Dynamical Systems

0. Dynamical Systems

A dynamical system is a set of eqns which describes the evolution of a set of variables with respect to a time-like variable. The possible states of the system define the state space / phase space.

Example Logistic map $x_{n+1} = \mu x_n (1 - x_n)$ with $0 \le \mu \le 4$.

Discrete in time, state phase $[0, 1]$.

Example Lotka - Volterra
$$\dot{r} = r(a - br - cs), \quad a, \dots, f > 0.$$
$$\dot{s} = s(d - er - fs)$$

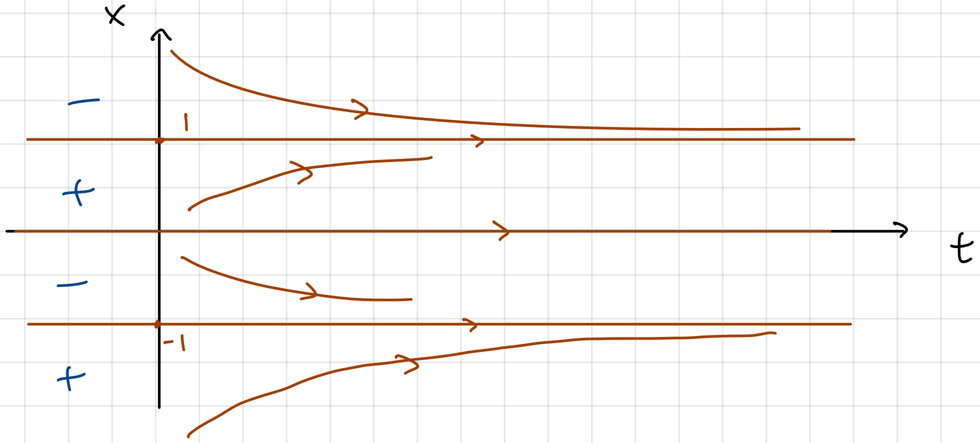
continuous in time, state phase $[0, \infty) \times [0, \infty)$.

For this course, we focus on ODEs, and maps in a small number of dimensions. It is not usually possible to write down a closed form solution. Instead, we aim to say something about long term behaviour using a mixture of geometric and analytic arguments. This is the "dynamical systems approach".

1D Example

Example $\dot{x} = x(1 - x^2)$

Formal sol^n: $x = \dfrac{\pm e^{t - t_0}}{\sqrt{1 + e^{2(t - t_0)}}}$, $x(0) = x_0 = \pm \dfrac{e^{-t_0}}{\sqrt{1 + e^{-2t_0}}}$



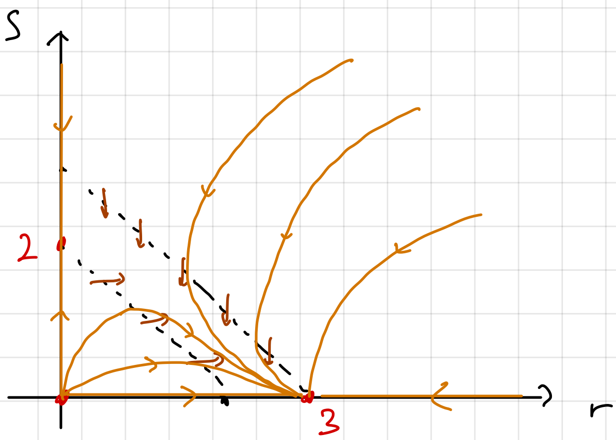
We can quickly see $X(0) > 0$, $x(t) \to 1$ as $t \to \infty$

$X(0) < 0$, $x(t) \to -1$ as $t \to \infty$

$X(0) = 0$. $x(t) = 0$ $\forall t$.

2D Example

Example $\dot{r} = r(3-r-s)$, $\dot{s} = s(2-r-s)$, $r, s \geq 0$.



Fixed points : $(r,s) = (0,0)$, $(0,2)$, $(3,0)$.

As $r, s \to \infty$, $\dot{r} = -r(r+s) < 0$. $\dot{s} = -s(r+s) < 0$.

At $r = 0$, $\dot{s} = s(2-s)$, At $s = 0$, $\dot{r} = r(3-r)$

So the long term behaviour :

$s \geq 0$, $r > 0$. $(r,s) \to (3,0)$ as $t \to \infty$

$s > 0$, $r = 0$ $(r,s) \to (0,2)$

$r = s = 0$ $(r,s) \to (0,0)$

Example $\dot{x} = -y + \varepsilon x(\mu - x^2 - y^2)$

$$\dot{y} = x + \varepsilon y(\mu - x^2 - y^2)$$

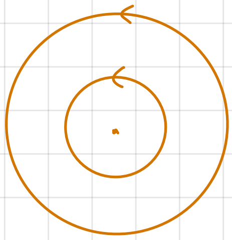
Change to polars : $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}, \quad \dot{\theta} = \frac{x\dot{y} - \dot{x}y}{r^2}.$$

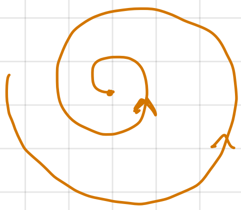
$$\Rightarrow \quad \dot{r} = \varepsilon r(\mu - r^2), \quad \dot{\theta} = 1.$$

Different cases depending on ε and μ. (parameters of the problem).

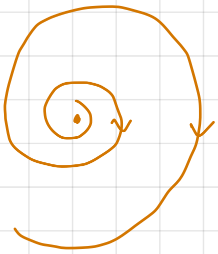
$\varepsilon = 0$:



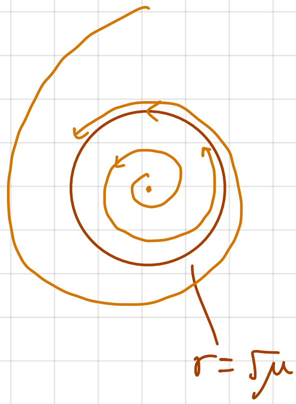
$\varepsilon > 0,\ \mu \le 0$



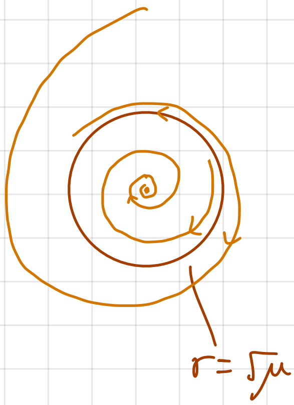
$\varepsilon < 0,\ \mu \le 0$:



$\varepsilon > 0,\ \mu > 0$:



$\varepsilon < 0,\ \mu > 0$:



For $\mu > 0$, there is a periodic orbit at $r = \sqrt{\mu}$

If $\varepsilon \neq 0$ and μ is increased, a periodic orbit is created at $\mu = 0$. this is Bifurcation.

At $\varepsilon = 0$, the system is sensitive to small changes in ε.

This is Structurally unstable.

Looking ahead into 3 or more dimensions, ODEs can have much more complicated behaviour including chaos. For maps, chaos is possible even in a 1-D map i.e.g. Logistic map.

1. Basic Definitions

1.1 Notation

Only consider ODEs for $\underline{x}$ in phase/state space $E \subseteq \mathbb{R}^n$ of the form

$$\dot{\underline{x}} = \frac{d\underline{x}}{dt} = \underline{f}(\underline{x}).$$

The n 1^{st} order eqn gives a dynamical system of order n. Since $\underline{f}$ does not depend on t, the system is autonomous.

Note: • A non-autonomous system $\dot{\underline{x}} = \underline{g}(\underline{x}, t)$ can be made autonomous by extending the def^n of $\underline{x}$. let $\underline{y} = (\underline{x}, t) \in \mathbb{R}^{n+1}$, and $T = t$,

$$\frac{d\underline{y}}{dT} = \begin{pmatrix} \underline{g}(\underline{x}, t) = \underline{F}(\underline{y}) \\ 1 \end{pmatrix}$$

• Any n th order ODE can be written as a series of n first order ODEs.

1.2 Initial Value Problems

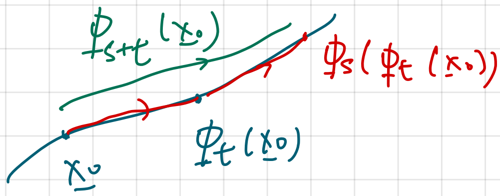
Typically, seek sol^{n}s to $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$. as an initial value problem (IVP).

"Given an initial condition $\underline{x}(t_0) = \underline{x}_0$ $(\underline{x}_0 \in E, t \in I \subseteq \mathbb{R})$, find a diff. f^n $\underline{x}(t)$ for $t \in I$ which remains in E and satisfies the initial condition and the diff. eqn".

For an autonomous system, we can alternatively define the solution in terms of a 'flow' ϕ_t.

Def^n (Flow) $\phi_t(\underline{x}_0)$, the sol^n at time t of $\dot{\underline{x}} = f(\underline{x})$ starting at $\underline{x}_0$ at $t=0$, is called a <u>flow</u> through $\underline{x}_0$ at $t=0$.

- $\phi_0(\underline{x}_0) = \underline{x}_0$
- $\phi_s(\phi_t(\underline{x}_0)) = \phi_{s+t}(\underline{x}_0)$



Key question: does a sol^n exist? is it unique?

<u>Thm</u> (Cauchy-Peano) If $f(\underline{x},t)$ cts in the domain $D := \{|t-t_0|<\alpha,\ |\underline{x}-\underline{x}_0|<\beta\}$ with $|\underline{f}|<M$ over D, then the IVP has a sol^n for $|t-t_0| < \min\{\alpha, \beta/M\}$.

<u>Remark</u>: uniqueness is guarenteed only for stronger conditions on f.

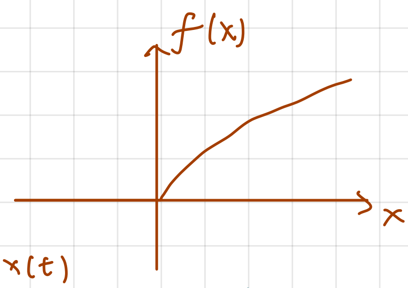
<u>Example</u> (unique sol^n) $\dot{x} = |x|$, $x(t_0) = x_0$

Sol^n:

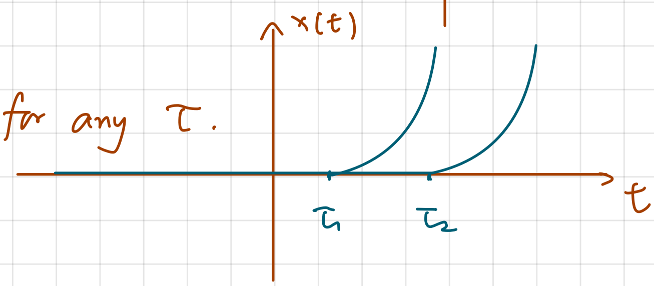
$$x(t) = \begin{cases} x_0 e^{t-t_0} & x>0 \\ 0 & 0 \\ x_0 e^{t_0-t} & x<0 \end{cases}$$

here f is cts but not diff. at $x=0$.

<u>Example</u> (Non-unique sol^n) $\dot{x} = \begin{cases} \sqrt{x} & x \geq 0 \\ 0 & x<0 \end{cases}$



Sol^n: $x = \begin{cases} 0 & t<\tau \\ \frac{1}{4}(t-\tau)^2 & t \geq \tau. \end{cases}$ for any τ.



Here f cts, not diff at $x=0$.

What's the difference? $\frac{d}{dx}(\sqrt{x})$ is not bounded at $x=0$.

To guarantee uniqueness, we need the stronger property that $\underline{f}$ is Lipschitz.

Def^n A f^n f defined on a subset of $\mathbb{R}^n$ satisfies a Lipschitz condition at point $\underline{x_0}$ with the Lipschitz constant L if $\exists (L, a)$ s.t. with $|\underline{x} - \underline{x_0}| < a$, $|\underline{y} - \underline{x_0}| < a$, then

$$|\underline{f}(\underline{x}) - \underline{f}(\underline{y})| \leq L\, |\underline{x} - \underline{y}|.$$

Note: differentiability $\Rightarrow$ Lipschitz $\Rightarrow$ continuity.

Thm (Picard-Lindelöf) Consider an IVP $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ with $\underline{x}(t_0) = \underline{x_0}$. If $\underline{f}$ satisfies a Lipschitz condition at $\underline{x_0}$, then the sol^n $\underline{\phi}_{t-t_0}(\underline{x_0})$ exists and is unique and cts in a neighbourhood of $(\underline{x_0}, t)$.

Note: Uniqueness and cty do not mean sol^n exist for all time.

Example (finite time blow up) $\dot{x} = x^2$, $x(0) = 1$.

sol^n is $x(t) = \frac{1}{1-t}$ has finite blow up as $t \to 1^-$.

From now on, we only consider diff $\underline{f}$ for rest of course.

1.3 Orbits, Interval sets and limit sets

Use the idea of a flow to make the following def^n.

Def^n The orbit / trajectory of $\underline{\phi}_t$ through $\underline{x_0}$ is the set

$$\theta(\underline{x_0}) = \{ \underline{\phi}_t(\underline{x_0}) : -\infty < t < \infty \}$$

The forward orbit is $\theta^+(\underline{x_0}) = \{ \underline{\phi}_t(\underline{x_0}) : t \geq 0 \}$.

The backward orbit is $\mathcal{O}^-(\underline{x}_0) = \{\phi_t(\underline{x}_0) : t \le 0\}$.

Def^n A set of points $\Lambda \subseteq E$ is an invariant set under $\underline{f}$ if $\underline{x} \in \Lambda \Rightarrow \mathcal{O}(\underline{x}) \subseteq \Lambda$

Clearly any orbit is invariant as is a union of orbits.

Special invariant sets:

- $\underline{x}_0$ is a fixed point (FP) (eqm / stationary point) if $\underline{f}(\underline{x}_0) = 0$, so $\mathcal{O}(\underline{x}_0) = \{\underline{x}_0\}$.
- $\underline{x}_0$ is a periodic point with period T if $\phi_T(\underline{x}_0) = \underline{x}_0$ for some T but $\phi_t(\underline{x}_0) \neq \underline{x}_0$ for $0 < t < T$. The set $\{\phi_t(\underline{x}_0) : 0 < t < T\}$ is called the periodic orbit (PO) through $\underline{x}_0$.

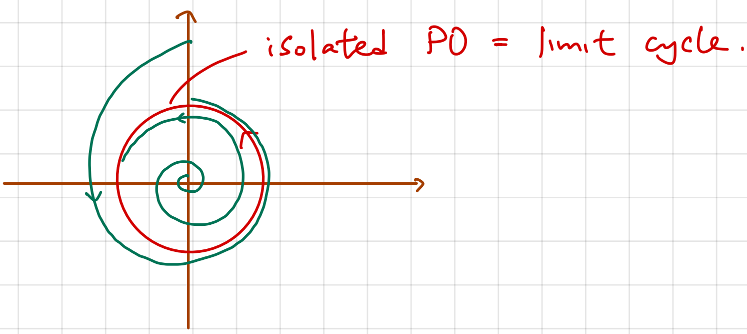
Def^n A limit cycle is an isolated periodic orbit (no other POs in its neighbourhood)

Example $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^3 \\ x^3 \end{pmatrix}$, sol^n: $x^4 + y^4 = $ const.

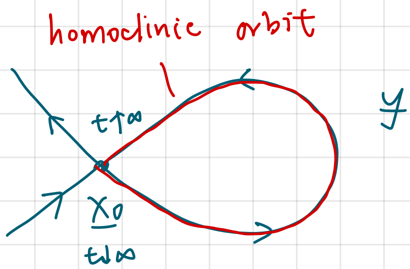
Continuum of POs (no limit cycles).

Now contrast with

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y + x(1-x^2-y^2) \\ x + y(1-x^2-y^2) \end{pmatrix} \Rightarrow \begin{array}{l} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 \end{array}$$



Def^n If $\underline{x}_0$ FP and $\exists\, y \neq \underline{x}_0$ s.t. $\phi_t(y) \to \underline{x}_0$ as $t \to \pm\infty$, then $O(y)$ is a <u>homoclinic orbit</u>.

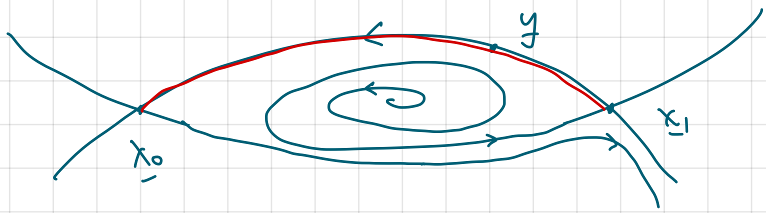


Def^n If $\underline{x}_0 \neq \underline{x}_1$ are both FPs and $\exists\, y$ s.t.

$$\phi_t(y) \to \underline{x}_0 \quad \text{as} \quad t \to +\infty$$

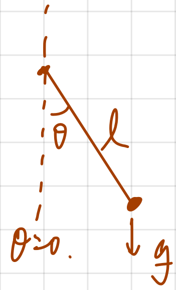
$$\phi_t(y) \to \underline{x}_1 \quad \text{as} \quad t \to -\infty$$

then $O(y)$ is called a <u>heteroclinic orbit</u>.



Example (Pendulum) $\ddot{\theta} = -\sin\theta$

$$\Rightarrow \tfrac{1}{2}\dot{\theta}^2 = c + \cos\theta \quad (c \text{ const.})$$



Heteroclinic orbit has $c=1$ so $\dot{\theta}=0$ at $\theta = \pm\pi$

$$\Rightarrow \tfrac{1}{2}\dot{\theta}^2 = 2\cos^2\tfrac{\theta}{2}$$

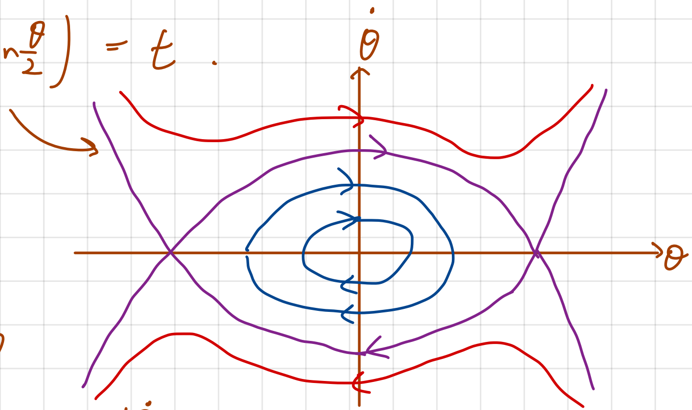
$$\Rightarrow \dot{\theta} = \pm 2\cos\tfrac{\theta}{2}$$

$$\Rightarrow \ln\left(\sec\tfrac{\theta}{2} + \tan\tfrac{\theta}{2}\right) = t.$$

When $t \to -\infty$, $\theta \to -\pi$

$t = 0$, $\theta = 0$

$t \to +\infty$ $\theta \to \pi$



Consider $\ddot{\theta} = -\sin\theta$ → $\ddot{\theta} \approx -\theta$

→ $\frac{d\theta}{dt} = \dot{\theta}$, $\frac{d\dot{\theta}}{dt} = -\sin\theta$.

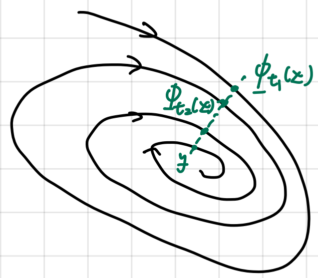
If we are interested in the long-term behaviour of the trajectories, it is not enough to just think of $\lim_{t\to\infty} \underline{\Phi}_t(\underline{x}_0)$ since this may not exist (e.g. periodic orbit).

Instead, we use these def^n:

The ω-limit set of $\underline{x}$ is

$$\omega(\underline{x}) = \{ \underline{y} : \exists \text{ infinite sequence } t_1, t_2, \ldots \to \infty \text{ with } \underline{\Phi}_t(\underline{x}) \to \underline{y} \}$$

α-limit set similarly defined with $t \to -\infty$.



The ω-limit set has some nice properties when $\theta(\underline{x})$ bounded, in particular, $\omega(\underline{x})$ is

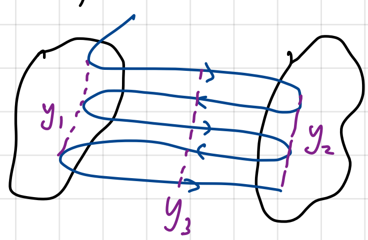
(a) Non-empty (every sequence in a closed bounded domain has at least one accumulation point)

(b) Invariant under $\underline{f}$

(c) Closed (think about sequence of limit points in $\omega(\underline{x})$)

(d) Bounded

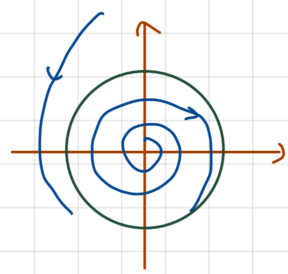
(e) Connected.



Example $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$ for $0 < |\underline{x}| < 1$

So $\omega(\underline{x}) = \{r = 1\}$,

$\alpha(\underline{x}) = \{0\}$



only consider autonomous maps

For maps $\underline{x}_{n+1} = \underline{F}(\underline{x}_n, \cancel{n})$, $\underline{x}_{n+2} = \underline{F}(\underline{x}_{n+1}, \underline{x}_n)$. Fixed points have $\underline{F}(\underline{x}) = \underline{0}$. Other def^{n}s are similar but we swap $\underline{\Phi}_t(\underline{x})$ with $\underline{F}^n(\underline{x})$. Periodic orbits now have discrete integer points where $\underline{x} = F^N(\underline{x})$ and $\underline{x} \neq F^n(\underline{x})$ with $n<N$ — called an

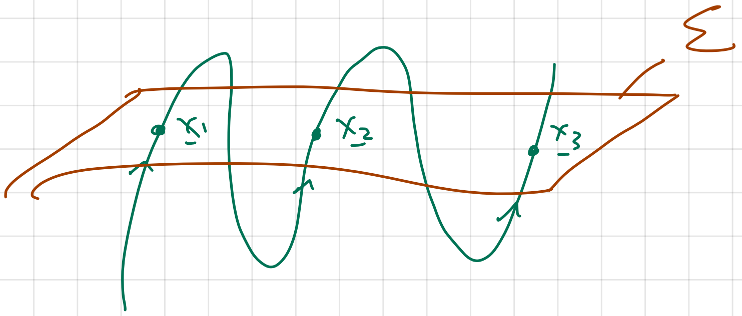
<u>N-cycle</u>.

<u>Note</u>: Can make maps from ODEs in 2 simple ways

(1) Take snapshots after a fixed time interval s.t.

$$\underline{x}_{n+1} = \underline{F}(\underline{x}_n) = \underline{\phi}_{\delta t}(\underline{x}_n)$$

(2) Poincaré return map — successive intersections of the trajectory with a surface Σ.



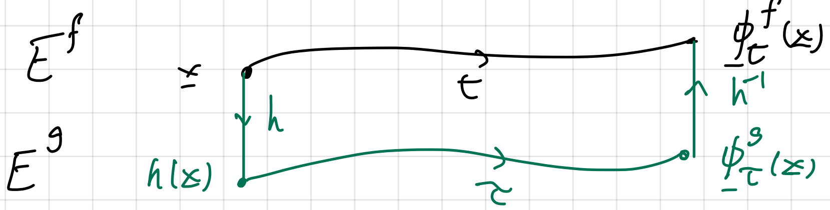
ODE dynamics 3D
map dynamics 2D

1.4 Topological Equivalence and Structural Stability of Flows

What do we mean by saying that 2 dynamical systems have "essentially" the same (topological / geometric) structures? Or that the structure of a system changes at a bifurcation?

<u>Defn</u> Two flows $\underline{\phi}^f_t(\underline{x})$ and $\underline{\phi}^g_t(\underline{x})$ are topologically equivalent if $\exists$ homeomorphism $h(\underline{x}) : E^f \to E^g$ (ie. a cts bij w/ cts inverse) and time-increasing f^n $\tau = \tau(\underline{x}, t)$ (ie a cts monotonic increasing f^n of $\underline{x}$ and t) with

$$\underline{\phi}^f_t(\underline{x}) = \underline{h}^{-1} \circ \underline{\phi}^g_\tau(\underline{x}) \circ h$$

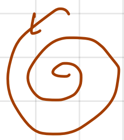


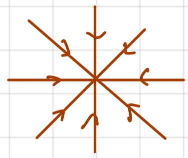
In other words, $\exists$ a "nice" map h from one phase space to the other phase space and a map τ from time in one to the other.

Clearly h maps FP's to FPs and POs to POs. If $\tau = t$, we say that the two systems (or flows) are topologically conjugate.

Example

$$\dot{r} = -r \qquad \text{and} \qquad \dot{\rho} = -2\rho$$
$$\dot{\theta} = 1 \qquad\qquad\qquad \dot{\varphi} = 0$$





are topologically equivalent with $\underline{h}(\underline{0}) = \underline{0}$ and

$$\underline{h}(r,\theta) = (r^2, \theta + \log r)$$

Check

$$h^{-1}(r,\theta) = (\sqrt{r}, \theta - \log\sqrt{r}).$$

$$\underline{\phi}_t^f(r_0,\theta_0) = (r_0 e^{-t}, \theta_0 + t)$$

$$\underline{\phi}_t^g(\rho_0,\varphi_0) = (\rho_0 e^{-2t}, \varphi_0)$$

$$\text{RHS} = \underline{\phi}_t^f(r_0,\theta_0) = h^{-1}\left(\underline{\phi}_t^g(h(r_0,\theta_0))\right)$$
$$= h^{-1}\left(\underline{\phi}_t^g(r_0^2, \theta_0 + \log r_0)\right)$$
$$= h^{-1}\left(r_0^2 e^{-2t}, \theta_0 + \log r_0\right)$$
$$= \left(r_0 e^{-t}, \theta_0 + \log r_0 - \tfrac{1}{2}\log(r_0^2 e^{-2t})\right)$$

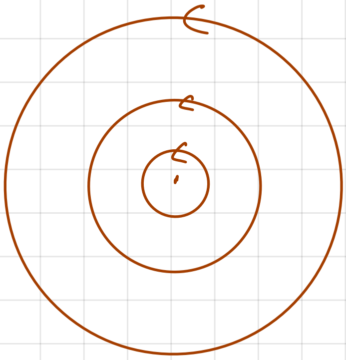
$$\text{LHS} = (r_0 e^{-t}, \theta_0 + t)$$

__Example__ $\dot{r} = 0, \ \dot{\theta} = 1$ and $\dot{r} = 0, \ \dot{\theta} = \rho + \sin^2\theta$

Same phase diagram for both.

So are topologically equivalent with $h(\underline{x}) = \underline{x}$ and the stretch in time

$$\frac{d}{d\tau} = (r + \sin^2\theta)\frac{d}{dt}$$



__Def__^{n} The vector field $\underline{f}$ (if $\dot{\underline{x}} = \underline{f}(\underline{x})$) is __structurally stable__ if $\exists \varepsilon > 0$ s.t. $\underline{f} + \underline{\delta}$ is topologically equivalent to $\underline{f}$ for all $\underline{\delta}(\underline{x})$ with $|\underline{\delta}| + \sum_i \left|\frac{\partial \underline{\delta}}{\partial x_i}\right| < \varepsilon$.

__Note__: The systems in first example are structurally stable, whereas those in second one are not. The periodic orbits are destroyed by small perturbations in $\dot{r}$.

2. Fixed Points

What happens in their neighbourhood ?

2.1 Linearisation

If $\underline{f}$ is sufficiently smooth near a fixed point $\underline{x}_0$, set $\underline{y} = \underline{x} - \underline{x}_0$ and Taylor expand $\dot{\underline{x}} = \underline{f}(\underline{x})$,

$$\frac{d}{dt}(\underline{x}_0 + \underline{y}) = \underline{f}(\underline{x}_0 + \underline{y})$$

$$\begin{matrix} \dot{\underline{x}}_0 = 0, \\ \underline{f}(\underline{x}_0) = 0 \end{matrix} \Rightarrow \quad \dot{\underline{y}} = \underline{y}(\underline{x}_0) + \underline{\underline{A}} \cdot \underline{y} + O(|\underline{y}|^2).$$

where $A_{ij} = \frac{\partial f_i}{\partial x_j}$ Jacobian.

The hope is that the dynamics near $\underline{x_0}$ in the full system is captured by the (reduced) linear system

$$y = \underline{\underline{A}} \cdot y$$

Consider 2D matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$D = \det A = ad - bc$$
$$T = \operatorname{tr} A = a + d$$

Evals of A satisfy $\lambda^2 - T\lambda + D = 0$.

$$\Rightarrow \lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

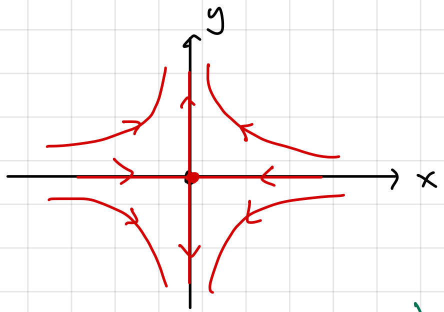
By suitable changes in basis, we can always put A into a standard form.

Possibilities :

$\lambda_1, \lambda_2 \in \mathbb{R}$, opposite signs

① Saddle point $(D < 0)$.

$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. WLOG $\lambda_1 < 0$, $\lambda_2 > 0$.



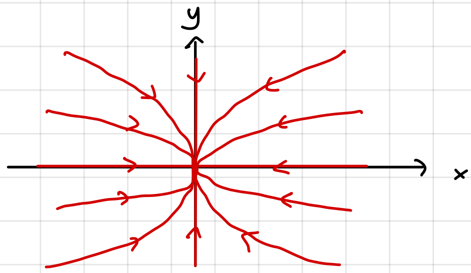
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 < 0$.

② Stable Node $(D > 0,\ T < 0,\ T^2 > 4D)$

$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, WLOG $\lambda_1 < \lambda_2 < 0$

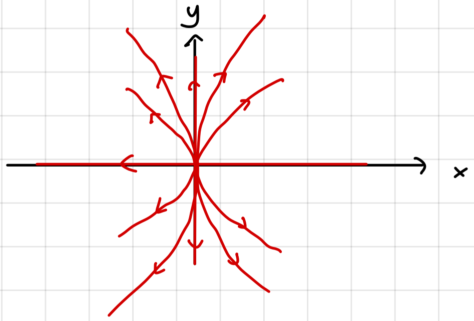
$$\left.\begin{aligned} x &= x_0 e^{\lambda_1 t} \\ y &= y_0 e^{\lambda_2 t} \end{aligned}\right| \Rightarrow y/x = \frac{y_0}{x_0} e^{(\lambda_2 - \lambda_1)t}$$

$$\Rightarrow y = \mu x^{\lambda_2/\lambda_1}$$



③ Unstable node ($D>0,\ T>0,\ T^2>4D$) — $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1, \lambda_2 > 0.$

$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ WLOG $0<\lambda_1<\lambda_2$

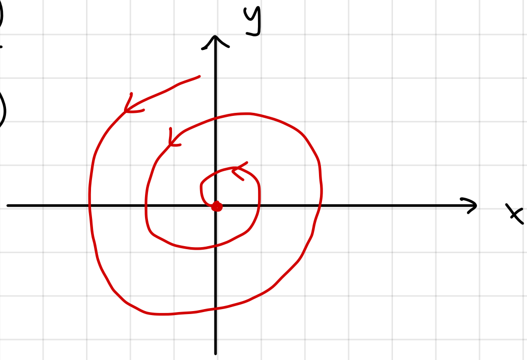


Same as ②, reversed arrow

④ Stable Focus ($D>0,\ T<0,\ T^2<4D$)

$\lambda = \rho \pm i\omega$, $\rho<0$, $\omega \neq 0$. (WLOG $\omega>0$)

$A = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \Rightarrow \begin{array}{l} \dot r = \rho r \\ \dot\theta = \omega . \end{array}$



⑤ Unstable focus ($D>0,\ T>0,\ T^2<4D$)

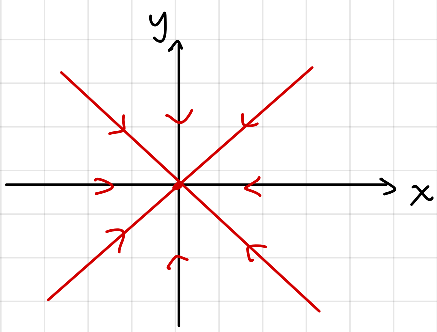
$\lambda = \rho \pm i\omega,\quad \rho>0.$

Same as ④ but spiral outward.

The rest are degenerate in some way

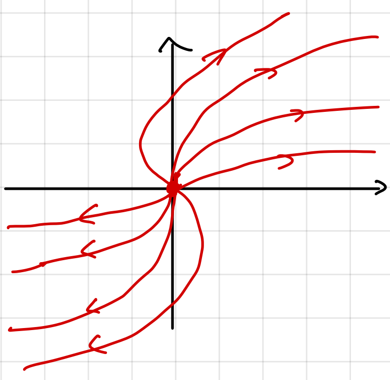
- $D>0,\ T^2=4D$ — boundary between a focus and a node

$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ Steller node, e.g. $\lambda<0$



$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ Improper node

$$\begin{cases} \dot x = \lambda x + y \\ \dot y = \lambda y \end{cases} \Rightarrow y = y_0 e^{\lambda t}.$$



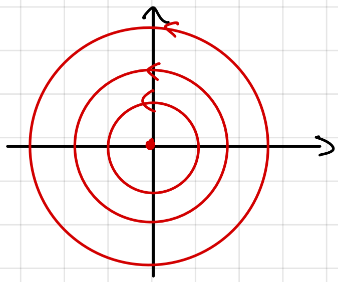
$\Rightarrow \dot x = \lambda x + y_0 e^{\lambda t}$

$\Rightarrow x = x_0 e^{\lambda t} + y_0 t e^{\lambda t}.$

$\Rightarrow \dfrac{y}{x} = \dfrac{y_0}{x_0 + y_0 t}$

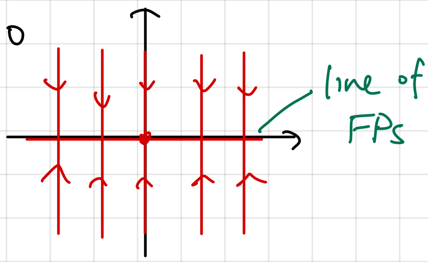
- $D > 0$, $T = 0$ — boundary between stable and unstable foci.

$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$ centre



- $D = 0$, $T \neq 0$ — between a saddle and a node

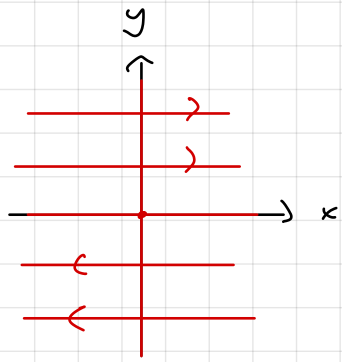
$A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$ $\dot{x} = 0$, $\dot{y} = \lambda y$ e.g. $\lambda < 0$



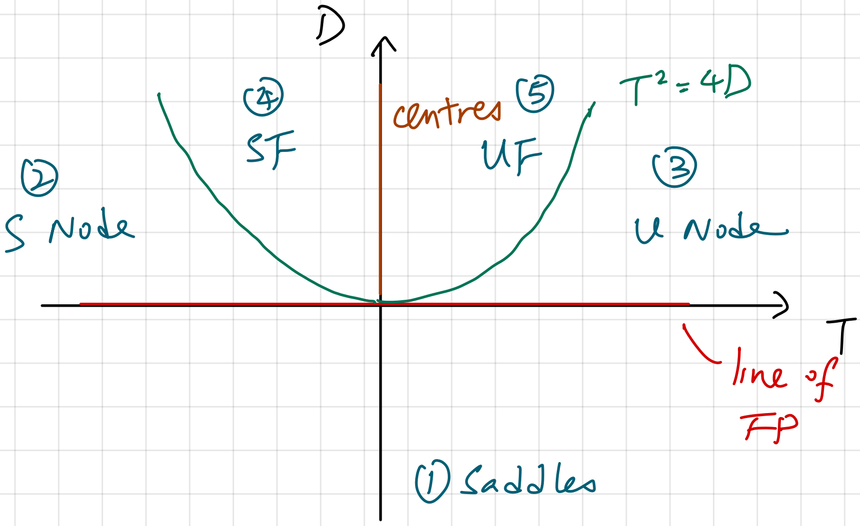
- $D = 0$, $T = 0$ — $\lambda = 0, 0$.

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ — plane of fixed points

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\dot{x} = y$, $\dot{y} = 0$ — line of FP with drift in x



Summary



These canonical forms are in a special basis

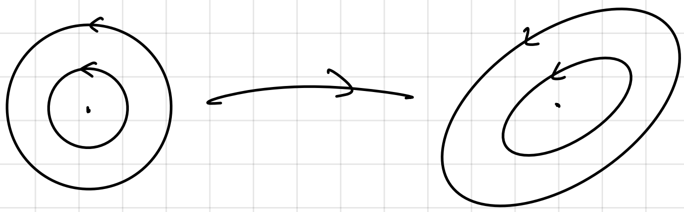
- $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ use evecs

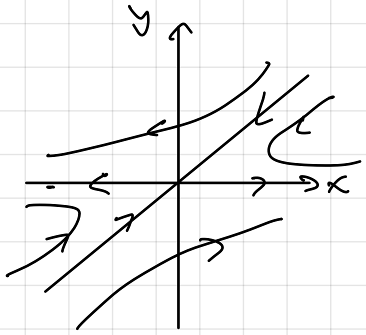
- $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ use generalised evecs

- $\begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}$ use Re and Im parts of evecs.

What happens in general? Do not need to change the basis to classify FP – just find T and D.

Diagrams above can be distorted.





Sometimes it is useful to know evecs for saddles and nodes. For a focus or a centre not worth doing. Often it is far easier to plug values in and plot/sketch locally.

<u>Final comment</u>: centres are rare but generic in certain type of systems, e.g. Hamiltonian systems.

Hamiltonian systems:

$$\dot{\underline{x}} = \begin{pmatrix} \partial H/\partial y \\ -\partial H/\partial x \end{pmatrix} = \underline{f}$$

where $H = H(x,y)$, $\underline{x} = (x,y)^T \in \mathbb{R}^2$.

$$\text{Jacobian} = \nabla \underline{f} = \begin{pmatrix} \partial_x\partial_y H & \partial_y^2 H \\ -\partial_x^2 H & -\partial_x\partial_y H \end{pmatrix}$$

So $\operatorname{tr}(\nabla \underline{f}) = 0 \Rightarrow$ always have centres $\lambda = \pm i\omega$ or saddles $\pm\lambda$.

2.2 Effect of non-linear terms

When does the linearisation $\dot{y} = Ay$ of the system $\dot{x} = f(x)$ about the fixed point x_0 tell us the behaviour of the non-linear system about x_0?

If (1) x_0 is hyperbolic and (2) the nonlinear corrections are $O(|x - x_0|^2)$, then the nonlinear system are topologically congruent.

Def^n A fixed point x_0 of $\dot{x} = f(x)$ is hyperbolic if none of the evals of the Jacobian at x_0 have zero real part, o/w. it is non-hyperbolic.

Classify hyperbolic points as follows:

- Sources if all $Re(\lambda) > 0$ (unstable nodes + unstable focus)
- Sinks if all $Re(\lambda) < 0$ (stable nodes + stable focus)
- Saddles o/w.

We discuss hyperbolic and non-hyperbolic FPs separately.

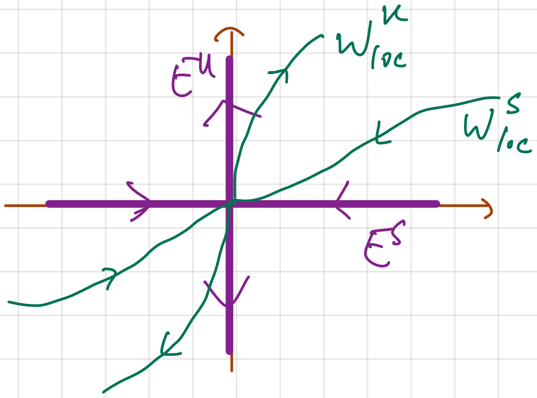
- Hyperbolic points

  For the linearised system, we can separate the phase space into different domains, corresponding to different behaviours in time.

Def^n The stable, unstable and centre subspaces of the linearisation of f at the fixed point x_0 are the 3 linear subspaces E^s, E^u, E^c spanned by the subsets of (possibly generalised) evecs whose evals have $Re(\lambda) < 0, > 0, = 0$ resp.

Note A hyperbolic point has an empty E^c.

Example $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



Thm (Stable Manifold thm) Suppose $\underline{0}$ is a hyperbolic fixed point of $\dot{\underline{x}} = \underline{f}(\underline{x})$, and that E^u, E^s are unstable and stable subspaces of the linearisation at $\underline{x}_0$. Then $\exists$ local stable and unstable manifolds $W^s_{loc}(\underline{0})$ and $W^u_{loc}(\underline{0})$ which have the same dimensions as E^s and E^u and are tangent to E^s and E^u at $\underline{0}$ s.t. for $\underline{x} \neq \underline{0}$ but in a sufficiently small nbd. of $\underline{0}$, then

$$W^u_{loc}(\underline{0}) = \{ \underline{x} : \phi_t(\underline{x}) \to \underline{0} \quad \text{as } t \to -\infty \}$$
$$W^s_{loc}(\underline{0}) = \{ \underline{x} : \phi_t(\underline{x}) \to \underline{0} \quad \text{as } t \to \infty \}.$$

Pf: is rather involved and outside course, but depends on convergence issues and can't be local.

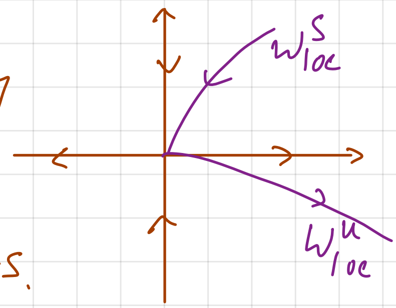
The point is that W^s_{loc} can be extended to a global invariant manifold W^s by following the flow backwards in time from points in W^s_{loc} (Do to W^u from W^u_{loc}. $t \to +\infty$).

This thm guarantees the existence of 2 trajectories that approach and leave a saddle in $\mathbb{R}^2$. Often easy in practice to find local approximations of stable and unstable manifolds to FPs.

Example For a saddle in $\mathbb{R}^2$, WLOG (can change basis), assume E^S is $x=0$, E^u is $y=0$.

Write W^s_{loc} as $x=S(y)$. Can immediately write $0=S(0)$ FP, $S'(0)=0$ tangent condition.

$\frac{dx}{dy}=0 \leftarrow E^S$.



Similarly, write W^u_{loc} as $y=U(x)$, then $U(0)=0$ FP, $U'(0)=0$ tangent condition.

Manifolds are invariant, so $\dot{x}=\frac{dS}{dx}\dot{y}$ must solve $\dot{\underline{x}}=\underline{f}(\underline{x})$. ditto, $\dot{y}=\frac{du}{dy}\dot{x}$ must be consistent with $\dot{\underline{x}}=\underline{f}(\underline{x})$.

So have ODEs of S and U usually can make progress by series expansion and comparing like terms.

Example $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}x(1-y)\\ -y+x^2\end{pmatrix}$.

$$A=\begin{pmatrix}1-y & -x\\ 2x & -1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Bigg|_{(0,0)}.$$

For W^u_{loc}, set $y=U(x)=a_2x^2+a_3x^3+a_4x^4+\dots$ (since $U(0)=0$, $U'(0)=0$).

Substitute this into $\dot{y}=-y+x^2=\frac{du}{dx}\dot{x}=\frac{du}{dx}x(1-y)$

$$\Rightarrow -U(x) + x^2 = \frac{du}{dx} x(1-U(x))$$

$$\Rightarrow (1-a_2)x^2 - a_3x^3 - a_4x^4 + \dots = (2a_2x + 3a_3x^2 + \dots)\, x(1-a_2x^2 - a_3x^3 \dots)$$

Comparing like terms of x,

$$x^2: \quad 1-a_2 = 2a_2 \Rightarrow a_2 = \tfrac{1}{3}$$

$$x^3: \quad -a_3 = 3a_3 \Rightarrow a_3 = 0$$

$$x^4: \quad -a_4 = -2a_2^2 + 4a_4 \Rightarrow a_4 = \tfrac{2}{5}a_2^2 = \tfrac{2}{45}$$

So $U(x) = \frac{1}{3}x^2 + \frac{2}{45}x^4 + \dots$

To find W^s_{loc}, let $x = S(y)$, $\dot{x} = \frac{dS}{dy}\dot{y} \Rightarrow x(1-y) = \frac{dS}{dy}(-y+x^2)$

let $S(y) = a_0 + a_1y + a_2y^2 + \dots$ ($a_0 = a_1 = 0$ by b.c.)

Plug this expansion and $a_n = 0 \ \forall n$. We can understand this as there is a symmetry in this system.

$$X = -x, \quad \dot{X} = X(1-y), \quad \dot{y} = -y + X^2$$

which is the same system. So symmetry of phase potrait about $x=0 \Rightarrow W^s(\underline{0})$ is $x=0$.

<u>Comments</u>: • If E^s and E^u not conveniently on axes, could transform axes to make them so, or work directly on relavant evec, e.g. $\binom{1}{k}$. then set $y = kx + a_2x^2 + a_3x^3 + \dots$

• Higher dimensions. e.g. $z = S(x,y)$, then $\dot{z} = \dot{x}\frac{\partial S}{\partial x} + \dot{y}\frac{\partial S}{\partial y}$.

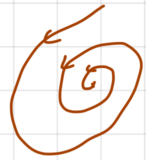
Non-linear terms in ~~non~~-hyperbolic Cases

Lots of possibilities. focus on $\mathbb{R}^2$.

(a) $\lambda = \pm i\omega$: linear system has a centre. Non-linear system?

Example (i) $\begin{cases} \dot{x} = -y - x^3 \\ \dot{y} = x - y^3 \end{cases}$

stable focus.



$$\Rightarrow \begin{cases} \dot{r} = -(x^4+y^4)/r \\ \dot{\theta} = 1 - (xy^3/yx^3)/r^2 \end{cases} \to 1 \text{ as } r \to 0.$$

(ii) $\begin{cases} \dot{x} = -y + x^3 \\ \dot{y} = x + y^3 \end{cases}$ $\Rightarrow$ unstable focus

(iii) $\begin{cases} \dot{x} = -y - 2x^2 y \\ \dot{y} = x + 2xy^2 \end{cases}$ Hamiltonian!

$$\dot{x} = \frac{\partial H}{\partial y} = -y - 2x^2 y \Rightarrow H = -\tfrac{1}{2}y^2 - x^2y^2 + A(x).$$

$$\dot{y} = -\frac{\partial H}{\partial x} = 2xy^2 - \frac{dA}{dx}$$

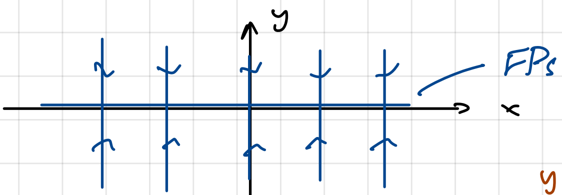
$$\Rightarrow -\frac{dA}{dx} = x \Rightarrow A = -\tfrac{1}{2}x^2 \Rightarrow H(x,y) = -\tfrac{1}{2}(x^2+y^2) - x^2y^2.$$

Notice that $\displaystyle \frac{dH}{dt} = \dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} + \cancel{\frac{\partial H}{\partial t}}^{\,0}$

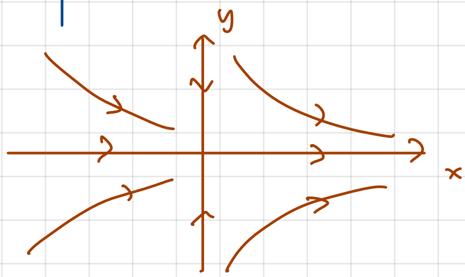
$$= \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} + \left(-\frac{\partial H}{\partial x}\right)\frac{\partial H}{\partial y} = 0.$$

Recall $H(x,y) = -\frac{1}{2}(x^2+y^2) - x^2y^2$. Contours of constant H are closed contours $\Rightarrow$ centre dynamics preserved.

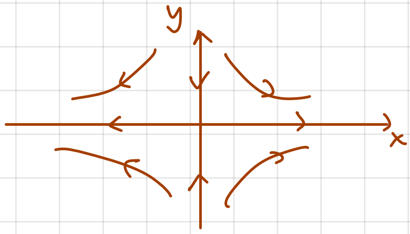
(b) $\lambda_1 = 0$, $\lambda_2 \neq 0$. (WLOG $\lambda_2 < 0$).



Example (i) $\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases}$



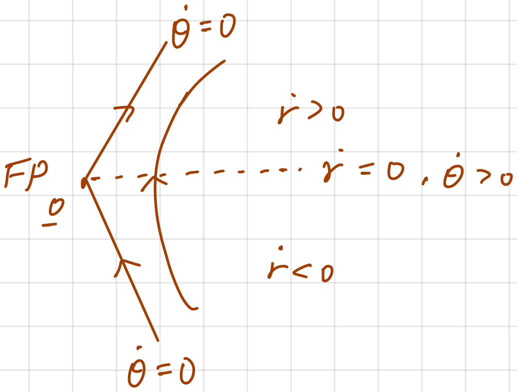
(ii) $\begin{cases} \dot{x} = x^3 \\ \dot{y} = -y \end{cases}$



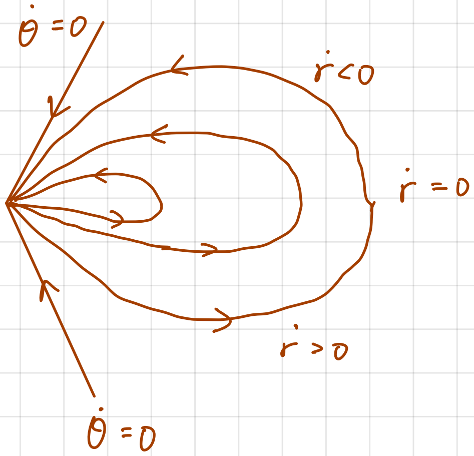
(c) $\lambda_1 = \lambda_2 = 0$ (almost anything is possible).

Work in polars. Consider signs of $\dot{r}$ and $\dot{\theta}$ as $r \to 0$.

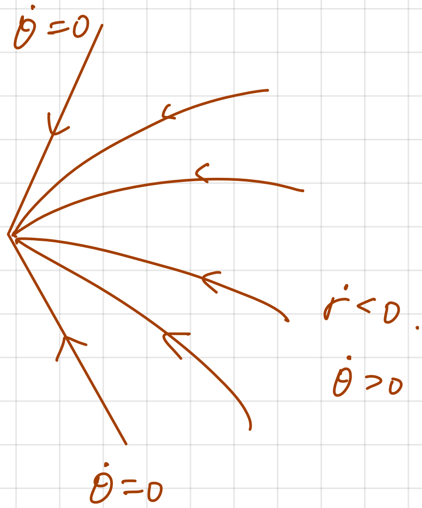
Example (i)



(ii)



(iii)



2.3 Sketching Phase Portrait

Go back to example $\begin{cases} \dot{x} = x(1-y) \\ \dot{y} = -y + x^2 \end{cases}$.

- First note $x \longrightarrow -x$ symmetry, so only need to work out what is happening for $x \geq 0$
- FPs $x(1-y) = 0 = -y + x^2$

  $\Rightarrow (x, y) = (0,0), (\pm 1, 1)$. For $(\pm 1, 1)$, just look at $(1,1)$

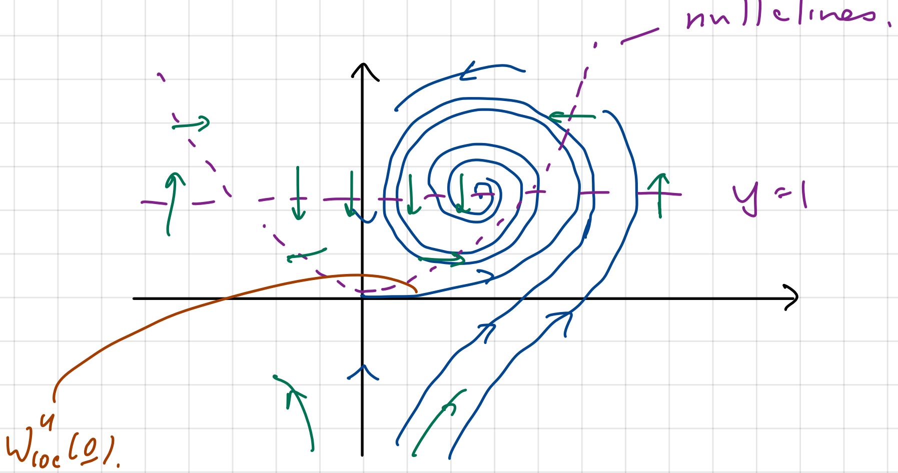
  Jacobian $A = \begin{pmatrix} 1-y & -x \\ 2x & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$ at $(-1, 1)$

  $T = -1$, $D = 2$, $T^2 - 4D < 0 \Rightarrow$ stable focus.

• Find nullclines

$W^u_{loc}(\underline{0}) = \frac{1}{3}x^2 + \frac{2}{45}x^4 + \dots$

Hence,



General Strategy for plotting out trajectories in phase space

1. Find FPs and check for symmetries.
2. Calculate Jacobian and classify FPs.
3. For saddles, find evec of Jacobian.
4. If useful, find nullclines
5. Construct global picture by joining up all these symmetries.
6. Remember trajectories cannot cross!
7. Are there any Periodic orbits? (See §4).

3. Stability

We want to know what happens given small perturbations away from an invariant set (FPs, POs, etc.). We are also interested in which invariant sets are approached at large times. Need the concept of stability and there are various defn's.

3.1 Stability Definitions

Def^n A fixed point $\underline{x}_0$ is <u>Lyapunov stable</u> (LS) if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

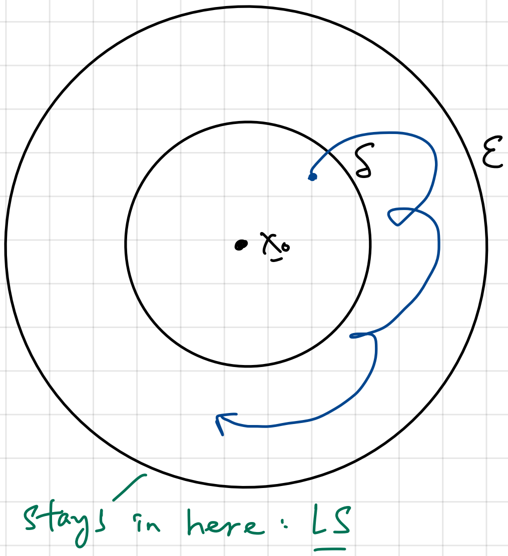
$$|\underline{x} - \underline{x}_0| < \delta \Rightarrow |\phi_t(\underline{x}) - \underline{x}_0| < \varepsilon \quad \forall t \geq 0$$

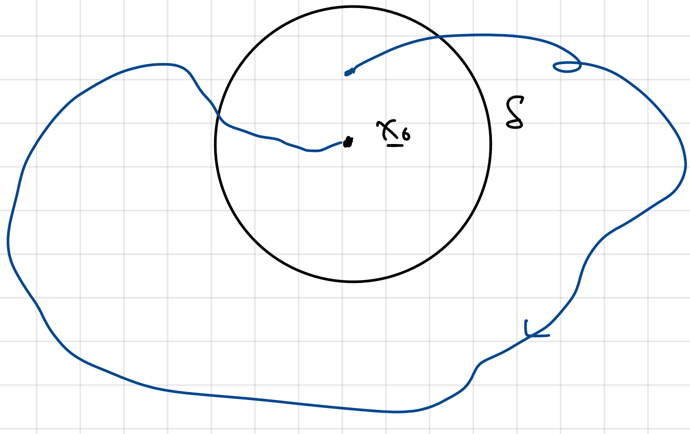
"Start near, stay close".

Def^n A fixed point $\underline{x}_0$ is <u>quasi-asymptotically stable</u> (QAS) if $\exists \delta > 0$ s.t. $|\underline{x} - \underline{x}_0| < \delta \Rightarrow \phi_t(\underline{x}) \to \underline{x}_0$ as $t \to \infty$

"gets close eventually".

Def^n A fixed point $\underline{x}_0$ is asymptotically stable (AS) if it is LS and QAS, i.e. LS + QAS = AS.

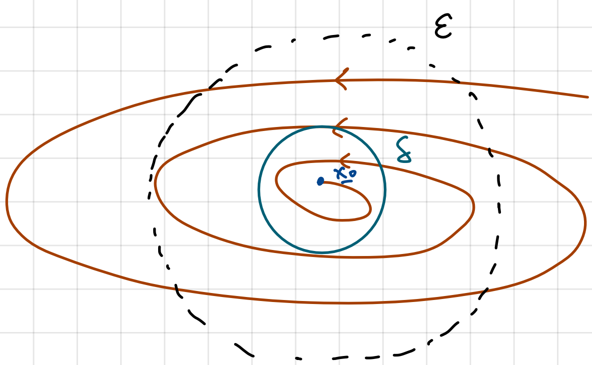




So AS — "stays near, and converges to $\underline{x}_0$".

Examples

1. Stable flows



LS ? Yes

QAS ? Yes

$\Rightarrow$ AS

Note: $|\phi_t(\underline{x}) - \underline{x}_0|$ doesn't have to $\to 0$ monotonically with $t \to \infty$.

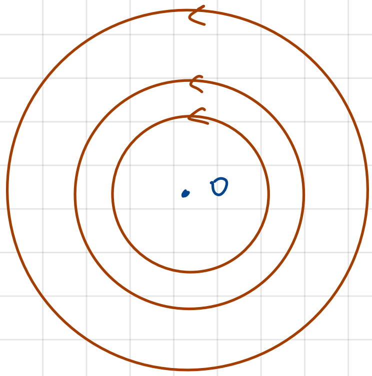
☆ <u>All hyperbolic sinks are AS.</u> ☆

2. Centres

$\dot{r} = 0$

$\dot{\theta} = 1$

FP at $\underline{0}$



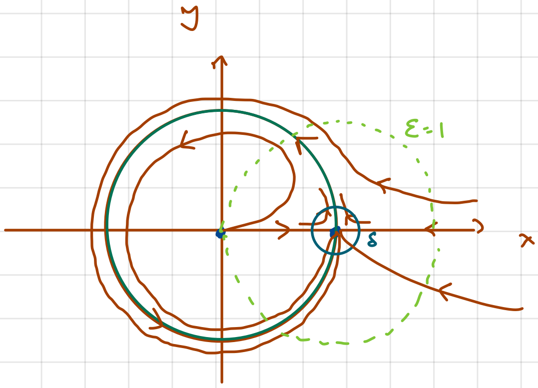
LS? Yes. choose $\delta = \varepsilon$.

QAS? No.

$\phi_t(\underline{x}) \to 0$ only for $\underline{x} = \underline{0}$.

$\Rightarrow$ Not AS.

3. Large excursions



$\dot{r} = r(1-r^2)$

$\dot{\theta} = \sin^2 \frac{1}{2}\theta$

FP at 0 and $r=1$, $\theta=0$.

LS? No. points just above x-axis takes a big excursion.
If given $\varepsilon < 2$, can't find δ.

QAS? Yes. $\phi_t(\underline{x}) \to$ FP $\forall\, \underline{x} \neq 0$.

Comments: $(1,0)$ is a saddle-node FP (§2.1)

$r=1$, $\theta \neq 0$ is a homoclinic orbit (§1.3).

For more general invariant sets, say Λ, e.g. a periodic orbit, let

$$N_\delta(\Lambda) = \{ \underline{x} : \exists\, \underline{y} \in \Lambda \text{ s.t. } |\underline{x} - \underline{y}| < \delta \}.$$

and say $\phi_t(\underline{x}) \to \Lambda$ if $\inf_{\hat{\underline{y}} \in \Lambda} |\phi_t(\underline{x}) - \hat{\underline{y}}| \to 0$ as $t \to \infty$,

then

- Λ is LS if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\underline{x} \in N_\delta(\Lambda)$

  $\Rightarrow \phi_t(\underline{x}) \in N_\varepsilon(\Lambda) \quad \forall t \geq 0$
- Λ is QAS if $\exists \delta > 0$ s.t. $\underline{x} \in N_\delta(\Lambda) \Rightarrow \phi_t(\underline{x}) \to \Lambda$.
- Λ is AS if both LS and QAS hold

If an invariant set is not LS or QAS, we say it is <u>unstable</u>.

$\rightleftharpoons$ AS $\Rightarrow$ stability

Thm (Stability of hyperbolic FP) If $\underline{x}_0$ is a hyperbolic stable focus / stable node, then it is AS. If $\underline{x}_0$ is a hyperbolic FP with at least one eval with $Re(\lambda) > 0$, then it is <u>unstable</u>.

3.2 Lyapunov Functions

We can prove much about stability of a fixed point $\underline{x}_0$ if we can find a suitable non-negative f^n $V(\underline{x})$ that is only zero at $\underline{x}_0$ and does not increase under the flow $\phi_t(\underline{x})$.

Def^n A continuously diff'ble (i.e. C^1) f^n $V(\underline{x}): \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov f^n for $\dot{\underline{x}} = \underline{f}(\underline{x})$ if

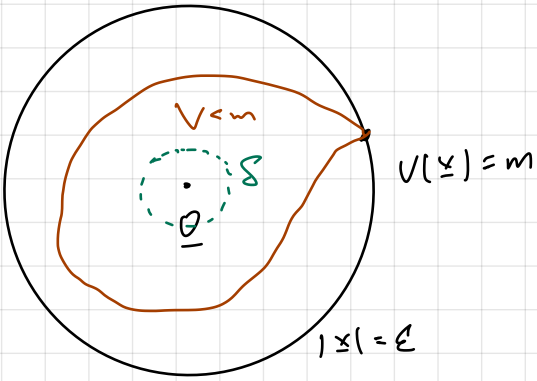
(i) $V(\underline{0}) = 0$ and $V(\underline{x}) > 0$ for $\underline{x} \neq 0$ in D

(ii) $\dot{V} = \underline{f} \cdot \nabla V \leq 0$ in D "non-increasing"

Notes (ii) is equivalent to $V(\phi_t(\underline{x})) \leq V(\phi_t(\underline{x}))$ for $t \geq 0$.

Strict Lyapunov f^n has $\dot{V} < 0$ except at $\underline{x} = 0$.

Thm (Lypanov's 1st thm (L1)) If a Lyaponov f^n exists, $\underline{0}$ is Lyaponov stable (LS).



Pf: WLOG, assume ε is small enough that $\{|x| < \varepsilon\} \subset D$.

Set $m = \min\{ V(\underline{x}) : |\underline{x}| = \varepsilon \}$. Note $m > 0$.

Let $C_{m,\varepsilon} = \{ \underline{x} : V(\underline{x}) < m, \ |\underline{x}| < \varepsilon \}$.

Now choose s.t. $\{|\underline{x}| < \delta\} \subset C_{m,\varepsilon}$, then have 3 nested sets

$$\{|\underline{x}| < \delta\} \subset C_{m,\varepsilon} \subseteq \{|\underline{x}| < \varepsilon\}$$

If $\underline{x} \in \{|\underline{x}| < \delta\}$, then $\underline{x} \in C_{m,\varepsilon}$ $\forall\, t \geq 0$, hence $\phi_t(\underline{x}) \in \{|\underline{x}| < \varepsilon\}$. $\square$

"Start in δ, stay in $C_{m,\varepsilon}$, so within ε".

Thm (Lyaponov's 2nd thm (L2)) If a strict Lyaponov f^n exists, $\underline{0}$ is asymptotically stable (AS).

Pf (sketch): For ε s.t. $\{|\underline{x}| < \varepsilon\} \subseteq D$, can find suitable δ as above so $O^+(\underline{x}_0) \subset D$ for all $|\underline{x}_0| < \delta$, so $V(\phi_t(\underline{x}_0))$ decreasing, bounded below by 0, so $\rightarrow V_0 \geq 0$.

Imagine $V_0 > 0$ $\Rightarrow$ $V(\phi_t(\underline{x}_0))$ is bounded away from 0

$\Rightarrow$ $\phi_t(\underline{x}_0)$ " $\underline{0}$

$\Rightarrow$ $\dot{V}(\phi_t(\underline{x}_0))$ " 0

Then we can always find a large enough time t s.t.

$$V(\underline{x}_0) + \int_0^t \dot{V}(\phi_t(\underline{x}_0))\, dt < 0. \quad ⨳$$

If $V_0 = 0$, then $\phi_t(\underline{x}_0) \rightarrow \underline{0}$ $\forall\, \underline{x}_0 \in \{|\underline{x}| < \delta\}$, so AS. $\square$

Even with non-strict Lyapunov f^ns, one can sometimes say something about the long-time limit of trajectories

La Salle's Invariance Principle

If V is a Lyapunov f^n on a domain D which is compact (closed + bounded) and forward invariant ($\underline{x} \in D$ at $t=0$ $\Rightarrow \phi_t(\underline{x}) \in D \ \forall t \geq 0$), then

$$\omega(\underline{x}) \subseteq \{ y : V(\phi_t(y)) = V_0 \ \forall t \}$$

Comment · May be more than one value of V_0.

E.g. $V(r,\theta) = r^2$, $\dot{r} = 0$, $\dot{\theta} = 1$.

More usefully: $\phi_t(\underline{x})$ tends to an invariant subset of $\{ y : \dot{V}(y) = 0 \} \cap D$.

Pf (outline): · $V(\phi_t(\underline{x}))$ is non-increasing, and bounded below, so tends to V_0.

· Know $\omega(\underline{x})$ is non-empty (forward orbit in bounded D.)
Cty of V gives $V(y) = V_0 \ \forall y \in \omega(\underline{x})$ as $\underline{x} \to y \in \omega(\underline{x})$.

· Know as $\omega(\underline{x})$ is invariant, $\phi_t(y) \in \omega(\underline{x}) \ \forall t$, so $V(\phi_t(y)) = V_0 \ \forall t$, so $\dot{V}(y) = 0$

· 'tends to' bit: can show that for $\underline{x} \in$ compact forward-invariant D, it is always true that $\phi_t(\underline{x}) \to \omega(\underline{x})$. $\square$

Note In practice, don't always need D closed so long as $\underline{x} \in D$ $\Rightarrow \omega(\underline{x}) \in D$, which is often the case, e.g. $D = \{ \underline{x} : V(\underline{x}) < V \}$ and $\dot{V} \leq 0$ on D.

Example If $\dot{\underline{x}} = \underline{\underline{A}} \cdot \underline{x} + o(|\underline{x}|)$ and $\underline{\underline{A}}$ has distinct eval λ_i, with $Re(\lambda_i) < 0$ and possibly complex evecs $\underline{v}_i$, then can write $\underline{x} = \sum_i a_i(t) \underline{e}_i$, where a_i complex coeff.

Then for any positive v_i, $V := \sum_i v_i |a_i|^2$ satisfies $\dot{V} < 0$. except at $\underline{x} = \underline{0}$. Noting $\dot{a}_i = \lambda_i a_i$.

$$\frac{dV}{dt} = \frac{d}{dt} \sum v_i (a_i^* a_i)$$

$$= \sum v_i (\dot{a}_i^* a_i + a_i^* \dot{a}_i)$$

$$= \sum v_i (\lambda_i^* a_i^* a_i + \lambda_i a_i^* a_i)$$

$$= \sum v_i \underbrace{(\lambda_i^* + \lambda_i)}_{2Re(\lambda_i)} |a_i|^2.$$

$$= 2 \sum v_i Re(\lambda_i) |a_i|^2 < 0$$

as long as $|a_i|^2 \neq 0 \; \forall i$. $\Rightarrow \underline{0}$ is AS.

We can use Lyaponuv f^n's to learn something about the Domain of Stability (DoS), or "basin of attraction".

Def^n The domain of stability (DoS) of an AS invariant set is $\{\underline{x} : \phi_t(\underline{x}) \to \Lambda\}$. In 2D, if DoS $= \mathbb{R}^2$, we say Λ is <u>globally stable</u>.

Example FP $\underline{x} = 0$ of $\dot{x} = y - xy^2$, $\dot{y} = -x^3$.

$$\begin{cases} \partial H/\partial y = y \\ -\partial H/\partial x = -x^3 \end{cases} \Rightarrow H = \frac{1}{2}y^2 + \frac{1}{4}x^4$$

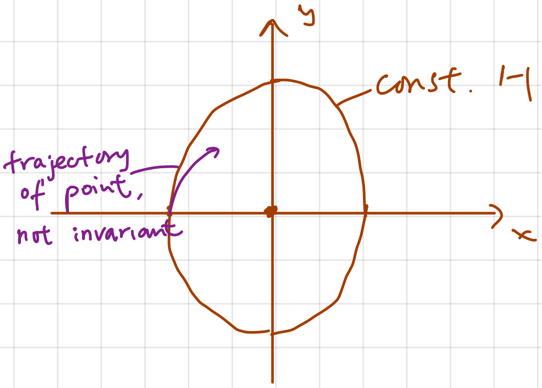
Notice that without $-xy^2$ term, the system is Hamiltonian with $H(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4$. With it,

$$\dot{H} = \dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y}$$
$$= (y - xy^2)x^3 + (-x^3)y = -x^4y^2.$$

So $\dot{H} \leq 0$, so H is a Lyapunov f^n over $\mathbb{R}^2$.

(Check: $H \geq 0$ and $H = 0$ only at $\underline{x} = \underline{0}$, fixed point of interest)

Note that $\{\dot{H} = 0\} = \{x = 0 \text{ or } y = 0\}$ but only invariant subset is the origin. E.g. $y = 0$, $x \neq 0$, then $\dot{y} \neq 0$, so leave $\{\dot{H} = 0\}$.



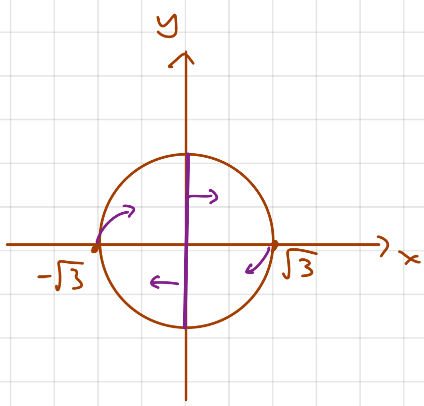
Now involve La Salle: for any $\underline{x}_0$, set
$$D = \{\underline{x} : H(\underline{x}) \leq H(\underline{x}_0)\}$$
Clearly bounded for our H, forward invariant $\phi_t(\underline{x}_0) \to 0$, so $\underline{0}$ is <u>globally stable</u>.

<u>Example</u> $\dot{x} = y + \mu(\frac{1}{3}x^3 - x)$, $\dot{y} = -x$, $\mu > 0$, FP at $\underline{0}$. Sink for $\mu > 0$.

Guess $V(x,y) = x^2 + y^2$ (because $\dot{x} = y$, $\dot{y} = -x$), then
$$\dot{V} = 2x\dot{x} + 2y\dot{y}$$
$$= 2\mu(\tfrac{1}{3}x^4 - x^2)$$

Hence $\dot{V} \leq 0$ for $x^2 \leq 3$.

Choose $D = \{\underline{x} : V(\underline{x}) \leq 3\}$. Compact and forward invariant. La Salle $\Rightarrow$ $\phi_t(\underline{x}) \to$ some invariant subset of $\underbrace{\{\dot{V} = 0\} \cap D}_{S}$.



But $S = \{(\pm\sqrt{3}, 0)$, line $x=0$ for $-\sqrt{3} \le y \le \sqrt{3}\}$.

Invariant set? Exclude $(\pm\sqrt{3}, 0)$, as $\dot{y} \neq 0$, so leaves S.

Exclude $(0, y)$ for $x \neq 0$, as $\dot{x} \neq 0$, so leaves S.

Only candidate left is $\underline{0}$, which is clearly invariant.

So $\phi_t(\underline{x}) \to 0 \quad \forall \underline{x} \in D$. By La Salle, $D \subseteq DoS(\underline{0})$.

<u>General method to establish stability</u>

1. Find V and domain D s.t.

   (i) $V \ge 0$ on D and $V = 0$ at $\underline{x} = \underline{0}$

   (ii) $\dot{V} \le 0$ on D (sometimes this will limit size of D)

2. Find k s.t. $C_k = \{\underline{x} : V(\underline{x}) < k\} \subseteq D$.

3. Adjust k (or V) so only invariant subset of $\{\dot{V} = 0\} \cap C_k$ is $\{\underline{0}\}$ $\Rightarrow$ La Salle gives $C_k \subseteq DoS(\underline{0})$.

(4. Take different V to maximise C_k to get best estimate of $DoS(\underline{0})$.)

<u>Example</u> $\dot{x} = -x + xy^2, \quad \dot{y} = -y + x^2 y$

Try $V = x^2 + a^2 y^2$, where a a general real number.

$$\dot{V} = 2x(-x + xy^2) + 2a^2 y(-y + x^2 y)$$
$$= -2(x^2 + a^2 y^2) + 2(1 + a^2) x^2 y^2$$

So $\dot{V} \le 0$ near $\underline{0}$, but how close?

Parameterise the curve of const. V with $x = \sqrt{V}\cos\phi$, $y = \frac{\sqrt{V}}{a}\sin\phi$.

On this contour,

$$\dot{V} = -2V + 2\,\frac{1+a^2}{a^2}\,V^2 \underbrace{\cos^2\phi \sin^2\phi}_{= \frac{1}{4}\sin^2 2\phi \,\le\, \frac{1}{4}}$$

$$\Rightarrow \dot{V} \le -2V + \frac{2(1+a^2)V^2}{4a^2} = -2V\left(1 - \frac{1+a^2}{4a^2}V\right)$$

So $\dot{V} < 0$ if $V < \frac{4a^2}{1+a^2}$.

So for a given "a", we have

$$V = x^2 + a^2y^2 < \frac{4a^2}{1+a^2}.$$

is in $DoS(\underline{0})$.

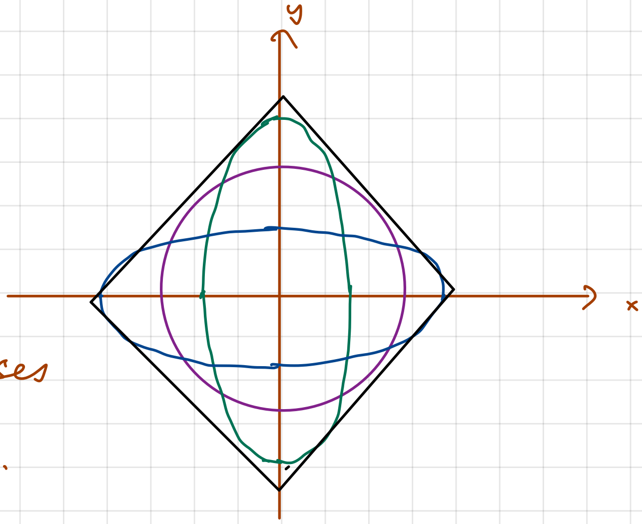
(Have a strict Lyapunov f^n so AS by L2).

Now take the union of those "a"-sets over a!

$a = 1 \Rightarrow x^2 + y^2 = 2$

$a \to 0 \Rightarrow x^2 + a^2y^2 = 4a^2$

$a \to \infty \Rightarrow x^2 + a^2y^2 = 4$.

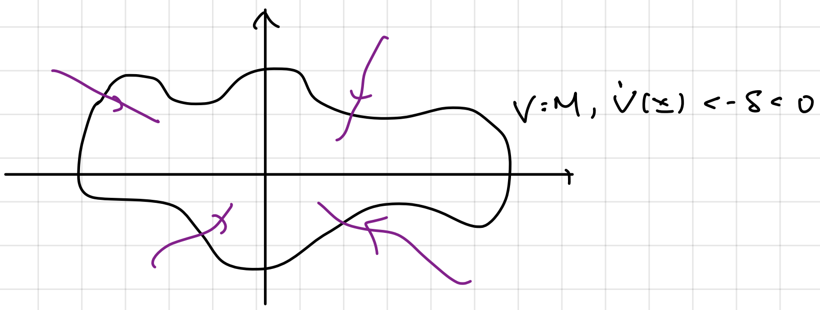


Can check, this family of ellipses fall in the set $\{|x| + |y| < 2\}$.

3.3 Bounding Functions

There exists another use of f^ns which decrease along the trajectories. If V is a f^n with <u>bounded</u> contours and we have $\dot{V}(\underline{x}) < -\delta < 0$ for $V(\underline{x}) \ge M$, then all trajectories (eventually) enter and remain in the set $\{V(\underline{x}) \le M\}$.

Such V is called a <u>bounding function</u>.



Example $\begin{cases} \dot r = r - r^3(1+k\sin\theta) - r^5 \\ \dot\theta = 1 \end{cases}$. $0<k<1$

Consider $V(r,\theta) = \frac{1}{2}r^2$. Then

$$\dot V = r\dot r = r(r - r^3(1+k\sin\theta) - r^5)$$
$$= r^2(1 - r^2(1+k\sin\theta) - r^4)$$

Notice $1+k\sin\theta \geq 1-k$.

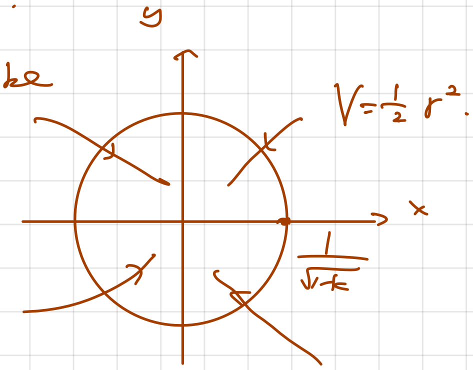
$$\dot V \leq r^2(1 - r^2(1-k) - r^4)$$

Now for $r^2 > \frac{1}{1-k}$,

$$\dot V < \frac{1}{1-k}\left(1 - 1 - \frac{1}{(1-k)^2}\right) < -\frac{1}{(1-k)^3} < 0.$$

So all trajectories enter disc $r \leq \frac{1}{\sqrt{1-k}}$.

But what happen next? limit set might be FP or PO — see §4.



4. Periodic Orbit Existence and Stability in $\mathbb{R}^2$

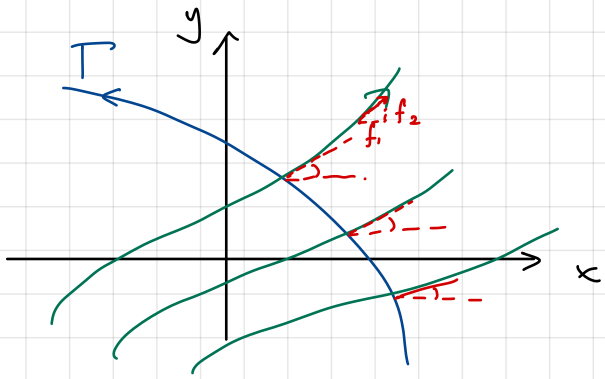
We first discuss techniques for detecting or discussing the possibility of periodic orbits.

4.1 The Poincaré Index

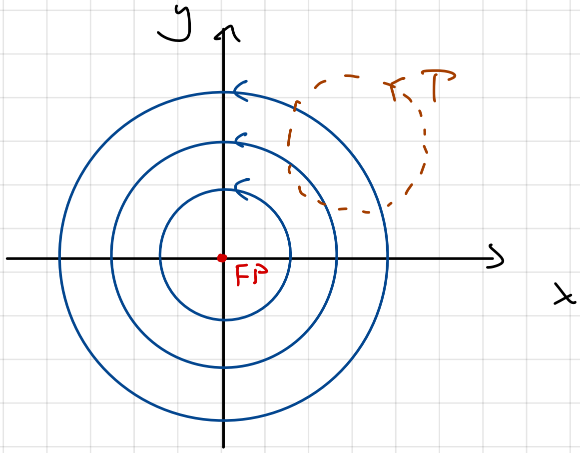
Consider $\begin{pmatrix}\dot x \\ \dot y\end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y)\end{pmatrix}$. Except at FPs, the trajectory at $\underline{x}$ makes an angle $\psi = \tan^{-1}(f_2/f_1)$ with the x-direction.

Let Γ be a simple closed curve, not necessarily a trajectory, that does not pass through any FPs (where $f_1 = f_2 = 0$ and ψ undefined).

Move once around Γ in an anticlockwise sense and consider $\psi = \tan^{-1}(f_2/f_1)$, ψ changes ctsly and returns to its initial value plus an integral multiple of 2π.



This multiple is called the Poincaré index of Γ, or $I(\Gamma)$



Properties of $I(\Gamma)$:

1. In integral form,

$$I(\Gamma) = \frac{1}{2\pi}\oint_\Gamma d\psi$$

$$= \frac{1}{2\pi}\oint_\Gamma d\left(\tan^{-1}(f_2/f_1)\right)$$

$$= \frac{1}{2\pi}\oint_\Gamma \frac{1}{1+(f_2/f_1)^2}\,\frac{f_1 f_2' - f_1' f_2}{f_1^2}.$$

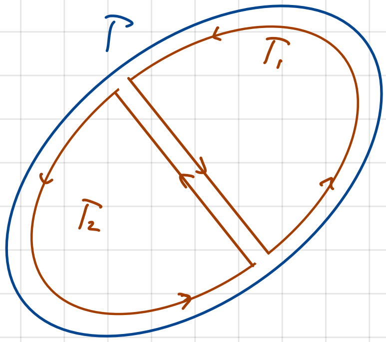
$$\Rightarrow I(\Gamma) = \frac{1}{2\pi}\oint \frac{f_1\,df_2 - f_2\,df_1}{f_1^2+f_2^2}.$$

2. $I(\Gamma)$ is unaffected by cts deformation of Γ provided this does not cross a FP (FP is a sing. as $f_1^2+f_2^2=0$).

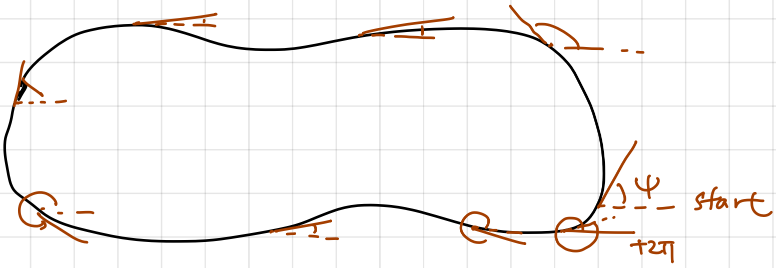
3. If $I(\Gamma)$ encloses no FP, then $I(\Gamma)=0$ (just shrink contour $\Gamma \to$ zero)

4. Index is additive

$$I(\Gamma) = I(\Gamma_1) + I(\Gamma_2)$$



5. Index of a closed trajectory is $+1$

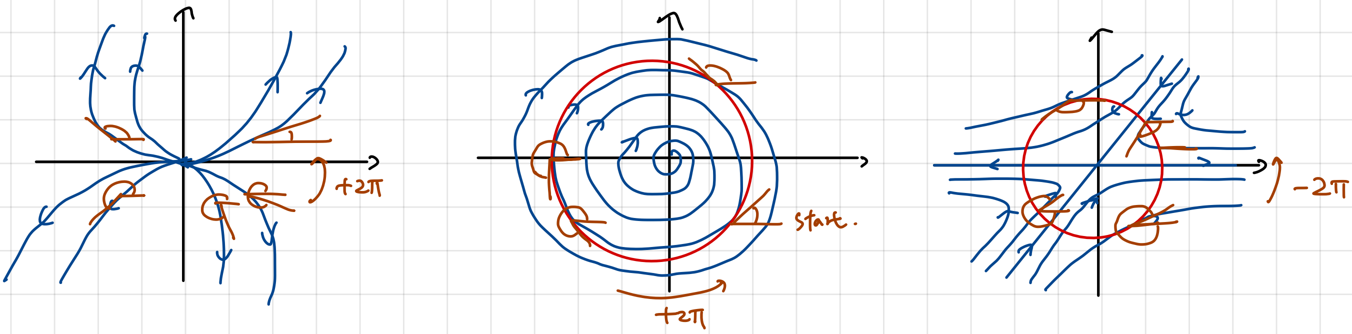


6. If time is reversed $I(\Gamma)$ is unaffected ($f_1 \mapsto -f_1$, $f_2 \mapsto -f_2$, f_2/f_1 unaffected)

7. The Poincaré index of an isolated FP $\underline{x_0}$ is the Poincaré index of any simple curve containing $\underline{x_0}$.

8. $I(\Gamma)$ is the sum of indices of all the FPs enclosed by Γ (by point 4 above)

9. The Poincaré index of hyperbolic sinks / sources is $+1$, and hyperbolic saddle is -1.



(non-hyperbolic — go case by case)

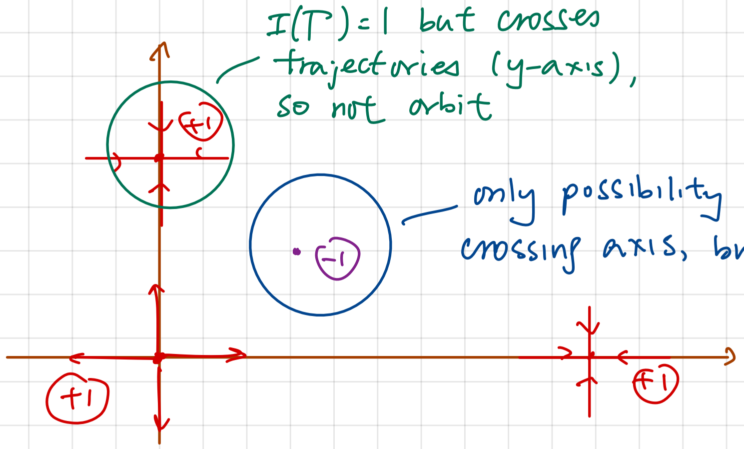
Cor Any PO must contain one or more FPs and the sum of their indices has to be $+1$.

This can be used to exclude the possibility of a PO — Poincaré Index Test.

Example $\begin{cases} \dot{r} = r(3-r-2s) \\ \dot{s} = s(2-r-s) \end{cases}$

Nodes at $(r,s) = (0,0), (0,2), (3,0)$,

saddle at $(1,1)$.



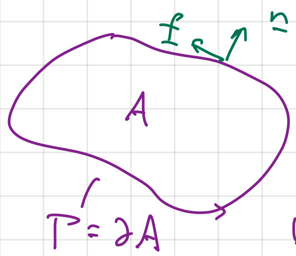
4.2 Dulac's criterion

Thm (Dulac's criterion) If there is a ctsly diff f^n $\phi(x,y)$ s.t $\nabla \cdot (\phi \underline{f}) > 0$ everywhere (or < 0 everywhere) on a simply connected domain $D \subset \mathbb{R}^2$, then there are no periodic orbits that are entirely in D.

Pf: WLOG $\nabla \cdot (\phi \underline{f}) > 0$. Suppose have PO Γ entirely in D enclosing an area A with outward normal $\hat{n}$. Then

$$0 = \oint_\Gamma \phi \underline{f} \cdot \hat{\underline{n}} \, dl = \int_A \nabla \cdot (\phi \underline{f}) \, dS > 0 \quad \#$$

with $\dot{\underline{x}} = \underline{f}(\underline{x})$

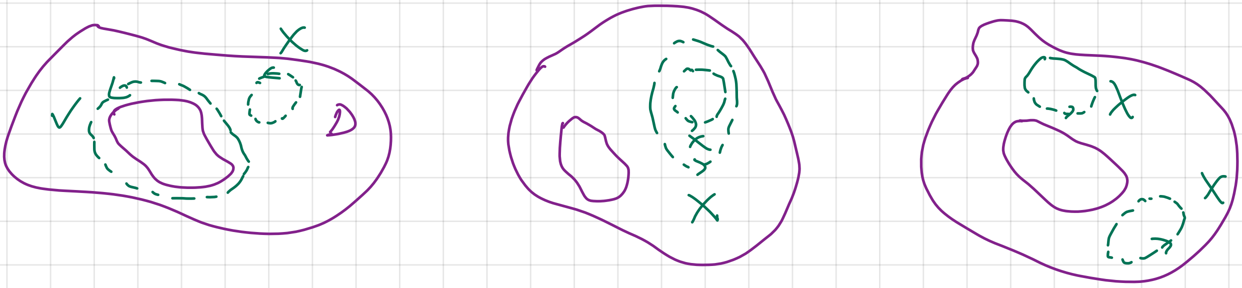


$\square$

In practice, often use $\phi = 1$. then if $\nabla \cdot \underline{f} > 0$ over D, then no PO. This is called the divergence test.

<u>Cor</u> If $\nabla \cdot (\phi \underline{f}) > 0$ everywhere on some doubly-connected domain $D \subset \mathbb{R}^2$, then there is at most one PO entirely in D (and it must enclose the hole).

<u>Pf</u>: By assumption, $\int_A \nabla \cdot (\phi \underline{f})\, dS > 0$, but $\int_{T_2 - T_1} \phi \underline{f} \cdot d\underline{l} = 0$ ※



<u>Example</u>

$$\dot{r} = r(a - br - cs) = f_r$$
$$\dot{s} = s(d - er - fs) = f_s \qquad , b, f > 0$$

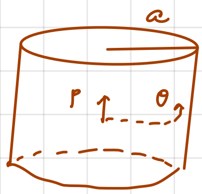
Choose $\phi = -1/rs$, then

$$\nabla \cdot (\phi \underline{f}) = \frac{\partial}{\partial r}(\phi f_r) + \frac{\partial}{\partial s}(\phi f_s)$$
$$= \frac{\partial}{\partial r}\left(\frac{a - br - cs}{s}\right) + \frac{\partial}{\partial s}\left(\frac{d - er - fs}{r}\right)$$
$$= -\frac{b}{s} - \frac{f}{r}$$

one signed over $s > 0, r > 0$, So by Dulac, no PO possible.

<u>Example</u> Damped pendulum + torque $\theta \in [0, 2\pi)$

$$\begin{cases} \dot{\theta} = p \\ \dot{p} = F - kp - \sin\theta, \quad k > 0 \end{cases}$$



Then $\nabla \cdot \underline{f} = \frac{1}{a} \frac{\partial p}{\partial \theta} + \frac{\partial}{\partial p}(F - kp - \sin\theta)$

$= -k < 0$.

So at most one PO and must encircle the entire cylinder, o/w excluded by Dulac.

Another useful negative test is the Gradient criterion.

If we have a +ve f^n $\rho(x,y)$ s.t. $\rho \underline{f} = \nabla \psi$ for some single-valued ψ on some simply connected domain D, there there are no POs enclosed by D.

Pf: If $\rho \underline{f} = \nabla \psi$ on a PO Γ, then

$$\oint_\Gamma \rho \underline{f} \cdot d\underline{l} = \oint_\Gamma \nabla \psi \cdot d\underline{l} = \int_\Gamma d\psi = 0$$

as Γ closed and ψ single valued.

But $\underline{f} \parallel d\underline{l}$, so $\underline{f} \cdot d\underline{l} > 0$, so $\oint \rho \underline{f} \cdot d\underline{l} > 0$ $\#$

So no PO. $\square$

Example
$$\begin{cases} \dot{x} = 2x + x^2 y - y^3 \\ \dot{y} = -2y - xy^2 + x^3 \end{cases}$$

Note that $\underbrace{e^{xy}}_{\rho > 0 \text{ over } \mathbb{R}^2} \underline{f} = \nabla(e^{xy}(x^2 - y^2))$ in $\mathbb{R}^2$, so no PO in $\mathbb{R}^2$.

4.3 Poincaré – Bendixson Theorem

Thm (PB) If the forward orbit $\mathcal{O}^+(\underline{x})$ of some point $\underline{x}$ remains in a compact set (closed and bounded) $K \subset \mathbb{R}^2$ that contains no FPs, then $\omega(\underline{x})$ is a periodic orbit.

Note: We know K cannot be simply connected as there must be at least one FP contained within the PO by §4.1, so there must be a hole.

Example
$$\begin{cases} \dot x = x - y - 2x(x^2+y^2) \\ \dot y = y + x - y(x^2+y^2) \end{cases}$$

linear terms give that there is unstable focus at $\underline{0}$ and

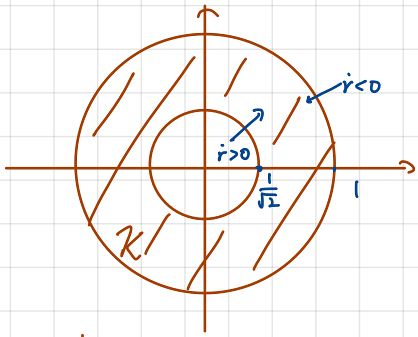
Cubic terms 'push' inwards.

Try to see if $\mathcal{K}$ is an annulus in polars

$$\dot r = \frac{x\dot x + y\dot y}{r} = \frac{1}{r}\left(x^2 - xy - 2x^2(x^2+y^2)\right) + \frac{1}{r}\left(y^2 + xy - y^2(x^2+y^2)\right)$$

$$= r - x^2 r - r^3$$

$$= r(1 - r^2 - r^2\cos^2\theta)$$



$$\Rightarrow \quad r(1-2r^2) < \dot r < r(1-r^2)$$

$\theta = 0$ $\theta = \pi/2$

So for $\dot r > 0$, need $0 < r < \frac{1}{\sqrt2}$. for $\dot r < 0$ need $r > 1$.

Define annulus $\mathcal{K} = \{\frac{1}{\sqrt2} < r < 1\}$. Once inside $\mathcal{K}$, can't leave.

Now examine if there is FP in $\mathcal{K}$.

$$\dot\theta = \frac{1}{r^2}(x\dot y - \dot x y) = 1 + \frac{1}{2}r^2\sin 2\theta > 0 \quad \text{in } \mathcal{K}$$

So no FPs in $\mathcal{K}$. Apply PB $\Rightarrow$ $\mathcal{K}$ contains a PO.

Bonus: $\quad \nabla\cdot \underline{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = 2 - 5r^2 - 2x^2 < 0$ in $\mathcal{K}$

So Dulac corollary $\Rightarrow$ exactly 1 PO.

4.4 Near-Hamiltonian Flows

Consider systems of the form

$$\dot x = f_1(x,y) + \varepsilon g_1(x,y)$$

$$\dot y = f_2(x,y) + \varepsilon g_2(x,y)$$

with $f_1 = \partial H/\partial y$, $f_2 = -\partial H/\partial x$.

If $\varepsilon = 0$, the system is Hamiltonian.

Trajectories lie on contours of H.

Lots of POs. (on closed contours of H)

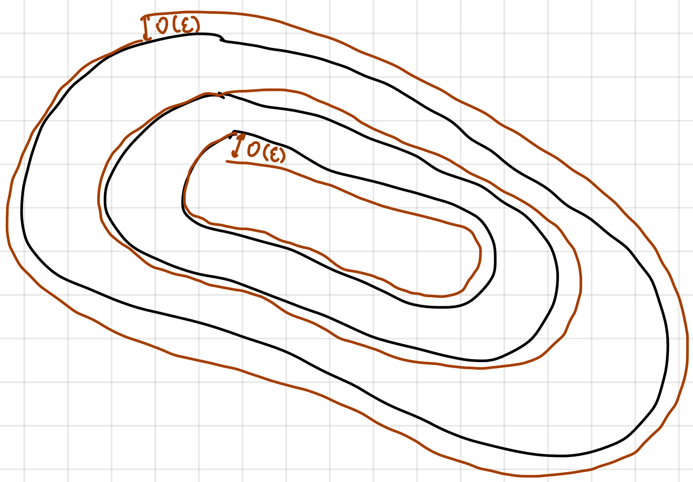
If $\varepsilon \neq 0$,
$$\dot{H} = \dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y}$$
$$= (f_1 + \varepsilon g_1)(-f_2) + (f_2 + \varepsilon g_2)(f_1).$$
$$= \varepsilon (f_1 g_2 - f_2 g_1),$$

If there is a PO, then
$$\int \dot{H} dt = \int_{T(PO)} dH = 0.$$

i.e.
$$\oint_T (f_1 g_2 - f_2 g_1)\, dt = 0$$

Hence, immediately, PO cannot entirely be in a region where $f_1 g_2 - f_2 g_1$ is singled value.

If $0 < \varepsilon \ll 1$, "near Hamiltonian". $\dot{H} = O(\varepsilon)$, so trajectories are similar to the contours of H. But we expect most if not all POs to be destroyed.

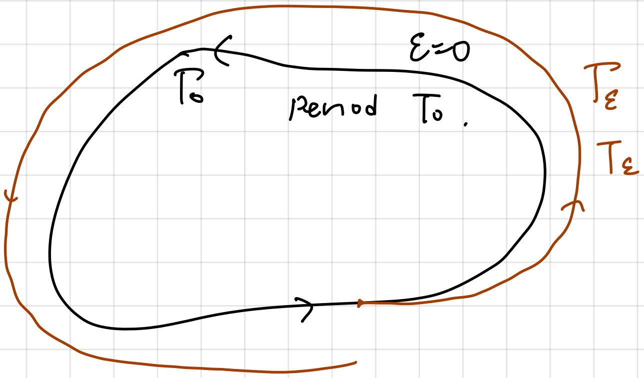


Use the energy balance method to see if any POs survive.

Estimate the change in H in the full system ($\varepsilon \neq 0$) over a trajectory by considering the change in H over $\varepsilon = 0$ contours of H.

$$\Delta H(\Gamma_\varepsilon) = \int_{\Gamma_\varepsilon}^{T_\varepsilon} \dot{H}\, dt$$

$$= \varepsilon \int_{\Gamma_\varepsilon}^{T_\varepsilon} (f_1 g_2 - f_2 g_1)\, dt$$



Key points: $T_\varepsilon = T_0 + O(\varepsilon)$.

"$\Gamma_\varepsilon = \Gamma_0 + O(\varepsilon)$".

So

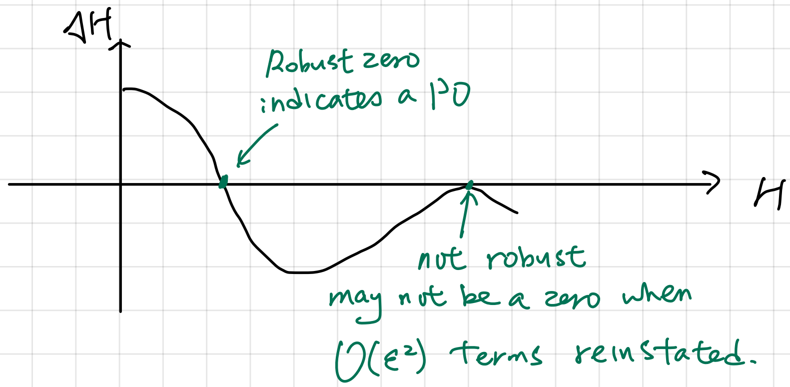
$$\Delta H(\Gamma_\varepsilon) = \varepsilon \int_{\Gamma_0}^{T_T} (f_1 g_2 - f_2 g_1)\, dt + O(\varepsilon^2).$$

So search for POs comes down to finding H_0 s.t.

$$\int_{\substack{0 \\ \Gamma_0 (H = H_0)}}^{T} (f_1 g_2 - f_2 g_1)\, dt = 0$$

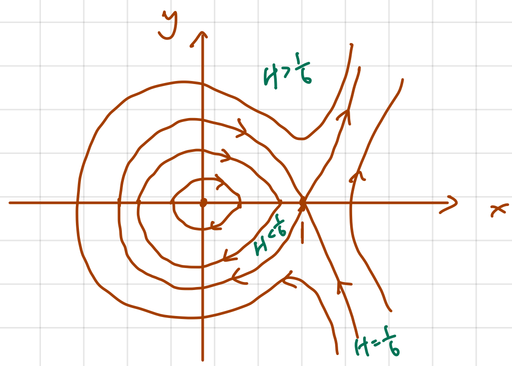
neglecting $O(\varepsilon^2)$ as $\varepsilon \ll 1$.

Remember criterion for $\Delta H = 0$ is only $O(\varepsilon)$.



<u>Example</u>
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x + x^2 + \varepsilon y (a - x) \end{pmatrix}$$

For $\varepsilon = 0$, $\quad H = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3$



For $\varepsilon \neq 0$,

1) $H' = \varepsilon y^2 (a-x)$, so any PO can't be entirely in $x<a$ or $x>a$

2) FPs remain $(0,0)$, $(1,0)$. By Poincaré index criterion,

PO must enclose FP at $(0,0)$, but not the saddle at $(1,0)$.

3) min/max of x on PO $\rightleftharpoons$ $\dot{x}=0$ or $y=0$

$\dot{x}>0$, $y>0$ and $\dot{x}<0$, $y<0$

$\Rightarrow$ $x_{min}<a$, $a< x_{max} <1$ ← don't enclose a saddle.

(need sign change for $y^2(a-x)$.

For $0<\varepsilon \ll 1$,

$$\Delta H(H_0) = \varepsilon \int_{H_0} f_1 g_2 - \cancel{f_2 g_1}^{=0}\, dt + O(\varepsilon^2)$$

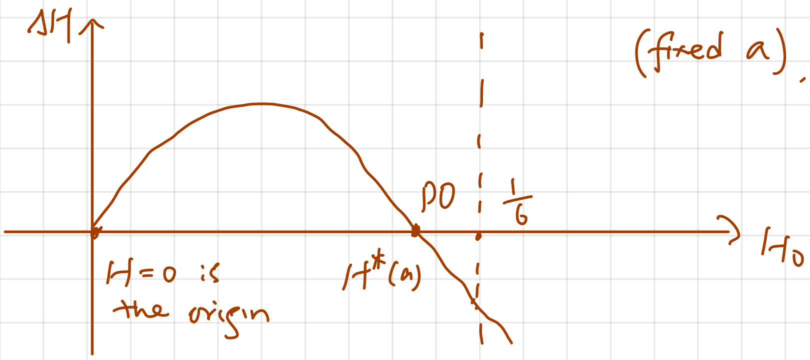
$$\approx \varepsilon \int_{H_0} g_2 \underbrace{f_1\, dt}_{\dot{x}\,dt = dx}$$

$$\approx 2\varepsilon \int_{x_{min}}^{x_{max}} y(a-x)\, dx.$$

$$= 2\varepsilon \int_{x_{min}}^{x_{max}} \left(2H_0 - x^2 + \frac{2}{3}x^3\right)^{1/2} (a-x)\, dx.$$

Note x_{min}, x_{max} given by $2H_0 - x^2 + \frac{2}{3}x^3 = 0$.

Can compute numerically. Varying a by a PO first appears at $a=0$ via a Hopf bifurcation and is destroyed when $a=\frac{1}{7}$ by a homoclinic bifurcation. (see §5).



4.5 Stability of Periodic Orbits in $\mathbb{R}^n$: Floquet Analysis

Suppose $\dot{\underline{x}} = \underline{f}(\underline{x})$ has a PO with period T and $\underline{x} = \underline{X}(t)$ being the PO $\quad \underline{X}(t+T) = \underline{X}(t)$.

Consider a small perturbation of $\underline{X}(t)$

$$\underline{x} = \underline{X}(t) + \underline{\eta}(t)$$

$$\dot{\underline{x}} = \dot{\underline{X}} + \dot{\underline{\eta}} = \underline{f}(\underline{X} + \underline{\eta}) = \underline{f}(\underline{X}) + \underline{\eta} \cdot \nabla \underline{f}(\underline{X}) + O(\eta^2)$$

So linear picture for how η evolves is just

$$\dot{\underline{\eta}} = \underline{\underline{A}}(t)\, \underline{\eta}$$

where $(A(t))_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\underline{X}(t)}$. Linear ODE for $\underline{\eta}$, where $\underline{\eta}(0)$ is arbitrary I.C.

Introduce a mapping $\quad \Phi_{ij}(t) : \underline{\eta}(0) \longrightarrow \underline{\eta}(t)$.

$$\eta_i(t) = \Phi_{ij}(t)\, \eta_j(0)$$

then

$$\dot{\Phi}_{ij}\, \eta_j(0) = A_{ij}(t)\, \Phi_{jk}(t)\, \eta_k(0).$$

Notice $\quad \Phi_{ij}(0) = \delta_{ij}$.

$$\dot{\Phi}_{ij}(t) = A_{ij}(t)\, \Phi_{kj}(t).$$

$$\Rightarrow \Phi_{ij}(T) = A_{ik}\, \Phi_{kj}(0) = A_{ij}$$

$$\Rightarrow \Phi_{ij}(2T) = A_{ik}\, \Phi_{kj}(T)$$

as $\underline{\underline{A}}$ is T-periodic, so

$$\underline{\underline{\Phi}}(nT) = \underline{\underline{A}}^n.$$

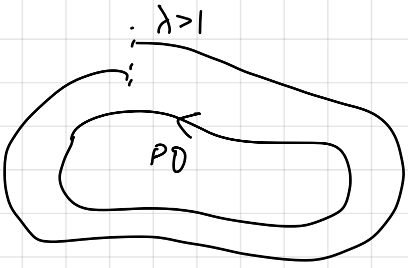
So

$$\underline{\eta}(nT) = [\underline{\underline{\Phi}}(T)]^n\, \underline{\eta}(0)$$

Thus whether or not an arbitrary perturbation $\underline{\eta}(0)$ grows or shrinks depends on the evals of $\underline{\underline{\Phi}}(T)$.

Specifically, if an eval has $|\lambda|>1$, then PO is unstable.

In $\mathbb{R}^2$,



Note that part of the perturbation is along the PO. This does not grow or shrink, i.e. $\exists$ always one eval $=1$.

Def^n Floquet multipliers of a PO are the evals λ of the matrix $\underline{\underline{\Phi}}(T)$. Floquet exponents are $\mu_i = \frac{1}{T}\log\lambda_i \Leftrightarrow \lambda_i = e^{\mu_i T}$.

We say a PO is hyperbolic if $|\lambda_i| \neq 1$ (ignoring the trivial λ along the trajectory. If all $|\lambda_i|<1$, PO is asymptotically stable. If any $|\lambda_i|>1$, PO is unstable.

POs are structurally stable $\Leftrightarrow$ they are hyperbolic.

In $\mathbb{R}^2$, there is only 1 non-trivial eval $\lambda \in \mathbb{R}$.

$$\det \underline{\underline{\Phi}}(t) = \lambda \times \underbrace{1}_{\text{trivial eval}} = \lambda$$

Recall $\dot{\underline{\underline{\Phi}}}(t) = \underline{\underline{A}}(t)\,\underline{\underline{\Phi}}(t)$

$$\Rightarrow \underline{\underline{\Phi}}(t) = e^{\int_0^t \underline{\underline{A}}(t')\,dt'}\,\underline{\underline{\Phi}}(0)$$

$$\Rightarrow \det \underline{\underline{\Phi}}(t) = \det\left(e^{\int_0^t \underline{\underline{A}}(t')dt'}\right) \underbrace{\det \underbrace{\underline{\underline{\Phi}}(0)}_{I}}_{=1}$$

Recall $\det e^{\underline{\underline{X}}} = e^{\operatorname{tr}\underline{\underline{X}}}$, since

$$\det e^{\underline{\underline{X}}} = \det e^{Q\Lambda Q^{-1}} \qquad , \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_k)$$

$$= \det \sum_{n=0}^{\infty} \frac{1}{n!}(Q\Lambda Q^{-1})^n$$

$$= \sum_{n=0}^{\infty} \det Q \det\left(\frac{\Lambda^n}{n!}\right) \det Q^{-1}$$

$$= \det \sum_{n=0}^{\infty} \frac{\Lambda^n}{n!}$$

$$= \det\left(\mathrm{diag}\left(e^{\lambda_1}, \dots, e^{\lambda_k}\right)\right)$$

$$= e^{\sum_{i=1}^{k} \lambda_i}$$

$$= e^{\mathrm{tr}\, \underline{X}}$$

$$\Rightarrow \quad \det \underline{\Phi}(t) = e^{\mathrm{tr} \int_0^t \underline{A}(t')dt'}$$

$$= e^{\int_0^t \mathrm{tr}(\underline{A}(t'))\, dt'}$$

Set $t=T$, then $LHS = \lambda$, $RHS = e^{\int_0^T \nabla\cdot \underline{f}(\underline{x}(t))\, dt}$, where $\dot{\underline{x}} = \underline{f}(\underline{x})$.

Since $\mathrm{tr}\, A = \mathrm{tr}\left(\partial f_i/\partial x_j \big|_{PO} \right) = \nabla\cdot \underline{f}(\underline{x}(t))\big|_{PO}$.

So if $\int_0^T \nabla\cdot \underline{f}\, dt < 0 \Rightarrow \lambda < 1$ STABLE

$> 0 \Rightarrow \lambda > 1$ UNSTABLE

$= 0 \Rightarrow$ PO not hyperbolic

Note in higher dimensions ($\mathbb{R}^n$),

$$e^{\int_0^T \nabla\cdot f\, dt} = \prod_{i=1}^{n} \lambda_i$$

Then $\int_0^T \nabla\cdot \underline{f}\, dt < 0$ is just a necessary but not sufficient condition for stability.

Example $\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = \frac{1}{r} \end{cases}$ PO at $r=1$ with $\dot{\theta} = 1$,

So $T = 2\pi$

Look at stability.

Let $r = \underbrace{1}_{} + \delta(t)$, then
$\theta = t + \varepsilon(t)$,

$\underline{x}(t) \quad \underline{y}(t)$

$$\dot r = (1+\delta)^{\cdot} = (1+\delta)(1-(1+\delta)^2).$$

$$\Rightarrow \dot\delta = (1+\delta)(-2\delta+\delta^2) \approx -2\delta$$

$$\Rightarrow \delta(t) = \delta(0)e^{-2t}.$$

$$\dot\theta = (t+\varepsilon)^{\cdot} = \frac{1}{1+\delta(t)} = 1-\delta(t)+O(\delta^2)$$

$$\Rightarrow \dot\varepsilon = -\delta(t) = -\delta(0)e^{-2t}$$

$$\Rightarrow \varepsilon(t) = \frac{1}{2}\delta(0)e^{-2t} + \varepsilon(0) - \frac{1}{2}\delta(0).$$

$$\Rightarrow \begin{pmatrix} \delta(t) \\ \varepsilon(t) \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-2t} & 0 \\ \frac{1}{2}(e^{-2t}-1) & 1 \end{pmatrix}}_{\underline{\underline{\Phi}}(t)} \begin{pmatrix} \delta(0) \\ \varepsilon(0) \end{pmatrix}$$

So $\underline{\underline{\Phi}}(t) = \begin{pmatrix} e^{-2t} & 0 \\ \frac{1}{2}(e^{-2t}-1) & 1 \end{pmatrix}$. $T = 2\pi$

Then evals are e^{-2T} and 1. ← evec $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so a shift along orbit

So the non-trivial Floquet number is $\lambda = e^{-4\pi} < 1$, so PO is $\underline{\text{stable}}$.

Alternatively, $$\nabla\cdot\underline{f} = \frac{1}{r}\frac{\partial}{\partial r}\left(r(1-r^2)\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(r\cdot\frac{1}{r}\right)$$
$$= 2-4r^2$$

On PO, $r=1$, so $\nabla\cdot\underline{f} = -2$.

Recall $\lambda = e^{\int_0^T \nabla\cdot\underline{f}\,dt} = e^{\int_0^{2\pi} -2\,dt} = e^{-4\pi}$.

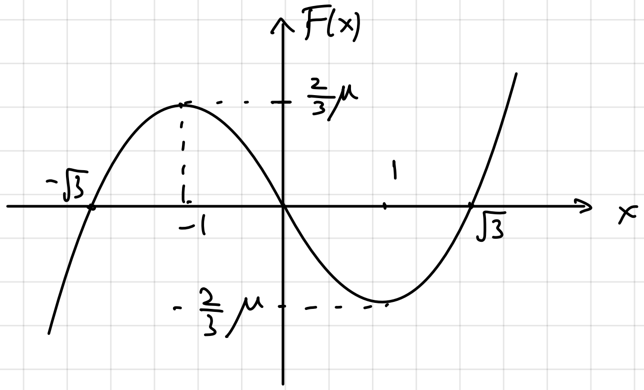
4.6 The van der Pol Oscillator

Defined by

$$\dot{x} = y - \mu x\left(\tfrac{1}{3}x^2 - 1\right) := y - F(x), \quad \mu > 0.$$
$$\dot{y} = -x$$

Note Symmetry $x \mapsto -x$, $y \mapsto -y$ (i.e. invariant under rotation of $\pm\pi$)

This looks like



Define $H_k(x,y) := \frac{1}{2}\left(x^2 + (y-k)^2\right)$.

Contours of constant H_k are circles centred at $(0,k)$.

Notice $\dot{H}_k = \dot{x}\frac{\partial H_k}{\partial x} + \dot{y}\frac{\partial H_k}{\partial y} = (y - F(x))\,x + (-x)(y-k)$

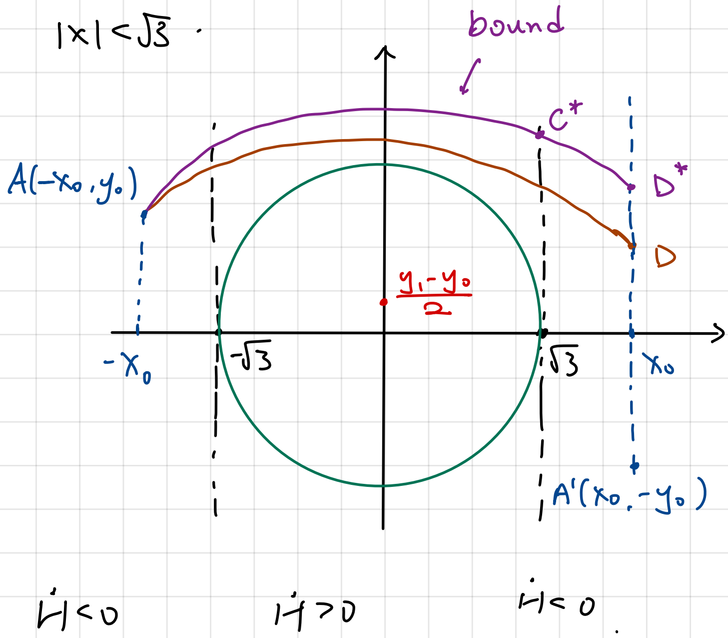
$$= kx - xF(x).$$

Hence, $\dot{H}_0 = -xF(x) \begin{cases} < 0 & \text{for } |x| > \sqrt{3} \\ > 0 & \text{for } |x| < \sqrt{3}. \end{cases}$

(i) $|x| < \sqrt{3} \Rightarrow \dot{H}_0 > 0$

Hence trajectories starting in $x^2+y^2<3$ eventually enter the region $x^2+y^2>3$.

Now, we show $\exists$ a region (containing $x^2+y^2<3$) that trajectories can't leave).



(ii) Let A be the point $(-x_0, y_0)$. As $-x_0 \to -\sqrt{3}$, $\dot{H}_0 < 0$.

So $B = (-\sqrt{3}, y_B)$. which is closer to the origin but $y_B > y_0$ as $\dot{y} > 0$

(iii) $B \to C = (\sqrt{3}, y_c)$. $\dot{H}_0 > 0$ so $y_c > y_B$. In particular,

$$\Delta H_0 = \int_{x=-\sqrt{3}}^{x=\sqrt{3}} -\mu x^2\left(\tfrac{1}{3}x^2 - 1\right) dt$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} -\mu x^2\left(\tfrac{1}{3}x^2 - 1\right) \frac{dt}{dx}\, dx$$

$\frac{dx}{dt} = y - F(x) > y - \frac{2}{3}\mu$, so $\frac{dt}{dx} < \frac{1}{y - \frac{2}{3}\mu} < \frac{1}{y_0 - \frac{2}{3}\mu}$

Also $y > y_0$ over $B \to C$ as $\dot{H}_0 > 0$. So

$$\Delta H_{0\,B\to C} < \frac{4\sqrt{3}\mu}{5(y_0 - \frac{2}{3}\mu)}$$

Hence, C lies within a circle of radius r, where

$$H_0(C) < H_0(A) + \frac{4\sqrt{3}\mu}{5(y_0 - \frac{2}{3}\mu)}$$

$$\Rightarrow \tfrac{1}{2}r^2 = \tfrac{1}{2}(x_0^2 + y_0^2) + \frac{4\sqrt{3}\mu}{5(y_0 - \frac{2}{3}\mu)} = H_0(C^*).$$

(iv) Continue circle of radius r (centre $\underline{O}$) from C^* to D^* at $x = x_0$.

Let $D^* = (x_0, y_1)$.

Now draw a circle centred on the y-axis from D^* to A'.

Centre of circle is at $(0, k)$, where $k = \frac{1}{2}(y_1 - y_0)$.

On this circle $\dot{H}_k = -\mu x^2\left(\frac{1}{3}x^2 - 1\right) + kx$. For this to be <0, need

$$\mu x_0^2\left(\tfrac{1}{3}x_0^2 - 1\right) > k x_0$$

as $x \geq x_0$ on circle.

[$k = \frac{1}{2}(y_1 - y_0)$, $\quad y_1^2 - y_0^2 = \frac{4\sqrt{3}\mu}{5(y_0 - \frac{2}{3}\mu)}$.

$$y_1 - y_0 = \frac{4\sqrt{3}\mu}{5(y_0 - \frac{2}{3}\mu)(y_0 + y_1)} < \frac{4\sqrt{3}}{5(y_0 - \frac{2}{3}\mu)\cdot 2y_0} \qquad (y_1 \geq y_0)$$

✓

So need

$$\mu x_0^2 \left(\frac{1}{3}x_0^2 - 1\right) > \frac{1}{2} \frac{4\sqrt{3}\mu}{5(y_0 - \frac{2}{3}\mu)(2\mu)} x_0 > k x_0$$

$$\Rightarrow \quad x_0^2 \left(\frac{1}{3}x_0^2 - 1\right) > \frac{\sqrt{3}}{5(y_0 - \frac{2}{3}\mu)\mu}$$

Hence, as long as x_0 is large enough for a given y_0, $\dot{H}_k < 0$. So trajectory stays within segment D^*A'. Hence, we deduce that there is a cts curve $ABCD + DD^* + D^*A'$ that trajectories cannot cross from the 'inside'.

Now complete this curve by reflecting about the origin (or rotate through $\pm\pi$) to generate a forward-invariant region. Then Poincaré – Bendixson gives us that there is at least one PO in the closed region.

(Note $x^2+y^2>3$ excludes only FP at $\underline{0}$)

<u>Two interesting limits</u>

1. For $\mu \ll 1$ – Near-Hamiltonian situations

$$H = \frac{1}{2}(x^2+y^2) \qquad x = \sqrt{2H_0}\cos t \qquad \dot{x} = y, \; \dot{y} = -x$$

$$y = \sqrt{2H_0}\sin t \qquad \text{period } 2\pi$$

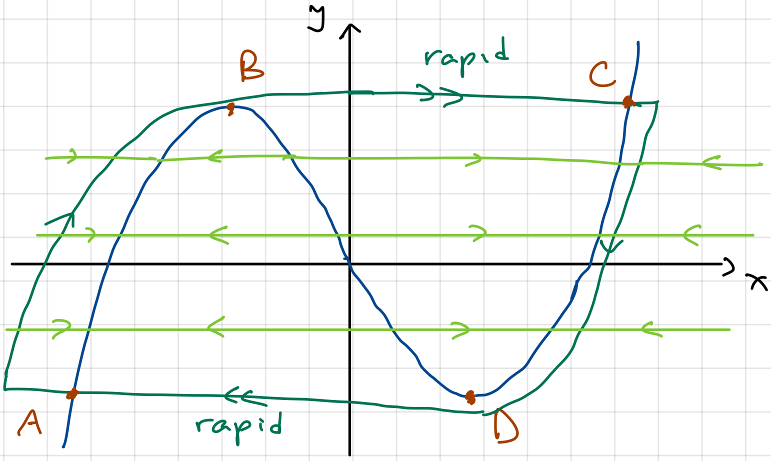
$$\begin{aligned}
\Delta H &= \mu \oint_{H=H_0} \dot{H}\, dt \\
&= \mu \oint_{H_0} -x F(x)\, dt \\
&= -\mu \int_0^{\pi} 2H_0 \cos^2 t \left(\frac{1}{3} \cdot 2H_0 \cos^2 t - 1\right) dt \\
&= \mu\pi H_0 (2 - H_0) + O(\mu^2)
\end{aligned}$$

So limit cycle is close to $H_0 = 2$, with period $= 2\pi + O(\mu)$.

2. $\mu \gg 1$

$$\dot{x} = y - \mu x\left(\tfrac{1}{3}x^2 - 1\right)$$

$$\dot{y} = -x$$



Time to go from A to B:

$\dot{y} = O(1)$, $\Delta y = O(\mu)$.

So time $= O(\mu)$ large.

From B to C: $\dot{x} = O(\mu)$, $\Delta x = O(1)$. time $= O(\frac{1}{\mu})$ small

So period will be dominated by the time taken on slow sections $\vec{AB}$ and $\vec{CD}$. Hence,

$$\begin{aligned} T &\approx 2\int_A^B dt \\ &= 2\int_{-2}^{1} \frac{dt}{dy}\cdot\frac{dy}{dx}\,dx \\ &= 2\int_{-2}^{1} \frac{1}{\dot{y}}\, F'(x)\,dx \qquad y \sim F(x). \\ &\approx 2\mu\int_{-2}^{1} \frac{1-x^2}{x}\,dx = \mu(3 - 2\log 2). \end{aligned}$$

5. Bifurcations

5.1 Introduction

Consider a system $\dot{\underline{x}} = \underline{f}(\underline{x};\mu)$ that depends continuously on a parameter μ.

A <u>bifurcation</u> is a change in the topological structure of the flow as μ passes through some critical value μ_0 (typically take as 0) — called a <u>bifurcation point</u>.

Examples 1. $\begin{cases} \dot{r} = r(\mu - r^2) \\ \dot{\theta} = 1 \end{cases}$ $\mu \le 0$ no POs
$\mu > 0$ have PO at $r = \sqrt{\mu}$

2. $\dot{x} = -\mu x + xy$ $\mu \le 0$ FP at $\underline{0}$
$\dot{y} = -y + x^2$ $\mu > 0$ FP at $\underline{0}$, $(\pm\sqrt{\mu}, \mu)$.

NL

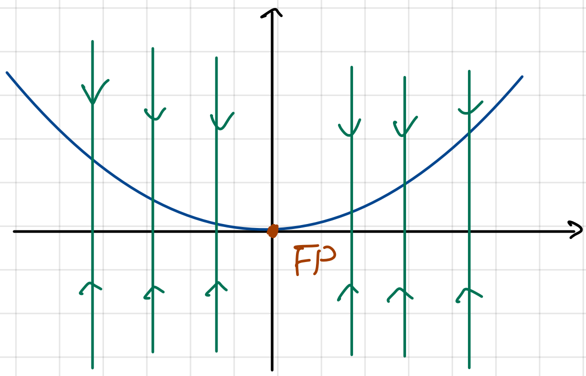
In both examples, the bifurcation is at $\mu = 0$ where the FP at $\underline{0}$ is non-hyperbolic. ($\lambda = \mu \pm i$, $\lambda = -\mu, -1$ resp.)

Bifurcations can also occur when there are non-hyperbolic POs or homoclinic or heteroclinic connections/orbits between saddles.
— these are features of the flow which are structurally unstable (see §1.4) and nonlinear terms influence the dynamics.

Consider example 2 again and consider x, y near the origin.
In particular, let $x, y = O(\mu)$,

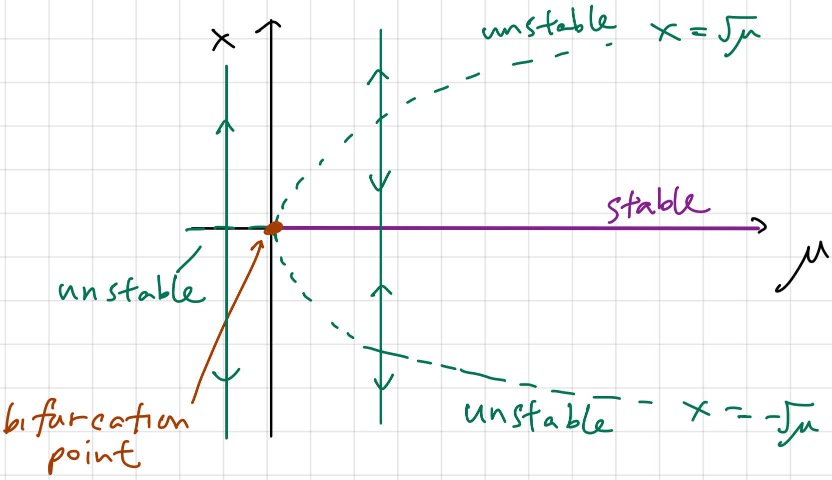
$$\dot{x} = 0 - \mu x + xy$$
$$\dot{y} = -y + x^2$$

Expect y to rapidly relax to the curve $y = x^2$ and then x varies slowly.



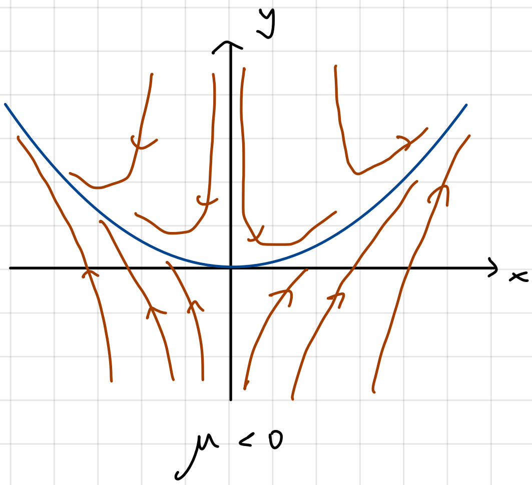
If $y \sim x^2$, $\dot{x} = -\mu x + x^3$
$= x(x^2 - \mu)$

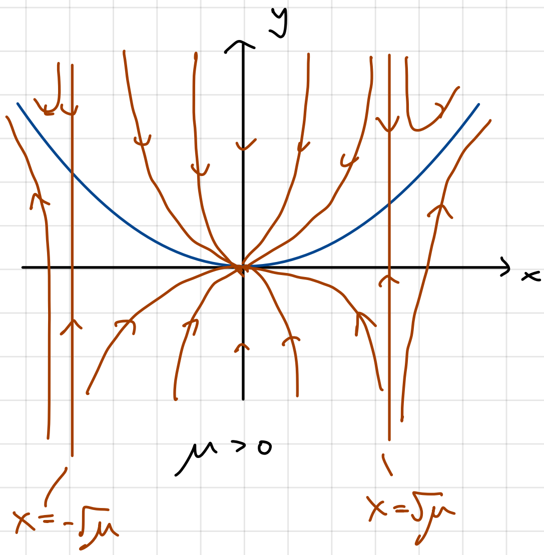
Thus captures the essence of the bifurcation.



Happens to be called a "pitchfork" bifurcation.

The key idea here is we have reduced the dynamics down to 1D (by working close to $y = x^2$) Then it is easier to explore the dynamics.





5.2 The centre manifold and extended centre manifold

Thm (Centre Manifold thm, CMT) If $\underline{0}$ is a non-hyperbolic FP of $\dot{\underline{x}} = \underline{f}(\underline{x})$ with linear stable, unstable and centre subspaces E^S, E^u and E^c (spanned by all λ with $Re(\lambda) = 0$). then $\exists$ stable, unstable and centre manifolds W^s, W^u and W^c that have the same dimensions as E^S, E^u, E^C and are tangent to them and are invariant.

__Example__ Consider in $\mathbb{R}^2$

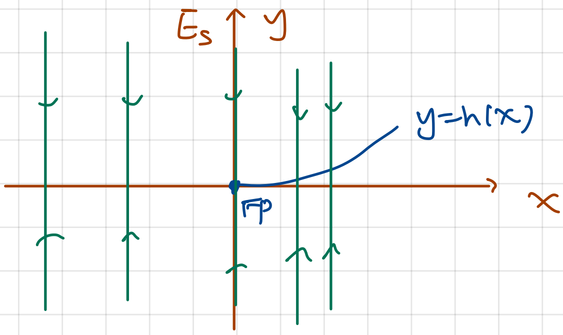
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -\lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}f(x,y)\\ g(x,y)\end{pmatrix}$$

$\lambda > 0$, $f, g = \mathcal{O}(|\underline{x}|^2) = \mathcal{O}(2)$, $f(0,0) = g(0,0) = 0$.

__Linearity__: For the full system, the centre manifold W^c is described by $y = h(x)$ s.t. $h(0) = h'(0) = 0$

goes through FP

E^c along x-axis



Seek a local approximation to $h(x)$.

$$h(x) = a_2 x^2 + a_3 x^3 + \dots$$

Recall W^c invariant, so $y = h(x)$ holds $\forall t$.

$$\Rightarrow \dot{y} = \frac{dh}{dx}\dot{x}$$

Choose $\lambda = 1$ $\Rightarrow$ $(-h + x^2) = \dfrac{dh}{dx}\, x h(x)$

Recall $\quad \dot{x} = xy = f, \quad \dot{y} = -y + x^2$

$$(-a_2x^2 - a_3x^3 - \dots + x^2) = (2a_2x + 3a_3x^2 + \dots)(a_2x^2 + a_3x^3 + \dots)$$

$$\Rightarrow (1 - a_2)x^2 - a_3x^3 - a_4x^4 + \dots = 2a_2^2x^4 + \dots$$

$$\left.\begin{aligned} x^2 &: \quad 1 - a_2 = 0\\ x^3 &: \quad -a_3 = 0\\ x^4 &: \quad -a_4 = 2a_2^2 \end{aligned}\right\} \Rightarrow a_2 = 1,\ a_3 = 0,\ a_4 = -2$$

$\Rightarrow y = h(x) = x^2 - 2x^4 + \dots$

(Recall we notice that $y \approx x^2$. Now see the leading correction is $-2x^4$.)

5.3 Extended Centre Manifold

As it stands, CMT does not allow us to deal with parameters. To include the effect of parameters and hence treat bifurcations (i.e. see what happens as a parameter pass through its BP value), we extend the centre manifold by augmenting the equations by the trivial eqn $\dot{\mu} = 0$. This adds one more dimension to the centre manifold and allows us to work in the neighbourhood of $\underline{0}$ and $\mu = 0$ in parameter space.

Assuming system is in <u>normal form</u> (E^c is $y=0$, $E^{s/u}$ is $x=0$), then the extended centre manifold can be parameterised by $y = h(x,\mu)$ with the conditions

(*) $h(0,0) = 0$, i.e. goes through FP.

(**) $Dh(0,0) = \underline{0}$, where $Dh = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial \mu}\right)$, i.e. manifold tangent to E^c.

<u>Example</u> $\dot{x} = -\mu x + xy$, $\dot{y} = -y + x^2$, $\dot{\mu} = 0$, or

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ \mu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ \mu \end{pmatrix} + \begin{pmatrix} x(y-\mu) \\ x^2 \\ 0 \end{pmatrix}$$

2 evals of 0 $\Rightarrow$ 2D centre manifold

Look for $h(x,\mu) = \underbrace{a_{0,0}}_{=0\ (*)} + \underbrace{a_{1,0}x + a_{0,1}\mu}_{=0\ (**)}$

$+ a_{2,0}x^2 + a_{1,1}x\mu + a_{0,2}\mu^2$

$+ a_{3,0}x^3 + \ldots$

ECM is invariant, so

$$\dot{y} = \dot{x}\frac{\partial h}{\partial x} + \cancelto{0}{\dot{\mu}\frac{\partial h}{\partial \mu}}$$

$$\Rightarrow -h + x^2 = (-\mu x + xh)\frac{\partial h}{\partial x}$$

$$\Rightarrow (1-a_{20})x^2 - a_{11}\mu x - a_{02}\mu^2 - a_{30}x^3 = (2a_{20}x + a_{11}\mu + 3a_{30}x^2 + \dots)$$
$$(-\mu x + x(a_{20}x^2 + a_{11}x\mu + a_{02}\mu^2))$$
$$= O(3)$$

So
$$\left.\begin{array}{ll} x^2: & 1-a_{20} = 0 \\ \mu x: & a_{11} = 0 \\ \mu^2: & a_{02} = 0 \end{array}\right| \Rightarrow a_{20} = 1,\ a_{11} = a_{02} = 0$$

$$\Rightarrow y = h(x,\mu) = x^2 + O(x^3)$$

So extended centre manifold is

pitchfork bifurcation as summarised before.

$$\dot{x} = -\mu x + xh(x,\mu) = -\mu x + x^3 + O(4)$$

<u>Example</u>
$$\begin{cases} \dot{x} = \mu + x^2 + xy + y^2 \\ \dot{y} = 2\mu - y + x^2 + xy \end{cases}$$
Non-hyperbolic FP at $\underline{0}$ when $\mu = 0$.

Extended system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ \mu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ \mu \end{pmatrix} + \text{non-linear terms}$$

E^c is the plane $y = 2\mu$. (deduce this by either *(a) find E^c directly, or (b) find E^s and generate dual to it)

$* A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ is defective.

$\lambda = 0$ has evec $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and generalised evec $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

E^c is $y = 2\mu$

So ECM $y = h(x,\mu)$ with $h(0,0) = 0$. $Dh(0,0) = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial \mu}\right) = (0,2)$.

So let $h(x,\mu) = 2\mu + a_{20}x^2 + a_{11}x\mu + a_{02}\mu^2 + O(x^3)$

h must satisfy $\dot{y} = \dot{x}\frac{\partial h}{\partial x} + \cancel{\dot{\mu}\frac{\partial h}{\partial \mu}}^{=0}$

$$\Rightarrow \underbrace{(2\mu - h + x^2 + xh)}_{\dot{y}} = \underbrace{(2a_{20}x + a_{11}\mu + \mathcal{O}(2))}_{\frac{\partial h}{\partial x}}\underbrace{(\mu + x^2 + xh + h^2)}_{\dot{x}}$$

Work to $\mathcal{O}(2)$:

$$2\mu - 2\mu - a_{20}x^2 - a_{11}x\mu - a_{02}\mu^2 + x^2 + 2\mu x + \mathcal{O}(3) = (2a_{20}x + a_{11}\mu)\mu + \mathcal{O}(3)$$

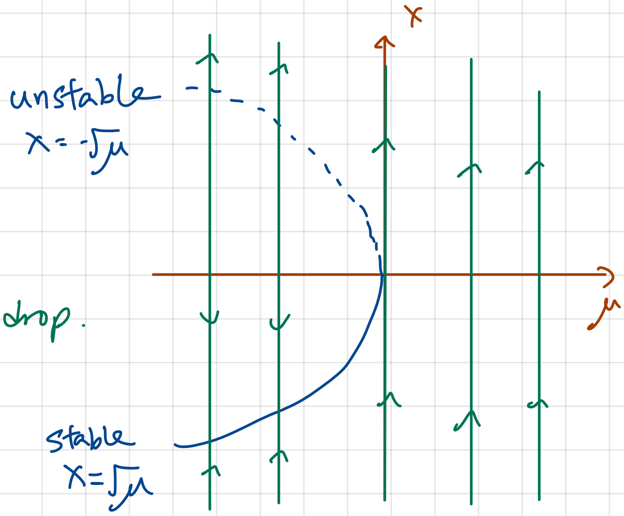
$$\left.\begin{aligned} x^2 &: \quad -a_{20} + 1 = 0 \\ x\mu &: \quad -a_{11} + 2 = 2a_{20} \\ \mu^2 &: \quad -a_{02} = a_{11} \end{aligned}\right\} \Rightarrow a_{20} = 1, \; a_{11} = a_{02} = 0.$$

So $\quad y = 2\mu + x^2 + \mathcal{O}(3)$.

So dynamics of ECM,

$$\dot{x} = \mu + x^2 + x(2\mu + x^2) + 4\mu^2 + \mathcal{O}(3)$$

$$= \underbrace{\mu + x^2}_{\substack{\text{interesting balance} \\ \text{has } x^2 = \mathcal{O}(\mu)}} + \underbrace{2x\mu}_{\mathcal{O}(\mu^{3/2}) \Rightarrow \text{drop}} + \underbrace{4\mu^2}_{\mu^2 \ll \mu \ll 1 \Rightarrow \text{drop.}} + \mathcal{O}(3)$$



Taking leading terms,

$$\dot{x} = \mu + x^2 + HOT$$

This is a "<u>saddle node bifurcation</u>".

<u>Example</u> (Higher D Lorentz equations)

$$\left.\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned}\right\} \text{FP at } \underline{0}, \qquad A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$\exists$ a bifurcation at $r = 1$.

E^c is the line $x = y$, $z = 0$, E^s is the plane $x + \sigma y = 0$.

Change variables $\mu = r-1$, $v = y - x$.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(1+\mu) & 0 \\ 0 & 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ v \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x(\mu - z) \\ x(v+x) \end{pmatrix}$$

For W^c set $v = V(x,\mu)$ and $z = Z(x,\mu)$

At $x = \mu = 0$. $V, V_x, V_\mu, Z, Z_x, Z_\mu$ all zero as E^c is $v = z = 0$.

So W^c invariant

$$\Rightarrow \quad \begin{aligned} \dot{V} &= \dot{x} V_x \\ \dot{Z} &= \dot{x} Z_x \end{aligned} \quad \Rightarrow \quad \begin{aligned} \sigma V V_x &= -(1+\sigma) V + x(\mu - Z) \quad &(1) \\ \sigma V Z_x &= -bZ + x(V+x) \quad &(2) \end{aligned}$$

So Z and $V = \{x^2, x\mu, \mu^2\} + O(3)$

(1), (2) to $O(2)$ is just

$$0 = -(1+\sigma) V_2 + x\mu \Rightarrow V_2 = \frac{x\mu}{1+\sigma}$$

$$0 = -b Z_2 + x^2 \Rightarrow Z_2 = \frac{x^2}{b}$$

(1) to $O(3)$:

$$\sigma \left(\frac{x\mu}{1+\sigma}\right)\left(\frac{\mu}{1+\sigma}\right) = -(1+\sigma) V_3 - x Z_2$$

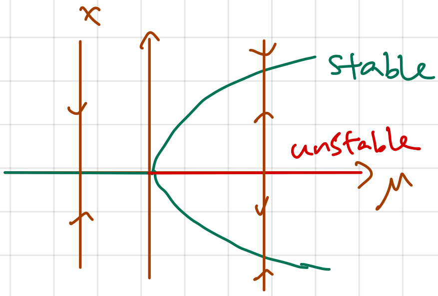
$$\Rightarrow \quad V_3 = -\frac{1}{b}\frac{x^3}{1+\sigma} - \frac{\sigma}{(1+\sigma)^3} x\mu^2 .$$

Hence, $$V = \frac{1}{1+\sigma} x\mu - \frac{1}{b(1+\sigma)} x^3 - \frac{\sigma}{(1+\sigma)^3} x\mu^2 + O(4)$$

Dynamics on ECM: $\dot{x} = \sigma V$

$$\Rightarrow \dot{x} = \sigma \left(\frac{1}{1+\sigma} x\mu + \frac{1}{b(1+\sigma)} x^3 \right) + HoT$$

$$= \frac{\sigma}{1+\sigma} x \left(\mu - \frac{x^2}{b}\right) + HoT$$



supercritical pitchfork bifurcation

$\dot{x} = \mu x - x^3$

5.3 Stationary Bifurcations

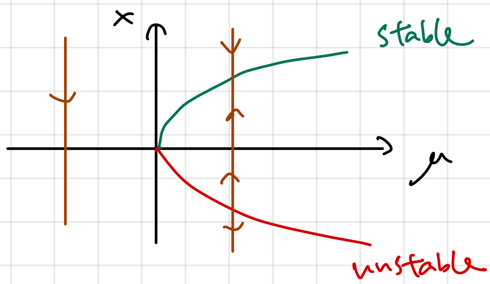
Consider $\dot{x} = f(x, \mu)$, $x \in \mathbb{R}$. Suppose non-hyperbolic FP at $x=0$ for $\mu = 0$, i.e. $f = f_x = 0$ at $\mu = x = 0$. We can classify the main types of bifurcations and give normal forms (= simple generic form)

1. Saddle node bifurcation (SN)

$$\boxed{\dot{x} = \mu - x^2}$$

$\mu > 0$: FPs at $x = \sqrt{\mu}$ (stable)

$x = -\sqrt{\mu}$ (unstable)



$\mu = 0$: FP at $x = 0$

$\mu < 0$: No FPs.

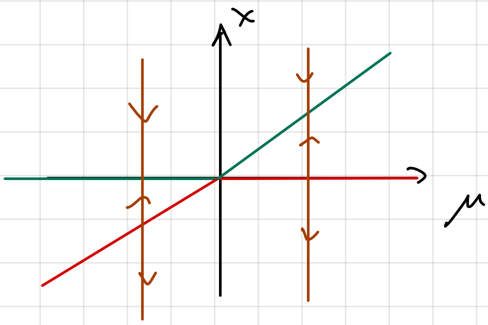
This is the most generic bifurcation

- structurally stable.

2. Transcritical bifurcation (TC)

$$\boxed{\dot{x} = \mu x - x^2}$$

FPs at $x = 0, \mu$. They cross each other and exchange stability

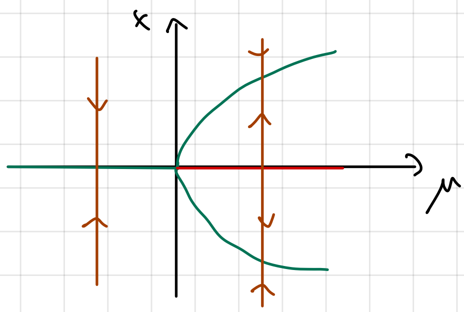


- Not structurally stable

3. Pitchfork bifurcation (PF)

$$\boxed{\dot{x} = \mu x - a x^3 \quad , a = \pm 1}$$

$a = 1$:

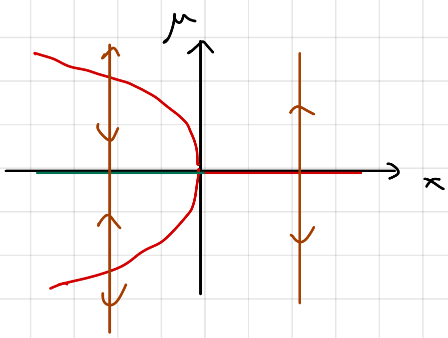


Supercritical pitchfork

FP at $x = 0$, $x = \pm\sqrt{\mu}$, $\mu > 0$

- Not structurally stable.

$a = -1$



Subcritical pitchfork

- Not structurally stable

More generally, consider Taylor series in x

$$\dot{x} = f(x,\mu) = a_0(\mu) + a_1(\mu)\,x + a_2(\mu)\,x^2 + \dots$$

with $f(0,0) = 0$ $\quad$ ($x=0$ is a FP and $\mu = 0$)

$f_x(0,0) = 0$ $\quad$ (BP at $\mu = 0$)

$\Rightarrow$ $a_0(0) = a_1(0) = 0$

$\Rightarrow$ $a_0, a_1 = O(\mu)$ and expect $a_2(0) \neq 0$.

So most <u>generic</u> case has $a_0'(0)$, $a_1'(0)$ and $a_2(0)$ all $\neq 0$.

$$\Rightarrow \dot{x} = A\mu + B\mu x + Cx^2 + \text{HOT}. \quad A, B, C \neq 0$$

$$= A\mu + C\left(x + \frac{B\mu}{2C}\right)^2 + \text{HOT}.$$

So now setting $X = x + B\mu/2C$, rescaling μ and t, get SN normal form

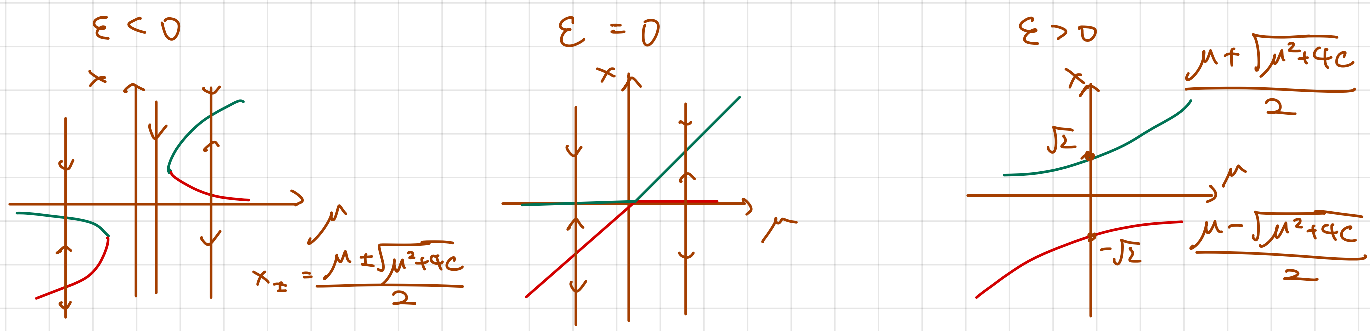
$$\boxed{\dot{X} = \hat{\mu} \pm X^2}$$

So saddle node is most generic bifurcation. Continuing through degenerate cases, find transcritical and pitchfork bifurcations are next common.

If system has some symmetry, then TC and PF may become "<u>generic</u>" within that context.

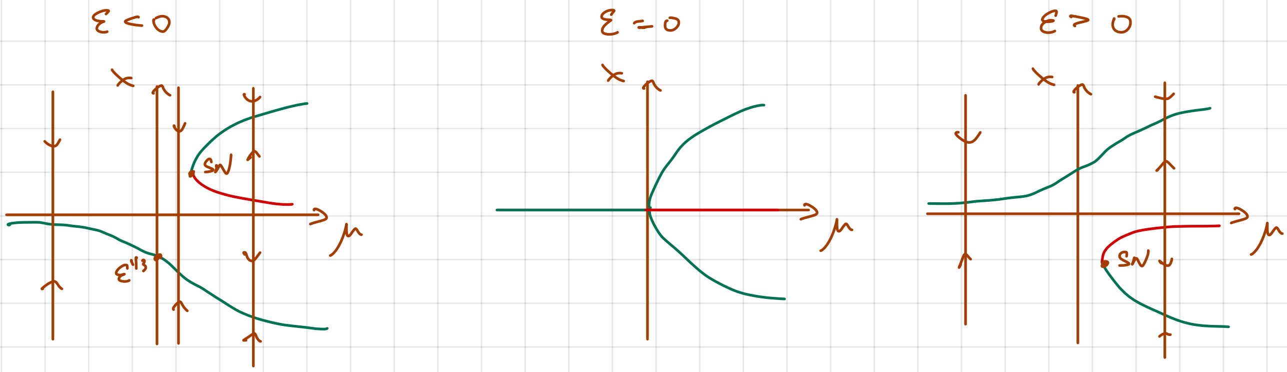
<u>Both TC and PF are structurally unstable</u>

<u>Example</u> TC + const. $\quad \dot{x} = \varepsilon + \mu x - x^2$

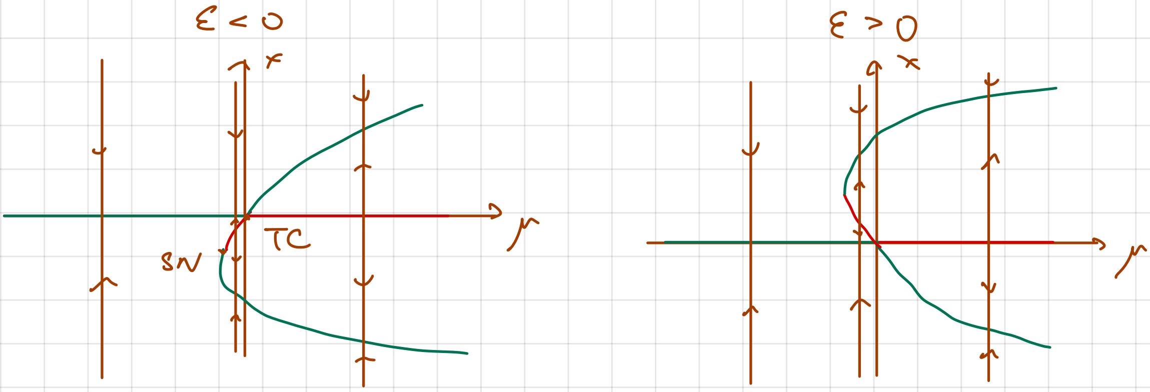


Saddle node at $\mu = \pm 2\sqrt{-\varepsilon}$, $x = \pm\sqrt{-\varepsilon}$.

Example PF + const. $\dot{x} = \varepsilon + \mu x - x^3$.



Example PF + const. x^2 $\dot{x} = \mu x + \varepsilon x^2 - x^3$



5.4 Oscillatory / Hopf bifurcations

The case that a pair of complex evals pass through $\mathrm{Re}(\lambda) = 0$ at $\mu = 0$ with $\lambda = \pm i\omega$ $(\omega \neq 0)$

CM is 2D. Dynamics has normal form

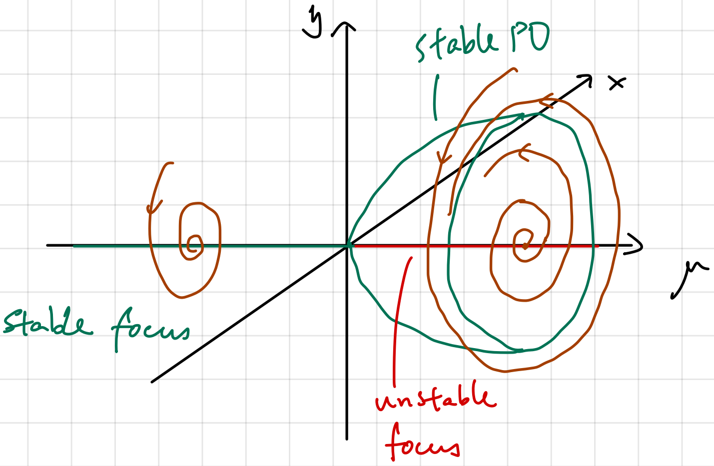
$$\dot{r} = (\mu - a r^2) r, \qquad a = \pm 1$$

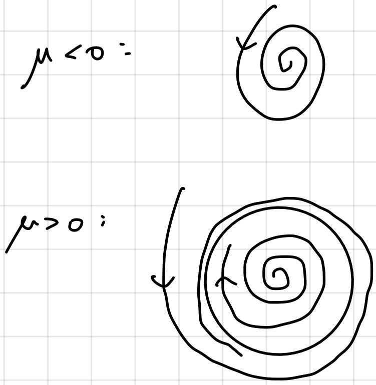
$$\dot{\theta} = \omega$$

Origin is a focus (stable for $\mu < 0$, unstable for $\mu > 0$).

PO at $r = \sqrt{\mu/a}$ when $\mu/a > 0$.

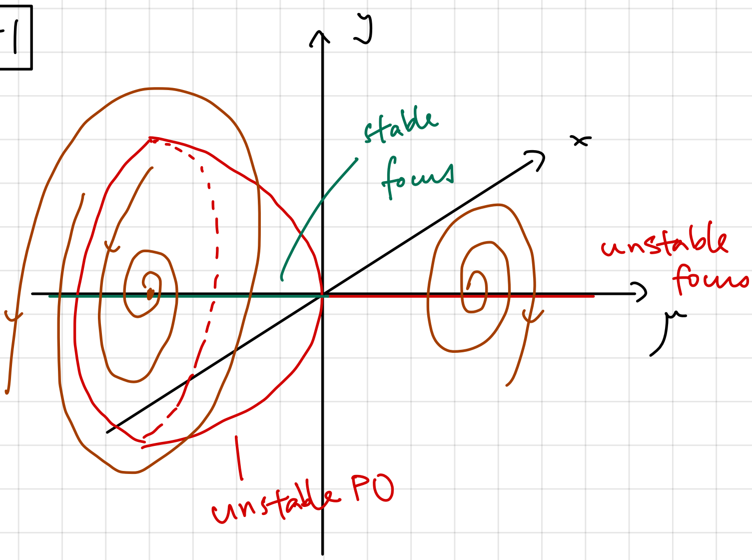
$\boxed{a=1}$

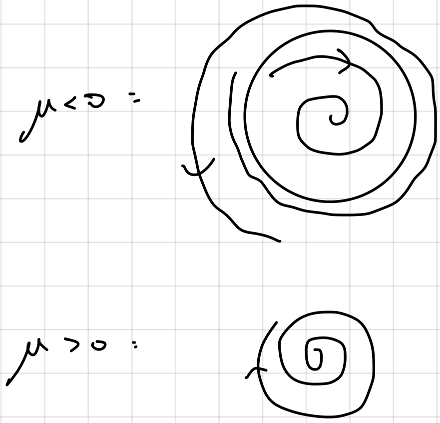




Supercritical Hopf

$\boxed{a=-1}$

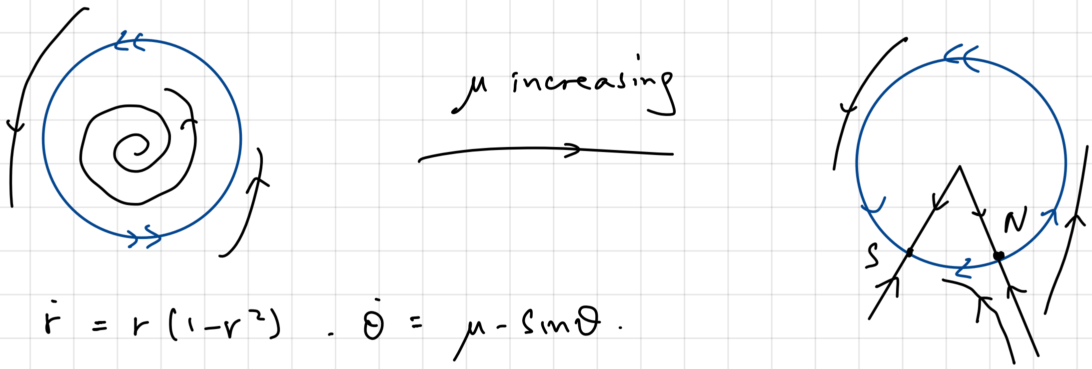




Subcritical Hopf

* 5.6 Bifurcations of POs *

(i) Saddle node bifurcation on a PO



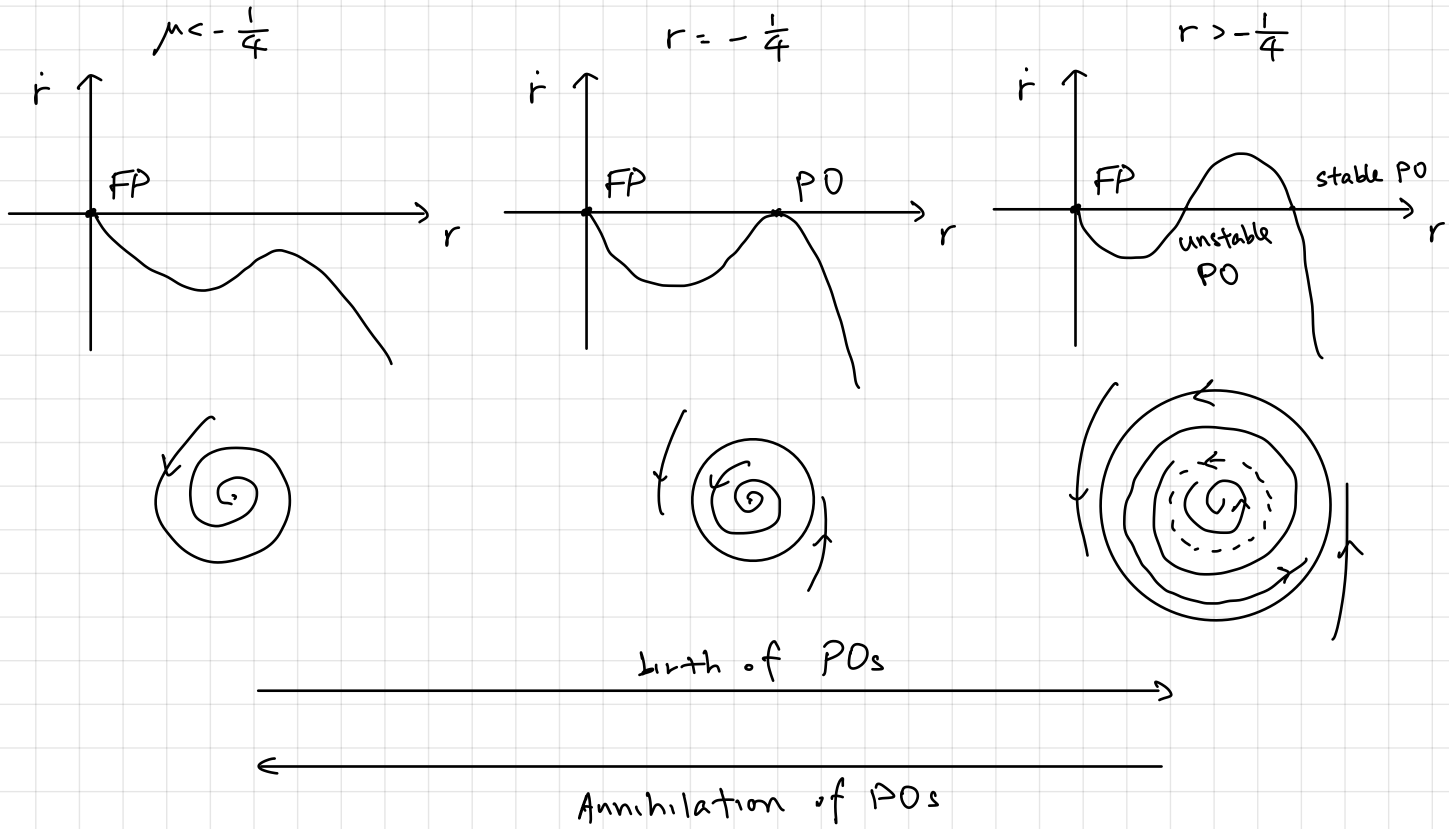
e.g. $\dot{r} = r(1-r^2)$. $\dot{\theta} = \mu - \sin\theta$.

BPs at $\mu = \pm 1$.

(2) Saddle node of POs

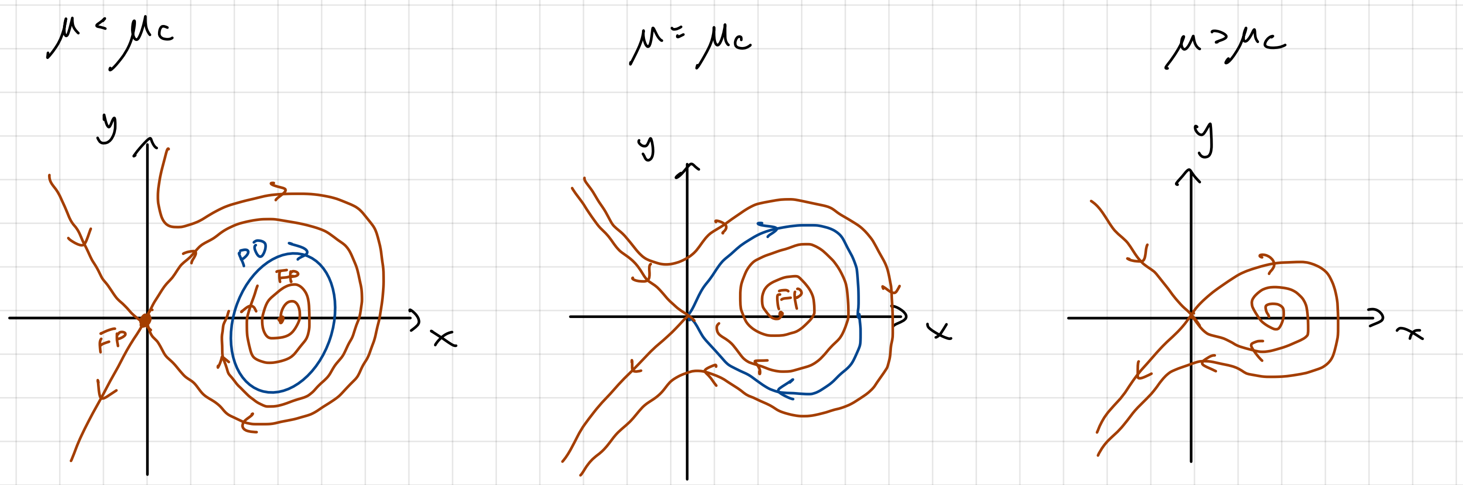
Two POs coalesce and annihilate (or vice versa)

e.g. $\dot{r} = \mu r + r^3 - r^5$, $\dot{\theta} = \omega + b r^2$



(3) Homoclinic bifurcations

e.g. $\dot{x} = y$, $\dot{y} = \mu y + x - x^2 + xy$, $\mu_c = -0.8645$



stable PO

limit cycle swells and bangs into the saddle creating homoclinic connection

No PO, no homoclinic connection

6. Maps

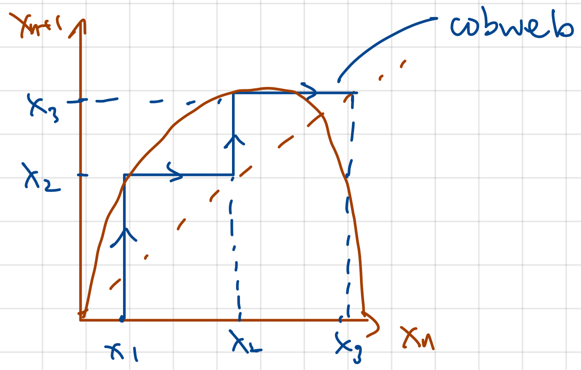
6.1 Key examples of maps

Maps emerge naturally by strobing ODE systems. Important in their own right (discrete time evolution) and provide simplest framework to consider chaos.

Common examples:

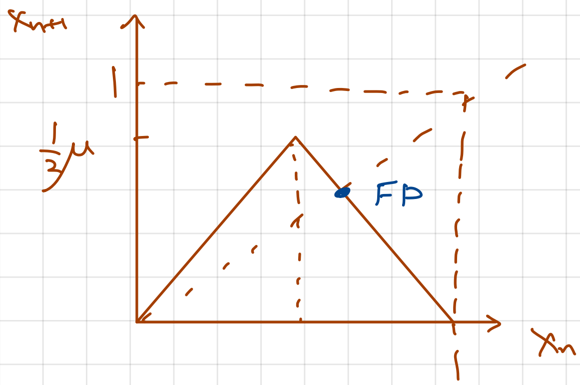
(1) Logistic Map $\quad x_{n+1} = F(x_n) = \mu x_n(1-x_n)$.

usually $0 \le \mu \le 4$, $\quad x \in [0,1]$



(2) Test map

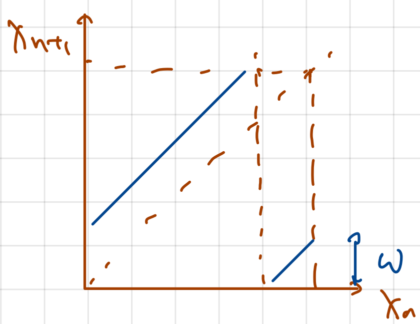
$$x_{n+1} = \begin{cases} \mu x_n & 0 \le x_n \le \frac{1}{2} \\ \mu(1-x_n) & \frac{1}{2} < x_n \le 1 \end{cases}, \quad 0 \le \mu \le 2, \ x \in [0,1]$$



(3) Rotation Map $\quad x_{n+1} = x_n + \omega \pmod 1$

If $\quad \omega \in \mathbb{Q}$, periodic "motion"

$\omega \notin \mathbb{Q}$. aperiodic "motion"



(4) Sawtooth / Bernoulli Shift map $\quad x_{n+1} = 2x_n \pmod 1$

Equivalent to a "shift" map. Express x_n as a binary number

$$x_n = 0.a_1a_2a_3\ldots, \quad a_i = 0,1$$

$$2x_n = 0.a_2a_3\ldots$$

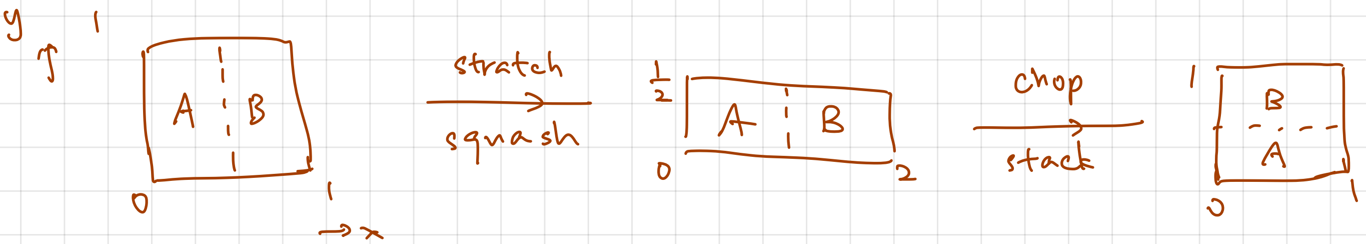
Shift to left and drop leading coeff.

Sol^n can
- hit zero eventually if $x_0 = p2^{-n}$, $n, p \in \mathbb{Z}$.
- be periodic (eventually) if $x_0 = \frac{p}{q} \in \mathbb{Q}$, $q \neq 2^n$ for any $n \in \mathbb{N}$
- be aperiodic if $x_0 \notin \mathbb{Q}$.

This map will be our prototype of chaos in Ch7.

(5) Baker's map (2D square)

$$x_{n+1} = 2x_n \pmod 1, \quad y_{n+1} = \begin{cases} \frac{1}{2}y_n & 0 \leq y_n \leq \frac{1}{2} \\ \frac{1}{2}(y_n+1) & \frac{1}{2} \leq y_n \leq 1 \end{cases}.$$



6.2 Fixed points, cycles, stability

Consider $\underline{x}_{n+1} = \underline{F}(\underline{x}_n)$

- FP $\underline{x}_0$ s.t. $\underline{F}(\underline{x}_0) = \underline{x}_0$
- PO with period n is $\underline{x}_0$ if $\underline{F}^n(\underline{x}_0) = \underline{x}_0$, with $\underline{F}^k(\underline{x}_0) \neq \underline{x}_0$ for $k < n$.
- $\underline{x}_0$ is period N, then $\{\underline{x}_0, \underline{F}(\underline{x}_0) = \underline{x}_1, \underline{F}^2(\underline{x}_0) = \underline{x}_2, \ldots, \underline{x}_{N-1}\}$ is called an N-cycle.

- Invariant sets $\Lambda \Rightarrow \underline{x} \in \Lambda \Rightarrow \underline{F}(\underline{x}) \in \Lambda$

If we write $\underline{F}(\Lambda) = \{\underline{x} : \exists\, y \in \Lambda \text{ s.t. } \underline{F}(y) = \underline{x}\}$, then invariant means $\underline{F}(\Lambda) \subseteq \Lambda$

- Forward orbit $\mathcal{O}^+(\underline{x}) = \{\underline{x}, \underline{F}(\underline{x}), \underline{F}^2(\underline{x}) \ldots\}$. Similarly backward.
- Stability: Lyapunov, QAS, AS of invariant sets all follow in similar way.
- Attractor: An invariant set Λ which is AS, and has no non-trivial subset with the same properties.

<u>Stability of FP</u> $\underline{x}_0 = F(\underline{x}_0)$ is determined by the Jacobian

$$A_{ij} = \left.\frac{\partial F_i}{\partial x_j}\right|_{\underline{x}_0}$$

Linearisation after setting $\underline{y}_n = \underline{x}_n - \underline{x}_0$ and assume $|\underline{y}_n| \ll |\underline{x}_0|$

$$\underline{x}_{n+1} = F(\underline{x}_n)$$

$$\Rightarrow \quad \underline{x}_0 + \underline{y}_{n+1} = F(\underline{x}_0 + \underline{y}_n)$$

$$\Rightarrow \quad \underline{x}_0 + \underline{y}_{n+1} = F(\underline{x}_0) + A \cdot \underline{y}_n + \mathcal{O}(|\underline{y}_n|^2)$$

Hence, a FP of a map is

- AS if <u>all</u> evals of A have $|\lambda| < 1$
- Unstable if <u>any</u> eval has $|\lambda| > 1$
- Non-hyperbolic if <u>any</u> eval is on the unit circle, i.e. $|\lambda| = 1$.

Can use this for <u>N-cycles</u>: if $\underline{x}_0$ is a period N point, $\underline{F}^N(\underline{x}_0) = \underline{x}_0$, so just consider $\underline{G} := \underline{F}^N$.

Work in 1D:

$$\frac{dG}{dx} = \frac{d}{dx} F^N(x) = \frac{d}{dx}\left(F(F\ldots(F(x)))\right)$$
$$= F'(F^{N-1}(x))\, F'(F^{N-2}(x)) \ldots F'(F(x))\, F'(x).$$
$$\Rightarrow \quad \frac{dG}{dx} = F'(x_{N-1})\, F'(x_{N-2}) \ldots F(x_1)\, F'(x_0)$$

So necessarily, get the same answer no matter which point on N-cycle we consider.

6.3 Bifurcations of 1D maps

Consider $x_{n+1} = F(x_n; \mu)$, $\lambda = \frac{\partial F}{\partial x}\Big|_{x^*}$. Bifurcation occurs when $\lambda = \pm 1$.

WLOG say FP is at $x^* = 0$ and the bifurcation is at $\mu = 0$.

$F(0,0) = 0$ $\qquad F_x(0,0) = \pm 1$.

Four common bifurcations:

(1) Saddle node ($\lambda = 1$) $\qquad x_{n+1} = F(x_n; \mu)$

x_{n+1}, x_n, μ all close to 0 (BP). Then

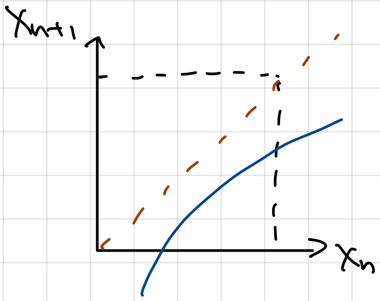
$$x_{n+1} = F(0,0) + F_x(0,0)\, x_n + F_\mu(0,0)\,\mu$$
$$+ \frac{1}{2} F_{xx}(0,0)\, x_n^2 + \underbrace{F_{x\mu}(0,0)\, x_n \mu + F_{\mu\mu}(0,0)\,\mu^2 + O(3)}_{HOT}$$

$$= x_n + F_\mu(0,0)\,\mu + \frac{1}{2} F_{xx}(0,0)\, x_n^2 + HOT.$$

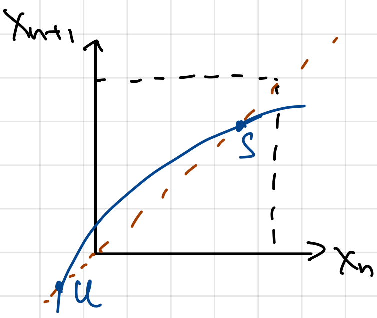
Rescale and redefine variables to get the normal form

$$\boxed{x_{n+1} = x_n + \mu - x_n^2}$$

(compare ODE $\dot{x} = \mu - x^2$)

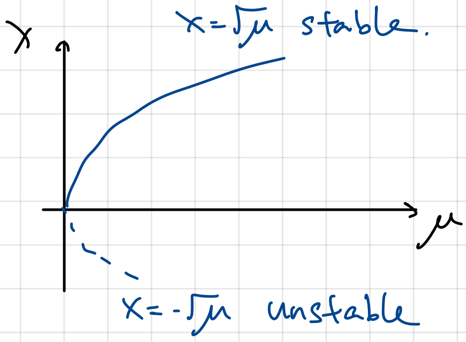


No FP



2 FPs

$\Rightarrow$



(2) Transcritical ($\lambda = 1$)

Keep all sol^n to $x=0$ $\forall \mu$

$$\Rightarrow F(0,\mu) = 0 \quad \forall \mu \Rightarrow F_\mu(0,0) = 0 \Rightarrow F_{\mu\mu}(0,0) = 0$$

So $$x_{n+1} \approx x_n + \tfrac{1}{2} x_n^2 F_{xx} + \mu x_n F_{x\mu} + O(\mu^3)$$

$$\approx x_n + x_n \left(\tfrac{1}{2} F_{xx} x_n + \mu F_{x\mu} \right) \quad \text{truncating}$$

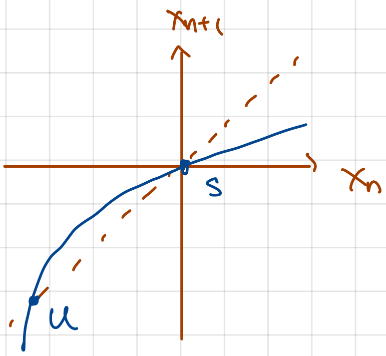
Example $F_{xx} < 0$, $F_{x\mu} > 0$.

$$x_{n+1} \approx x_n - x_n(x_n - a\mu) \qquad a > 0 \quad \text{(normal form)}$$

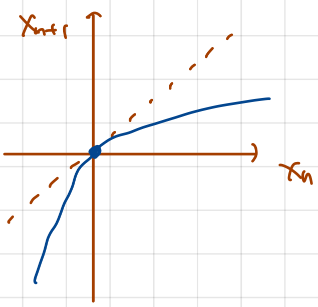
FPs at $x = 0$, $a\mu$.

$F'(0) = 1 - 2x + a\mu \big|_{x=0} = 1 + a\mu$ — S $\mu < 0$, U $\mu > 0$

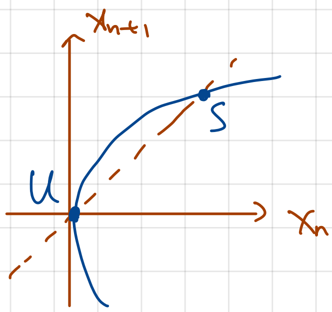
$F'(a\mu) = 1 - 2x + a\mu \big|_{x=a\mu} = 1 - a\mu$ — U $\mu < 0$, S $\mu > 0$



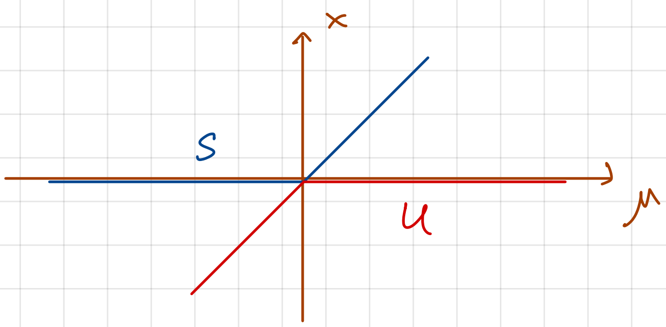
$\mu < 0$



$\mu = 0$



$\mu > 0$



(3) Pitchfork ($\lambda = 1$)

Have $F_\mu(0,0) = 0 = F_{xx}(0,0)$. Assume $x \sim \mu^{1/2}$.

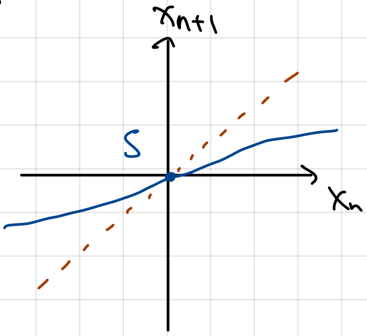
Then we have

$$x_{n+1} \approx x_n + x_n \mu F_{x\mu} + \frac{1}{6} x_n^3 F_{xxx} + O(\mu^2)$$

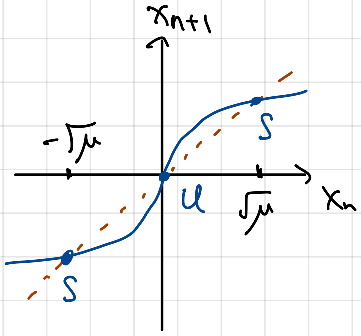
FPs at $x = 0$, $x = \pm\left(-6\mu F_{x\mu} / F_{xxx}\right)^{1/2}$ when $(\quad) > 0$.

Normal form is $\boxed{x_{n+1} = x_n + x_n(\mu - a x_n^2)}$, $a = \pm 1$.

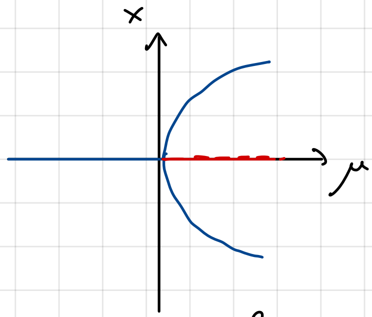
<u>$a = 1$</u>



$\mu < 0$

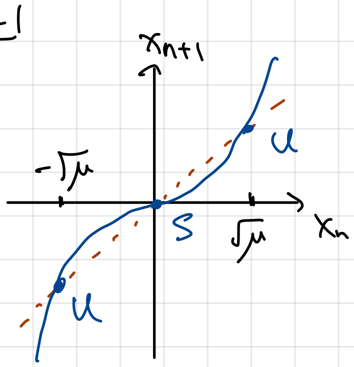


$\mu > 0$

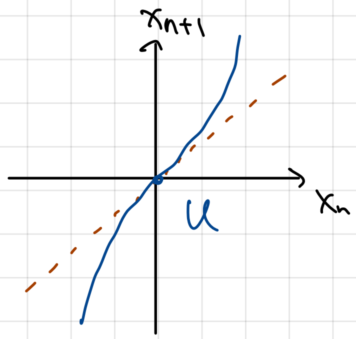


supercritical bifurcation.

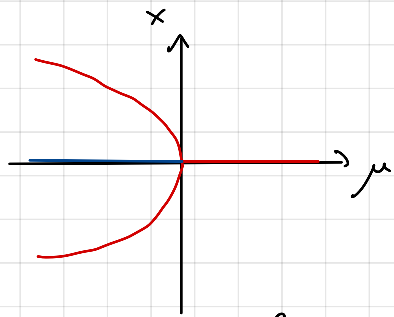
<u>$a = -1$</u>



$\mu < 0$

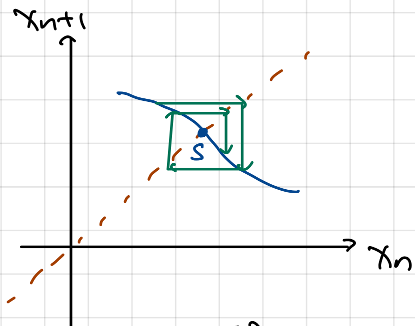


$\mu > 0$

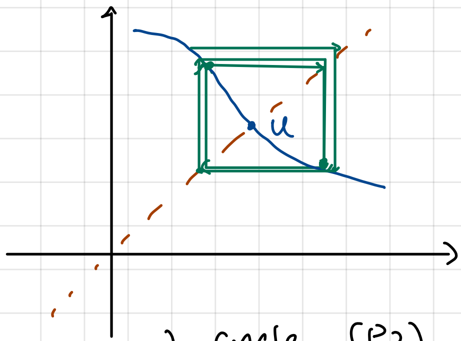


subcritical bifurcation.

(4) Period doubling bifurcation ($\lambda = -1$)



stable FP



2 cycle (P2)

FP changes stability and sheds a 2-cycle.

Suppose bifurcation is at $x=\mu=0$. Also keep FP at 0 $\forall \mu$. So

$F(0,\mu)=0$. $F_x(0,0)=-1$. Then assuming $x_n \sim \mu^{1/2}$.

$$x_{n+1} = (\underbrace{-1}_{F_x(0,0)=-1} + \underbrace{A}_{F_{x\mu}}\mu)\, x_n + \underbrace{B}_{\frac{1}{2}F_{xx}} x_n^2 + \underbrace{C}_{\frac{1}{6}F_{xxx}} x_n^3 + O(\mu^2)$$

$$\Rightarrow \quad x_{n+1} = -\underbrace{x_n}_{O(\mu^{1/2})} + \underbrace{B x_n^2}_{O(\mu)} + \underbrace{(A\mu x_n + C x_n^3)}_{O(\mu^{3/2})} + O(\mu^2)$$

Now consider x_{n+2} (because of "$-$" x_n)

$$\begin{aligned} x_{n+2} &= -x_{n+1} + B x_{n+1}^2 + (A\mu x_{n+1} + C x_{n+1}^3) + O(\mu^2) \\ &= -(-x_n + B x_n^2 + A\mu x_n + C x_{n+1}^3) \\ &\quad + B(-x_n + B x_n^2 + \dots)^2 \\ &\quad + A\mu(-x_n + \dots) + C(-x_n + \dots)^3 + O(\mu^2) \end{aligned}$$

$$\begin{aligned} x_{n+2} &= x_n - B x_n^2 - A\mu x_n - C x_n^3 \\ &\quad + B x_n^2 - 2B^2 x_n^3 - A\mu x_n - C x_n^3 + O(\mu^2) \\ &= x_n - 2A\mu x_n - 2(B^2+C)\, x_n^3 + O(\mu^2) \qquad (*) \end{aligned}$$

Look for FPs of F^2 $x_{n+2} = x_n$ $(=x)$.

$$\Rightarrow \quad x\,(A\mu + (B^2+C)x^2) = 0$$

So $x=0$ is a FP, and

$$x = \pm\sqrt{-\frac{A\mu}{C+B^2}} \qquad (x \sim \mu^{1/2}) \text{ which forms a 2-cycle.}$$

A period doubling bifurcation of F is a pitchfork bifurcation of F^2.

Check the stability of the 2-cycle:

$$\frac{d}{dx}F^2 = 1 - 2A\mu - 6(B^2+C)x^2 + O(\mu^{3/2})$$

At P_2, $= 1 - 2A\mu - 6(-A\mu) = 1 + 4A\mu$

At $x=0$. $\left.\frac{dF}{dx}\right|_0 = -1 + 2Bx + A\mu + 3Cx^2\Big|_{x=0} = -1 + A\mu$.

Hence have opposite stability

If $A > 0$, P_2 stable $\mu < 0$. unstable $\mu > 0$

FP $(x=0)$ unstable $\mu < 0$, stable $\mu > 0$.

We don't usually need the normal form to discuss a period-doubling bifurcation — the fact that $\lambda = -1$ is enough.

7. Chaos

7.1 Introduction

Two key idea for chaos:

- arbitrarily close orbits separate from each other
- Some mixing of even the smallest sets

This means any tiny error in the initial conditions will magnify, making long term prediction impossible.

Many def^n of chaos — consider 2.

Consider a map F and an invariant set Λ $(F: \Lambda \to \Lambda)$

(i) F has sensitive dependence on initial conditions (SDIC) on Λ if $\exists\, \delta > 0$ s.t. for any $x \in \Lambda$, $\varepsilon > 0$, $\exists\, y \in \Lambda$ and $n > 0$ s.t.

$$|y - x| < \varepsilon \quad \text{and} \quad |F^n(y) - F^n(x)| > \delta.$$

"However close I start, I can iterate out a finite separation".

(ii) F is <u>topologically transitive</u> (TT) on Λ if for any non-empty open set $U, V \subseteq \Lambda$, $\exists n > 0$ s.t. $F^n(U) \cap V \neq \phi$

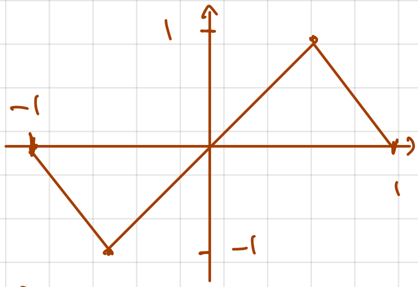
"There are orbit dense in Λ, i.e. they come arbitrarily close to every point in Λ"

<u>Example</u> (TT but not SDIC) Rotation map on $[0,1)$

$x_{n+1} = x_n + \omega \mod 1$, $\omega \notin \mathbb{Q}$. forward orbit is dense but not SDIC.

<u>Example</u> (SDIC but not TT)

$$x_{n+1} = \begin{cases} 2(-1-x_n) & x_n < -\frac{1}{2} \\ 2x_n & -\frac{1}{2} \leq x_n \leq \frac{1}{2} \\ 2(1-x_n) & x > \frac{1}{2} \end{cases}$$



Not TT as can split into $x \geq 0$ and $x \leq 0$, but SDIC as $|f'| = 2$ everywhere.

<u>Def^n</u> (Devaney, D-chaos) $F: \Lambda \to \Lambda$ is chaotic if

(i) F has SDIC on Λ

(ii) F has TT on Λ

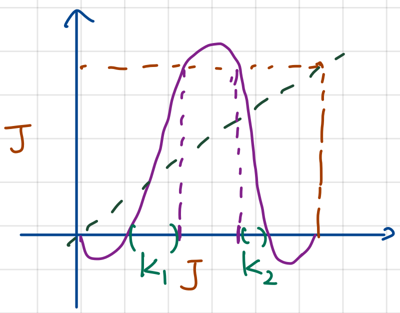
(iii) Periodic points are dense in Λ.

<u>Note</u> not proved but (ii). (iii) $\Rightarrow$ (i), but we check (i) nonetheless.

<u>Def^n</u> (Glending, G-chaos) Use the concept of <u>horseshoe</u>. Say $F: I \to I$ has a <u>horseshoe</u> if $\exists$ an open interval $J \subseteq I$ and <u>disjoint</u> open subset K_0 and $K_1 \subset J$ s.t.

$$F(K_0) = F(K_1) = J$$

F is <u>chaotic</u> if F^n has a horseshoe for some $n \geq 1$.

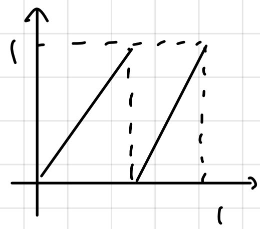


Note: G-chaos $\Rightarrow$ D-chaos.

7.2 Sawtooth map (§6.1)

Now show that it is G-chaotic and D-chaotic.

Recall map is $x_{n+1} = 2x_n \mod 1$, $x \in [0,1]$



Recall can view as a binary shift

$$0.a_1a_2a_3 \ldots \to 0.a_2a_3 \ldots$$

- G-chaos: $J = (0,1)$, $K_0 = (0,\frac{1}{2})$. $K_1 = (\frac{1}{2},1)$. Horseshoe ✓.
- D-chaos: 3 things to check

(i) SDIC: Choose any $\delta < \frac{1}{2}$. Given x, ε, pick n large enough s.t. $2^{-(n+1)} < \varepsilon$. Construct y to be the same as x but with the $(n+1)^{th}$ binary place changed. Then $|x-y| = 2^{-(n+1)} < \varepsilon$, and $|F^n(x) - F^n(y)| = \frac{1}{2} > \delta$

(ii) TT: Given $u = 0.a_1a_2 \ldots \in \mathcal{U}$, $v = 0.b_1b_2 \ldots \in V$.

Set $u_N = 0.a_1a_2 \ldots a_N b_1 b_2 \ldots$, then $F^N(u_N) = v \in V$

Choose N large enough that $u_N \in \mathcal{U}$ ($\mathcal{U}, V$ open sets).

(iii) Dense POs: Given $\omega = 0.a_1a_2 \ldots$.

Set $W_N = 0.\underbrace{a_1a_2 \ldots a_N}\underbrace{a_1a_2 \ldots a_N}a_1 \ldots$ This is periodic

Now choose N large to get arbitrarily close to w.

$\Rightarrow$ D-chaos.

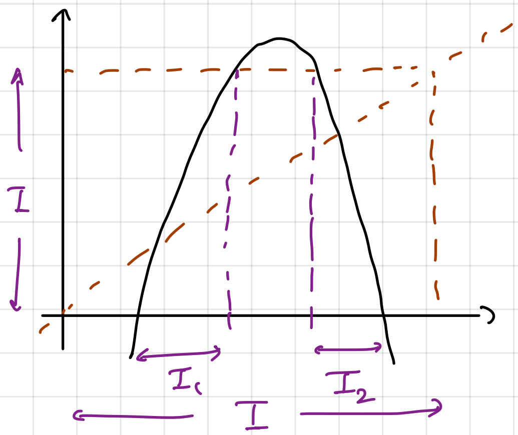
7.3 Horseshoes and Symbolic Dynamics

Aim to show that a horseshoe gives equivalent dynamics to the shift / sawtooth map.

Suppose f has a horseshoe on $J \subset \mathbb{R}$. (and accompanying open subsets K_1, K_2).

Define closed interval $I = \bar{J}$, $I_1 = \bar{K}_1$, $I_2 = \bar{K}_2$. Assume f is monotonic over I_i. $I_1 \cap I_2 = \phi$.

$f(x) \in I \Rightarrow x \in I_1$ or I_2 (simplest set-up, others possible)



Define $\Lambda = \{x : f^n(x) \in I \ \forall n \geq 0\}$. Clearly

$$x \in \Lambda \Rightarrow f(x) \in \Lambda \Rightarrow f(\Lambda) \subseteq \Lambda$$

Also if $x \in \Lambda$, $\exists\, y$ s.t. $f(y) = x$ with $y \in I$ since

$$F^N(y) \in I, \ y \in \Lambda \Rightarrow \Lambda \subseteq f(\Lambda)$$

So $\Lambda = f(\Lambda)$, and Λ invariant.

Now the "construction": for each $x \in \Lambda$, $f^n(x) \in I$

$$\Rightarrow f^{n-1}(x) \in I_1 \text{ or } I_2.$$

Define $a_n = 0$ if $f^{n-1}(x) \in I_1$, $a_n = 1$ if $f^{(n-1)}(x) \in I_2$

Then x corresponds to a sequence

$$x = 0.a_1 a_2 \dots a_N$$

$f^0(x)$ $f(x)$ $f^{N-1}(x)$

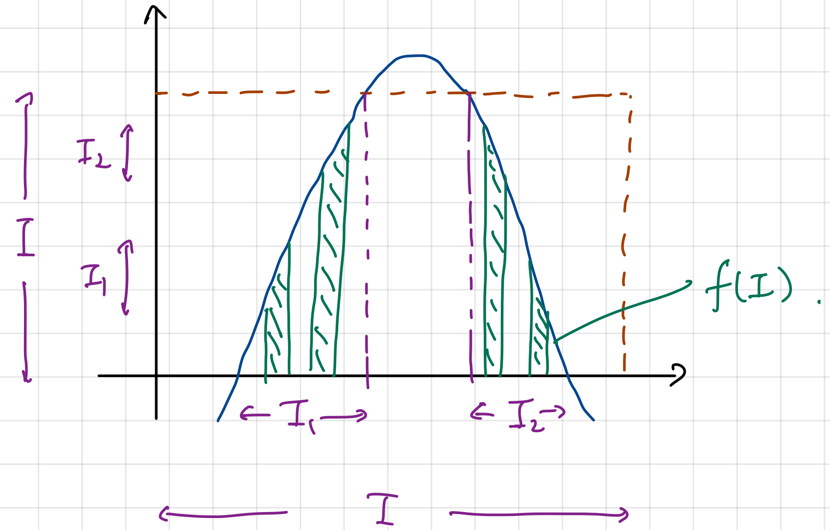
$f(x) = 0.a_2 \ldots a_N$ (left shift)

Hence, map on Λ ≡ action of a "left" shift on a binary expansion in $[0,1]$

$\Rightarrow f$ is TT, SDIC, POs dense in Λ. Thus

$$\boxed{\text{horseshoe / G-chaos} \longrightarrow \text{shift map} \longrightarrow \text{D-chaos}}$$

What does Λ look like? Cantor set



$\downarrow n$

Cantor set

Fact: $\Lambda = \bigcap_{k=1}^{\infty} f^k(I)$. It is closed, but uncountably many points, might have measure 0.

7.4 Period 3 implies chaos

Thm If a cts map on $I \subseteq \mathbb{R}$ has a 3-cycle, then f^2 has horseshoe and therefore f is chaotic.

Pf: Let $x_1 < x_2 < x_3$ be elts of a 3-cycle.

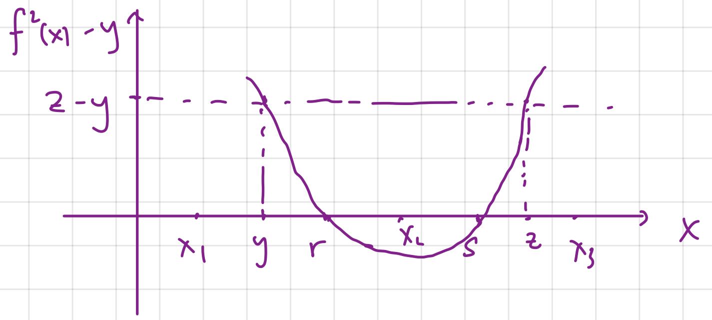
WLOG suppose $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_1$.

Now $f(x_2) = x_3 > x_2$, $f(x_3) = x_1 < x_3$ so $f(x) - x$ changes sign over $[x_2, x_3]$. IVT $\Rightarrow \exists z \in (x_2, x_3)$ s.t. $f(z) = z$.

Also, $f(x_1) = x_2 < z$ $f(x_2) = x_3 > z$. So $f(x) - z$ changes sign over $[x_1, x_2]$. IVT $\Rightarrow \exists y \in (x_1, x_2)$ s.t. $f(y) = z$.

$f^2(y) = f(z) = z > y$, $f^2(x_2) = f(x_3) = x_1 < y$, so $f^2(x) - y$ changes sign in $[y, x_2]$. IVT $\Rightarrow \exists$ smallest $r \in (y, x_2)$ s.t. $f^2(r) = y$.

$f^2(z) = z > y$, $f^2(x_2) = x_1 < y$, so f^2 changes sign over $[x_2, z]$
IVT $\Rightarrow$ $\exists$ largest $s \in [x_2, z]$ s.t. $f^2(s) = y$



Notice $f^2(y) = z$, $f^2(s) = y$.
$f^2(r) = y$, $f^2(z) = z$

So have a horseshoe for f^2 with $K_1 = (y, r)$, $K_2 = (s, z)$, $J = (y, z)$
($K_1 \cap K_2 = \phi$, $K_1, K_2 \subseteq J$). So have chaos. $\square$

7.5 Existence of N-cycles

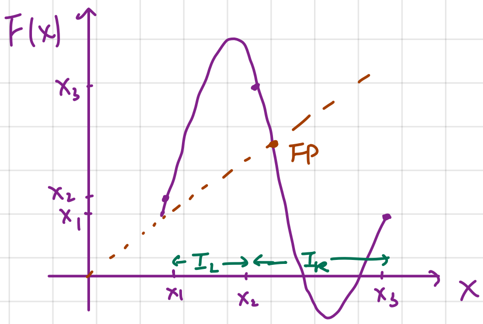
Have shown that F^2 has a horseshoe if there is a 3-cycle.
which implies cycle for F^2 of all periods.
In fact we can show that F has cycles of all periods.

<u>Lem</u> Recall if F cts, $V \subseteq F(U)$, where U, V are closed intervals, then $\exists$ a closed interval $K \subseteq U$ s.t. $F(K) = V$.

Thm If a cts map F on $I \subseteq \mathbb{R}$ has a 3-cycle, then there is an N-cycle for all $N \geq 1$.

Pf: Assume $x_1 = F(x_3) < x_2 = F(x_1) < x_3 = F(x_2)$.

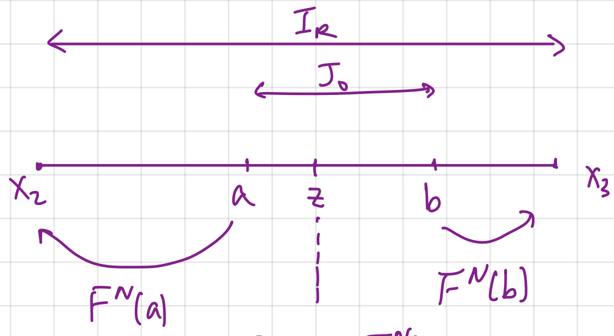
(a) $N=1$ $F(x)-x$ changes sign over $[x_2, x_3]$, so IVP $\Rightarrow \exists\, x^*$ s.t. $F(x^*) - x^* = 0$, $x^* \in (x_2, x_3)$, so $\exists$ a FP.



(b) $N>1$ let $I_L = [x_1, x_2]$, $I_R = [x_2, x_3]$, have $I_R \subseteq F(I_L)$, $I_L \cup I_R \subseteq F(I_R)$.

Choose $J_N = I_R$. Define J_{N-1} s.t. $F(J_{N-1}) = J_N$ and $J_{N-1} \subseteq I_L$ (exist by lemma). Now work backwards $F(J_i) = J_{i+1}$ back to $J_0 \subseteq I_R$ with $J_i \subseteq I_R$, $i \leq N-2$. $(\underbrace{0, \dots, N-2}_{R}, \underbrace{N-1}_{L}, \underbrace{N}_{R})$

So $F^N(J_0) = J_N = I_R \Rightarrow \exists\, a, b \in J_0$ s.t. $F^N(a) = x_2$, $F^N(b) = x_3$.



z s.t. $F^N(z) = z$, by IVT $F^N(x) - x$ changes sign over $[a, b]$.

FP of F?

The only way all iterates could be the same is if $x \in I_L \cap I_R = \{x_2\}$. but $F(x_2) = x_3 \neq x_2$ ※

Same argument holds to eliminate smaller period cycles, so must exist N-cycle. $\square$

The statements $I_R \subseteq F(I_L)$ and $I_L \cup I_R \subseteq F(I_R)$ can be shown / displayed as a directed graph.

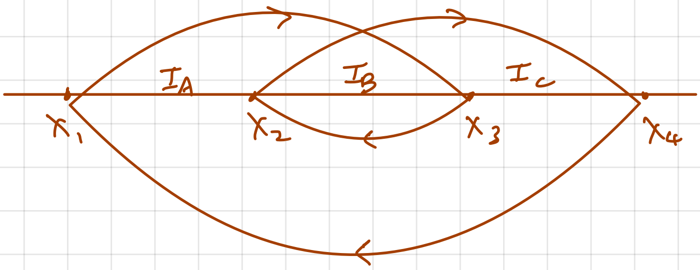
$$\begin{tikzcd} I_L \arrow[r, shift left, "F(I_L) \supseteq I_R"] & I_R \arrow[l, shift left, "I(I_R) \supseteq I_L"] \arrow[loop right, "F(I_R) \supseteq I_R"] \end{tikzcd},$$

and cycles are implied when there are closed path in the diagram.

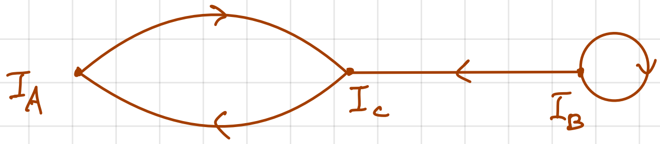
e.g. expect FP only in I_R,

no periodic orbit / N-cycle just involve I_L.

<u>Example</u> Suppose there is a 4-cycle.



⇒ Directed graph



Expect only FPs to be in I_B, and 2 cycles between I_A and I_C.

<u>Thm</u> (Sharkovski) If $f: \mathbb{R} \to \mathbb{R}$ cts, f has a k-cycle and '$l \lhd k$' in the following ordering, then f also has an l-cycle.

$$1 \lhd 2 \lhd 2^2 \lhd 2^3 \lhd \ldots$$

$$\uparrow n \uparrow \infty$$

$$\ldots \lhd 2^n \cdot 9 \lhd 2^n \cdot 7 \lhd 2^n \cdot 5 \lhd 2^n \cdot 3 \lhd \ldots$$

$$\ldots$$

$$\ldots \lhd 2^2 \cdot 9 \lhd 2^2 \cdot 7 \lhd 2^2 \cdot 5 \lhd 2^2 \cdot 3 \lhd \ldots$$

$$\ldots \lhd 2 \cdot 9 \lhd 2 \cdot 7 \lhd 2 \cdot 5 \lhd 2 \cdot 3 \lhd \ldots \lhd 9 \lhd 7 \lhd 5 \lhd 3.$$

Comments:

- If $k=3$, then all other N-cycles exist (recover p3 ⇒ any period cycle)
- $k=1$ (FP) $\not\Rightarrow$ N-cyclic, but $1 \triangleleft N$, any N. so any cycle ⇒ FP.
- For any k not a power of 2, ∃ small l s.t. $3l \triangleleft k$.

  ⇒ have a 3-cycle in F^l

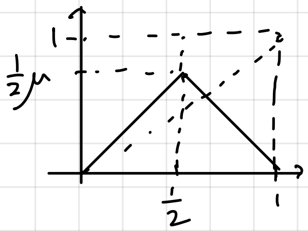
  ⇒ horseshoe in F^{2l} ⇒ chaos

☆ Any non-power-of-2 cycle ⇒ chaos.

7.6 The Trent Map (recall § 6.1)

More interesting than the sawtooth map as it depends on a parameter.

$$F(x) = \begin{cases} \mu x & 0 \le x \le 1/2 \\ \mu(1-x) & 1/2 \le x \le 1 \end{cases}$$



For $0<\mu<1$ $F:[0,1] \to [0,1]$.

$F(x) \le \mu x$ with $\mu<1$, so $x_n \to 0$ as $n \to \infty$.

For $\mu=1$

$$F(x) = \begin{cases} x & 0 \le x \le 1/2 \\ (1-x) & 1/2 \le x \le 1 \end{cases}.$$

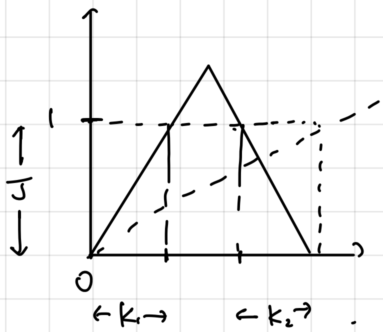
Very degenerate, line of FPs are $x \in [0, \frac{1}{2}]$.

For $x \in (\frac{1}{2}, 1]$, hit FP at $1-x$ after 1 iterate.

For $\mu \ge 2$ F has a horseshoe.

Let $J=(0,1)$, $K_1 = (0, \frac{1}{\mu})$, $K_2 = (1-\frac{1}{\mu}, 1)$

$F(K_1) = F(K_2) = J$, $K_1 \cap K_2 = \phi$, $K_1, K_2 \subseteq J$.

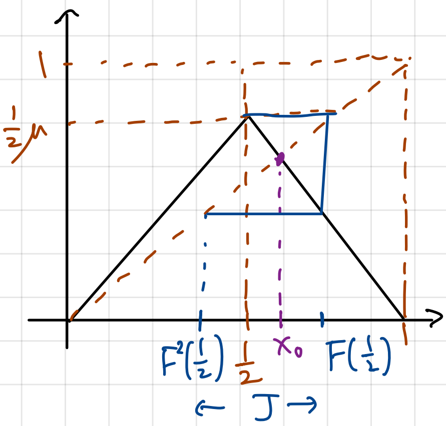


For $1<\mu<2$ Fix points 0 and $x_0 = 1 - \frac{1}{1+\mu} = \frac{\mu}{1+\mu}$.

$|F'|$ at both FP $= \mu > 1$, so unstable.

Consider interval $J = [F^2(\tfrac{1}{2}), F(\tfrac{1}{2})]$

$= [\mu(1-\tfrac{1}{2}\mu), \tfrac{1}{2}\mu]$

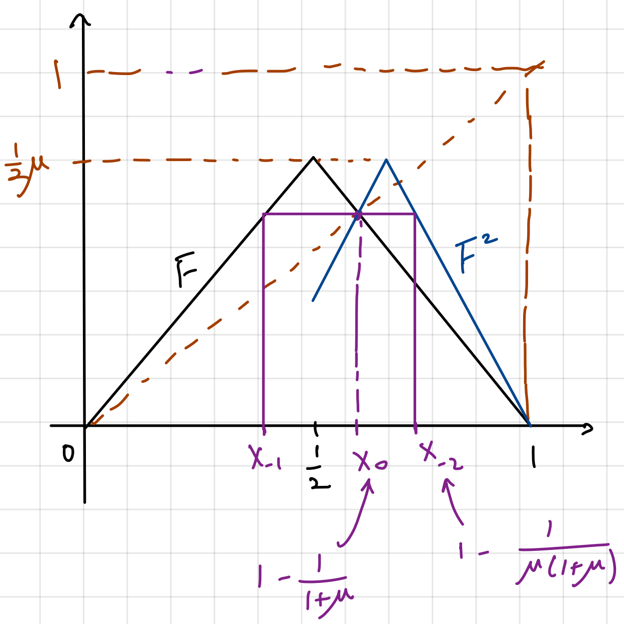


- points above J map to below J
- points below J grow geometrically $(\times\mu)$ until they enter J
- Once in J can't leave.

$\Rightarrow$ So all interesting dynamics in J.

$$F(x) = \begin{cases} \mu x & 0 \le x \le 1/2 \\ \mu(1-x) & 1/2 \le x \le 1 \end{cases}$$

$$F^2(x) = \begin{cases} \mu^2 x & 0 \le x \le 1/2\mu \\ \mu(1-\mu x) & 1/2\mu \le x \le 1/2 \\ \mu(1-\mu(1-x)) & 1/2 \le x \le 1 - 1/2\mu \\ \mu^2(1-x) & 1 - 1/2\mu \le x \le 1 \end{cases}$$



$x_0 = F^2(x_0)$ $\begin{cases} F(x_{-1}) = x_0 \\ F^2(x_{-2}) = x_0. \end{cases}$

$F^2(x_*) = \tfrac{1}{2}\mu$

$F(x_*) = \tfrac{1}{2}$

$F(\tfrac{1}{2}) = \tfrac{1}{2}\mu.$

Know that $\mu \geq 2$ gives a horseshoe in F. We've just demonstrated that $\mu \geq \sqrt{2}$ gives a horseshoe in F^2.

Can show

$\Rightarrow$ F^4 has a horseshoe for $\mu \geq 2^{1/4}$

$\vdots$

F^{2^n} has " $\mu \geq 2^{1/2^n}$.

So chaos for all $\mu > 1$.

In summary, $\cdot\ \mu < 1$. $x \to 0 \quad \forall\ x \in [0,1]$

$\cdot\ \mu = 0$ line of FPs $[0, \frac{1}{2}]$

$\cdot\ \mu > 1$ chaos.

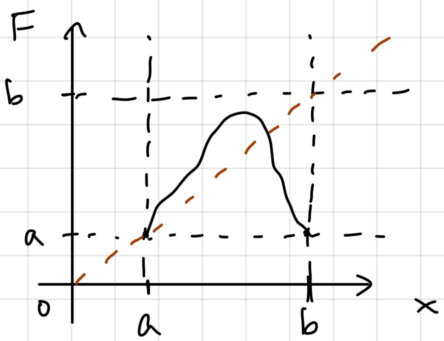
The route of chaos is unusual. Normally get period-doubling cascade (see next section and logistic map).

7.7 Unimodal maps and Feigenbaum's constant

Def^n A unimodal map on the interval $[a,b]$ is a continuous map $F: [a,b] \to [a,b]$ s.t.

(i) $F(a) = F(b) = a$

(ii) $\exists\ c \in (a,b)$ s.t. F is strictly increasing on $[a,c)$ and strictly decreasing on $(c,b]$.



Most famous example is the logistic map

$$x_{n+1} = r x_n (1 - x_n) \quad , \quad x \in [0,1]$$

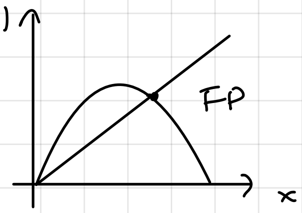
- Parabolic with max $\frac{r}{4}$ at $c = \frac{1}{2}$.
- Usual to consider $0 < r < 4$ ($r \geq 4$ not interesting)
- Behaviour $F = rx(1-x)$

$\underline{0 < r < 1}$: $x_n \to 0$, only FP.

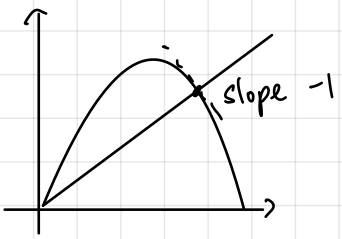
$\underline{2^0 = r_0 = 1}$: FP at 0 is non-hyperbolic ($F'(0) = 1$)

(actually stable due to non-linear terms)

$\underline{1 < r < 3}$: New FP $\neq 0$.



$\underline{r_1 = 3}$: p2 bifurcation $\quad 2^1$



period doubling cascade

$r_2 = 3.449$ p4 ($= 2^2$) bifurcation

$r_3 = 3.54409$ p8 ($= 2^3$) bifurcation

$\vdots$

r_n $\quad p2^n$

$\vdots$

$r_\infty = 3.569946$ $\quad p2^\infty$

Note successive bifurcation come faster and faster

Feigenbaum (1978) noticed that

$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.6692\ldots \qquad \text{Feigenbaum's const.}$$

This number turns out to be <u>universal</u> for all one humped (unimodal) map with <u>quadratic maximum</u>.

N.B. The tent map is not one of these.