# Cosmology

## 1. Expanding Universe

### 1.1 The FLRW metric

The CMB is isotropic (same in every direction) on average. Large scale structure (LSS) homogeneous (same) at all point on average.

Cosmological principle: our universe is homogeneous and isotropic on large scales ($\gg$ Mpc).

In GR, invariant interval

$$ds^2 = g_{\mu\nu}(\underline{x})\, dx^\mu dx^\nu.$$

By isotropy, $g_{i0} = 0$. By homogeneity, $g_{00}(t) \stackrel{!}{=} 1$ by redefining cosmic time

$$ds^2 = -dt^2 + dl^2.$$

With $dl^2$ the line elt maximally symmetric space, this is FLRW metric.

#### Maximally Symmetric Space

In 3d, only 3 possibilities — curvature $K = +1, 0, -1$.

Flat ($K=0$): $dl^2 = dx^i \delta_{ij} dx^j$. — Euclidean space.

Positive curvature ($K=+1$): Embed 3-sphere into $\mathbb{R}^4 = (\underline{x}, u)$

$$dl^2 = d\underline{x}^2 + du^2$$

s.t. $\underline{x}^2 + u^2 = a^2$.

Negative curvature ($K=-1$): Embed hyperboloid into $\mathbb{R}^{3,1}$.

$$d\ell^2 = d\underline{x}^2 - du^2$$

s.t. $\underline{x}^2 - u^2 = -a^2$.

To write them all in a unified form, rescale $\underline{x} \mapsto a\underline{x}$, $u \mapsto au$, then

$$d\ell^2 = a^2\left(d\underline{x}^2 + K\, du^2\right)$$

with $\underline{x}^2 \pm u^2 = \pm 1$ for $K = 0, \pm 1$.

Differentiate the constraint $\underline{x}^2 \pm u^2 = \pm 1$

$$\Rightarrow \quad 2\underline{x}\cdot d\underline{x} \pm 2u\, du = 0$$

$$\Rightarrow \quad u\, du = \mp\, \underline{x}\cdot d\underline{x}$$

$$\Rightarrow \quad du^2 = \frac{(\underline{x}\cdot d\underline{x})^2}{u^2} = \frac{(\underline{x}\cdot d\underline{x})^2}{\pm(1-\underline{x}^2)}.$$

Hence

$$\boxed{d\ell^2 = a^2\left[d\underline{x}^2 + K\frac{(\underline{x}\cdot d\underline{x})^2}{1-K\underline{x}^2}\right]}$$

Using sph. coord. $(r, \theta, \phi)$

$$d\underline{x}^2 = dr^2 + r^2 d\Omega_2^2,$$

with $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. Then FLRW metric

some 3d metric

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\, dx^i \gamma_{ij}\, dx^j \\ &= -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega_2^2\right], \end{aligned}$$

and $a(t)$ is the scale factor.

- FLRW in cartesian coord has $\gamma_{ij} = \delta_{ij}$.

• $\underline{x}$ are called comoving coordinate – related to physical distance by

$$\Delta X_{ph} = (\Delta X^\mu g_{\mu\nu} \Delta X^\nu)^{1/2}$$

comoving

$$\overset{\text{some } t}{=} (a^2 \Delta X^i \delta_{ij} \Delta X^j)^{1/2} = a |\Delta X| .$$

It's convenient to use conformal time $\tau$ s.t.

$$\boxed{a \, d\tau = dt .}$$

Then

$$ds^2 = a^2 \left[ -d\tau^2 + \gamma_{ij} \, dx^i dx^j \right]$$

Flat FLRW $(K=0)$ is conformally flat.

Covariant derivatives :

$$\nabla_\mu A = A_{;\mu} = \frac{\partial A}{\partial x^\mu} = \partial_\mu A = A_{,\mu} .$$

$$\nabla_\nu A^\mu = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma_\alpha{}^\mu{}_\nu A^\alpha$$

$$\nabla_\nu A_\mu = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma_\mu{}^\alpha{}_\nu A_\alpha .$$

where $\Gamma_\alpha{}^\mu{}_\beta = \frac{1}{2} g^{\mu\lambda} ( g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda} )$

For FLRW,

$$\Gamma_i{}^0{}_j = H a^2 \delta_{ij} \qquad \Gamma_i{}^j{}_0 = \Gamma_j{}^i{}_0 = H \delta_{ij}$$

$$\Gamma_j{}^i{}_k = \frac{1}{2} \gamma^{il} ( \partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk} )$$

where $H(t) = \dot{a}(t) / a(t)$ is the Hubble parameter.

## 1.2 Dynamical Relations

From the Einstein - Hilbert action

$$S = \int d^4x \sqrt{-g} \frac{M_p^2}{2} R + S_m$$ "matter" action

We derive

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = - \frac{1}{M_p^2} T_{\mu\nu} .$$

with $M_p$ the Planck mass $M_p = \frac{1}{\sqrt{2\pi G_N}}$ $(\hbar = c = 1)$,

Newton const.

$R_{\mu\nu}$ is the Ricci Tensor

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$$

Riemann Tensor $\sim$ "$\partial\partial g$".

$T_{\mu\nu}$ is the energy-momentum tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}$$

By contracted Bianchi identities,

Einstein eqn.

$$\nabla^{\mu} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \nabla^{\mu} G_{\mu\nu} = 0$$

Hence

$$\nabla_{\mu} T^{\mu\nu} = \partial_{\mu} T^{\mu\nu} + \Gamma_{\alpha}{}^{\mu}{}_{\mu} T^{\alpha\nu} + \Gamma_{\alpha}{}^{\nu}{}_{\mu} T^{\mu\alpha} = 0$$

This is the continuity eqn.

By homog. and isotropy, the most generic $T^{\mu}{}_{\nu}$ is

$$T^{\mu}{}_{\nu} = \mathrm{diag}\left(-\rho, P, P, P\right)$$ $T_{\mu\nu} = g_{\mu\alpha} T^{\alpha}{}_{\nu}$

where $\rho(t, x) = \rho(t)$ is the energy density, and

$P(t,x) = P(t)$ is the pressure.

Conservation $\nabla_\mu T^{\mu 0} = 0$

$$\Rightarrow \boxed{\dot{\rho} + 3H(\rho + P) = 0}$$

Hubble param $\frac{\dot{a}}{a}$

The type of substance is specified by an eqn of state.
A simple choice is

$$P(t) = w\,\rho(t).$$

Example : $w = 0$ : NR matter $(v \ll c)$, a.k.a. dust

$w = \frac{1}{3}$ : Relativistic matter, a.k.a. radiation.

$w \leq -\frac{1}{3}$ : Dark energy.

$w = -1$ : Cosmological const.

Now solve the cty eqn for $\rho = \rho(a)$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + w\rho) = 0$$

$$\Rightarrow \frac{d}{dt}\log\rho = -3(1+w)\frac{d\log a}{dt}$$

$$\Rightarrow \boxed{\rho = \rho_0\left(\frac{a(t_0)}{a(t)}\right)^{3(1+w)}}$$

with $\rho(t_0) = \rho_0$ is integration const.

Def$^n$ (Perfect Fluid) A medium for which $\forall\, \underline{x}, t$, $\exists$ frame in which $T^\mu{}_\nu = \mathrm{diag}(-\rho, P, P, P)$. Then in any frame,

$$T^\mu{}_\nu = (\rho + P)\,U^\mu U_\nu + P\,g^\mu{}_\nu$$

$g^\mu{}_\nu = \delta^\mu{}_\nu$

with velocity $U^\mu$ a normalised time-like vector

$$U^\mu U_\mu = -1$$

Frame in which $U^\mu = (-1, 0, 0, 0)$

We'll often assume perfect fluids.

## Friedmann equation

Direct calculation of the 00 cpt of Einstein's eqn

$$R_{00} - \frac{1}{2} g_{00} R = M_p^{-2} T_{00}$$

$$\Rightarrow \boxed{3M_p^2 \left( H^2 + \frac{K}{a^2} \right) = \sum_i \rho_{ii}} \qquad \text{(Friedmann eqn)}$$

all substances

Find sol$^n$ for a single field and $K=0$.

$$H = \frac{\dot{a}}{a} = \sqrt{\frac{\rho}{3M_p^2}} = H_0 \left(\frac{a_0}{a}\right)^{3(1+w)/2}$$

with $H_0 = H(t_0)$.

Using ansatz $a = t^q$, we find sol$^n$

$$a(t) = \left(\frac{3}{2}(1+w) \; H_0 t\right)^{\frac{2}{3(1+w)}}$$

- Dusts $(w=0)$ $\Rightarrow$ $a(t) \propto t^{2/3}$
- Radiation $(w = \frac{1}{3})$ $\Rightarrow$ $a(t) \propto t^{1/2}$
- Vacuum energy $(w=-1)$ $\Rightarrow$ $a(t) \propto e^{H_0 t}$.

Have $a(t) \propto t^{2/3(1+w)}$

$$\Rightarrow H(t) = \frac{2}{3(1+w)} \cdot \frac{1}{t}.$$

The age of universe is

$$t_{age} = \frac{2}{3(1+w)} \cdot \frac{1}{H(t_{age})}$$

The acceleration eqn is $\partial_t(\text{Fried})$, and $\dot{\rho}$ from cty eqn,

$$\Rightarrow M_{pl}^2 \frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3P)$$

## Charge conservation

Charge conservation is

$$(nU^\mu)_{;\mu} = 0.$$

Homog. and isotropy gives $U^\mu = (1,0,0,0)$

$$\Rightarrow \dot{n} + 3\frac{\dot{a}}{a} n = 0$$

$$\Rightarrow n(t) \propto a^{-3}.$$

# 2. Distance and Redshifts

## 2.1 Geodesics and redshift

For a particle along a worldline $X^\mu(s)$, the velocity is

$$U^\mu = \frac{dX^\mu}{ds}$$

with

$$U^\mu U_\mu = \begin{cases} 1 & \text{massive} \\ 0 & \text{massless} \end{cases}$$

For massive particles, $P^\mu = mU^\mu$.

Geodesic eqn is

$$\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0 .$$

$$\Rightarrow \boxed{P^\alpha \partial_\alpha P^\mu + \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta = 0}$$

Focus on $\mu = 0$ eqt $(E = P^0)$

$$E \frac{dE}{dt} = -\frac{\dot{a}}{a} p^2$$

since $E^2 - p^2 = m^2$, $E\,dE = p\,dp$

$$\Rightarrow \frac{dp}{dt} p = -\frac{\dot{a}}{a} p^2$$

$$\Rightarrow p \propto \frac{1}{a} .$$

## Redshift

For $m = 0$,

$$E = p \propto \frac{1}{a}$$

Since $E = h/\lambda \Rightarrow \lambda \propto a$

$$\Rightarrow \lambda_0 = \frac{a(t_0)}{a(t_1)} \lambda_1$$

Define the redshift

$$z := \frac{\lambda_0 - \lambda_1}{\lambda_1}$$

$$\Rightarrow 1 + z = \frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)}$$

We'll adopt the convention that $a(t_0) = 1$.

$$\Rightarrow 1 + z = \frac{1}{a(t_1)} .$$

## 2.2 Distances

### Comoving Distance

Curved spacetime has many "distances". — comoving dist., luminosity dist., angular diameter dist. ...

In sph. coords,

$$ds^2 = a^2 \left[ -d\tau^2 + d\chi^2 + \chi^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right]$$

Photons follow null geodesics at $d\tau = d\chi$.

Comoving distance is

$$\chi(t_i, t_f) = \int d\chi = \int d\tau$$

$$= \int_{t_i}^{t_f} \frac{dt}{a(t)}$$

$$= \int \frac{da}{a^2 H(a)} = \int \frac{dz}{H(z)}$$

If $t_f = t_0$, $a(t_0) = a_0 = 1$,

$$\chi(z) = \int_0^z \frac{dz}{H(z)} .$$

Particle horizon is the distance travelled by light since $t_i$ (as opposed to since $z$) until $t$

$$d_{p.h.}(z) = a(t)\, \chi(t_i, t) = a(t) \int_{t_i}^{t} \frac{dt'}{a(t')}$$

Let's compute this for single component universe $w > -1$.

$$a(t) = (t/t_0)^{\frac{2}{3(1+w)}}$$

$$\Rightarrow d_{p.h.}(t) = t^\alpha \int_{t_i}^{t} \frac{dt'}{t'^\alpha} = \frac{t^\alpha}{1-\alpha} \left[ t^{1-\alpha} - t_i^{1-\alpha} \right]$$

with $\alpha = \frac{2}{3(1+w)}$

For $w > -1$, the Big Bang is when $a(t_{BB}) = 0$, namely $t_{BB} = 0$.

- $d_{p.h.}$ diverge for $\alpha > 1 \Rightarrow w < -\frac{1}{3}$
- $d_{p.h.}$ converges for $\alpha < 1$ $(w > -\frac{1}{3})$ to $d_{p.h.} = \mathcal{O}(1)\, t$.

For example, radiation $w = \frac{1}{3} \Rightarrow d_{p.h.} = 3t$.

NR matter $w = 0 \Rightarrow d_{p.h.} = 2t$.

## Luminosity Distance

Let $L$ be the intrinsic luminosity $[L] = \frac{\text{Energy}}{\text{Time}}$.

What you measure is flux $f$, a.k.a. apparent brightness,

$[f] = \frac{\text{Energy}}{\text{Time} \cdot \text{Area}}$.

In a Euclidean universe,

$$f = \frac{L}{4\pi d_L^2}$$

where $d_L$ is the luminosity distance.

In a flat expanding universe with emission at $a(z) = \frac{1}{1+z}$

and observation now $a_0 = 1$,

- Energy redshifts by a factor

$$\frac{a(z)}{a_0} = \frac{1}{1+z}$$

- Rate of arrival of photon decreases by

$$\frac{a(z)}{a_0} = (1+z)^{-1}.$$

So in an expanding universe,

$$f = \frac{\text{Energy}}{\text{Time} \cdot \text{Area}} = \frac{L(a/a_0)^2}{4\pi \chi^2} = \frac{L}{4\pi(\chi/a)^2} = \frac{L}{4\pi d_L^2}.$$

So luminosity distance

$$d_L(z) = \frac{\chi}{a} = (1+z)\,\chi(z).$$

Astronomers use apparant magnitude $m$ and absolute $M$

$$m = -2.5 \log_{10}(f/f_0),$$

$$M = -2.5 \log_{10}(L/L_0).$$

where $f_0$ and $L_0$ are reference flux and lumen.

For example, $m_{sun} = -27$, $m_{M31} = 0.1$.

Their difference is called distance modulus

$$\mu = m - M = 5 \log_{10}(d_L / 10pc).$$

In 1920's, Hubble observed nearby galaxies

$$V_{ph} = H_0 \, X_{ph}$$

where $H_0$ called Hubble const., $H_0 > 0$.

In our model,

$$V_{ph} = (\dot{ax}) = \dot{a}\, x_{com} + a\, \dot{x}_{com} = x_{ph} H + x_{peculiar}$$

Modern GR version is $\mu$ v.s. $z$. This was done by Supernovae in 1998 and they discovered $\ddot{a}(t_0) > 0$.

## Age of universe

Use chain rule

$$t_{age} = \int dt = \int \frac{da}{\dot{a}} = \int \frac{da}{aH}$$

$$= \int \frac{da}{a} \cdot \left[ \sum_i \rho_i / 3M_{pl}^2 \right]^{-1/2}.$$

We'll use fractional energy density

$$\Omega_{i0} = \frac{\rho_i(t_0)}{3H_0^2 M_p^2}$$

From sol$^n$ of cty eqn,

$$\rho_i(a) = \rho_{i,0} \frac{1}{a^{3(1+w)}} = 3M_p^2 H_0^2 \frac{\Omega_{i,0}}{a^{3(1+w)}}$$

In our best model of the universe, we have

$$\Omega_{\Lambda,0} = 0.7, \quad \Omega_m = 0.3, \quad \Omega_r = 9\times 10^{-5}$$

So one finds

$$t_{age} = \frac{1}{H_0}\int_0^1 \frac{da}{a}\left[\Omega_{\Lambda,0} + \Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4}\right]^{-1/2} \approx 13.7\times 10^9 \, yr$$

Only particles with $t_{life} > t_{age} \sim H_0^{-1}$ are around. Stable known particles: $e^-$, $p^+$, $\gamma$, $\nu$.

Note: Currently there are $O(7\%)$ discrepancies in the value of $H_0$.

So better use physical fractional densities (dim-less) $h^2\Omega$.

$$\Omega_{\Lambda,0} = \frac{\rho_\Lambda(t_0)}{3M_p^2 H_0^2} \quad \text{v.s.} \quad h^2\Omega_{\Lambda,0} = \omega_{\Lambda,0} = \frac{\rho_\Lambda(t_0)}{3M_p^2(100\,km/s)^2}.$$

## 3. Constituents of Universe

### Photons

Most photons are in CMB. Only 1% from stars. CMB is a gas of photons with a thermal distribution at $T_{CMB}(t_0) = 2.72548 \pm 0.00057\,K$. Hence,

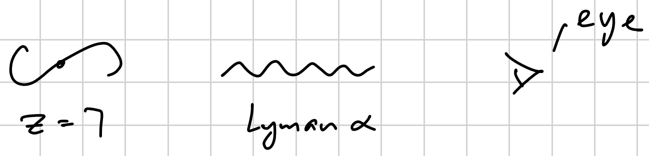
$$\rho_\gamma = \frac{\pi^2}{15} T_{CMB}^4$$

$$\Rightarrow h^2\Omega_{\gamma,0} = 2.5\times 10^{-5}.$$

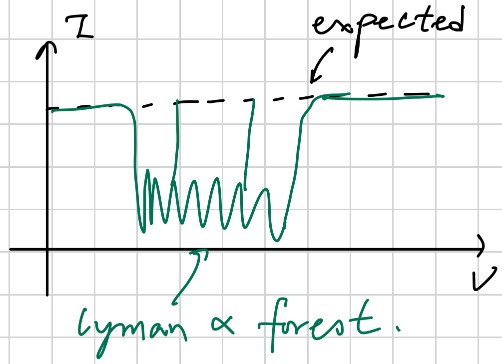
**Baryons** (Atoms) : protons + $e^-$ + neutrons.

- 75% (energy density of baryons) Hydrogen, 25% Helium, traces of heavier elements.

These probes:

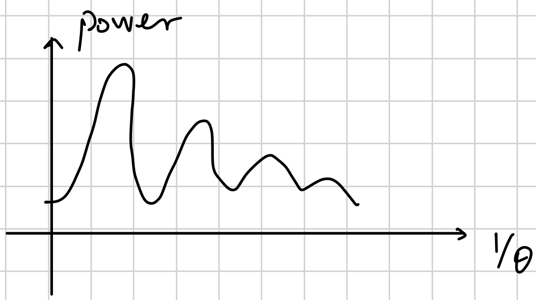
- 


AGN Quasar




$$\Rightarrow h^2 \Omega_b = 0.021 \pm 0.002$$ (Intergalactic medium) IGM.

- Power spectrum




$$\Rightarrow h^2 \Omega_b = 0.02225 \pm 0.00016$$

$$\Rightarrow \Omega_b \sim 0.04 \quad (4\%).$$

- Big Bang Nucleosynthesis (BBN): formation of deuterium $^2H$, Helium $^3He$, $^4He$ and Lithium $^7Li$.

$$\Rightarrow h^2 \Omega_{b,0} = 0.022 \pm 0.006.$$

**Dark matter**

- F. Zwicky observed Coma cluster. Assume virialised.

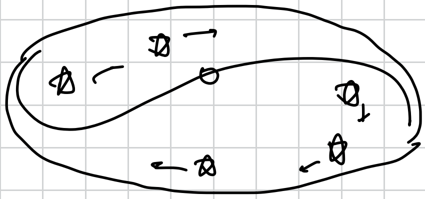
Virial theorem :

$$\langle T \rangle = \frac{1}{2} \sum_i m_i \underline{v}_i^2 = -\frac{1}{2}\langle V \rangle = \frac{GM}{2} \sum_{i \neq j} \frac{m_i m_j}{|\underline{x}_i - \underline{x}_j|} \approx \frac{GM}{2} \cdot \frac{M^2}{R_c}$$

He deduced

$$M_{\text{virial}} \approx 2\times 10^{15} M_\odot \gg M_{IGM} \sim 2\times 10^{14} M_\odot \gg M_{☆} \sim 3\times 10^{13} M_\odot$$

- V. Rubin and K. Ford



expected keplerian orbits

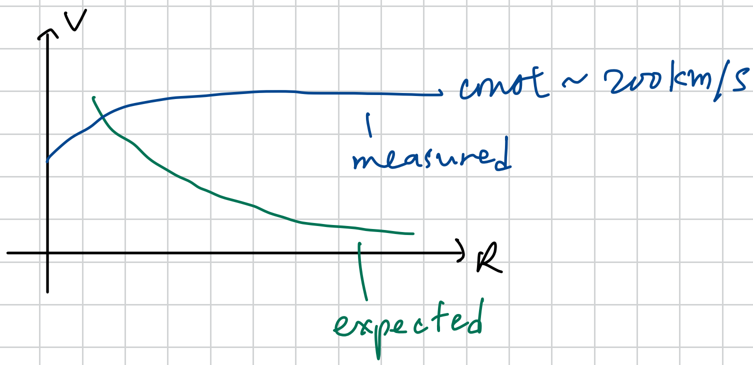
$$\frac{v^2}{R} = G_N \frac{M(R)}{R^2}$$

$$\Rightarrow v(R) = \sqrt{G_N \frac{M(R)}{R}}.$$

If only luminous matter, $R \gg 10\,\text{Kpc}$.

$$\Rightarrow v(R) \propto \sqrt{\frac{1}{R}}.$$

Instead, they measured flat galaxy rot$^n$ curves.




They estimated galaxies inside a puffin and big dark matter. halo.

- CMB power spectrum : $\Omega_{DM,0} h^2 = 0.1198 \pm 0.0015$

$$\Rightarrow \Omega_{DM,0} \simeq 0.26.$$

- Number of galaxy clusters give similar results.

## Neutrinos

Charge e-states $\nu_e, \nu_\mu, \nu_\tau \Rightarrow$ mass estates $\nu_1, \nu_2, \nu_3$.

with $V_{mass} = U \, V_{charge}$.

$$|\Delta m_{21}|^2 \sim 8 \times 10^{-5}\, eV, \qquad |\Delta m_{31}|^2 \sim 2 \times 10^{-3}\, eV.$$

Two possibilities


Cosmology on probe

$$\sum_{i=1}^{3} m_i < 0.15\,eV$$

Neutrinos also have a thermal dist$^n$ at

$$T_\nu \equiv T_{CMB} \left(\frac{4}{11}\right)^{1/3}.$$

$$\Rightarrow \rho_\nu = \rho_r \cdot 3 \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{4/3}$$

$N_{eff} \overset{\text{data}}{=} 3.04 \pm 0.18.$

$$\Rightarrow h^2 \Omega_{\nu,0} = O(10^{-5}).$$

## Dark Energy

In 1998, observation of supernova Ia observed accelerated expansion $\ddot{a}(0) > 0$.

Accel. eqn : $$\frac{\ddot{a}}{a} \propto -(\rho + 3P)$$

$$\Rightarrow \rho + 3P < 0.$$

So need $w < -\frac{1}{3}$. Anything with $w < -\frac{1}{3}$ is called dark energy.

A leading candidate is a cosmological const. ($w=-1$).

$$\Rightarrow \dot{\rho}_{cc} = 0 \quad \Rightarrow \rho_{cc} \cdot \text{const.}$$

In GR.

$$S = \int d^4x \sqrt{-g} \left[ \Lambda_{cc} + \frac{M_p^2}{2} R + \text{matter} \right]$$

Problems associated with a cosmo. const.

- Naturalness problem:

$$\Lambda = \int^{\Lambda_{cutoff}} d^3k \; \omega \approx \Lambda_{cutoff}^4 > (10 \text{ TeV})^4$$

Measure: $\rho_{DD} \sim (10^{-3} eV)^4$, off by $10^{60}$.

- Maybe $\Lambda_{cc} = 0$ b/c symmetry? No.
- Today: $\Omega_{DE} \sim O(10^2) \Omega_{m,0}$. Why now?

## 4. Inflation : Motivations

Decelerated cosmologies, i.e. $\ddot{a}(t) < 0 \;\; \forall t$, has some problems.

The hot Big Bang model is dominated by matter ($w \approx 0$) or radiation ($w = \frac{1}{3}$) until $z \simeq 0.5$. Both satisfy $\rho + 3P \geq 0$, hence by accel. eqn, $\ddot{a} \propto -(\rho + 3P) < 0$

### 4.1 Old background problems

#### Curvature Problem

Current bound on spatial curvature is

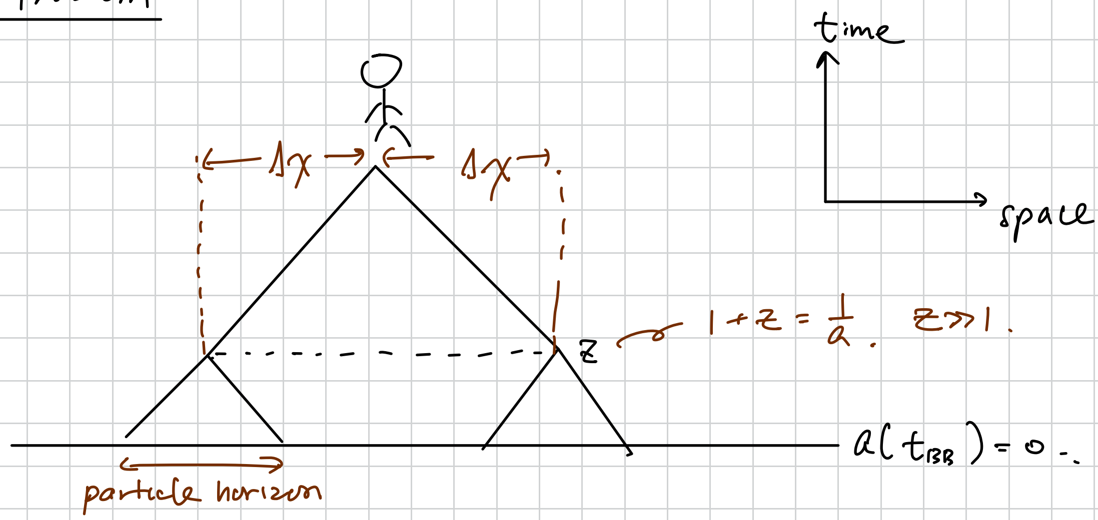
$$\Omega_K \equiv \frac{K}{a^2 H^2} = 0.000 \pm 0.05.$$

However, $\Omega_K$ grows in decelerated expanding cosmologies

$$\dot{\Omega}_K = \frac{-2K\ddot{a}}{\dot{a}^3} \propto -\ddot{a} \propto \underbrace{(\rho + 3P)}_{\text{strong energy condition}} \geq 0$$

Problem: why was $\Omega_K(a_i) \ll 1$ ? Equivalently, $\sum_i \Omega_i(a_i) \simeq 1$.
leading $\mathrm{sol}^n$ is a dynamical expansion: $\ddot{a}, \dot{a} > 0$ at early times, before hot big bang.

## Horizon Problem




Neglect dark energy.

$$\Delta\chi = \chi(a_0, a) = \int_1^a \frac{da}{a^2 H}$$

$$= \frac{1}{a_0 H_0} \frac{2}{3w+1} \left(1 - a_i^{(3w+1)/2}\right)$$

$a_i = a_{\text{initial}}$

$$\overset{a \ll 1}{=} \frac{\mathcal{O}(1)}{a_0 H_0}$$

Particle horizon until $a(t) = \frac{1}{1+z}$

$$\chi_{ph}(a) = \chi(a=0, a) = \frac{1}{aH(a)} \frac{2}{(3w+1)} \sim \frac{\mathcal{O}(1)}{aH(a)}$$

Using Fried. $H \propto \sqrt{\rho} \propto a^{-3(1+w)/2}$, we evaluate the ratio

$$\frac{2\Delta\chi}{\chi_{ph}} \sim \frac{aH(a)}{a_0 H_0} = \left(\frac{1}{a}\right)^{\frac{3w+1}{2}} \gg 1.$$

Thus no causal explanation of isotropy of galaxies $z \sim 10$ or CMB $z = 10^3$.

## Scale Invariance

Correlations of matter, galaxies, or photons. all come from initial conditions satisfying

$$\langle \zeta(\underline{x}) \zeta(\underline{y}) \rangle \simeq \langle \zeta(\lambda \underline{x}) \zeta(\lambda \underline{y}) \rangle \quad \forall \lambda .$$

with deviation $\sim 3\%$, with $\zeta$ the <u>curvature perturbation</u>. Why?

## de Sitter Spacetime

This spacetime is maximally symmetric, i.e. it has $\frac{D(D+1)}{2}$ isometries, with $D = d+1 = 3+1$, so 10 isometries.

Sol$^n$ of Einstein eq. w/ a cosmological const.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0$$

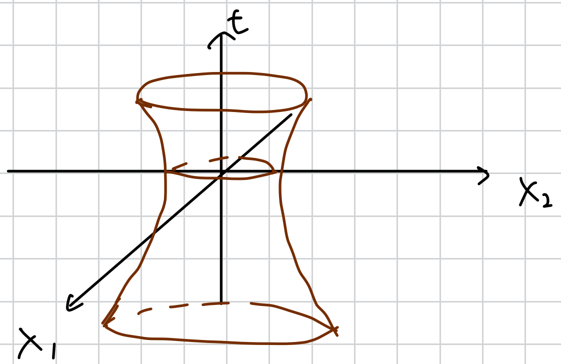
$$\overset{\text{trace}}{\Rightarrow} \quad R = 6\Lambda$$

$$\Rightarrow \quad R_{\mu\nu} = 2\Lambda g_{\mu\nu} .$$

By uniqueness thm, we can find dS coord. by embedding $dS_{3+1}$ into $\mathrm{Mink}_{4+1}$.

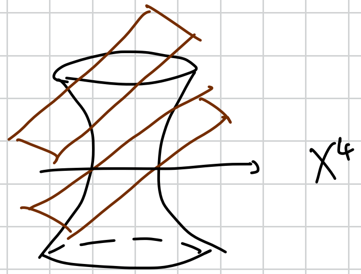
$$-(X^0)^2 + \sum_{a=1}^{d} X^a X^a = L^2$$

with $\Lambda = \frac{1}{L^2}$.




We can slice the hyperboloid into noncompact and spatially flat.

$$X^0 = L \sinh(Lt) - \frac{1}{2} \frac{X^a X_a}{L} e^{-Lt} ,$$



$$X^i = x^i e^{-Lt}$$

$$X^4 = L\cosh(Lt) - \frac{1}{2}\frac{x^i x_i}{L} e^{-Lt},$$

with $i = 1, 2, 3$, $t = X^0$. So that

$$ds^2_{3+1} = -dt^2 + e^{2Ht} dx^i \delta_{ij} dx^j = \frac{-d\tau^2 + dx^i \delta_{ij} dx^j}{H^2\tau^2}$$

where $dt = -d\tau / H\tau$

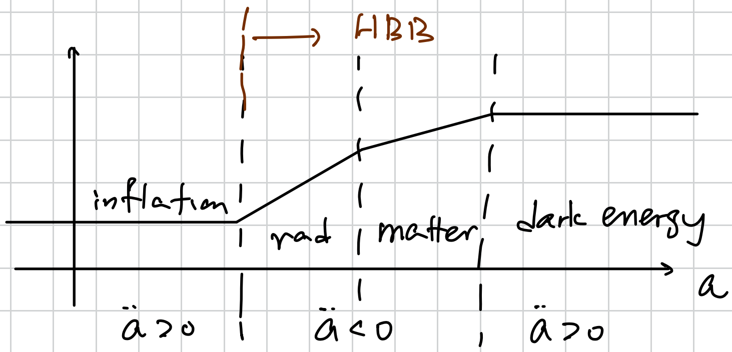
In particular, dilation isometry,

$$(\underline{x}, \tau) \longmapsto (\lambda \underline{x}, \lambda\tau)$$

hence correlation in dS for $\tau$-indpt. fields are scale invariance.

## 5. Slow-Roll Inflation

Inflation is a phase of accelerated expansion before the hot big bang, $\dot{a}, \ddot{a} > 0$




Define the <u>first slow-roll param</u> $\varepsilon$ by

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 \equiv H^2(1-\varepsilon).$$

where $\varepsilon = -\dot{H}/H^2$. and $\varepsilon < 1 \Leftrightarrow \ddot{a} > 0$.

Scale invar. can be explained by de Sitter spacetime where $H = \text{const.}$, $\varepsilon_{dS} = 0 \Rightarrow a_{dS} = e^{Ht}$.

So we'll require $\varepsilon \ll 1$. equivalently $w = -1$.

End of inflation ~ reheating into standard model particles

$$3M_p^2 H_{reh}^2 = g_* \frac{\pi^2}{30} T_{reh}^4 ,$$

where $T_{reh}$ is reheating temp. (unknown)

To talk about duration of inflation, we talk about e-foldings of expansion

$$dN = \frac{da}{a} = H\,dt.$$

$$\Rightarrow N(t_2) - N(t_1) = \log\left(\frac{a(t_2)}{a(t_1)}\right).$$

$$\Rightarrow a_2 = a_1 e^{N_2 - N_1}$$

To solve HBB problem (homog + isotropy + curvature).

$$\Delta N_{inf} > 50 + \log\left(\frac{T_{reh}}{10^{10}\,\text{GeV}}\right).$$

$$\Rightarrow a_{\text{end of infl}} \sim e^{50} a_{beg} \simeq 10^{23} a_{beg}.$$

Require $\varepsilon \ll 1$ during inflation

$$\varepsilon(N) = \varepsilon(N_*) + \left.\frac{\partial \varepsilon}{\partial N}\right|_{N_*} (N - N_*) + \cdots$$

$$= \varepsilon_* \left[1 + \eta_* (N - N_*) + \cdots\right]$$

where $\eta \equiv \dot{\varepsilon}/\varepsilon H$ is the <u>second slow-roll param</u>.

To higher orders, $\eta = \dfrac{\partial \log \varepsilon}{\partial N}$, then for $n \geq 3$, $\xi_n = \dfrac{\partial \log \xi_{n-1}}{\partial N}$.

Slow-roll inflation is $\varepsilon, \eta, \xi_n \ll 1$.

# Single Field Inflation

What drives inflation? A cosmological const. $\Lambda$ would dS forever. We introduce a clock $\phi(x,t)$ to stop inflation, then energy density $\supset V(\phi)$.

For simplicity, we take a canonical single minimally-coupled scaler theory

$$S = -\int d^4x \sqrt{-g}\cdot\frac{1}{2}\left[M_p^2 R + \partial_\mu\phi\partial^\mu\phi + \underbrace{2V(\phi)}_{\text{potential}}\right]$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + V(\phi)\right]$$

For homog. and isot. background, $\phi(x,t) = \phi(t)$., then

$$T_{\mu\nu} = (\rho + P)\,u_\mu u_\nu + g_{\mu\nu} P$$

with $\rho = \frac{1}{2}\dot\phi^2 + V$, $P = \frac{1}{2}\dot\phi^2 - V$, $u_\mu = (1,0,0,0)$

$$\Rightarrow T^\mu{}_\nu = \mathrm{diag}(-\rho, P, P, P)$$

Fried: $3H^2 M_p^2 = \frac{1}{2}\dot\phi^2 + V$

E.o.m. of $\phi$: $\frac{\delta S}{\delta\phi} \propto \ddot\phi + \underbrace{3H\dot\phi}_{\text{Hubble friction}} + V' = 0$.

Useful 2nd Fried: $-\dot H M_p^2 = \frac{1}{2}(\rho + P) = \frac{1}{2}\dot\phi^2$.

There is no general analytic sol$^n$., but there are approximate analytic sol$^n$ — "slow-roll sol$^n$s".

What $V(\phi)$ gives inflation, i.e. when are $\varepsilon, \eta, \ldots \ll 1$ ?

Fried:
$$V = (3-\varepsilon) H^2 M_p^2 \qquad (\ddagger)$$

Then, we want $\frac{1}{2}\dot{\phi}^2 \ll V$ to be "potential dominated".
Also, for this to remain true, we want small accel,
$$\ddot{\phi} \ll 3H\dot{\phi}, V ,$$
then to leading order,
$$\begin{cases} 3H^2 M_p^2 \simeq V \\ 3H\dot{\phi} = -V' \end{cases}$$

These are valid as long as
$$V \gg \frac{1}{2}\dot{\phi}^2 \approx \frac{1}{2}\left(\frac{-V'}{3H}\right)^2 = \frac{1}{2}\cdot\frac{1}{3}\frac{(V')^2}{3H^2} = \frac{M_p^2}{6}\frac{(V')^2}{V}$$

$$\Rightarrow \varepsilon_V := \frac{1}{2}\left(\frac{V'}{V}\right)^2 M_p^2 \ll 1 .$$

where $\varepsilon_V$ is called <u>potential slow-roll param.</u>

Define also
$$\eta_V := M_p^2 \frac{V''}{V} , \quad \xi_{3V} := M_{pl}^4 \frac{V'V'''}{V^2}$$

By taking $\partial_t(\ddagger)$, we find exact eqn
$$\varepsilon_V = \frac{\varepsilon(\eta - 2\varepsilon + \sigma)^2}{4(\varepsilon-3)^2}$$
$$\eta_V = \frac{\eta(\eta + 2\xi/3 + 6) - 2(5\eta + 12)\varepsilon + 8\varepsilon^2}{4(\varepsilon-3)}$$

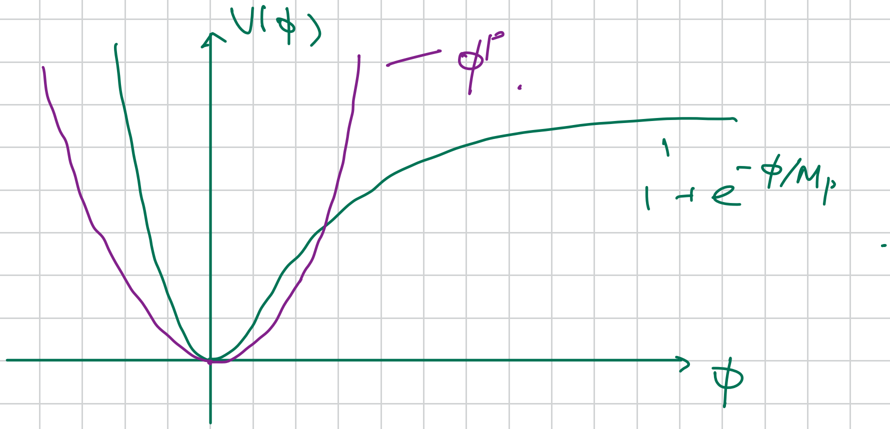
To leading order in $\varepsilon, \eta$ and $\xi$, we find the necessary conditions
$$\varepsilon \approx \varepsilon_V + \ldots , \quad \eta \approx 4\varepsilon_V - 2\eta_V + \ldots$$

So the potential has to be "slow-roll flat"

$$\varepsilon_V = \left(\frac{V'}{V}\right)^2 \frac{M_p^2}{2} \ll 1 ,$$

$$\eta_V = M_p^2 \frac{V''}{V} \ll 1$$

Example




To find approximate sol$^n$, use shorthand not$^n$ $X = \frac{1}{2}\dot{\phi}^2$.

Slow-roll condition means "potential domination"

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{3X}{V+X} \ll 1$$

$$\Rightarrow X \ll V$$

$$\Rightarrow 3H^2 M_p^2 \simeq V.$$

and "small accel"

$$\eta = \frac{\dot{\varepsilon}}{\varepsilon H} = 2\varepsilon + \frac{\dot{X}}{XH} \ll 1$$

$$\Rightarrow \dot{X} \ll XH$$

$$\Rightarrow 2\ddot{\phi} \ll \dot{\phi} H$$

$$\Rightarrow 3H\dot{\phi} \simeq V'$$

Slow-roll approx:

$$\dot{\phi} = \frac{V' M_p}{\sqrt{3V}} \Rightarrow t = \int d\phi \frac{\sqrt{3V}}{V' M_p} + \text{const}.$$

Also, $N$ (e-fold)

$$N = \int dN = \int H\, dt = \int \frac{d\phi\, H}{\dot\phi}$$
$$= \int \frac{d\phi}{M_p \sqrt{2\varepsilon}}$$
$$= \int_{\phi_i}^{\phi_f} d\phi \, \frac{V}{M_p^2 V'}$$

There are infinitely many potentials for which $N \gg 50$ of slow-roll inflation.

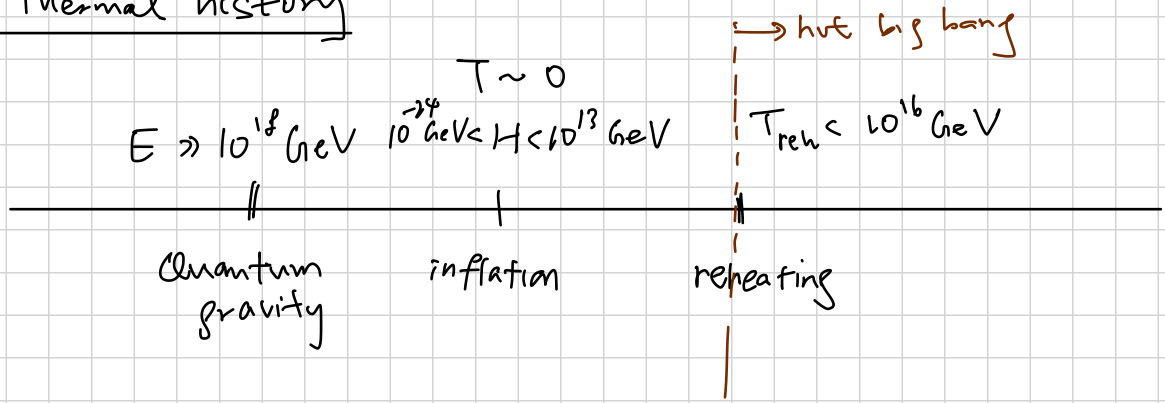
<u>Comment</u>: large and small-field models

$$\frac{\Delta\phi}{M_p} \gg 1 \quad \text{or} \quad \ll 1 .$$

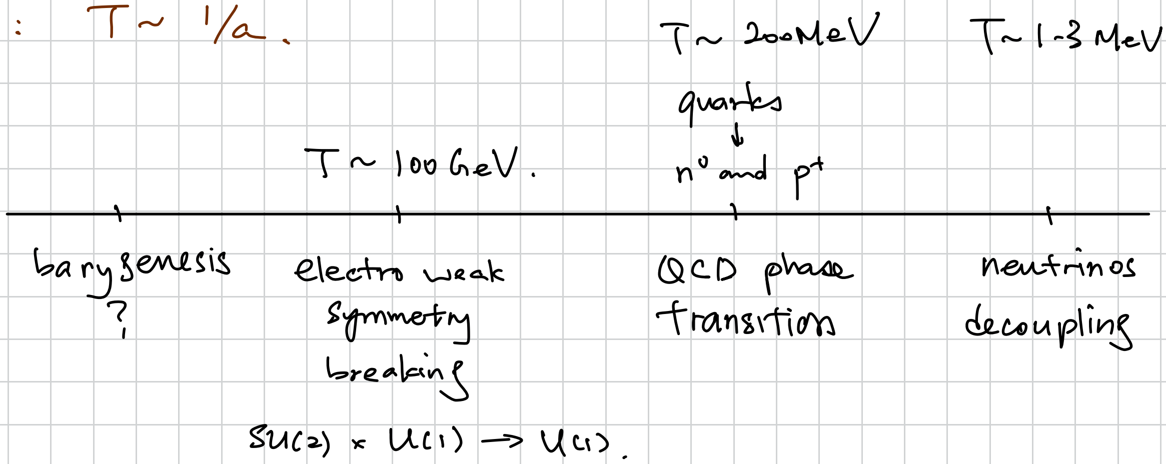
This is approximated by $\Delta\phi / M_p \approx \Delta N \cdot \frac{M_p V'}{V}$ (very rough).

# 6. Thermal history

## 6.1 Thermal history




Note : $T \sim 1/a$.



| $T \sim 0.5\,\text{MeV}$, $t \sim 5$ sec. | $T \sim 0.1\,\text{MeV}$, $t \sim 3$ mins | $T \sim 0.5\,\text{keV}$, $t \sim 2$ months |
|---|---|---|
| $e^- - e^+$ annihilation | Big Bang nucleosynthesis | Blackbody time |
| some $e^-$ are left over | ${}^1H$, ${}^2H$, ${}^3H$, ${}^4He$, ${}^7Li$ | $N_{\text{photons}} \sim$ conserved |
| $\#e^- = \#p^+$ | | |

| $T \sim 1\,\text{eV}$, $z \approx 3300$ | $T \sim 0.3\,\text{eV}$, $z \approx 1100$, $t \approx 370$ kyrs | $z \sim 10$ | $t =$ few Gyrs, $z \sim 0.5$ | now |
|---|---|---|---|---|
| mat = rad | recombination | re-ionisation | acceleration starts | |
| | $e^- + p^+ \to {}^1H$ | H ionised by stars | mat $\sim$ dark-energy | |
| | universe become transparent | | | |

## 6.2 Kinetic Theory

Probability $d\,\text{Prob}$ of finding a particle inside a physical volume $a^3 d^3x$ at a comoving position $\underline{x}$ with physical momentum $\underline{q}$ is

$$d\,\text{Prob} = f(\underline{x}, \underline{q}, t)\, a^3\, d^3x\, d^3q.$$

where $f(\underline{x}, \underline{q}, t)$ is the <u>phase-space density</u> and

$$p^\mu = (E, \underline{P}) = \left(\sqrt{m^2 + q^2},\ \underline{q}/a\right).$$

Then

$$T^{\mu\nu}(\underline{x}, t) = g \int \frac{d^3q}{(2\pi)^3} \frac{p^\mu p^\nu}{E} f(\underline{x}, \underline{q}, t)$$

$$n(\underline{x}, t) = g \int \frac{d^3q}{(2\pi)^3} f(\underline{x}, \underline{q}, t)$$

where $g$ is the no. of degree of freedom, e.g. $g = 2$ for photon, graviton, each neutrino.

We will assume <u>kinetic equilibrium</u>, i.e. Bose-Einstein or Fermi-Dirac dist$^n$

$$f(\underline{x}, q, t) = f_{BE, FD}(\underline{x}, q, t) = \frac{1}{e^{(E-\mu)/T} \mp 1}$$

where $E = \sqrt{m^2 + q^2}$, $\mu = \mu(\underline{x}, t)$ is chemical potential, $T = T(\underline{x}, t)$ temp.

Energy mom. tensor $T^0{}_0 = -\rho$ and $T^i{}_j = p(\underline{x}, t)\, \delta^i{}_j$. Then

$$\rho = \frac{g}{2\pi^2} \int dq\, q^2 \frac{E(q)}{e^{(E-\mu)/T} \mp 1},$$

$$p = \frac{g}{2\pi^2} \int dq\, q^2 \cdot \frac{q^2}{3E} \cdot \frac{1}{e^{(E-\mu)/T} \mp 1}$$

$$n = \frac{g}{2\pi^2} \int dq \frac{q^2}{e^{(E-\mu)/T} \mp 1}.$$

<u>Relativistic particles</u>

For $T \gg m$, approximate $E(q) = \sqrt{m^2 + q^2} \approx q$. Also $\mu$ small, then

$$\rho = 3p = g \cdot \frac{\pi^2}{30} T^4 \cdot \begin{cases} 1 & \text{Boson} \\ 7/8 & \text{Fermion} \end{cases},$$

$$n = g \cdot \frac{\zeta(3)}{\pi^2}, \quad \begin{cases} 1 & \text{Boson} \\ 3/4 & \text{Fermion} \end{cases}$$

Note that $w = P/\rho = 1/3$.

From 1st law of thermodynamics ($\mu = 0$),

$$dE = TdS - pdV$$

We get "Euler's relation"

$$E = TS - pV.$$

Dividing by volume, entropy density is

$$s := \frac{S}{V} = \frac{\rho + P}{T} = g \cdot \frac{2\pi^2}{45} T^3 \begin{cases} 1 & \text{Boson} \\ 7/8 & \text{Fermion} \end{cases}$$

Non-relativistic Particles

At low $T$, $m - \mu \gg T$. Both quantum dist$^n$ reduce to Boltzmann dist$^n$. Using $E = \sqrt{m^2 + q^2} = m + q^2/2m$, we find

$$n = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{(\mu - m)/T}$$

$$\rho = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{(\mu - m)/T} \left( m + \frac{3}{2} T \right) = n \left( m + \frac{3}{2} T \right).$$

$$p = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{(\mu - m)/T} \, T = nT$$

$$s = \frac{\rho + p}{T} = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{(\mu - m)/T} \left( \frac{m}{T} + \frac{5}{2} \right).$$

Note that $\rho \approx nm$ and $p = nT \Rightarrow w = \frac{p}{\rho} \sim \frac{T}{m} \ll 1 \Rightarrow w = 0$.

In thermal eqm and for $\mu \simeq 0$, NR particles are negligible

$$\frac{\rho_{NR}}{\rho_{rel}} \sim e^{-m/T} \left( \frac{T}{m} \right)^{5/2} \ll 1.$$

6.3 Number of relativistic particles

It's useful to define effective no. of rel. particles $g_*$ by

$$\rho = g_* \frac{\pi^2}{30} T^4$$

with

$$g_* \equiv g_{Bos} + \frac{7}{8} g_{Fer}$$

where

$$g_* = g_*^{ther} + g_*^{dec}, \qquad g_*^{th}(T) = \sum g_{bos} + \frac{7}{8} \sum g_{fer}$$

For example, at $T \gg 100$ MeV, $g_* = g_*^{ther} = 106.75$, while at $T \sim 10$ MeV, $g_* = \underset{\gamma}{2} + \frac{7}{8}(\underset{e^+}{2} + \underset{e^-}{2} + \underset{\nu_e, \nu_\mu, \nu_\tau, \bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau}{2 \times 3}) = 10.75$,

As universe expands, $T(t) \sim \frac{1}{a(t)}$ decreases. Away from $T \approx m_i$,

$$g_*(T) = \text{const.}$$

Around $T \sim m_i$, the $i$-particle becomes NR. This is called a threshold.

Species out-of-eqm at different temp $T \neq T_i$

$$g_*^{dec}(T) = \sum_{Bos} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{Fer} g_i \left(\frac{T_i}{T}\right)^4$$

## 6.4 Conservation of entropy

In eqm, for $\mu \simeq 0$. can be shown

$$\frac{\partial P}{\partial T} = \frac{\rho + P}{T}$$

1st law with $E = \rho V$

$$\Rightarrow \quad T dS = d(\rho V) + P dV$$

$$\Rightarrow \quad dS = \frac{1}{T}\Big( d[(\rho + P)V] - V dP \Big)$$

$$= \frac{1}{T} d((\rho + P)V) - \frac{V}{T^2}(\rho + P) dT.$$

$$= d\left[ \frac{\rho + P}{T} V \right]$$

Then

$$\frac{dS}{dt} = \frac{V}{T}\left(\dot{\rho} + 3H(\rho + P)\right) + \frac{V}{T}\left(\dot{P} - \frac{\rho + P}{T}\dot{T}\right) = 0$$

So $S$ conserved in eqm.

Recall entropy density $s = \frac{S}{V} = \frac{\rho + P}{T}$. For multiple species,

$$s = \sum_{\text{species}} \frac{\rho_i + P_i}{T_i} \approx \frac{2\pi^2}{45} g_{*s}(T)\, T^3$$

where

$$g_{*s}(T) = g_{*s}^{\text{thermal}}(T) + g_{*s}^{\text{dec}}(T).$$

In eqm, $g_{*s}^{th}(T) = g_*^{th}(T)$. For decoupled particles

$$g_{*s}^{\text{dec}}(T) = \sum_{\text{Bos}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{Fer}} g_i \left(\frac{T_i}{T}\right)^3 \neq g_*^{\text{dec}}(T).$$

Entropy conservation implies

$$a^3 S \propto g_{*s}(T)\,(Ta)^3 = \text{const}.$$

$$\Rightarrow T \propto \frac{1}{g_{*s}^{1/3}\, a}$$

## 6.5 Neutrinos

Three periods:

① Neutrinos in thermal eqm. w/ $e^\pm, \gamma, \ldots$

② Neutrinos decouple before "electron threshold" a.k.a. electron-positron annihilation. After this

$$T_\nu(a)\, a = T_{\nu,\text{dec}}\, a_{\text{dec}}.$$

③ Neutrinos become NR. ($z < 500$) and cluster.

After $\nu$ decouple, below electron threshold $T \sim m_e$,

$$e^+ + e^- \to \gamma + \gamma$$

heats photons but not neutrinos $\Rightarrow T_\gamma > T_\nu$.

Before $e^+ - e^-$ annihilation,

$$s_1 = \frac{2\pi^2}{45} T_1^3 g_{*s} = \frac{2\pi^2}{45} T_1^3 \left[ \underbrace{g_{bos}}_{2} + \frac{7}{8} \underbrace{g_{fer}}_{\substack{2\times(1+1+3) \\ e^+,\, e^-,\, \nu}} \right]$$

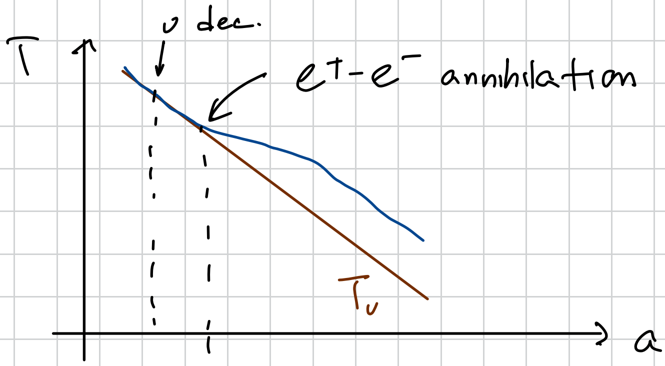
After annihilation,

$$s_2 = \frac{2\pi^2}{45} \left[ 2 T_\gamma^3 + \frac{7}{8} (2\times 3) T_\nu^3 \right]$$

Conservation of entropy : $a_1^3 S_1 = a_2^3 S_2$ and $a_1 T_1 = a_2 T_\nu$.

$$\Rightarrow \quad T_\nu = T_\gamma \left(\frac{4}{11}\right)^3$$

$$T_\nu(a_0) \approx \left(\frac{4}{11}\right)^3 (2.7\,K) \approx 1.96\,K \approx 1.7\times 10^{-4}\ eV.$$




Neglect unknown neutrino masses

$$\rho_\nu = \rho_\gamma \cdot 3 \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{1/3} \Rightarrow h^2 \Omega_\nu = 1.7\times 10^{-5}.$$

To test this and probe beyond standard model, people define $N_{eff}$ :

$$\underbrace{\rho_r}_{\text{radiation}} = \rho_\gamma \left( 1 + N_{eff} \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{1/3} \right)$$

where : SM. $N_{eff} = 3.04$. CMB best fit is $N_{eff} = 3.04 \pm 0.18$.

# 7. Thermal History: Out of Equilibrium

## 7.1 Boltzmann Equation

General idea:
$$\frac{d f(x(t), q(t), t)}{dt} = \text{collisions}$$

For 2-body process $1+2 \leftrightarrow 3+4$, the integrated Boltz eqn reads

QFT: scattering amplitude

$$a^{-3}\frac{d(a^3 n_1)}{dt} = \int \prod_{i=1}^{4} \frac{d^3 p_i}{(2\pi)^3 \cdot 2E_i}\, \delta_D^{(3)}\Big(\sum_{i=1}^{4} q_i\Big)\, \delta_D\Big(\sum_{i=1}^{4} E_i\Big)\, |M|^2 .$$
$$\Big[f_3 f_4 (f_1 \pm 1)(f_2 \pm 1) - f_1 f_2 (f_3 \pm 1)(f_4 \pm 1)\Big].$$

Bose enhancement (+), Pauli blocking (−).

We'll assume kinetic eqm (not chemical eqm)

$$f_{1,2,3,4} \longrightarrow f_{BE,FD} = \frac{1}{e^{(E-\mu)/T} \mp 1}.$$

We'll also assume $T \ll E-\mu$, so $f_{BE,FD} \sim e^{(\mu - E)/T}$ and $f \pm 1 \approx \pm 1$.

Define $n^{(0)} = n(\mu=0) = n(\mu)\, e^{-\mu/T}$.

Then,

$$\Big[f_3 f_4 (f_1 \pm 1)(f_2 \pm 1) - f_1 f_2 (f_3 \pm 1)(f_4 \pm 1)\Big]$$
$$= e^{-(E_1+E_2)/T}\left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right] \qquad (E_1 + E_2 = E_3 + E_4)$$

Define the thermally average cross-section.

$$\langle \sigma v \rangle := \frac{1}{n_1^{(0)} n_2^{(0)}} \int \prod_{i=1}^{4} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_i} \cdot \delta_D^{(4)}\Big(\sum_{j=1}^{4} P_j\Big) |M|^2 e^{-(E_1+E_2)/T}$$

Obtain

$$\boxed{a^{-3}\frac{d(a^3 n_1)}{dt} = \langle\sigma v\rangle n_1^{(0)} n_2^{(0)}\left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right]}$$

Two time scales : $\Gamma$ and $H$

- $\Gamma := \langle\sigma v\rangle n_2^{(0)}$ (time) is the interaction rate.

Two regimes:

- Chemical eqm : $\Gamma \gg H$. Solution is

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \quad \text{up to } \mathcal{O}(H/\Gamma).$$

Equivalently,

$$\mu_3 + \mu_4 = \mu_1 + \mu_2$$

- Freeze out : $\Gamma \ll H$, then sol$^n$ is

$$n a^3 \sim \text{const},$$

or

$$n(t) = n(a_*)\left(\frac{a_*}{a}\right)^3.$$

## 7.2 Big Bang Nucleosynthesis

Today, atoms are 75% H and 25% $^4$He, others traces.

Since binding energy is 2-8 MeV, at $T \gg 8$MeV, only have protons and neutrons. How do atoms form as $T$ decreases?

Two steps:

1. Neutron abundance, around $T \sim 0.1$ MeV (out of eqm)
2. Instantaneous formation $n + p \longrightarrow$ Deuterium $\longrightarrow$ $^4$He.

① Neutron abundance is controlled by weak interactions

$$n + \nu_e \longleftrightarrow p^+ + e^- ,$$

$$n + \bar{e}^+ \longleftrightarrow p^+ + \bar{\nu}_e$$

$$n \longrightarrow p^+ + e^- + \bar{\nu}_e$$

neglect this

We'll also use that $\mu_e \sim \mu_\nu \sim 0$. Then in chemical eqm,

$$\frac{n_n}{n_p} \approx \frac{n_n^{(0)}}{n_p^{(0)}} = e^{(m_p - m_n)/T} \equiv e^{-Q/T}$$

where $Q = m_n - m_p \approx 1.3$ MeV.

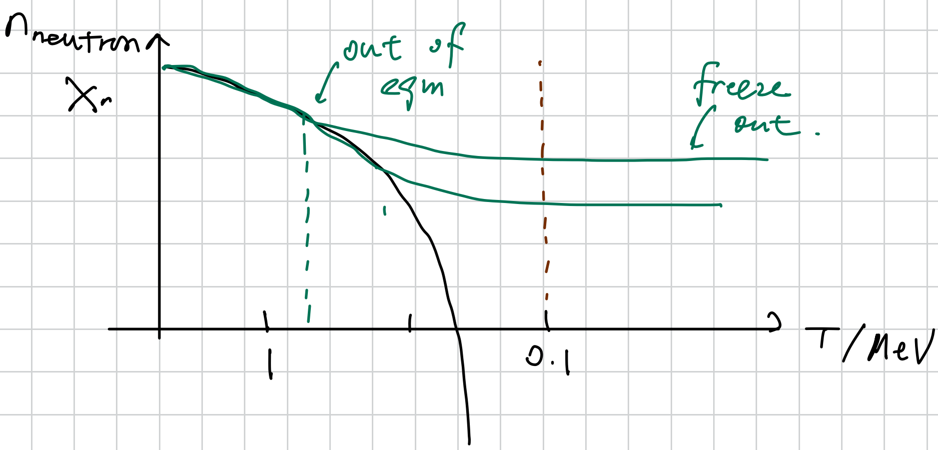
Boltzmann eqn with $n_1 = n_n$, $n_{2,3} \approx n_e$, $n_3 = n_p$. For leptons $n_e = n_e^{(0)}$.

$$\Rightarrow a^{-3} \frac{d(a^3 n_n)}{dt} = \underbrace{\langle \sigma v \rangle n_e^{(0)}}_{\lambda_{np}} \left[ \frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n \right]$$

Change variables to $\lambda_{np} = \langle \sigma v \rangle n_e^{(0)}$ and $(n_p + n_n) a^3 = \text{const}$, so that

$$\frac{d X_n}{dt} = \lambda_{np} \left[ (1 - X_n) e^{-Q/T} - X_n \right]$$

where $X_n = \dfrac{n_n}{n_n + n_p} = \dfrac{n_n}{n_b}$.




Particle physics scattering amplitude

neutron-to-proton conversion rate

$$\lambda_{np} = \frac{255}{\tau_{life}\, x^5} \left( 12 + 6x + x^2 \right).$$

with $\tau_{life} \approx 15$ mins $\approx 900$ sec. and $x = Q/T$.

Using $\rho_{free} \propto T^4 \propto a^{-4}$

$$\Rightarrow T = \frac{1}{a} \Rightarrow \frac{dx}{dt} = -x\frac{\dot{T}}{T} = xH.$$

Finally,

$$\frac{dX_n}{dx} = \frac{\lambda_{np}(x)}{x\,H(x)}\left[e^{-x} - X_n(1+e^{-x})\right].$$

More intuitive to use $X_n(T)$ instead of $X_n(t)$.

$$\frac{d}{dt} = \dot{T}\frac{d}{dT} = -HT\frac{d}{dT}$$

For $H(T)$, use the Fried.

$$H = \sqrt{\frac{\rho}{3M_p^2}} = \sqrt{\frac{\pi^2}{30}g_* T^4 / 3M_p^2} = \frac{\pi}{3}\sqrt{\frac{g_*}{10}}\cdot\frac{T^2}{M_p} \approx \frac{T^2}{M_p}.$$

From numerical sol$^n$ $X_n(0.1\,MeV) = 0.11$.

Small correction: neutron decay $n \to p + e + \bar{\nu}_e$, so

$$n_n \to n_n e^{-t/t_{life}}$$

(II) $n + p \leftrightarrow D + \gamma$, then $D + D \leftrightarrow {}^4He + \gamma$.

Since $\mu_\gamma = 0$, $n_\gamma = n_\gamma^{(0)}$ and assuming eqm.,

$$\frac{n_D \cancel{n_\gamma}}{n_n n_p} = \frac{n_D^{(0)} \cancel{n_\gamma^{(0)}}}{n_n^{(0)} n_p^{(0)}}$$

Using $n^{(0)} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$ and binding energy of Deuterium,

$$B_D = m_p + m_n - m_D = 2.2\,MeV$$

Then

$$\frac{n_D}{n_n n_p} = \frac{3}{4}\left(\frac{2\pi m_D}{m_n m_p T}\right)^{3/2} e^{(m_n+m_p-m_D)/T} \approx \frac{3}{4}\cdot\left(\frac{4\pi}{m_p T}\right)^{3/2} e^{B_D/T}.$$

Rough approx: drop all $O(1)$ factors: $n_n \sim n_p \sim n_b$.

$$\frac{n_D}{n_b} \approx n_b\cdot\frac{T^3}{n_\gamma}\frac{1}{(m_p T)^{3/2}}\, e^{B_D/T}$$

Assumed eqm

Baryon-to-photon ratio.

Have

$$\eta_b \equiv \frac{n_b}{n_\gamma} = \frac{\rho_b \pi^2}{m_b \cdot 2\zeta(3) \cdot T_\gamma^3} = 5\times 10^{-10}.$$

Eqm. is broken around $T_{nuc}$ with $n_D / n_b \approx \mathcal{O}(1)$.

$$\frac{B_D}{T_{nuc}} + \log \eta_b + \log \left(\frac{T_{nuc}}{m}\right)^{3/2} = 0.$$

$$\Rightarrow T = \frac{B_D}{\log(5\times10^{-10})} \cdot \frac{2.2\,\mathrm{MeV}}{20} = 0.1\,\mathrm{meV}.$$

All free neutrons have formed Deuterium by $T = 0.1\,\mathrm{MeV}$.

Now $D + D \longrightarrow {}^4He + \gamma$. This happens very quickly at $0.1\,\mathrm{MeV}$.

Helium mass fraction today is

$$X_4 = \frac{4 m_p \, n_{He}}{m_p n_b} = \frac{4(n_{He}/2)}{n_b} = 2\frac{n_n}{n_b} = 2X_n(0.1\,\mathrm{MeV}) = 0.22.$$

This is very closed to measured value of 25°C ${}^4He$ and 75% H = 0.22.

Traces of Deuterium are left over. This is a good "barymmeter" giving

$$\text{BBN}: \quad \Omega_b h^2 = 0.0205 \pm 0.0018.$$

# 8. Cosmological Perturbation Theory

## 8.1 Linearised equations of motion

Now, we move on to describe the inhomg. universe. On large scales, inhomogeneities are small and we can treat them perturbatively.

Define perturbation by

$$g_{\mu\nu}(\underline{x},t) = \bar{g}_{\mu\nu}(t) + h_{\mu\nu}(\underline{x},t).$$

$$T_{\mu\nu}(\underline{x},t) = \bar{T}_{\mu\nu}(t) + \delta T_{\mu\nu}(\underline{x},t).$$

Recall the background

$$\bar{g}_{\mu\nu}(t) = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{pmatrix}$$

$$\bar{T}_{\mu\nu}(t) = \begin{pmatrix} -\bar{\rho}(t) & & & \\ & \bar{p}(t) & & \\ & & \bar{p}(t) & \\ & & & \bar{p}(t) \end{pmatrix}$$

where $|h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|$ and $|\delta T_{\mu\nu}| \ll |\bar{T}_{\mu\nu}|$.

Our dynamical eqn are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{M_p^2} T_{\mu\nu}.$$

$$\Rightarrow \nabla^\mu T_{\mu\nu} = 0.$$

Now we perturb

$$\delta R_{\mu\nu} - \frac{1}{2}\delta g_{\mu\nu} \bar{R} - \frac{1}{2}\bar{g}_{\mu\nu}\,\delta R = -\frac{1}{M_p^2}\delta T_{\mu\nu}$$

Sometimes useful to note

$$-R = -\frac{1}{M_p^2} T$$

and

$$R_{\mu\nu} = -\frac{1}{M_p^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \qquad \left(\begin{matrix}\text{trace-reversed}\\ \text{EE}\end{matrix}\right).$$

For example, the 00 cpt

$$\frac{1}{2a^2}\nabla^2 h_{00} + \frac{3}{2} H \dot{h}_{00} - \frac{1}{a^2}\partial_i \dot{h}_{i0} + \frac{1}{2a^2}\left[\ddot{h}_{ii} - 2H\dot{h}_{ii} + 2\left(H^2 - \frac{\ddot{a}}{a} h_{ii}\right)\right]$$
$$+ 3\left(H^2 + \frac{\ddot{a}}{a}\right) h_{00} = -\frac{1}{M_p^2}\left(\delta T_{00} + \frac{1}{2}\delta T^\lambda{}_\lambda\right).$$

A bit too hard to solve on this. Three simplifications

① Fourier transform: different 3-momenta $\underline{k}$ decouple from each other

② Scaler - Vector - Tensor (SVT) decomposition

③ Coordinate invariance: choose coords to eliminate some perturbations

## 8.2 Fourier transform

Let $\mathrm{Pert}_A = \{ h_{\mu\nu}(\underline{x}, t), \delta T_{\mu\nu}(\underline{x}, t) \} \Rightarrow$ e.o.m. is

$$\sum_A \mathcal{O}_A \, \mathrm{Pert}_A(\underline{x}, t) = 0$$

$$\mathcal{O}_A = \mathcal{O}_A(\nabla_\mu, \bar{g}_{\mu\nu}, \bar{T}_{\mu\nu}).$$

$$\xRightarrow{FT} \quad \sum_A \mathcal{O}_A(\partial_t, \partial_i \to ik_i, \bar{g}_{\mu\nu}, \bar{T}_{\mu\nu}) \, \mathrm{Pert}_A(\underline{k}, t) = 0$$

PDE in $\partial_t, \partial_x$ $\to$ ODE in $\partial_t$ (infinitely-many decoupled).

## 8.3 SVT decomposition

Theory and FLRW are not$^n$ invar., so e.o.m. are not$^n$ covariant.

At linear order, fields can decouple only if they have same transf$^n$. $x^i \to R^i{}_j x^j$.

Say $S, V_\mu$ and $h_{\mu\nu}$ are 4-diff scaler, vector and tensor.

- 4-diff scaler is also a rotation - scaler:

$$S(\underline{x}, t) \to S(\underline{x}', t') = S(\underline{x}, t).$$

where $x'^i = R^i{}_j x^j$.

For example, pressure $P$, energy density $\rho$ etc.

Also, time cpt of 4-diff tensor, e.g. $V_0$, $h_{00}$.

$$V_0(x) \to V_0(x') = J_0{}^\mu V_\mu(x) ,$$

where $J_\nu{}^\mu = \begin{pmatrix} 1 & \\ & R_i{}^j \end{pmatrix}$.

- Diff-vectors with spatial indices, e.g. $V_i(x)$, can be decomposed as in E and M (Helmholtz decomp.)

$$V_i = w_i + \partial_i \theta ,$$

where $\partial_i \partial_i \theta = \partial_i V_i$ and $W_i = V_i - \partial_i \theta$, $\partial_i W_i = 0$.

Here $\theta$ is a rot$^n$-scaler and $w_i$ a "transverse vector"

$$W_i'(x') = R_i{}^j W_j(x).$$

- Diff-tensors: generalised Helmholtz decomposition

$$h_{00} = -E, \quad h_{0i} = a[\partial_i F + G_i],$$

$$h_{ij} = a^2 \left[ \delta_{ij} A + \partial_i \partial_j B + \partial_{(i} C_{j)} + D_{ij} \right]$$

where $\partial_i G_i = \partial_i C_i = 0$ are (transverse) vector, and

$$D_{ij} = D_{ji}, \quad \partial_i D_{ij} = 0, \quad D_{ij}\delta_{ij} = D_{ii} = 0.$$

$D_{ij}$ is a symmetric transverse tensor, a.k.a. TT-tensor/tensor.

$h_{\mu\nu}$ has 10 cpt: 4 scaler $(A, B, E, F)$, $2\times 2$ vectors $(C_i, G_i)$ and $1 \times 2$ tensor $(D_{ij})$.

For $T_{\mu\nu}$, we start with a perfect fluid

$$u_\mu u^\mu = -1$$

future pointing normalised vector

$$T_{\mu\nu}^{(\text{perfect})} = (\rho + p)\, u_\mu u_\nu + P g_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a^2 p & & \\ & & a^2 p & \\ & & & a^2 p \end{pmatrix}$$

$u^\mu = [-1, 0, 0, 0]$

$$= \rho\, u_\mu u_\nu + P \underbrace{(g_{\mu\nu} + u_\mu u_\nu)}$$

$$u^\mu (g_{\mu\nu} + u_\mu u_\nu) = u_\nu + u^\mu u_\mu u_\nu = 0.$$

The SVT decomposition follows from that of $\rho, P, u_\mu$, namely,

$$\rho(x,t) = \bar\rho(t) + \delta\rho(x,t)$$

$$p(x,t) = \bar p(t) + \delta p(x,t)$$

vector

$$u_\mu = (\delta u_0,\ \partial_i \delta u + \delta u_i^V).$$

where $\partial_i u_i^V = 0$.

Note $u^\mu u_\mu = -1 \Rightarrow \delta u_0 = \delta u^0 = h_{00}/2$. So only 3 scalers $\delta\rho$, $\delta p$, $\delta u$, 1 transverse vector $\delta u_i^V$.

For a more generic $T_{\mu\nu} \neq T_{\mu\nu}^{\text{perfect}}$, we need 5 more cpt. These are

$$\delta T_{ij} = \bar p\, h_{ij} + a^2 \delta p\, \delta_{ij} + a^2 [\partial_i \partial_j \Pi^S + \partial_i \Pi_j^V + \Pi_{ij}^T]$$

where $\partial_i \Pi_i^V = 0$ (transverse), $\Pi_{ii}^T = 0 = \partial_i \Pi_{ij}^T$.

The $\Pi$'s are called anistropic inertia, or viscosity and describe dissipative fluids.

## 8.4 Gauge transformations

We consider only diff-invar. theories $\Rightarrow$ e.o.m. are covariant.

$\Rightarrow$ choose coords to simplify eqns.

We know $g_{\mu\nu}$ and $T_{\mu\nu}$ transforms as tensors. How do $\delta T_{\mu\nu}$ and $h_{\mu\nu}$ transform?

It's convenient to choose: $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$.

fixed

soaks up all transf$^n$ of $T_{\mu\nu}$.

Consider a change of coords: $x^\mu \to x'^\mu = x^\mu + \varepsilon(x)^\mu$

$$x^\mu = x'^\mu - \varepsilon^\mu.$$

We'll assume $|\varepsilon^\mu| \ll 1$, and $|\varepsilon^\mu| \sim |h_{\mu\nu}| \sim |\delta T_{\mu\nu}|$. and

we'll drop $\mathcal{O}(\varepsilon^2, \varepsilon h, \varepsilon \delta T)$.

Instead of a change of coords, we define <u>gauge transformation</u> of the fields.

① Transform covariant tensors, e.g. $\rho$, $p$, $u_\mu$, $g_{\mu\nu}$, $T_{\mu\nu}$

② Attribute all the change $\Delta$ to perturbations $\delta\rho$, $\delta p$, $\delta u$, $h_{\mu\nu}$, $\delta T_{\mu\nu}$

For example, any pert. to a scaler $S(\underline{x},t) = \bar{S} + \delta S(\underline{x},t)$.

$$S'(x') = S(x),$$

equivalently,

$$S'(x) = S(x-\varepsilon) \qquad \forall\, x \in \mathbb{R}^4$$

$$\Rightarrow \Delta\,\delta S = S'(x) - S(x) = S(x-\varepsilon) - S(x).$$

$$= -\varepsilon^\mu \partial_\mu S(x) + \mathcal{O}(\varepsilon^2).$$

$$\underset{\text{order}}{\overset{\text{lowest}}{=}} -\varepsilon^\mu \partial_\mu \bar{S}(t)$$

$$= -\varepsilon^0 \dot{\bar{S}}(t).$$

Similarly, for 4-vec: $V^\mu = \bar{V}^\mu + \delta V^\mu$

$$\Delta\,\delta V^\mu = \delta V'^\mu - \delta V^\mu(x)$$

$$= V'^\mu(x) - V^\mu(x)$$

$$= \frac{\partial x^\mu}{\partial (x-\varepsilon)^\lambda} V^\lambda(x-\varepsilon) - V^\mu(x).$$

$$= V^\mu(x-\varepsilon) + V^\lambda \partial_\lambda \varepsilon^\mu - V^\mu(x) + O(\varepsilon^2)$$
$$= -\varepsilon^\lambda \partial_\lambda V^\mu(x) + V^\lambda \partial_\lambda \varepsilon^\mu + O(\varepsilon^2).$$
$$= -\varepsilon^\lambda \nabla_\lambda V^\mu(x) + V^\lambda \nabla_\lambda \varepsilon^\mu$$

Also for the metric,

$$\Delta h_{\mu\nu}(x) = h'_{\mu\nu}(x) - h_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x)$$

Using

$$g'_{\mu\nu}(x) = \frac{\partial(x+\varepsilon)^\alpha}{\partial x^\mu} \frac{\partial(x+\varepsilon)^\beta}{\partial x^\nu} g_{\alpha\beta}(x-\varepsilon) = \dots$$

$$\Rightarrow \Delta h_{\mu\nu}(x) = -\nabla_\mu \varepsilon_\nu - \nabla_\nu \varepsilon_\mu \qquad (\ddagger).$$

Note the general rule, for tensor $T$,

$$\Delta \delta T = -\mathcal{L}_\varepsilon T \approx -\mathcal{L}_\varepsilon \bar{T}.$$

Recall the SVT decomposition of $h_{\mu\nu}$, and use $(\ddagger)$, the cpts transform as

$$\Delta A = 2H\varepsilon_0, \qquad \Delta B = -\frac{2}{a^2}\varepsilon^S, \qquad \Delta F = \frac{1}{a}\left(-\varepsilon_0 - \dot{\varepsilon}^S + 2H\varepsilon^S\right)$$
$$\Delta E = 2\dot{\varepsilon}_0, \qquad \Delta G_i = \frac{1}{a}\left(-\dot{\varepsilon}_i^V + 2H\varepsilon_i^V\right) \qquad \Delta C_i = -\frac{1}{a^2}C_i^V$$
$$\Delta D_{ij} = 0.$$

when we SVT-decompose the param $\varepsilon_\mu = (\varepsilon_0, \partial_i \varepsilon^S + \varepsilon_i^V)$ with $\partial_i \varepsilon_i^V = 0$.

Other transf$^n$ of SVT cpts of $\delta T_{\mu\nu}$ ($\bar{u}_\mu = (-1, 0, 0, 0)$, $\delta u_\mu = (\delta u_0, \partial_i \delta u + \delta u_i^V)$).

$$\Delta \delta\rho = \varepsilon_0 \dot{\bar{\rho}} \qquad \Delta \delta p = \varepsilon_0 \dot{\bar{p}}, \qquad \Delta \delta u = -\varepsilon_0$$

$$\Delta \pi^S = 0, \qquad \Delta \pi^V = 0, \qquad \Delta \pi^T = 0, \qquad \Delta \delta u_i^V = 0$$

## 8.5 Equations

### Vector Equations

To solve for $\delta u^v_i$, $C_i$, $G_i$, we may set scalers and tensors to 0. Can work with

$$G_{\mu\nu} + \frac{1}{M_p^2} T_{\mu\nu} = 0 \quad \text{and} \quad \nabla_\mu T^\mu{}_\nu = 0$$

where $T^\mu{}_\nu = (\rho + p) u^\mu u_\nu + p\,\delta^\mu{}_\nu$.

$$0 = \nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha{}_i - \Gamma^\alpha_{\mu\ i} T^\mu{}_\alpha$$

$$= \cdots$$

$$\Rightarrow \partial_t\left[(\bar\rho + \bar p)\,\delta u^v_i\right] + 3H(\bar\rho + \bar p)\,\delta u^v_i = -\cancel{\partial_i^2 \pi^v}^{\,0} = 0$$

Hence,

$$(\bar\rho + \bar p)\,\delta u^v_i \propto \frac{1}{a^3}$$

so decaying. Moreover, the only gauge invariant comb$^n$ of $G_i$ and $C_i$ is $(G_i - a\dot C_i)$ so that

$$\Delta(G_i - a\dot C_i) = 0$$

The $0i$-cpt of EEs gives

$$\frac{1}{M_p^2}(\bar\rho + \bar p)\,\delta u^v_i a = \frac{1}{2}\partial_j\partial_j(G_i - a\dot C_i) \propto \frac{1}{a^2}.$$

Vector perturbations decay and don't play much of a role in cosmology.

### Tensor Equations

$D_{ij}$ are gravitational waves and are gauge invar (at linear order). 15+ cosmo expt. looking for $D_{ij}$. The Pulsar Timing Arrays (PTA) detected a stochastic background of $D_{ij}$, presumably from super massive BH. No primordial $D_{ij}$ detected.

Take transverse trace part of $ij$ - EEs gives

$$\ddot{D}_{ij} + 3H D_{ij} - \frac{\partial_k \partial_k}{a^2} D_{ij} = \cancelto{0}{\frac{1}{M_p^2} \Pi^T_{ij}} \qquad (*)$$

In Fourier space, $\partial_k \partial_k \to -\underline{k}^2$. Introduce polarisation tensors $\varepsilon^{(s)}_{ij}(\underline{k})$.

$$D_{ij}(t, \underline{k}) = \sum_s \varepsilon^{(s)}_{ij}(\underline{k}) \, D_s(t, \underline{k})$$

where $s = \pm 2$ for circular polarisations, or $s = +, \times$ polarisation.

with

$$\varepsilon^s_{ii}(\underline{k}) = k_j \varepsilon^s_{ij}(\underline{k}) = 0$$

$$\varepsilon^s_{ij}(\underline{k})^* = \varepsilon^s_{ij}(-\underline{k}) \quad , \quad \varepsilon_{ij} = \varepsilon_{ji} \Rightarrow D_{ij}(x) \in \mathbb{R}.$$

$$\varepsilon^s_{ij}(\underline{k}) \, \varepsilon^{s'}_{ij}(\underline{k})^* = 2 \delta_{ss'}.$$

We can solve $(*)$ numerically or approximate:

- SuperHubble $k \ll aH$:

$$\ddot{D}_s + 3H D_s \simeq 0 \Rightarrow D \simeq C_1 + \underbrace{C_2 \frac{1}{a^{3(1+w)/2}}}_{\text{decaying}}$$


- SubHubble $k \gg aH$ : use WKB with ansatz

$$D_s(t) = X(t) \, e^{ik \int dt'/a(t')}$$

and solve for $X(t)$ to leading order in $aH/k$ to find:

$$D_s(k, t) = \frac{\tilde{C}_1 \cos(k\tau) + \tilde{C}_2 \sin(k\tau)}{a},$$

with $\tau$ conformal time.

## Scaler equation of motion

We need the scaler eqn insider $EE_{\mu\nu}=0$ — $EE_{00}, {}_{ii}$ and longitudinal part of $EE_{0i}$, $EE_{ij}$.

Have 8 unknowns $A, B, E, F, \delta p, \delta u, \delta\rho, \pi^S$.

We'll remove 2 by specifying the fluid, e.g. $\pi^S=0$ and $P=P(\rho)$.

Remove 2 more unknowns using gauge parameters $\varepsilon_0$ and $\varepsilon^S$.

Two options:

1. Use only gauge invariant variables
2. Fix the gauge by specify $\varepsilon_0$, $\varepsilon^S$.

## Newtonian Gauge

Recall that

$$\Delta B = -\frac{2}{a^2}\varepsilon^S \quad , \quad \Delta F = \frac{1}{a}\left(-\varepsilon_0 - \dot{\varepsilon}^S + 2H\dot{\varepsilon}^S\right)$$

For any solution $B(x)$ and $F(x)$, choose

$$\begin{cases} \varepsilon^S = a^2 B/2 \\ \varepsilon_0 = aF - \frac{a^2}{2}\dot{B} \end{cases} \Rightarrow \quad \begin{aligned} B' &= B + \Delta B = B - B = 0 \\ F' &= F + \Delta F = 0 \end{aligned}$$

Traditionally, $E = 2\Phi$, $A = -2\Psi$.

Perturbed metric in N. gauge

$$ds^2 = -(1+2\Phi)\,dt^2 + a^2 dx^i \delta_{ij} dx^j (1-2\Psi).$$

with $\Phi$, $\Psi$ the Newtonian potentials, e.g. in non-rel Newton,

$$\ddot{x}^i = -\partial^i \Phi.$$

In this gauge,

$$\delta T_{00} = 2\bar{\rho}\Phi + \delta\rho$$

$$\delta T_{0i} = -(\bar{\rho}+\bar{P})\,\partial_i \delta u$$

$$\delta T_{ij} = a^2 \partial_i\partial_j \pi^S + \delta_{ij} a^2(\delta P - \bar{P}\Psi).$$

Then EE become

$$-\frac{1}{2M_p^2}\left[\delta\rho + \delta P - \nabla^2\Pi^s\right] = H\dot{\Phi} + \left(4H^2 + 2\frac{\ddot{a}}{a}\right)\Phi - \frac{\nabla^2\Psi}{a^2} + \ddot{\Psi} + 6H\dot{\Psi}$$

$$-\frac{a^2}{M_p^2}\partial_i\partial_j\Pi^S = \partial_i\partial_j(\Phi - \Psi).$$

$$\frac{1}{2M^2}(\bar{\rho} + \bar{P})\partial_i\delta u = -H\partial_i\Phi - \partial_i\dot{\Psi}$$

$$\frac{1}{2M^2}\left[\delta\rho + 3\delta P + \nabla^2\Pi^s\right] = \frac{\nabla^2\Phi}{a^2} + 3H\dot{\Phi} + 3\ddot{\Psi} + 6H\dot{\Psi} + 6\frac{\ddot{a}}{a}\Phi.$$

The conservation eqn $\nabla_\mu T^\mu{}_\nu = 0$:

$$\delta P + \nabla^2\Pi^s + \partial_0\left((\bar{\rho} + \bar{P})\delta u\right) + 3H(\bar{\rho} + \bar{P})\delta u + (\bar{\rho} + \bar{P})\Phi = 0.$$

$$\delta\dot{\rho} + 3H(\delta P + \delta\rho) + \nabla^2\left[\frac{(\bar{\rho} + \bar{P})}{a^2}\delta u + H\Pi^s\right] - 3(\bar{\rho} + \bar{P})\dot{\Psi} = 0.$$

All cpts of $\Lambda$CDM: $\Pi^S = 0 \Rightarrow \Phi = \Psi$.

Many other gauges:
- synchronous gauge: $E = 0 = F$
- Comoving gauge: $\delta u = 0 = F$
- Const.-density gauge: $\delta\rho = 0 = F$.

## 8.6 Adiabatic Modes

Def^n Gauge invariant variables (at linear order)

$$\mathcal{R} = \frac{A}{2} + H\,\delta u, \quad \zeta = \frac{A}{2} - H\frac{\delta\rho}{\dot{\bar{\rho}}}$$

$\Rightarrow \Delta\mathcal{R} = \Delta\zeta = 0$

$\mathcal{R}/\zeta$ are curvature perturbation on comoving / const.-density hyperslices.

$\delta u = 0$ $\qquad$ $\delta\rho = 0$

$g_{ij} = a^2\delta_{ij}(1+2\mathcal{R})$ $\qquad$ $g_{ij} = a^2\delta_{ij}(1+2\zeta)$.

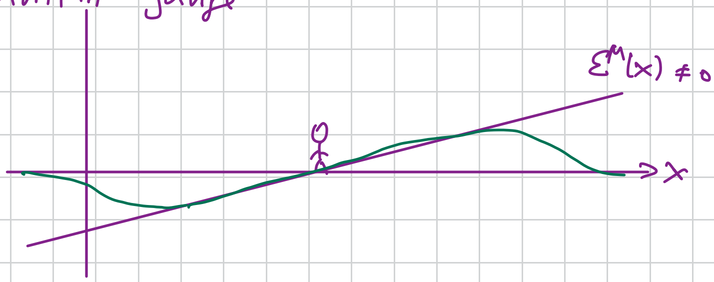
They are the same as $k \ll aH$ (superHubble).

$$\zeta - \mathcal{R} = \frac{M^2}{3a^2(\bar{\rho} + \bar{P})}k^2 A(\bar{k}, t) \propto \left(\frac{k}{aH}\right)^2 \quad \text{(Newt. gauge)}.$$

Thm Whatever the constituent of the Universe, at $k \ll aH$, there are two conserved scaler adiabatic modes, i.e. $\dot{\mathcal{R}} = 0$, one of which has $\mathcal{R} \neq 0$, and one tensor mode $\dot{D}_{ij} = 0$.

Pf: Fix the small gauge : Newtonian gauge

$\epsilon^\mu(x \to \infty) = 0$




Perform a large gauge transf$^n$ $\epsilon^\mu(x \to \infty) \neq 0$ on unperturbed FLRW

$$\epsilon_\mu = (\epsilon(t),\ a^2 \lambda x^i)$$

Use $\Delta h_{\mu\nu} = -\nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu$.

$$\Phi = -\dot{\epsilon}, \quad \Psi = H\epsilon - \lambda$$

$$\Rightarrow \quad \delta p = -\dot{\bar{p}}\epsilon, \quad \delta\rho = \dot{\bar{\rho}}\epsilon, \quad \delta u = \epsilon, \quad \Pi^S = 0$$

Still a sol$^n$, in fact FLRW.

Large diff

$$\epsilon(\underline{k}) = \int_{\underline{k}} \epsilon(t)\, e^{-i\underline{k}\cdot\underline{x}} \quad \propto \epsilon(t)\, \delta(\underline{k})$$

only supported at $\underline{k} = 0$. We want to extend this to finite $\underline{k} \neq 0$.

Look at the $ij$-cpt of EE

$$\partial_i \partial_j (\Phi - \Psi) \propto \Pi^S = 0.$$

$$\overset{FT}{\Rightarrow} \quad k_i k_j (\Phi - \Psi) = 0$$

For $\underline{k} \neq 0$, need to further acquire $\Phi = \Psi$. Thus gives us an ODE

$$\dot{\epsilon} + H\epsilon = \lambda \Rightarrow \epsilon(t) = \frac{\lambda}{a} \int^t a(t')\, dt'.$$

This solves all EE to $\mathcal{O}((k/aH)^0)$

$$\Phi = \Psi = \lambda(\underline{k})\left[-1 + \frac{H}{a}\int^t a(t')\,dt'\right]$$

$$\frac{\delta s}{\dot{s}} = \frac{\delta\rho}{\dot{\bar\rho}} = -\delta u = -\frac{\lambda}{a}\int^t a(t')\,dt'.$$

$$\Rightarrow \mathcal{R} = \frac{A}{2} + H\delta u = -\Psi + H\delta u = \lambda(\underline{k}). \quad \square$$

Our universe obeys this sol$^n$ on super Hubble scales to $\mathcal{O}(k|\eta_0)$.

# 9. Correlators and Initial Conditions

## 9.1 Intro to Observables

Deviations from homog. and isotropy in dist$^n$ of matter

$$\delta_m(t,\underline{x}) = \frac{\rho_m(t,\underline{x}) - \bar\rho_m(t)}{\bar\rho_m(t)} = \frac{\delta\rho_m(t,\underline{x})}{\bar\rho_m(t)}. \qquad \text{(dim-less)}$$

Similarly for other substances.

$\delta_m$ is measured from the dist$^n$ of galaxies. The galaxy number density

$$\delta_g = \frac{n_g(t,\underline{x}) - \bar n_g(t)}{\bar n_g(t)}.$$

On very large scales $\delta_g$ traces $\delta_m$, ie.

$$\delta_g(t,\underline{x}) = b(t)\,\delta_m(t,\underline{x}) + \mathcal{O}(\delta_m^2) + \mathcal{O}(\nabla^2\delta_m).$$

where $b(t)$ is called linear local bias.

### The CMB (Proxy for $\delta\rho_\gamma$)

This is a black body radiation in all direction $\hat{\underline{n}}$. This has small anistrophies

$$T_{CMB}(\hat{n}) = \bar{T}_0 \left[ 1 + \Theta(\hat{n}) \right]$$

$\bar{T}_0$: 2.7K $\qquad \Theta(\hat{n})$: $\sim 10^{-5}$

We'll see $\Theta(\hat{n}) \propto \delta_{p\gamma}(t, \underline{x})$.

## 9.2 Statistical Initial Conditions

We can't predict $\delta_m(\underline{x}, t)$ or $\delta_{p\gamma}(\underline{x}, t)$, but we can predict correlations

$$\langle \delta(t_1, \underline{x}_1)\, \delta(t_2, \underline{x}_2) \ldots \delta(t_n, \underline{x}_n) \rangle .$$

Here $\langle \ldots \rangle$ means an average. Originally, this is a quantum expectation value, in the state $|\Omega\rangle \in \mathcal{H}$, where $|\Omega\rangle$ is the state of universe. After inflation, $\langle \ldots \rangle$ is well approximated by a classical average.

We'll mostly work in F-space and at equal time.

$$\left\langle \prod_{a=1}^{n} \delta(t, \underline{x}_a) \right\rangle = \int \prod_{a=1}^{n} d\underline{k}_a \; e^{i \sum_{a=1}^{n} \underline{k}_a \cdot \underline{x}_a} \left\langle \prod_{b=1}^{n} \delta(t, \underline{x}_b) \right\rangle$$

FLRW background and laws of nature are inv. under rot$^n$ $R: \underline{x} \longrightarrow R\underline{x}$ and trans$^n$ $\underline{x} \longrightarrow \underline{x} + \underline{a}$ for $\underline{a} \in \mathbb{R}^3$. We expect

$$\left\langle \prod_{a=1}^{n} \delta(t_a, R\underline{x}_a + \underline{a}) \right\rangle = \left\langle \prod_{b} \delta(t_b, \underline{x}_b) \right\rangle$$

This is called <u>statistical homog. and isotropy</u>.

Infinitesimally,

$$\sum_{a=1}^{n} \frac{\partial}{\partial \underline{x}_a} \langle \delta(\underline{x}_1) \ldots \delta(\underline{x}_n) \rangle = 0 .$$

$$\sum_{a=1}^{n} \left( x^i_a \frac{\partial}{\partial x^j_a} - x^j_a \frac{\partial}{\partial x^i_a} \right) \langle \delta(\underline{x}_1) \ldots \delta(\underline{x}_n) \rangle = 0 .$$

In F-space,

$$\sum_a \underline{k}_a \langle \delta(\underline{k}_1) \dots \delta(\underline{k}_n) \rangle = 0$$

$$\dots$$

Solved by

$$\langle \prod_a \delta_a(\underline{k}_a) \rangle = (2\pi)^3 \delta_D\left(\sum_a \underline{k}_a\right) B_n(t_1, \dots, t_n; \underline{k}_b \cdot \underline{k}_{b'}, \underline{k}_b \times \underline{k}_{b'} \cdot \underline{k}_{b''}).$$

Dirac delta

## 9.3 Gaussian Random Fields

Let's warm up with a Gaussian r.v. $X$. The PDF is (assuming $\langle X \rangle = 0$, o/w $X \to X - \langle X \rangle$)

$$P(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{X^2}{2\sigma^2}\right).$$

It is fully specified by 2-point $f^n$

$$\langle X^{2n+1} \rangle = 0, \quad \langle X^{2n} \rangle = \sigma^{2n} (2n-1)!!$$

A Gaussian random field $\delta(x)$ for any set of points $(\underline{x}_1, \dots, \underline{x}_N)$ has a PDF

$$P_{DF}(\delta(\underline{x}_1), \dots, \delta(\underline{x}_N)) = \frac{1}{\sqrt{(2\pi)^N \det C}} \exp\left(-\frac{1}{2} \delta_i C_{ij}^{-1} \delta_j\right)$$

Fully specified by 2-point correlation $f^n$

$$\langle \delta(\underline{x}_1) \delta(\underline{x}_2) \rangle = \xi(\underline{x}_1, \underline{x}_2) = \xi(\underbrace{|\underline{x}_1 - \underline{x}_2|}_{r}).$$

In F-space,

$$\langle \delta(\underline{k}_1) \delta(\underline{k}_2) \rangle = (2\pi)^3 \delta(\underline{k}_1 + \underline{k}_2) P(|\underline{k}_1|)$$

where $P(k)$ is the power spectrum.

$$\xi(r) = \int_{\underline{k}_1} \int_{\underline{k}_2} \delta_D(\underline{k}_1 + \underline{k}_2) P(k_1) e^{i(\underline{k}_1 \cdot \underline{x} + \underline{k}_2 \cdot (\underline{x} + \underline{r}))}$$

$$\int_{\underline{k}} = \int \frac{d^3k}{(2\pi)^3}$$

$$= \int \frac{dk\, k^2}{(2\pi)^3}\, d(\cos\theta)\, d\phi\, P(k)\, e^{i\Omega k\cos\theta}$$

$$= \int \frac{dk\, k^2}{2\pi^2}\, \frac{\sin(k\Omega)}{ik}\, P(k).$$

Note that $\langle \delta(\underline{k})\, \delta(\underline{k}') \rangle = 0$ for $\underline{k} \neq \underline{k}'$.

## 9.4 Initial Conditions of the universe

Remarkably, pert$^n$ are adiabatic, ie. all prop. to $\mathcal{R}(\underline{k})$.

$$\underbrace{\delta_a(t, \underline{k})}_{\text{random field}} = \underbrace{F_a(k, t)}_{\text{deterministic } f^n \text{, not r.v.}}\, \underbrace{\mathcal{R}(\underline{k})}_{\text{random field.}} + \mathcal{O}(\mathcal{R}^2).$$

for some transfer $f^n$ $F(k, t)$. $\mathcal{R}$ determines all statistics to linear order

$$\left\langle \prod_{a=1}^{n} \delta(t_a, \underline{k}_a) \right\rangle = \left[ \prod_{b=1}^{n} F(t_b, k_b) \right] \left\langle \prod_{c=1}^{n} \mathcal{R}(\underline{k}_c) \right\rangle + \mathcal{O}(\mathcal{R}^2).$$

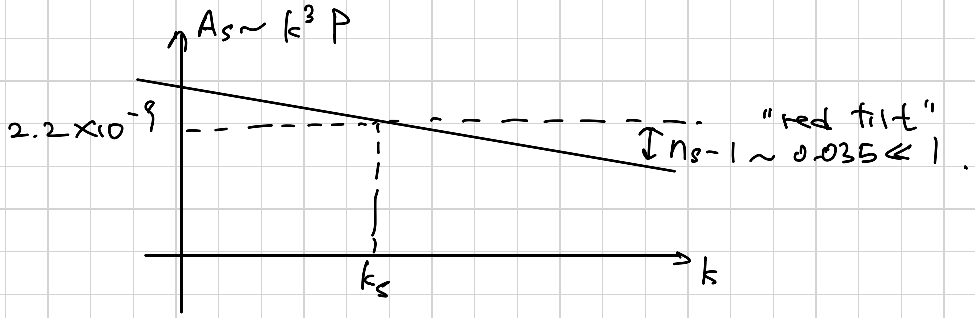
We observe $\mathcal{R}$ to be a Gaussian random field with power spectrum

$$P_{\mathcal{R}}(k) = \frac{A_s(k)}{k^3}$$

where

$$A_s(k) = A_s(k_*) \left(\frac{k}{k_*}\right)^{n_s - 1}.$$

Where at
- $k_* = 0.005\ \text{Mpc}^{-1}$ is an arbitrary pivot scale
- $A_s(k_*) \approx 2.2 \times 10^{-9} \ll 1$ is amplitude of primordial curvature pert.
- $n_s \simeq 0.965 \pm 0.005$ is spectral tilt



People looked for $\langle R(k_1)\, R(k_2)\, R(k_3)\rangle$. This is not seen.
non-Gaussianity
$R$ is Gaussian to 1 part in $10^4$.

## 10. Newtonian Perturbation Theory

We'll study structure formation neglecting relativity in the linear regime.

$$\ddot{\delta}_m = (\underbrace{\text{grav. attraction}}_{\text{collapse}} - \underbrace{\text{pressure} - \text{expansion}}_{\text{stability / waves}})\, \delta_m .$$

### 10.1 Continuity and Euler equation

The non-rel cty eqn and Euler eqn in physical coords $\underline{r}$ are

$$\dot{\rho} + \nabla_r (\rho \underline{u}) = 0 .$$

$$\dot{\underline{u}} + \underline{u} \cdot \nabla_r \underline{u} = -\frac{1}{\rho} \nabla_r p - \nabla_r \Phi$$

The Poisson eqn

$$\nabla_r^2 \Phi = 4\pi G_N \rho = \frac{1}{2M_p^2} \rho .$$

For $|r| \ll 1/H$, we can use these eqn by accounting for expansion of universe

$$\underbrace{\underline{r}(t)}_{\text{physical}} = a(t)\, \underbrace{\underline{x}(t)}_{\text{comoving}}$$

and hence $\nabla_r = \frac{1}{a} \nabla_x$ and write $\nabla_x = \nabla$.

Derivatives transform as

$$\left(\frac{\partial}{\partial t}\right)_{\underline{r}} = \left(\frac{\partial}{\partial t}\right)_{\underline{x}} + \left(\frac{\partial \underline{x}}{\partial t}\right)_{\underline{r}} \cdot \nabla = \left(\frac{\partial}{\partial t}\right)_{\underline{x}} - H \underline{x} \cdot \nabla$$

hence $\partial t = \partial t|_{\underline{x}}$.

We focus on pert$^n$.

$$\rho(x,t) = \bar{\rho}(t)\,(1+\delta(x,t)).$$
$$p(x,t) = p(t) + \delta p(x,t).$$
$$\underline{u} = Ha\underline{x} + \underline{v}$$
$$\Phi = \bar{\Phi} + \phi.$$

with $\underline{v}$ the peculiar velocity.

The linearised eqn become

$$(\partial_t - H\underline{x}\cdot\nabla)(\bar{\rho}(1+\delta)) + \frac{1}{a}\nabla\left(\bar{\rho}(1+\delta)\,(Ha\underline{u} + \underline{v})\right) = 0$$

$$\Rightarrow (\partial_t - H\underline{x}\cdot\nabla)(\bar{\rho}\delta) + \frac{1}{a}\nabla\left(\bar{\rho}\,\delta a H\underline{x} + \bar{\rho}\underline{v}\right) = 0.$$

$$\Rightarrow \dot{\bar{\rho}}\delta + \bar{\rho}(\partial_t - H\underline{x}\cdot\nabla)\delta + \frac{1}{a}3\bar{\rho}\delta a H + \frac{1}{a}\bar{\rho}\nabla\cdot\underline{v} = 0$$

Using $\dot{\bar{\rho}} + 3H\bar{\rho} = 0$ and $\nabla\cdot\underline{x} = 3$.

$$\Rightarrow \quad \dot{\delta} = -\frac{1}{a}\nabla\cdot\underline{v} \qquad (*).$$

The linearised Euler

$$(\partial_t - Ha\underline{x}\cdot\nabla)(Ha\underline{x} + \underline{v}) + (Ha\underline{x} + \underline{v})\cdot\frac{\nabla}{a}(Ha\underline{x} + \underline{v}) = -\frac{\nabla p}{a\bar{\rho}(1+\delta)} - \frac{\nabla\phi}{a}.$$

$$\Rightarrow \quad \dot{\underline{v}} + H\underline{v} = -\frac{1}{a\bar{\rho}}\nabla\delta p - \frac{\nabla\phi}{a} \qquad (\ddagger).$$

Finally, we have

$$\nabla^2\phi = 4\pi G\,\bar{\rho}\,\delta a^2, \quad \partial_i v_i = \theta.$$

We SVT-decompose $(\ddagger)$:

$$\partial_i(\ddagger) \Rightarrow \quad \dot{\theta} + H\theta = -\frac{\nabla^2\delta p}{a\bar{\rho}} - \frac{\nabla^2\phi}{a}.$$

Comment: the eqns are the Newtonian limit of EG.

For example,

$$-\frac{1}{2M_p^2}\left[\delta\rho - \delta p - \nabla^2\pi^S\right] = H\dot{\Phi} + \left(4H^2 + 2\frac{\ddot{a}}{a}\right)\Phi + \ddot{\Psi} + 6H\dot{\Psi}$$

For subhubble scales $k \gg aH$, $\left(\frac{a}{k} \ll \frac{1}{H}\right)$.

$$\Rightarrow \frac{\nabla^2 \bar{\Phi}}{a^2} = \frac{1}{2M_p^2} \underbrace{\left(\frac{\delta\rho}{\bar{\rho}}\right)}_{=\delta} \bar{\rho}\, a^2 \qquad \left(\frac{1}{M_p^2} = 8\pi G\right)$$

and $\nabla_\mu T^{\mu i} = 0$

$$\Rightarrow \delta p + \nabla^2 \Pi^S + \partial_0\left[(\bar{\rho} + \bar{p})\,\delta u\right] + 3H(\bar{\rho} + \bar{p})\,\delta u + (\bar{\rho} + \bar{p})\Phi = 0$$

$\left(a\partial^2 \delta u = a\partial_i v_i = \theta\right)$ $\Rightarrow$ $\bar{\rho}\,\delta\dot{u} + \bar{\rho}\bar{\Phi} = 0$

$$\Rightarrow \delta\dot{u} + \bar{\Phi} = 0.$$

Using Poisson to remove $\phi$ and $\theta = -a\dot{\delta}$ into Euler we get

$$\ddot{\delta} + 2H\dot{\delta} - \frac{1}{a^2\bar{\rho}} \nabla \delta p - 4\pi G \delta = 0$$

For barotropic fluid $p = p(\rho)$

$$\Rightarrow \delta p = \frac{\partial p}{\partial \rho}\,\delta\rho = c_s^2\, \bar{\rho}\, \delta.$$

with $c_s$ speed of sound. In F-space

$$\ddot{\delta} + 2H\dot{\delta} + \left(\frac{c_s^2}{a^2} k^2 - 4\pi G \bar{\rho}\right) \delta = 0.$$

Let $k_J$ be the Jeans wavenumber

$$k_J \equiv \frac{\sqrt{4\pi G a^2 \bar{\rho}}}{c_s}$$

For $k > k_J$, damped oscillations (short distances)

For $k < k_J$, gravitational collapse, unstable growth.

## Matter Dom

$\delta$ can be dark matter or baryons. For $z < 1100$ (recombination $e^- + p^+ \to H$). Here $\delta \to \delta_m = \delta_{dm} + \delta_b$ and $p = 0$ $c_s^2 = 0$.

Also $3H^2 M_p^2 = \bar{\rho}_m = \frac{1}{a^3} \Rightarrow H = \frac{2}{3t}$, $a = (t/t_0)^{2/3}$.

So
$$\ddot{\delta}_m + \frac{4}{3t}\dot{\delta}_m - \frac{2}{3t^2}\delta_m = 0$$

Ansatz $\delta_m = t^c \Rightarrow c = 2/3$ or $c = -1$.

The growing sol$^n$ is $\delta_m \propto t^{2/3} \propto a$.

This is good approx for DM $3300 > z > 0.3$.

Radiation Dom

Again, for DM, $c_s = 0$. $(z > 3300)$. We focus on $\delta_{DM}$.

- Expansion rate is faster $a = (t/t_0)^{1/2}$ by $3H^2 M_p^2 = \rho_r \propto a^{-4}$.
- Poisson eqn

$$\nabla^2 \phi = 4\pi G a^2 (\bar{\rho}_{dm}\delta_{dm} + \bar{\rho}_b \cancelto{0}{\delta_b} + \bar{\rho}_r \cancelto{0}{\delta_r} + \bar{\rho}_\nu \cancelto{0}{\delta_\nu})$$

So
$$\ddot{\delta}_{dm} + 2H\dot{\delta}_{dm} - 4\pi G \bar{\rho}_{dm}\delta_{dm} = 0$$

Approx. sol$^n$: expect $\partial_t \sim H$, so

$$\ddot{\delta}_{dm} \sim H^2 \delta_{dm} \sim H\dot{\delta}_{dm} \gg \frac{\bar{\rho}_{dm}}{M_p^2}\delta_{dm}$$

$$\Rightarrow \ddot{\delta}_{dm} + \frac{1}{t}\dot{\delta}_{dm} = 0 \qquad \left(H = \frac{\dot{a}}{a} = \frac{1}{2t}\right)$$

Two sol$^n$: $\dot{\delta}_{dm} = 0 \Rightarrow \delta_{dm} = \text{const}$ and a slowly growing sol$^n$ $\delta_{dm} \propto \ln t \propto \ln a$.

Dark energy

$z < 0.3$. We neglect rad.

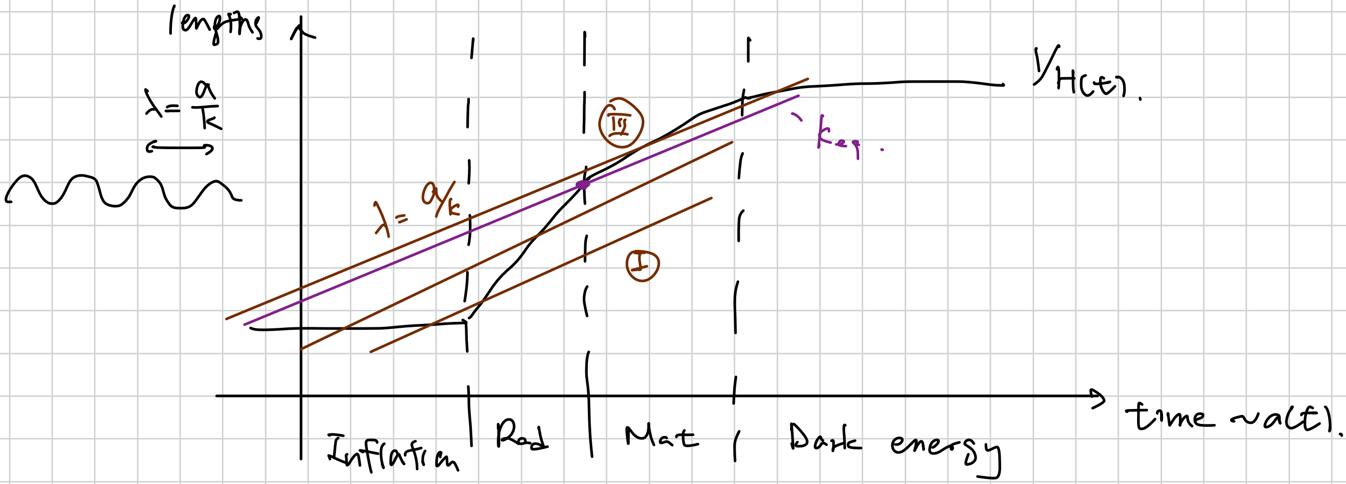
$$H^2 = \frac{8\pi G}{3}\bar{\rho}_\Lambda \approx \text{const.}$$

$$\Rightarrow \ddot{\delta}_m + 2H\dot{\delta}_m = 0$$

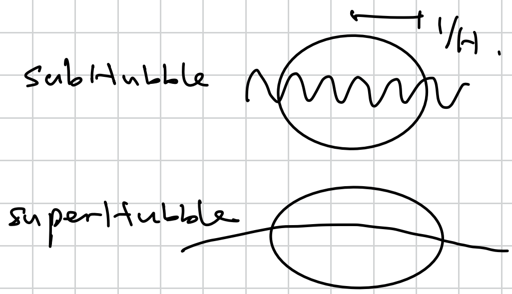
$$\Rightarrow \delta_m = \text{const.} + \frac{1}{a^2}$$

So structure stop growing.

# 11. Large Scale Structures




$$\left(\frac{a/k}{1/H}\right)^{\cdot} \propto (aH)^{\cdot} = \ddot{a}$$




We'll work in Newtonian gauge and conformal $a\,d\tau = dt$

$$ds^2 = a^2(-d\tau^2 + dx^2).$$

also $\quad H = \dot{a}/a, \quad \mathcal{H} = a'/a = aH = \dot{a}$

For single fluid, $\delta p = w\,\delta\rho$. (for $p = w\rho$), $\Pi^S = 0$.

$ij$-eqn $\Rightarrow \Phi = \Psi$

$0i$-eqn $\Rightarrow$ $\delta u$ in terms of $\Phi$.

$\Rightarrow$ Solve $EE_{00}$ for $\delta\rho$, and $EE_{ii}$ to find

$$\Phi'' + 3(1+w)\mathcal{H}\Phi' + k^2 w\Phi = 0 \qquad (*)$$

## 11.1 SuperHubble limit and Initial Conditions

Recall the adiabatic mode $k \ll aH$.

$$\Phi = \Psi = \mathcal{R}\left[-1 + \frac{H}{a}\int^t a(t')\,dt'\right]$$

$$\frac{\delta s}{\dot{s}} = -\delta u = -\frac{\mathcal{R}}{a}\int^t a(t')\,dt'.$$

This is always true. In particular, for $S = \rho_a$ for $a = \gamma, \nu, b$, dark matter, ...

$$\frac{\delta\rho_a}{\dot{\rho}_a} = -\frac{\delta\rho_a}{3H(\bar{\rho}_a + \bar{p}_a)} = -\frac{\delta\rho_a}{\bar{\rho}_a}\frac{1}{3H(1+w_a)}$$

$$\Rightarrow \frac{\delta_a}{1+w_a} = \frac{\delta_b}{1+w_b} \qquad \forall\, a, b$$

For example, $\delta_m = \frac{3}{4}\delta_r$.

For a single-cpt universe with e.o.s. param $w$, we can solve for $a(t)$.

$$\mathcal{R} = -\frac{5+3w}{3+3w}\Phi$$

In rad., $\mathcal{R} = -\frac{3}{2}\Phi_{RD}$. In mat., $\mathcal{R} = -\frac{5}{3}\Phi_{MD}$

Since $\mathcal{R} =$ const., we know

$$\mathcal{R} = -\frac{3}{2}\Phi_{RD} = -\frac{5}{3}\Phi_{MD}.$$

Also

$$H\frac{\delta\rho_a}{\dot{\bar{\rho}}_a} = -\frac{\delta_a}{3(1+w_a)} = \Phi\frac{2}{3(1+w)}.$$

The dominating substance has $w_a = w \Rightarrow \delta_{dom} = -2\Phi$.

E.g. In rad., $\delta_r = -2\Phi_{RD} = \frac{4}{3}\delta_m$

In mat., $\delta_m = -2\Phi_{MD} = \frac{3}{4}\delta_r$.

## 11.2 Potential Evolution

We solve $(*)$ for $\Phi$, then find sol$^n$

Rad-dom: $\mathcal{H} = 1/\tau$, $a = (t/t_0)^{1/2} = \tau/\tau_0$.

$$(*) \Rightarrow \quad \Phi'' + \frac{4}{\tau}\Phi' + \frac{k^2}{3}\Phi = 0$$

Sol$^n$ are sph. Bessel and Newmann fcts

$$\Phi(k) = A(k) \frac{j_1(k\tau/\sqrt{3})}{k\tau/\sqrt{3}} + B(k) \frac{n_1(k\tau/\sqrt{3})}{k\tau/\sqrt{3}}.$$

where

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \xrightarrow{x\to 0} \frac{x}{3} + O(x^2).$$

$$n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \xrightarrow{x\to 0} -\frac{1}{x^2} + O(1).$$

So matching I.C., require $B=0$, $A = -3\Phi(k) = -2R(k)$.

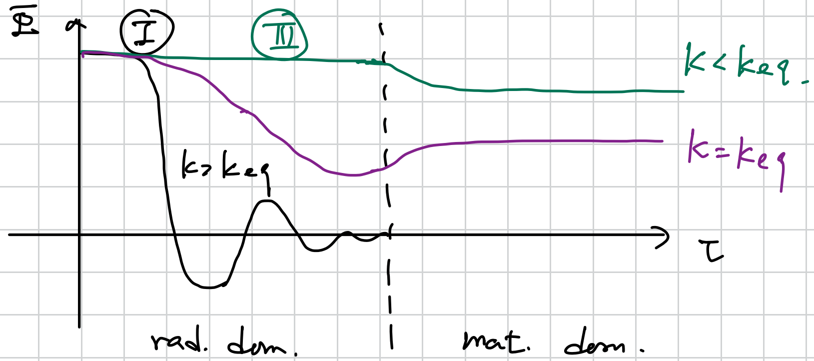
So we found the sol$^n$

$$\Phi(\tau, k) = -2R(k) \frac{\sin(k\tau/\sqrt{3}) - k\tau\cos(k\tau/\sqrt{3})}{(k\tau/\sqrt{3})^3}.$$

For $k\tau \gg 1$ (Same as $k \gg aH = \mathcal{H} = 1/\tau$).

$$\Phi(\tau, k) = 6R(k) \frac{\cos(k\tau/\sqrt{3})}{(k\tau)^2}.$$

const on superHubble scales, then oscillating with a decaying envelop.




Mat. dom. : $\mathcal{H} = aH = a'/a = 2/\tau$, $a = (t/t_0)^{2/3} = (\tau/\tau_0)^2$.

Using $w=0$,

$$(*) \Rightarrow \Phi'' + \frac{6}{\tau}\Phi' = 0.$$

The non-decaying sol$^n$ is $\Phi = $ const. (all scales).

Let $k_{eq} = (aH)_{eq} = 10^{-2}\ \mathrm{Mpc}^{-1}$ be wavenumber entering Hubble at mat-rad. equality $\bar{\rho}_m = \bar{\rho}_r$.

## 11.3 Radiation Evolution

From $EE_{00}$ or $EE_{ii}$, we can find dominating $\delta$ as $f^n$ of $\Phi$.

Rad. dom: $EE_{00} \Rightarrow \delta_r = -\frac{2}{3}\frac{k^2\Phi}{\mathcal{H}^2} - \frac{2\Phi'}{\mathcal{H}} - 2\Phi$.

$$\Rightarrow \delta_r = -\frac{2}{3}(k\tau)^2\Phi - 2\tau\Phi - 2\Phi. \quad (\mathcal{H} = 1/\tau).$$

Super-Hubble: $\delta_r = -2\Phi$

Sub-Hubble ($k\tau \gg 1$)

$$\Rightarrow \delta_r \approx -\frac{2}{3}(k\tau)^2\Phi = -4\mathcal{R}(k)\cos(k\tau/\sqrt{3}).$$

So $\delta_r$ oscillates around $\delta_r = 0$.

Mat. dom.: We must return to $\nabla_\mu T^{\mu\nu} = 0$, i.e. cty and Euler.

$$\begin{cases} \delta\dot{u}_r - H\,\delta u_r + \frac{1}{4}\delta_r + \Phi = 0. & (\text{used } p = \frac{1}{3}\rho) \\ \dot{\delta}_r - \frac{4}{3}\frac{k^2}{a^2}\delta u - 4\dot{\Phi} = 0 \end{cases}$$

Using $\dot{\Phi} = 0$ to solve $\delta u = \frac{a}{k^2}\frac{3}{4}\delta_r'$.

$$\Rightarrow \delta_r'' - \frac{1}{3}\nabla^2\delta_r = \frac{4}{3}\nabla^2\Phi = \text{const}$$

Equivalently,
$$\left(\frac{\delta_r}{4} + \Phi\right)'' + \frac{k^2}{3}\left(\frac{\delta_r}{4} + \Phi\right) = 0.$$

So $\delta_r$ oscillates around $\delta_r \sim -4\Phi_{MD}(k)$.

These acoustic oscillations are imprinted in CMB and Baryon Acoustic Oscillation (BAO). in large scale structures.

## 11.4 Matter Evolution

We ignore Baryons.

Subhubble in mat. and rad. dom.: Cty and Euler eqn $\nabla_\mu T^{\mu\nu}=0$

$$\begin{cases} \dot{\delta}_m + \frac{\nabla^2}{a^2}\delta u - 3\dot{\Phi} = 0. \\ \delta \dot{u} + \Phi = 0 \end{cases} \qquad (p=0).$$

Solve for $\delta u$:

$$\delta u_m = -\frac{a^2}{k^2}(3\dot{\Phi} - \dot{\delta}_m)$$

$$\Rightarrow \quad \delta'' + \mathcal{H}\delta' = \nabla^2\Phi + 3(\Phi'' + \mathcal{H}\Phi').$$

On Subhubble scale, $\Phi''$. $\Phi'\mathcal{H} \sim \mathcal{H}^2\Phi \ll k^2\Phi$. Using the Poisson eqn,

$$\underbrace{\cancel{\delta_r}}_{\text{neglect}} + \delta_m = \frac{1}{4\pi G}\frac{\nabla^2}{a^2}\Phi.$$

We find the Meszaros eqn

$$\delta_m'' + \mathcal{H}\delta_m' - 4\pi G\bar{\rho}_m a^2 \delta_m = 0$$

From Fried,

$$\mathcal{H}^2 = \frac{H_0 \Omega_{m,0}}{\Omega_{r,0}}\left(\frac{1}{y} + \frac{1}{y^2}\right).$$

where $y = a/a_{eq}$, $a_{eq} = \Omega_{r,0}/\Omega_{m,0}$, and

$$\frac{\rho_r}{3H_0^2M_p^2} = \frac{\Omega_{r,0}}{a_{eq}^4}. \quad \frac{\rho_m}{3H_0^2M_p^2} = \frac{\Omega_{m,0}}{a_{eq}^3}.$$

$$\Rightarrow \quad \frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} - \frac{3}{2y(1+y)}\delta_m = 0$$

Two sol$^n$: $\delta_m \propto 2+3y$. $\delta_m \propto (2+3y)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 6\sqrt{1+y}$.

Rad. dom. $y \ll 1$, $\delta_m \approx \ln y \propto \ln a$

Mat dom. $y \gg 1$, $\delta_m \propto a$.

## 11.5 Matter Power Spectrum.

Recall $\langle \delta_m(\underline{k},t)\, \delta_m(\underline{k}',t)\rangle = (2\pi)^3 \delta_D^{(3)}(\underline{k}+\underline{k}')\, P_\delta(k,t)$.

The initial condition are fixed by the Gaussian random field $\mathcal{R}(k)$ with

$$P_{\mathcal{R}}(k) = \frac{A}{k^{3-(n_s-1)}}$$

scale invariant

with $n_s - 1 = -0.03 \ll 1$.

Two regimes :

① $\underline{k < k_{eq}}$ : large scales, enter during mat. dom.

$\Phi \approx$ const. in mat. dom. $= -\frac{3}{5}\mathcal{R}(k)$.

When $\Phi(\underline{k})$ is subhorizon, use Poisson eqn

$$-k^2\, \Phi_{MD}(\underline{k}) = 4\pi G\, a^2 \bar{\rho}_m\, \delta_m .$$

$$\Rightarrow \quad \delta_m(\underline{k}) = -\frac{k^2}{4\pi G a^2 \bar{\rho}_m}\, \Phi_{MD}(\underline{k}) .$$

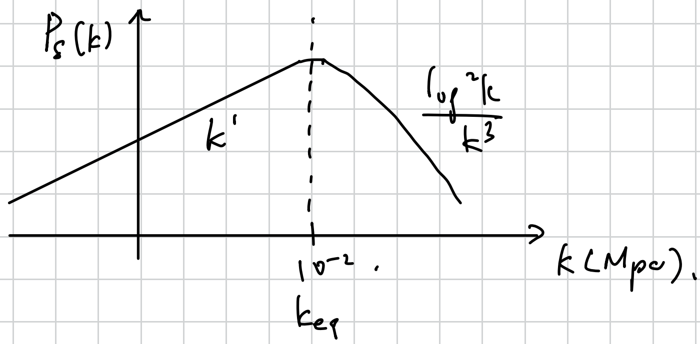
Using $\bar{\rho}_m = \frac{\bar{\rho}_{m,0}}{a^3} = 3H_0^2 M_p^2 \frac{\Omega_{m,0}}{a^3}$

$$\Rightarrow \quad \frac{\langle \delta_m(\underline{k},t)\, \delta_m(\underline{k}',t)\rangle}{(2\pi)^3\, \delta_D^{(3)}(\underline{k}+\underline{k}')} = \left(\frac{k^2}{(4\pi G) 3H_0^2 M_p^2 \cdot \Omega_{m,0}}\right)^2 . a^2 . \left(\frac{3}{5}\right)^2 P_{\mathcal{R}}(k) .$$

$$= \left(\frac{2k^2}{5H_0^2\Omega_{m,0}}\right)^2 a^2\, P_{\mathcal{R}}(\underline{k})$$

<u>Comment</u>:

- Valid to linear order for any $P_{\mathcal{R}}(k)$. For $P_{\mathcal{R}} \propto \frac{1}{k^{3-(n_s-1)}}$, we get $P_\delta \sim k^{n_s} \sim k^1$.



- Note $P_\delta(\underline{k},t) \sim a(t)^2 \cdot k^1$.

② $k > k_{eq}$ : short scales, entering during rad. dom. when $\Phi$ decays as $\frac{1}{\tau^2} \propto \frac{1}{a^2}$, so by time of mat. dom. when $\Phi$ becomes const., these modes has decayed by (using $H^2 \propto \rho_r \propto \frac{1}{a^4}$).

$$\left(\frac{a(k)}{a_{eq}}\right)^2 \overset{k=a(k)H(k)}{=} \left(\frac{a_{eq}H_{eq}}{a(k)H(k)}\right)^2 = \left(\frac{k_{eq}}{k}\right)^2$$

Extra enhancement $\log(a_{eq}/a(k)) = \log(k/k_{eq})$. So the power spectrum is

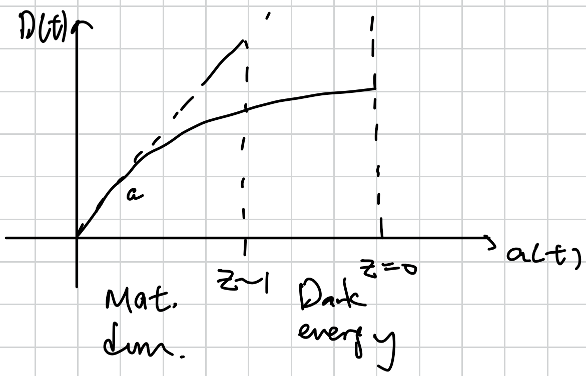
$$P_\delta = P_{\mathcal{R}}(k)\,(k_{eq}/k)^4 \log^2(k/k_{eq})\left(\frac{2k^2}{5H_0^2\Omega_{m,0}}\right)^2 a^2$$

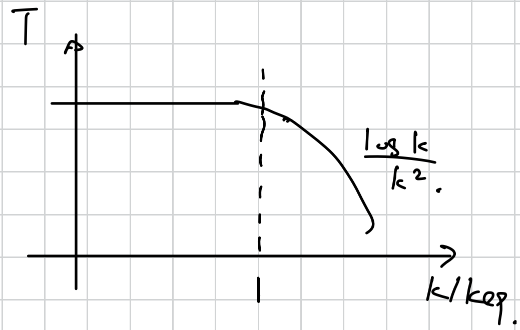
Generally.

$$P_\delta(k,t) = \left(\frac{2k^2}{5H_0^2\Omega_{m,0}}\right)^2 D^2(t)\; T(k/k_{eq})\; \underbrace{P_{\mathcal{R}}(k)}_{\frac{A_s}{k^3}}.$$

where $D$ is growth $f^n$, $T$ is transfer $f^n$.

Approximately,



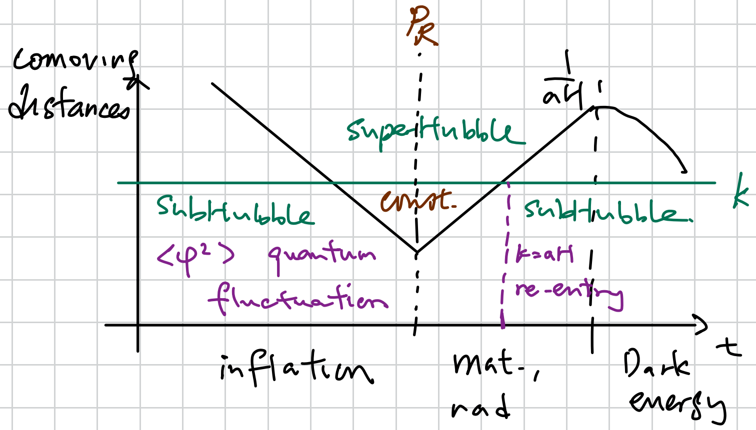

Comments : • higher orders, e.g.

$$\langle \delta(\underline{k}_1)\,\delta(\underline{k}_2)\,\delta(\underline{k}_3)\rangle \propto \langle \mathcal{R}^3\rangle + \text{non-linear structure formation}$$

primordial non-Gaussianity

- Non-linear evolution from $k > 1\,\text{Mpc}^{-1}$.

## 12. Primordial Perturbations




Now we promote pert.$^n$ to operators on Hilbert space.

$$g_{\mu\nu}(x,t) = \bar{g}_{\mu\nu}(t) + \hat{h}_{\mu\nu}(x,t).$$

$$\phi(x,t) = \underbrace{\bar{\phi}(t)}_{\text{classical}} + \underbrace{\hat{\varphi}(\underline{x},t)}_{\text{quantum}}.$$

For now, neglect $\hat{h}_{\mu\nu} \to 0$ and keep $\hat{\varphi}$.

### 12.2 Quantum Fluctuations

Start with action

$$S = \int d^4x \sqrt{-g}\, \frac{M^2}{2} R + \underbrace{\int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right)}_{\text{compute } T_{\mu\nu} \text{ and takes form of a perfect fluid}}.$$

Expand in $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\phi = \bar{\phi} + \varphi$.

generate non-gaussianity.

$$\Rightarrow S = -\int d^4x \sqrt{-g}\, \frac{1}{2} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V'' \varphi^2 + O(\varphi^3) \right).$$

$$= \int dt\, d^3x\, a^3 \cdot \frac{1}{2} \left( \dot{\varphi}^2 - \frac{\partial_i \varphi \partial_i \varphi}{a^2} - \cancelto{0}{V'' \varphi^2} \right).$$

slow roll param.

Since $V'' \varphi^2 = \eta_V H^2 \varphi^2 \ll H^2 \varphi^2$, so $\varphi$ is a massless scaler field in de Sitter.

In $F$-space,
$$\varphi(\underline{x}) = \int_{\underline{k}} \varphi(\underline{k})\, e^{i\underline{k}\cdot\underline{x}}, \qquad \int_{\underline{k}} = \int \frac{d^3k}{(2\pi)^3}$$

then

$$S = \int_{\underline{k}} dt\; a^3 \cdot \frac{1}{2}\left( \dot\varphi(\underline{k})\,\dot\varphi(-\underline{k}) - \frac{k^2}{a^2}\varphi(\underline{k})\,\varphi(-\underline{k})\right).$$

Each $\underline{k}$ is an indpt. QHO. We quantize as usual

$$\varphi(\underline{k},t) = f_k(\tau)\, a_{\underline{k}} + f_k^*(\tau)\, a^\dagger_{-\underline{k}}$$

Note $\varphi(\underline{x}) \in \mathbb{R} \Rightarrow \varphi(\underline{k})^\dagger = \varphi(-\underline{k})$, with

$$[a_{\underline{k}},\, a^\dagger_{\underline{k}'}] = (2\pi)^3\, \delta_D^{(3)}(\underline{k}-\underline{k}'),$$

and $f_k(t)$ are called <u>mode functions</u>.

The $f$'s are determined by classical e.o.m.:

$$\Box_{FLRW} f = 0 \Rightarrow \ddot f_k + 3H\dot f_k + \frac{k^2}{a^2} f_k = 0.$$

Introduce conformal time $' = \partial_\tau$, and

$$(a\cdot f_k)'' + \left(k^2 - \frac{a''}{a}\right)(a f_k) = 0$$

In dS, $\frac{a''}{a} = \frac{2}{\tau^2}$. $\left(a = -\frac{1}{H\tau}\right)$.

So QHO with $t$-dependent freq.

Two indpt sol$^n$:

$$f_k = \alpha\,(1+ik\tau)\,e^{-ik\tau} + \beta(1-ik\tau)e^{ik\tau}.$$

To determine $\alpha, \beta$,

I $[\varphi, \pi] = i\delta_D^{(4)}$ and $a_{\underline{k}}|0\rangle = 0$. $\Rightarrow \alpha, \beta$.

(II) Notice at $\tau \to -\infty$, $k \gg \frac{a''}{a}$, we get Klein-Gordon for $af$. The modes $k\tau \gg 1$ are effectively in Mink., so we can quote QFT.

$$\varphi(x,t) = \int \frac{d^3 k_p}{(2\pi)^3} \frac{e^{i k_p \cdot x}}{\sqrt{2k_p}} \left( e^{-ik_p t} a_{\underline{k_p}} + e^{ik_p t} a^\dagger_{-\underline{k_p}} \right)$$

$$= \int_{\underline{k}} e^{-i\underline{k_p}\cdot x} \varphi(\underline{k_p}, t)$$

Match this with $af_k$ at some early time $|k\tau_*| \gg 1$.

$$(af_k)_{\tau=\tau_*} = \left.\frac{e^{-ik_p t}}{\sqrt{2k_p}}\right|_{t=t_*}$$

and

$$\partial_t (af_k)\Big|_{\tau=\tau_*} = \partial_t \left( \frac{e^{-ik_p t}}{\sqrt{2k_p}} \right)\Bigg|_{t=t_*}.$$

Solving for $\alpha$ and $\beta$,

$$\alpha = i e^{-ik\tau_*(1-Ht_*)} \frac{H}{\sqrt{2k^3}} \left( 1 + \frac{i}{k\tau_*} + \dots \right)$$

$$\beta = i e^{-ik\tau_*(1-Ht_*)} \frac{H}{\sqrt{2k^3}} \frac{1}{2(k\tau_*)^2}$$

Matching at $|k\tau_*| \gg 1$,

$$\lim_{|\tau_*| \to +\infty} |\alpha| = \frac{H}{\sqrt{2k^3}}, \qquad \lim_{\tau_* \to -\infty} \beta = 0.$$

up to small phase. This fixes

$$f_k(\tau) = \frac{H}{\sqrt{2k^3}} (1 + ik\tau) e^{-ik\tau}. \quad \left( \begin{matrix} \text{massless} \\ \text{dS mode fn} \end{matrix} \right).$$

For $|k\tau| \gg 1$, resembles Mink (subHubble)

For $|k\tau| \ll 1$, it stops oscillating and freezes out.

The Fock vacuum $a_{\underline{k}}|0\rangle = 0$ is called Bunch-Davies States.

We are interested in expectation values

$$\lim_{\tau\to 0} \langle 0| \varphi(\underline{k}_1, \tau) \dots \varphi(\underline{k}_n, \tau)|0\rangle$$

Clearly this vanishes for $n$ odd. For $n$ even, they are all fixed by $n=2$.

$$\langle \varphi(\underline{k},\tau)\, \varphi(\underline{k}',\tau)\rangle$$

$$= \langle (f_k a_{\underline{k}} + f_k^* a^\dagger_{-\underline{k}})(f_{k'} a_{\underline{k}'} + f_{k'}^* a^\dagger_{-\underline{k}'})\rangle$$

$$= f_k f_{k'}^* \,(2\pi)^3\, \delta_D(\underline{k}+\underline{k}')$$

$$= |f_k|^2 (2\pi)^3\, \delta_D(\underline{k}+\underline{k}').$$

$$\Rightarrow \quad P_\varphi = |f_k(\tau)|^2$$

$$\Rightarrow \quad P_\varphi(k,\tau) = \frac{H^2}{2k^3}\left[1+(k\tau)^2\right] \overset{|k\tau|\to 0}{=} \frac{H^2}{2k^3}.$$

Comments:

- Dirac delta preserves momentum ← homog.
- $P(\underline{k},\tau) = P(k,\tau)$ ← isotropy.
- $P(\lambda k, \tau/\lambda) = \lambda^{-3} P(k,\tau)$. In particular, $\tau\to 0$, then

$$P(\lambda k) = \lambda^{-3} P(k).$$

← dilation invar. of dS for $m=0$.

Indeed, $\langle \varphi(\underline{x})\,\varphi(0)\rangle = \int_{\underline{k}\underline{k}'} e^{i\underline{k}\underline{x}} \langle \varphi(\underline{k})\,\varphi(\underline{k}')\rangle$

$$= \frac{H^2}{2}\int_{\underline{k}} \frac{e^{i\underline{k}\underline{x}}}{k^3}$$

$$\sim \frac{H^2}{(2\pi)^2}\int_0^\infty d\tilde{k}\, \frac{\sin\tilde{k}}{\tilde{k}^2}.$$

Compare with Mink.,

equal time
$$\begin{cases} \langle \varphi(\underline{k})\, \varphi(\underline{k}')\rangle = (2\pi)^3\, \delta_D(\underline{k}+\underline{k}')\cdot \frac{1}{2k} & (\text{Mink}). \\ \langle \varphi(\underline{x})\, \varphi(\underline{0})\rangle = \int_{\underline{k}\underline{k}'} \langle \varphi(\underline{k})\, \varphi(\underline{k}')\rangle \sim \frac{1}{x^2} & (\text{Mink}). \end{cases}$$

## 12.5 Gauge Invariant Variable

What about $\mathcal{R}$? Recall

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g_{\mu\nu}}$$

$$= \partial_\mu\phi\, \partial_\nu\phi + g_{\mu\nu}\left(-\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - V(\phi)\right).$$

This is actually a perfect fluid

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + g_{\mu\nu}\, p.$$

with

$$\rho = -\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + V$$

$$p = -\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - V$$

$$u_\mu = \frac{\partial_\mu\phi}{\sqrt{-\partial_\rho\phi\,\partial^\rho\phi}}.$$

Hence, SVT decomposition.

$$u_i = \partial_i \delta u + \cancel{u_i^{V}}^{\,0} = -\frac{\partial_i\phi}{\sqrt{\partial_\rho\phi\partial^\rho\phi}}.$$

Expand around $\phi = \bar{\phi} + \varphi$ to find

$$\delta u = -\frac{\partial_i\varphi}{|\dot{\bar{\phi}}|}$$

Then we know

$$\mathcal{R} = \frac{A}{2} + H\,\delta u = \frac{A}{2} - \frac{H}{|\dot{\bar{\phi}}|}\varphi.$$

Our calculation of $\varphi$ was performed using $A=0$ (flat gauge).

$$\Rightarrow \mathcal{R} = -\frac{H\varphi}{|\dot{\bar\phi}|}$$

## 12.6 Primordial Power Spectrum

On superHubble scale, $\mathcal{R}$ = const. Since $\frac{H}{|\dot{\bar\phi}|}$ evolves, then $\varphi$ evolves. So we convert from $\varphi \to \mathcal{R}$ at $k = aH$.

drop $S_D$

$$\langle \mathcal{R}(\underline{k})\, \mathcal{R}(\underline{k}')\rangle' = P_{\mathcal{R}}(k) = \left(-\frac{H}{\dot{\bar\phi}}\right)^2 P_\varphi = \frac{H^2}{\dot{\bar\phi}^2}\frac{H^2}{2k^3}$$

Note 2nd Fried: $\quad 2M_p^2 \dot{H} = \dot{\bar\phi}$

$$\Rightarrow \frac{H^2}{\dot{\bar\phi}^2} = -\frac{H^2}{2\dot{H} M_p^2} = \frac{1}{2\varepsilon M_p^2}$$

$$\Rightarrow \langle \mathcal{R}(k)\, \mathcal{R}(k')\rangle = \frac{H^2}{4\varepsilon M_p^2}\cdot\frac{1}{k^3}$$

To match observations, we define

possible deviation

$$\Delta^2_{\mathcal{R}}(k) = \frac{k^3 P_{\mathcal{R}}(k)}{(2\pi)^2} = A_s\,(k/k_*)^{n_s-1}.$$

For CMB, the pivot scale is $k_* = 0.05\ \mathrm{Mpc}^{-1}$. Then CMB measures

$$A_s = (2.10 \pm 0.03)\cdot 10^{-9}$$

- $A_s \ll 1$, so pert$^n$ thy in $\mathcal{R}$ is good.
- This fixes $\left(\frac{H}{M_p}\right)^2\cdot\frac{1}{4\varepsilon}$ but <u>not</u> $\frac{H}{M_p}$ or $\varepsilon$ separately. Each is uncertain by $\sim 10^{50}$.

## 12.7 The Spectral Tilt

When do we evaluate $\frac{H^2}{\varepsilon}$? At horizon crossing $k=aH$.

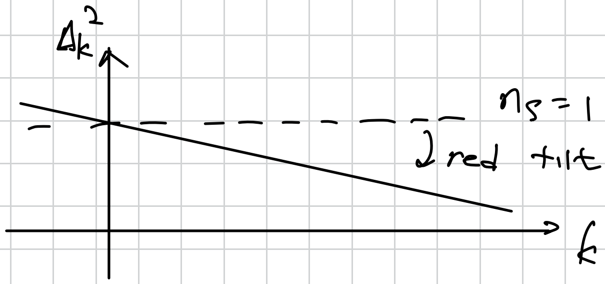
$$\Delta_R^2(k) = \frac{1}{8\pi^2}\left(\frac{H^2}{\varepsilon M_p^2}\right)_{aH=k}.$$

Since

$$n_s - 1 = \frac{d\ln \Delta_R^2(k)}{d\ln k}$$

Nomenclature:
$$n_s \begin{cases} <1 & \text{red tilt} \\ =1 & \text{scale inv} \\ >1 & \text{blue tilt.} \end{cases}$$

spectral tilt.




To predict $n_s$, use chain rule.

$$n_s - 1 = \frac{d\ln\Delta_R^2/dt}{d\ln k/dt}$$

Denom:
$$k=aH \Rightarrow \ln k = \ln a + \ln H$$
$$\Rightarrow \frac{d\ln k}{dt} = H + \frac{\dot H}{H} = H(1-\varepsilon) \simeq H + O(\varepsilon).$$

Num:
$$\frac{d\ln\Delta_R^2}{dt} = 2\frac{\dot H}{H} - \frac{\dot\varepsilon}{\varepsilon} = -2\varepsilon H - H\eta \propto O(\varepsilon,\eta).$$

Combining,
$$n_s - 1 = -2\varepsilon - \eta$$

Comments:
- $n_s - 1 \ll 1$ is prediction.
- Measured: $n_s = 0.9649 \pm 0.0042$ (68% CL).

In slow-roll, we can convert to

$$\varepsilon_V = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta_V = M_p^2 \frac{V''}{V}.$$

$$\Rightarrow \quad n_s - 1 \simeq -6\,\varepsilon_V + \eta_V^2.$$

Flatter potential $\Rightarrow$ closer to scale invariant.