u>Dispersion</u>

More realistic treatment: response of medium (e.g. polarisability) is <u>frequency dependent</u>.

$$\varepsilon = \varepsilon(\omega)$$

$$\nabla \cdot \underline{D} = 0 \qquad \nabla \times \underline{H} = \partial \underline{D} / \partial t$$

$$\nabla \cdot \underline{B} = 0 \qquad \nabla \times \underline{E} = -\partial \underline{B} / \partial t$$

$$\underline{E} = \underline{E}_0 \, e^{i\underline{k} \cdot \underline{x} - i\omega t}, \qquad \underline{B} = \underline{B}_0 \, e^{i\underline{k} \cdot \underline{x} - i\omega t}$$

$$\underline{D} = \varepsilon(\omega) \underline{E}, \qquad \underline{B} = \mu \underline{H}.$$

Polarisation vector

$$\underline{E}_0 = \underline{E}_0(\omega), \qquad \underline{B}_0 = \underline{B}_0(\omega)$$

$$\nabla \cdot \underline{D} = \nabla \cdot \underline{B} = 0 \Rightarrow \varepsilon(\omega)\, \underline{k} \cdot \underline{E}_0 = \underline{k} \cdot \underline{B}_0 = 0$$

$$\nabla \times \underline{H} = \partial \underline{D} / \partial t \Rightarrow \underline{k} \times \underline{E}_0 = \omega \underline{B}_0(\omega)$$

Note that $k^2 = |\underline{k}|^2 = \mu \varepsilon(\omega) \omega^2$. Useful to invert sol^n.

$$\boxed{\omega = \omega(k)} \longleftarrow \text{dispersion relation}$$

Waves propagate at speed

$$v_{phase} = \omega(k) / k.$$

Form <u>wavepackets</u> by linear superposition of different wave numbers.

$$E_{wp}(\underline{x}, t) = \int \frac{dk}{\sqrt{2\pi}} \underline{E}_0(k) \sin\left(kz - \omega(k)t\right)$$

Suppose wave packets sharply peaked around $k = k_0$.

$$W(k) = \omega(k_0) + \frac{d\omega}{dk}\Big|_{k=k_0}(k-k_0) + \mathcal{O}((k-k_0)^2)$$

Phase factor

$$kz - \omega(k)t = kz - \omega(k_0)t - \frac{d\omega}{dk}\Big|_{k=k_0}(k-k_0)t$$

$$= -\left(\omega(k_0) - \frac{d\omega}{dk}\Big|_{k=k_0} k_0\right)t + k\left(z - \frac{d\omega}{dk}\Big|_{k=k_0} t\right)$$

The wavepacket moves at speed

$$V_{group}(k) = \frac{d\omega(k)}{dk}$$

<u>The Drude Model</u>

Simple model of moving electrons, velocity $\underline{v}(t)$, in conductor.

$$m_e \frac{d\underline{v}}{dt} = q\underline{E} - \frac{m_e}{\tau}\underline{v}$$

τ is <u>scattering time</u> for e^-.

Solve in homogeneous time-dependent field.

$$\underline{E} = \underline{E}(\omega)e^{-i\omega t}, \quad \underline{v} = \underline{v}(\omega)e^{-i\omega t},$$

$$\Rightarrow \left(-i\omega + \frac{1}{\tau}\right)\underline{v}(\omega) = \frac{q}{m_e}\underline{E}(\omega)$$

$$\Rightarrow \underline{v}(\omega) = \frac{q\tau}{m_e}\frac{1}{1-i\omega\tau}\underline{E}(\omega)$$

The resulting current density is

$$\underline{J} = n_e q \underline{v} = \underline{J}(\omega)e^{-i\omega t}.$$

$$\Rightarrow \underline{J}(\omega) = \sigma(\omega)\underline{E}(\omega) \qquad (*)$$

where

$$\sigma(\omega) = \frac{\sigma_{DC}}{1-i\omega\tau}.$$

direct current, i.e. $\omega = 0$.

Conductivity

$$\sigma_{DC} = n_e q^2 \tau / m_e .$$

This is a frequency-dependent form of <u>Ohm's law</u>.

EM waves in conductors

$$\nabla \cdot \underline{D} = \rho_{free} , \qquad \nabla \times \underline{H} = \underline{J}_{free} + \frac{\partial D}{\partial t}$$

$$\nabla \cdot \underline{B} = 0 \qquad \nabla \times \underline{E} = - \frac{\partial B}{\partial t} .$$

with $\underline{D} = \varepsilon(\omega) \, \underline{E}$, $\underline{B} = \mu \underline{H}$.

Complex wave-form

$$\underline{E}(\underline{x}, t) = \underline{E}_0(\omega) \, e^{i(\underline{k} \cdot \underline{x} - \omega t)}$$

$$\underline{B}(\underline{x}, t) = \underline{B}_0(\omega) \, e^{i(\underline{k} \cdot \underline{x} - \omega t)}$$

Then

$$\underline{J}_{free} = \sigma(\omega) \, \underline{E}(\underline{x}, t)$$

$$= \sigma(\omega) \, \underline{E}_0(\omega) \, e^{i(\underline{k} \cdot \underline{x} - \omega t)}$$

where $\sigma(\omega) = \sigma_{DC} / (1 - i\omega\tau)$.

mem free speed.

Drude model is good is $\lambda = \dfrac{2\pi}{|\underline{k}|} \gg \rho = \langle v \rangle \tau$

Solve for charge density, $\rho_{free}(\underline{x}, t)$:

$$\nabla \cdot \underline{J}_{free} + \frac{\partial \rho_{free}}{\partial t} = 0$$

$$\Rightarrow \frac{\partial \rho_{free}}{\partial t} = - i \underline{k} \cdot \underline{J}_{free} = - i \underline{k} \cdot \left(\sigma(\omega) \, \underline{E}(\underline{x}, t) \right) e^{i(\underline{k} \cdot \underline{x} - \omega t)}$$

$$\Rightarrow \rho_{free} = \frac{\sigma(\omega)}{\omega} \, \underline{k} \cdot \underline{E}_0(\omega) \, e^{i(\underline{k} \cdot \underline{x} - \omega t)}$$

Now impose Maxwell eqn:

- $\nabla \cdot \underline{D} = \rho_{free} \Rightarrow i \, \varepsilon(\omega) \, \underline{k} \cdot \underline{E}_0(\omega) = \dfrac{\sigma(\omega)}{\omega} \, \underline{k} \cdot \underline{E}_0(\omega)$.

$$\Rightarrow i \left(\varepsilon(\omega) + i \frac{\sigma(\omega)}{\omega} \right) \underline{k} \cdot \underline{E}_0(\omega) = 0 . \qquad (*)$$

- $\nabla \cdot \underline{B} = 0 \quad \Rightarrow \quad \underline{k} \cdot \underline{B}_0(\omega) = 0$
- $\nabla \times \underline{H} = \underline{J}_{free} + \frac{\partial D}{\partial t}$

$$\Rightarrow \quad \frac{1}{\mu}\left(i\underline{k} \times \underline{B}_0(\omega)\right) = \sigma(\omega)\, \underline{E}_0(\omega) - i\omega\, \varepsilon(\omega)\ \underline{E}_0(\omega)$$

$$\Rightarrow \quad -i\mu\omega\left(\varepsilon(\omega) + i\,\frac{\sigma(\omega)}{\omega}\right) \underline{E}_0(\omega) = i\underline{k} \times \underline{B}_0(\omega).$$

- $\nabla \times \underline{E} = -\partial B/\partial t \quad \Rightarrow \quad \underline{k} \times \underline{E}_0(\omega) = \omega\, \underline{B}_0(\omega).$

Hence, contribution of free charges / current is equivalent to

$$\varepsilon(\omega) \longrightarrow \varepsilon^{eff}(\omega) = \varepsilon(\omega) + i\,\frac{\sigma(\omega)}{\omega}$$

The wave eqn:

$$\frac{1}{V^2}\frac{\partial \underline{E}}{\partial t} - \nabla^2 \underline{E} = 0$$

solved with

$$V = V_{eff} = \left(\varepsilon^{eff}(\omega)\,\mu\right)^{-1/2}.$$

<u>High-frequency limit</u>:

$$\sigma(\omega) = \frac{\sigma_{DC}}{1 - i\omega\tau} \approx \frac{i\,\sigma_{DC}}{\omega\tau}, \qquad \sigma_{DC} = n_e q^2 \tau / m_e$$

$$\varepsilon^{eff}(\omega) \approx \varepsilon(\omega) - \frac{\sigma_{DC}}{\omega^2 \tau}, \qquad \varepsilon(\omega) \xrightarrow{\omega \to \infty} \varepsilon_0$$

$$\Rightarrow \quad \varepsilon^{eff} \approx \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right).$$

where ω_p is the plasma frequency

$$\omega_p^2 = n_e q^2 / m_e \varepsilon_0 .$$

Then (*) $\Rightarrow$

$$i\,\varepsilon^{eff}(\omega)\ \underline{k} \cdot \underline{E}_0 = 0$$

For $\omega = \omega_p$, get new modes which are not transverse.

$$\underline{k} \cdot \underline{E}_0 \neq 0 .$$

These are called <u>plasma oscillations</u>.