Cosmology

1. The Expanding Universe

1.1 The Cosmological Principle (Copernicus)

Our position in the Universe — with respect to the largest scales — is in no sence preferred, i.e. our universe is spatially homogeneous and isotropic.

Isotropy means the same in all dimensions about a point P.

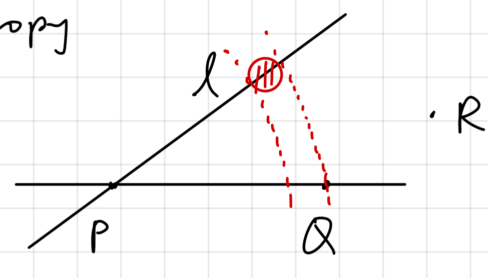
Homogeneous means space is the same about all points.

Homogeneity does not imply isotropy (e.g. crystals). But a homogeneous universe that is isotropic about a point P is also isotropic about all points.

Isotropy does not imply homogeneity (e.g. Ptolemaic view). But isotropy about theree non-collinear points does imply hom.

Take points P, Q, R. Consider line ℓ through P but not Q. Any inhom along ℓ become an isotropy viewed from Q. ↯

Space is hom except for ℓ_{PQ}, now view from R.

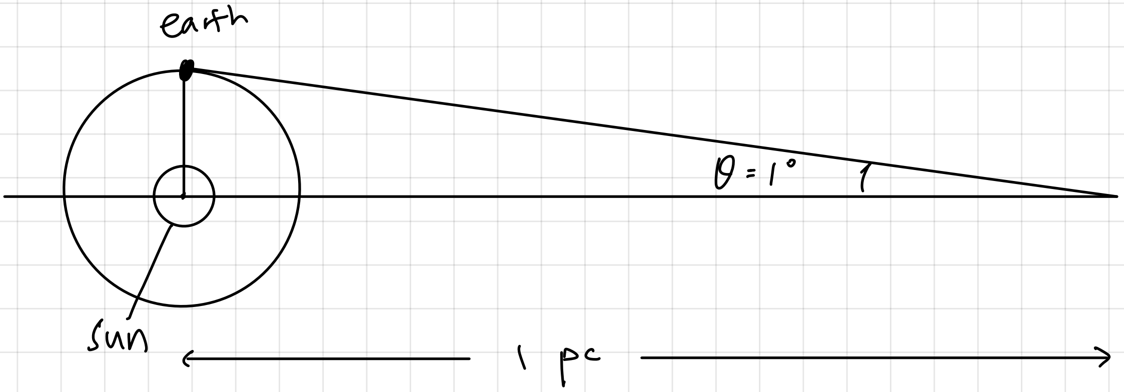


Astronomical units

Time : 1 year = 3.16×10^7 s

Distance : 1 lightyear = 9.5×10^{12} km.

Earth-sun 1 a.u = 1.5×10^8 km



1 pc (1 parallax of 1 arcsecond) = 326 lightyear = 3.1×10^{13} km

1 Mpc = 10^6 pc.

<u>Mass</u> : 1 solar mass = $1 M_\odot$ = 2×10^{30} kg.

1.2 Kinematics of an expanding universe

Hubble - Lemaitre Law

The recessional velocity of a galaxy $\underline{v}$ is proportional to its distance $r = |\underline{r}|$

$$\underline{v} = H_0 \underline{r} \tag{1.1}$$

where $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ is the Hubble's const. (canonical value). The units are inverse time

$$H_0^{-1} \approx 4.4 \times 10^{17} \text{s} \approx 14.1 \text{ billion years.}$$

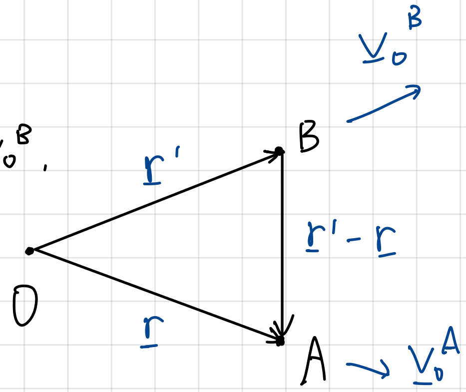
(linear approximation to the age of the universe)

Why Hubble's Law ?

Consider 3 galaxies O, A, B, with velocities from O of $\underline{v} = \underline{0}$, $\underline{v}_0^A$, $\underline{v}_0^B$.

View from A, B has velocity

$$\underline{v}_A^B = \underline{v}_0^B - \underline{v}_0^A \quad (*)$$



• <u>Homogeneity</u> implies that relative velocities can only depend on relative position. From (*),

$$\underline{v}(\underline{r}_B - \underline{r}_A) = \underline{v}(\underline{r}_B - \underline{r}_0) - \underline{v}(\underline{r}_A - \underline{r}_0),$$

or

$$\underline{v}(\underline{r}' - \underline{r}) = \underline{v}(\underline{r}') - \underline{v}(\underline{r})$$

i.e. the vector field $\underline{v}$ (coords v_i) is <u>linearly</u> related to $\underline{r}$ (coords r_i):

$$v_i = H_{ij} r_j \qquad (\dagger).$$

with const. matrix H_{ij} ($\underline{r}$ indpt).

• <u>Isotropy</u> implies $H_{ij} = L_{ik} H_{kl} L_{lj}$ for any rotation matrix $[L_{ij}]$. The only rank 2 isotenser is $\lambda \delta_{ij}$. So we must have $H_{ij} = H_0 \delta_{ij}$

$$\Rightarrow \qquad \underline{v} = H_0 \underline{r} \qquad \square$$

<u>Comments</u>: (1) Uniform expansion is consistent with the cosmological principle.

(2) H_0 is indpt of $\underline{r}$, but it can be time dependent.

$$\underline{v} = H(t)\,\underline{r} \qquad (1.3)$$

H_0 is the value measured today at $t = t_0$.

<u>The scale factor of the universe</u>

Integrate (1.3) $\quad \dfrac{dr}{dt} = H(t)\, r \qquad (r = |\underline{r}|)$,

$$\Rightarrow \frac{dr}{r} = H\,dt$$

$$\Rightarrow \ln r = \int H\,dt \equiv \ln a + \text{const};$$

where we define the "scale factor" $a(t)$ by

$$\frac{a(t)}{a(t_0)} = \exp\left(\int_{t_0}^{t} H(t')\, dt' \right)$$

So we have

$$\boxed{H \equiv \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt}} \qquad (1.4)$$

The hom and iso sol^n when $r(t) = a(t) \dfrac{r(t_0)}{a(t_0)}$, or

$$\underline{r}(t) = a(t)\, \underline{x} \qquad (1.5)$$

Overall scaling motion delocalised by <u>scale factor</u> $a(t)$ with const. component $\underline{x}$ labelling galaxies in the uniform Hubble flows (Lagrangian).

<u>Note</u>: • Often we take $a_0 \equiv a(t_0) = 1$ (1.6), so $|\underline{x}|$ is the physical distance to a galaxy today $(t = t_0)$.

• In many cases, $a(t) = (t/t_0)^\alpha$. e.g. $\alpha = \frac{2}{3}$ for "dust".

$$\Rightarrow H(t) = \frac{\alpha}{t}$$

Then at $\alpha = 0$, when $t = 0$. "big bang".

• Most galaxies have a <u>peculiar velocity</u> $\underline{v}_{pec}$ relative to the comoving frame $(\dot{\underline{x}} \neq 0)$:

$$\underline{v} = \dot{a}\underline{x} + a\dot{\underline{x}} = H\underline{r} + \underline{v}_{pec}$$

Hubble flow.

deviation

<u>Aside</u>: A first approx to scale factor

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \ldots \Rightarrow \frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) + \ldots$$

So there is an apparent singularity when $t_0 - t = H_0^{-1}$ at which $a = 0$, ie. a finite age of the universe and a big bang.

Cosmological Redshifts.

Recessional velocities v are inferred from the redshift z of galaxy spectral lines.

$$1 + z \equiv \frac{\lambda_o}{\lambda_e} , \tag{1.8}$$

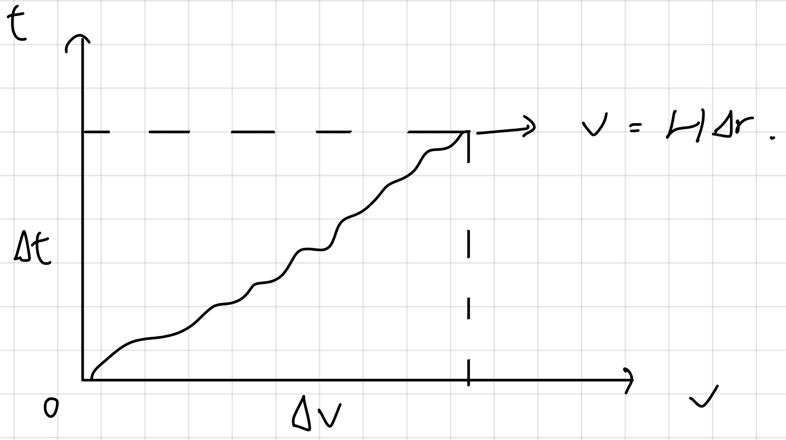
where λ_e is the emitted wavelength and λ_o is the observed today (freq $\nu = c/\lambda$)

In special relativity, Doppler shift seen by a moving observer (velocity v)

$$\lambda_o = \sqrt{\frac{1+v/c}{1-v/c}}\, \lambda_e \approx \lambda_e \left(1 + \frac{v}{c} + \ldots\right) \tag{1.9}$$

So $cz \approx v$ for $v \ll c$. Consider photon emitted at $t=0$ at $r=0$, reading a comoving galaxy at Δr receding at velocity

$$v = H \Delta r = Hc\,\Delta t \tag{t}$$



From (1.9), with $\lambda_o \mapsto \lambda + \Delta\lambda$.

$$\frac{\lambda + \Delta\lambda}{\lambda} = 1 + z \approx 1 + \frac{v}{c} \Rightarrow \frac{\Delta\lambda}{\lambda} = \frac{v}{c} = H\Delta t .$$

Hence, with $H = \dot{\alpha}/\alpha$, we have

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{a}}{a} \Rightarrow \ln \lambda = \ln a + \text{const.}$$

Integrating from emission to today t_0.

$$\frac{\lambda(t_0)}{\lambda(t_e)} = \frac{a(t_0)}{a(t_e)} = 1+z \qquad (1.10)$$

or for frequency

$$\frac{\nu(t_0)}{\nu(t_e)} = \frac{a(t_e)}{a(t_0)}$$

1.3 Newtonian Gravity Revisited

Universal law of Gravitation states force on one object (mass m, pos^n $\underline{r}$) due to another object (mass M, pos^n $\underline{r}'$) is

$$\underline{F} = m\ddot{\underline{r}} = -\frac{GMm\,(\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|^3} \qquad (1.12)$$

with $G = 6.67 \times 10^{-11}\ \text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$. Acceleration due to a mass dist^n with mass density $\rho(\underline{r})/c^2$ ($\rho(\underline{r})$ is <u>energy density</u>) in a volume V.

$$\ddot{\underline{r}} = -G \int d^3\underline{r}' \, \frac{\rho(\underline{r}')}{c^2} \, \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} \qquad (1.13)$$

Equivalently, we introduce grav potential Φ satisfying Poisson eqn

$$\nabla^2 \Phi = \frac{4\pi G}{c^2}\rho \qquad (1.14)$$

with general sol^n

$$\Phi(\underline{r}) = -\frac{G}{c^2}\int \frac{\rho(\underline{r})}{|\underline{r}-\underline{r}'|}\, d^3\underline{r}' \qquad (1.15)$$

<u>Exercise</u>: Verify force law using

$$\underline{F} = -m\nabla\Phi \qquad (1.16)$$

Gauss' law for gravity: Mass M in a volume V enclosed by surface S.

$$M = \int_V d^3\underline{r} \, \frac{\rho(\underline{r})}{c^2} = \frac{1}{4\pi G} \int_V d^3\underline{r} \, \nabla^2 \Phi$$

$$= \frac{1}{4\pi G} \int_S \nabla\Phi \cdot d\underline{S} \qquad (1.17)$$

by Gauss law.

Isotropic configuration (Newton's thm)

The grav. force outside a sph. sym. mass dist (radius R) is same as if all of the mass were concentrated at its centre $r=0$. (Principle I).

Pf: Consider a sph shell (mass M) at $r=R$ with

$d\underline{S} = \hat{\underline{e}}_r \, R^2 \sin\theta \, d\theta \, d\phi$ and $\nabla\Phi = -\underline{\ddot{r}} = -\ddot{r}\,\underline{e}_r$

So

$$\int_S \nabla\Phi \cdot d\underline{S} = -\int_S \underline{\ddot{r}} \cdot d\underline{S} = -\ddot{r} \int R^2 \sin\theta \, d\theta \, d\phi = -4\pi R^2 \ddot{r}$$

But $\int_S \nabla\Phi \cdot d\underline{S} = 4\pi G M$ by Gauss' law (1.17), where

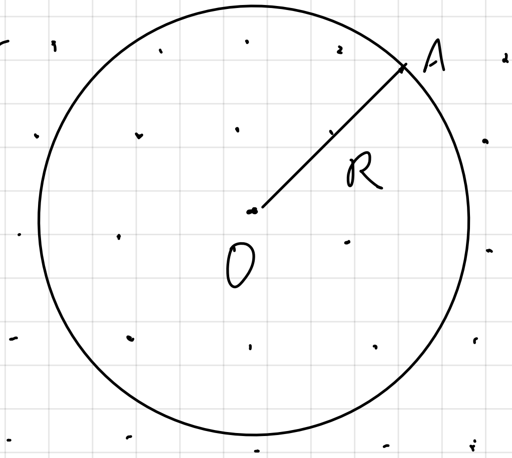
$M = \int_0^R \frac{\rho(r)}{c^2} \, d^3\underline{r}$, i.e. acceleration from shell at $r=R$

$\ddot{r} = -\frac{GM}{r^2}$ is the same as from a point mass at $r=0$. $\square$

Newton's 2nd thm states that sph. shells at $r>R$ exert no net grav force at R (Exercise: prove using previous result).

Why Newton's static universe was wrong?

Newton claimed that uniform matter is in an infinite universe would be in static unstable equilibrium.



Consider force at point A (mass δm, distance R) in frame O at rest. Hom and iso implies $\underline{F}$ at O vanishes ("control star") but not $\underline{F}$ at A.

For uniform ρ, there is a net force at A.

$$F_{r<R} = \delta_m \ddot{R} = -\frac{G\,\delta m M}{R^2} \qquad (1.18)$$

with $M = 4\pi \int_0^R \frac{\rho}{c^2} r^2 dr = \frac{4\pi}{3c^2} \rho R^3$. by (1.17).

Is this cancelled for $F_{r>R}$? Sph. shells $r>R$ produce no net force (Birkhoff's thm). $F_{r>R} = 0$. Net inward pointing force at A.

$$\underline{F} = -\frac{G\,\delta m M}{R^2} \hat{\underline{e}}_r \neq 0$$

So Newton's static universe will collapse.

1.4 The Friedmann Equation

Consider force (1.18) on a particle (galaxy mass δm) in a spherical shell at R(t).

$$\delta m \ddot{R} = -\frac{G\,\delta m\, M}{R^2} .$$

with $M = \frac{4\pi}{3c^2}\rho R^3$. Multiply by $\dot{R}$ and integrate

$$\underbrace{\frac{1}{2}\delta m \dot{R}^2}_{\text{KE}} = \frac{G\,\delta m\, M}{R} \quad \leftarrow \text{-Gravitational PE.}$$

For a comoving shell $R(t)$, we have $\dot{M} = 0$ by mass conservation (no. of particles inside R is const.). Integrating,

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = \text{const.} \qquad (\ddagger)$$

$$\Rightarrow \underbrace{\frac{1}{2}\dot{R}^2}_{\text{KE}} - \underbrace{\frac{4\pi G}{3c^2}\rho R^2}_{\text{PE}} = \underbrace{\text{const.}}_{\text{total energy}}$$

Now substitute with a scale factor $R(t) = a(t)\,x_0$ (1.5)

and find

comoving coord labelling shell

$$\frac{1}{2}\dot{a}x_0^2 - \frac{4\pi G}{3c^2}\rho a^2 x_0^2 = \text{const.}$$
$$\equiv -\frac{kc^2}{2}x_0^2.$$

where k const. has dimensions of inverse length square (same as curvature). This yields the

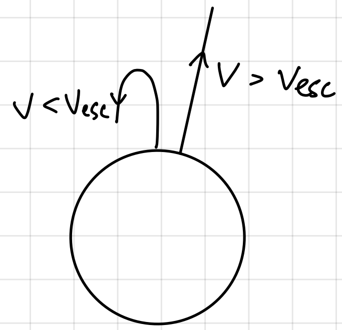
<u>Friedmann equation</u>

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho} \qquad (1.20)$$

Derived in 1924 from Einstein's full GR eqn.

<u>How do we interpret k?</u>

Projectile thrown from earth satisfies the analogue (1.19) with $v = \dot{R}$.



Take const. $= -k = 0$, then $\frac{1}{2}v^2 = \frac{GM}{R}$

$$\Rightarrow \quad v = v_{esc} = \sqrt{\frac{2GM}{R}} \, .$$

the escape velocity

Exercise Find v_{esc} for $M_\oplus = 6 \times 10^{24}$ kg, $R_\oplus = 6000$ km

In Friedmann (1.20), $H^2 = (\dot{a}/a)^2$ is either above ($k<0$), equal to ($k=0$), or below ($k>0$) the escape velocity of the Universe. Usually described by comparing to the critical density for $k=0$ in (1.20) defined as

$$\rho_{crit}(t) \equiv \frac{3c^2}{8\pi G} H^2 \qquad (1.21)$$

The density parameter is

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{crit}(t)} \qquad (1.22)$$

Today, $\Omega_0 = \frac{8\pi G}{3c^2 H_0^2} \rho(t_0)$. Hence.

modified by Λ

$$\begin{cases} \Omega > 1, & k > 0 \quad \text{turnaround recollapse}^* \\ \Omega = 1, & k = 0 \quad \text{continued expansion} \\ \Omega < 1, & k < 0 \quad \text{continued expansion} \end{cases}$$

1.5 Fluid Conservation Equation

A small fluid element in a comoving volume V_0 obeys

$$V(t) = a^3(t) V_0$$

so rate of change of vol. is

$$\frac{dV}{dt} = 3\dot{a}a^2 V_0 = 3\frac{\dot{a}}{a} V \qquad (1.23)$$

The energy of comoving fluid $E = \rho V$ changes as

$$\frac{dE}{dt} = \dot\rho V + \rho \dot V .$$

or

$$\dot\rho = \frac{\dot E}{V} - 3\frac{\dot a}{a}\rho \qquad (*)$$

For NR matter (galaxies or "dust", number conserved $dN = 0$, $V_{pec} \ll c$), so we have $\dot E \approx 0$.

<u>But</u> if we have a <u>uniform pressure</u> $P(t)$ then any adiabatic (slow) change in Vol. means work is done.

$$dE = -P\,dV \qquad (1.24)$$

This is the first law of thermodynamics. (§4, $dN = dS = 0$).

So (1.23) $\Rightarrow$ $$\frac{dE}{dt} = -P\frac{dV}{dt} = -3\frac{\dot a}{a} PV \qquad (\dagger)$$

Combining ($\dagger$) and ($*$), we obtain the continuity eqn.

$$\boxed{\dot\rho = -3\frac{\dot a}{a}(\underbrace{\rho}_{\text{dilution}} + \underbrace{P}_{\text{work against expansion}})} \qquad (1.25)$$

If we now specify an eqn of state

$$P = P(\rho) \qquad (1.26)$$

then a closed system with 3 eqn and 3 unknowns a, ρ, P. A <u>linear</u> EoS relation is sufficient in many cases

$$P = w\rho \qquad (1.27)$$

where w is a const. (ordinary matter $0 < w \leq 1$).

- Dust (NR matter) $w = 0$
- Radiation $w = 1/3$ ($P = \rho/3$ – see §4).

Exercise For the linear EoS (1.27), solve (1.25) to show

$$\boxed{\rho = \rho_0 \, a^{-3(1+w)}} \tag{1.28}$$

with $\rho_0 = \rho(t_0)$, $a(t_0) = 1$.

1.6 Einstein-de Sitter solution ($P=0$, $k=0$)

Pressure-free ($w=0$) matter const. at critical density $\Omega = 1$ ($k=0$).

Continuity eqn (1.25) becomes

$$\frac{\dot\rho}{\rho} = -3\frac{\dot a}{a} \Rightarrow \frac{d\rho}{\rho} = -3\frac{da}{a} \Rightarrow \rho \propto a^{-3},$$

i.e.

$$\rho = \rho_0/a^3 \tag{1.29}$$

Friedmann (1.20) becomes

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\frac{\rho_0}{a^3}$$

$$\Rightarrow \sqrt{a}\, da = \sqrt{\frac{8\pi G}{3c^2}}\, dt$$

$$\Rightarrow \frac{2}{3}a^{3/2} = \sqrt{\frac{8\pi G}{3c^2}}\, t + \text{const.}$$

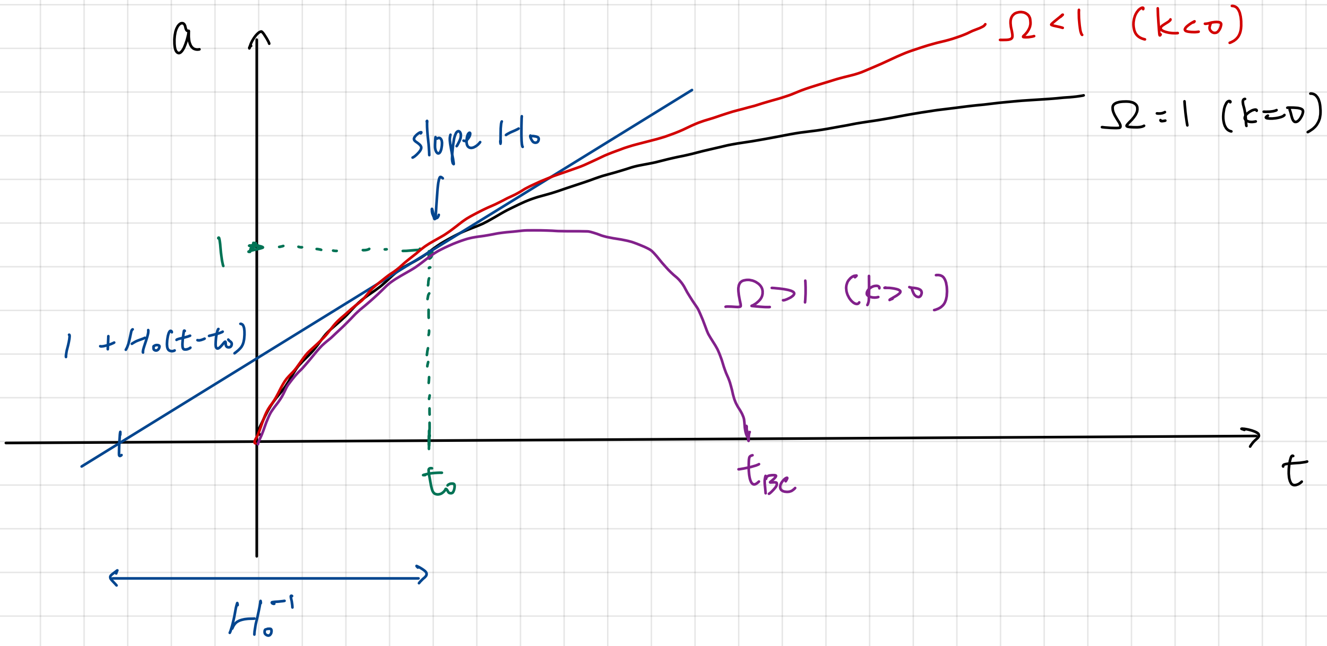
with $a \to 0$, $t \to 0$ $\Rightarrow$ const. $= 0$, so

$$a^3 = 6\pi G \frac{\rho_0}{c^2} t^2,$$

or

$$a = (t/t_0)^{2/3} \tag{1.30}$$

with $t_0 = (6\pi G \rho_0/c^2)^{-1/2}$.



So using measured H_0 (1.2), we have age of universe

$$t_0 = \frac{2}{3} H_0^{-1} \approx 9 \times 10^9 \text{ yrs}.$$

Mass density from (1.29) is

$$\boxed{\rho / c^2 = \frac{1}{6\pi G t^2}} \qquad (1.31)$$

1 H atom / m^3

with $\rho_0 \approx 10^{-26}$ kg m^{-3} $\sim 10^{11} M_\odot / (1 \text{Mpc})^3$ avg. galaxy separation.

1.7 Acceleration (or Raychaudhuri) Equation

Differentiate Friedmann (1.20) as $\dot{a}^2 + kc^2 = \frac{8\pi G}{3c^2} \rho a^2$

$$\Rightarrow \quad 2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2} \left(\dot{\rho} a^2 + 2\dot{a} a \rho \right)$$

$$\Rightarrow \quad 2 \frac{\ddot{a}}{a} \cdot \frac{\dot{a}}{a} = \frac{8\pi G}{3c^2} \left(-3 \frac{\dot{a}}{a} (\rho + P) + 2 \frac{\dot{a}}{a} \rho \right)$$

$$\Rightarrow \quad \boxed{\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2} (\rho + 3P)} \qquad (1.32)$$

which is a simplified form of Raychaudhuri eqn in GR

Exercise Using $H = \dot{a}/a$, show that (1.32) takes the form

$$\dot{H} + H^2 = -\frac{4\pi G}{3c^2} (\rho + 3P) \qquad (1.33)$$

* 1.8 General Isotropic Force Law *

What is the most general force satisfying Newton's sph. thm? i.e. when does $\Phi_{shell}(r) = \Phi_{point}(r)$?

See lecture notes to derive linear ODE Φ must satisfy. Seek power sol^{n}s $\Phi = \alpha r^n$, and we find 2 sol^n $n=-1$, $n=2$.

$$\Phi = Ar^{-1} + Br^2 = -\frac{GM}{r} - \frac{1}{6}\Lambda c^2 r^2 \qquad (1.34)$$

where Λ has units of inverse length squared.

The general force law is $\underline{F} = -m\nabla\Phi$.

$$F = -\underbrace{\frac{GMm}{r^2}}_{\text{Newton's inverse square law}} + \underbrace{\frac{1}{3}\Lambda c^2 mr}_{\text{Hooke's law for elastic media}} \qquad (1.35)$$

Cosmological constant Λ

For $\Lambda > 0$, repulsive force opposing grav. attraction (anti-gravity). Dominates if $\Lambda c^2 > 8\pi G\rho/c^2$.

For $\Lambda < 0$. we have attractive force with only elliptical orbits. (no escape)

Exercise Start with general potential (1.34), and re-derive Friedmann (1.20) to find

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho + \frac{\Lambda c^2}{3}} \qquad (1.36)$$

2. Relativistic Cosmology

2.1 Special Relativity generalised

Newtonian gravity in Euclidean space $\mathbb{R}^3$ is inadequate for strong gravity or the global geometry on large scales.

In SR, we measure spacetime distance with the Minkowski metric (summation convention)

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{00}(dx^0)^2 + \underbrace{g_{ij}^{(3)} dx^i dx^j}_{=dl^2}$$

$$= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{2.1}$$

In cosmology, seek a similar diagonal time and space shift consistent with hom. and isotropy. In GR, Einstein eqns relate the curvature tensor (Ricci $R_{\mu\nu}$, trace R) to the energy tensor $T_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.2}$$

curvature

matter content

"Geometry tells matter how to move, matter tells geometry how to curve." Geometry becomes empirical.

Curvature R shifts into intrinsic part (spatial curvature $k = \frac{R^{(3)}}{6}$) plus an extrinsic part (expansion $\chi = -3H$).

2.2 Three spaces of constant curvature

2D analogues in R^2, S^2, H^2.

- Flat Euclidean space R^2 has metric

$$dl^2 = dx^2 + dy^2 = dr^2 + r^2 d\phi^2$$

where $x = r\cos\phi$, $y = r\sin\phi$, $r^2 = x^2+y^2$, $\phi \in [0, 2\pi]$

Here, curvature $=0$, but we can have a non-zero k provided it is constant everywhere, ie. $k=0$, $k>0$, $k<0$.

- Two-sphere S^2 (radius R) has positive curvature

$K = \frac{1}{R^2}$. We can embed S^2 into 3D Euclidean space.

$$x^2+y^2+z^2 = R^2 \qquad (2.4)$$

Take $r^2 = x^2+y^2$ again in Euclidean 3-metric

$$dl^2 = dx^2+dy^2+dz^2 = dr^2 + r^2 d\phi^2 + dz^2$$

From (2.4), $r^2+z^2 = R^2 \Rightarrow r\,dr = -z\,dz$

$$\Rightarrow dl^2 = dr^2 + r^2 d\phi^2 + \frac{r^2}{z^2} dr^2$$

$$= dr^2 + r^2 d\phi^2 + \frac{r^2}{R^2-r^2} dr^2$$

$$= \frac{1}{1 - r^2/R^2} dr^2 + r^2 d\phi^2$$

$$\Rightarrow dl^2 = \frac{1}{1-kr^2} dr^2 + r^2 d\phi^2 \qquad (2.5)$$

where $K = \frac{1}{R^2}$, $0 \le r \le R$ (compact)

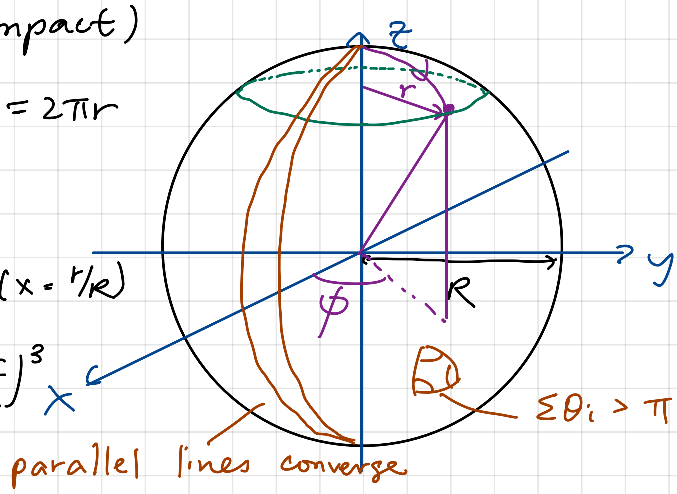
From (2.5), circle has circum$(S^1) = 2\pi r$

but radial path length

$$\text{Radius}(S^1) = \int_0^r \frac{dr'}{\sqrt{1 - r'^2/R^2}} = R\int \frac{dx}{\sqrt{1-x^2}} \quad (x = r/R)$$

$$= R\sin^{-1}\left(\frac{r}{R}\right) \approx r + \frac{R}{6}\left(\frac{r}{R}\right)^3$$

$> r$.



Exercise : Define polar coord

$$\theta = \frac{1}{R}\int \frac{dr}{\sqrt{1-(r/R)^2}} \tag{2.6}$$

to find the usual S^2 metric

$$dl^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi) = R^2 d\Omega^2_{(2)} \tag{2.7}$$

or directly from (2.5).

$$x = R\cos\phi\sin\theta, \quad y = R\sin\phi\sin\theta, \quad z = R\cos\theta.$$

• Hyperbolic space H^2 ($K<0$). found by embedding in 3D space with

$$x^2+y^2-w^2 = -R^2$$

with metric

$$dl^2 = dx^2+dy^2-dw^2 = \frac{dr^2}{1+r^2/R^2} + r^2 d\phi^2,$$

with $K = -\frac{1}{R^2}$ and $r \geq 0$.

$$\text{Circum}(S^1) = 2\pi r$$

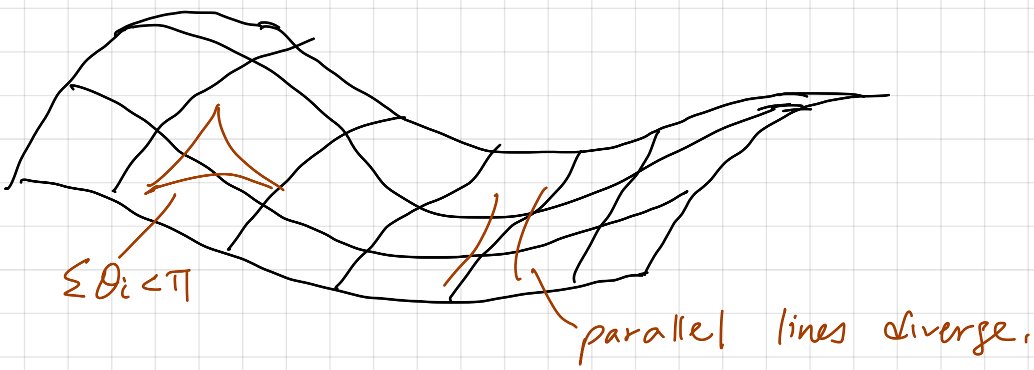
$$\text{Radius}(S^1) = \int_0^r \frac{dr'}{\sqrt{1+r'^2/R^2}} = R\sinh^{-1}\left(\frac{r}{R}\right) < r.$$

$$\text{Radial coord } \chi = \frac{1}{R}\int \frac{dr'}{\sqrt{1+Kr'^2}} = \sinh^{-1}\left(\frac{r}{R}\right).$$

We find the metric

$$dl^2 = R^2(d\chi^2 + \sinh^2\chi\, d\phi^2) \tag{2.9}$$

Best represented with a saddle.



Three-space S^3 metric

Embed S^3 in $\mathbb{R}^4$ $\quad x^2+y^2+z^2+w^2 = R^2$ with $r^2 = x^2+y^2+z^2$ and metric becomes

$$\begin{aligned} dl^2 &= dx^2 + dy^2 + dz^2 + dw^2 \\ &= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{r^2 dr^2}{R^2 - r^2} \\ &= \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \end{aligned} \tag{2.10}$$

Area $(S^2) = 4\pi r^2$, $\quad$ Radius $(S^2) = R\sin^{-1}\left(\frac{r}{R}\right)$ with angular coord

$$d\chi = \frac{1}{R}\frac{dr}{\sqrt{1-Kr^2}}. \tag{2.11}$$

where $\chi = \sin^{-1}\left(\frac{r}{R}\right)$

Constant curvature 3-metrics

$$dl^2 = \frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{2.12}$$

with

$$K = \begin{cases} \frac{1}{R^2} & S^3 \text{ (closed)} \\ 0 & \mathbb{R}^3 \text{ (flat)} \\ -\frac{1}{R^2} & H^3 \text{ (open)} \end{cases} \tag{2.13}$$

Using the def^n $d\chi^2 = \frac{1}{R^2}\frac{dr^2}{1-Kr^2}$, we find

$$dl^2 = R^2\left(d\chi^2 + f^2(\chi)\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

with

$$f(\chi) = \begin{cases} \sin\chi & 0 \le \chi \le \pi & S^3 \\ \chi & 0 \le \chi < \infty & \mathbb{R}^3 \\ \sinh\chi & 0 \le \chi < \infty & H^3 \end{cases} \tag{2.14}$$

2.3 Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetimes

Now generalise Mink (2.1) to describe expanding spaces of const. curvature (2.12)

$$ds^2 = -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right) \quad (2.15)$$

with

$$k = \begin{cases} 1/R_0^2 & , S^3 \\ 0 & , \mathbb{R}^3 \\ -1/R_0^2 & , H^3 \end{cases} \quad (2.16)$$

where r is the comoving radial coord and k is the comoving curvature with $K = k/a^2$. and $R(t) = a(t) R_0$.

The proper radial distance dl^2 between two points separated by coord. distance r $(d\theta = d\phi = 0)$

$$l_r = a(t)\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = a(t)\begin{cases} R_0 \sin^{-1}(r/R_0) & , k>0 \\ r & k=0 \\ R_0 \sinh^{-1}(r/R_0) & k<0 \end{cases} \quad (2.17)$$

Note: • Metric (2.15) in invariant under rescaling $a \mapsto \lambda a$, $r \mapsto r/\lambda$, $R_0 \mapsto R_0/\lambda$.

• For $k \neq 0$, we can set $k = \pm 1$ (common convention for FLRW metrics) but can't choose $a(t_0) = 1$ with a physical length today.

• Metric (2.15) is not invariant under Lorentz boosts, unlike (2.1) because there is a preferred rest-frame. (ie. we can measure motion relative to CMB).

Conformal structure

Often convenient to use conformal time defined by

$$d\tau = dt/a(t).$$

then
$$\tau = \int_0^t \frac{dt'}{a(t')} \tag{2.19}$$

Time derivatives $\dot{a} = \frac{da}{dt}$, $a' = \frac{da}{d\tau}$.

Together with rescaled χ variable from (2.14), we have elegant form

$$ds^2 = a^2(\tau)\left(-c^2 d\tau^2 + d\chi^2 + f^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right) \tag{2.20}$$

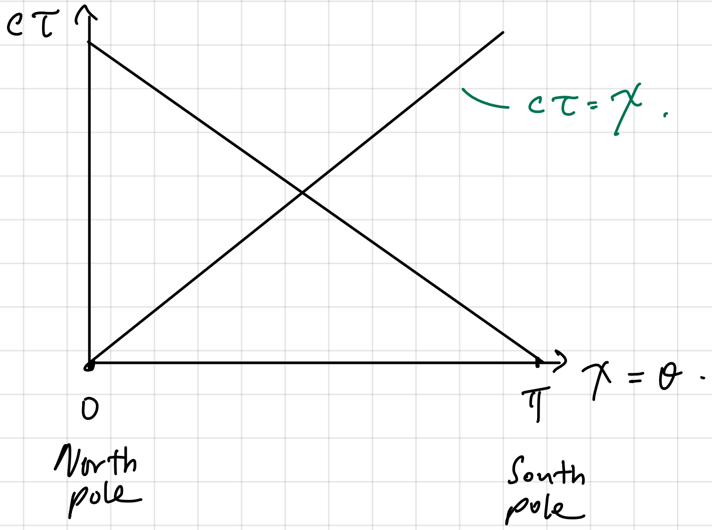
where
$$f(\chi) = \begin{cases} \sin\chi & S^3 \\ \chi & R^3 \\ \sinh\chi & H^3 \end{cases}$$

(Here $R_0 = 1$. $a(t_0) \neq 1$)

Light propagates along null paths $ds^2 = 0$, i.e. radial $c\,d\tau = \pm d\chi$ ($d\theta = d\phi = 0$) with $45°$ trajectory

$$\tau = \pm\chi + \underbrace{\chi_i}_{\text{const.}} \tag{2.21}$$

Take S^3.



Cosmological Horizons

Radial light rays in (2.15) obey

$$c\,dt = a(t)\frac{dr}{\sqrt{1-kr^2}},$$

and so since the Big Bang, light has travelled comoving distance

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_0^t \frac{c\,dt'}{a(t')}$$

or physical distance

$$d_H(t) = a(t)\int_0^t \frac{c\,dt'}{a(t')} \qquad (1.22)$$

Today $t=t_0$, this finite distance $d_H(t_0)$ is the max. distance light can travel, or cosmological horizon.

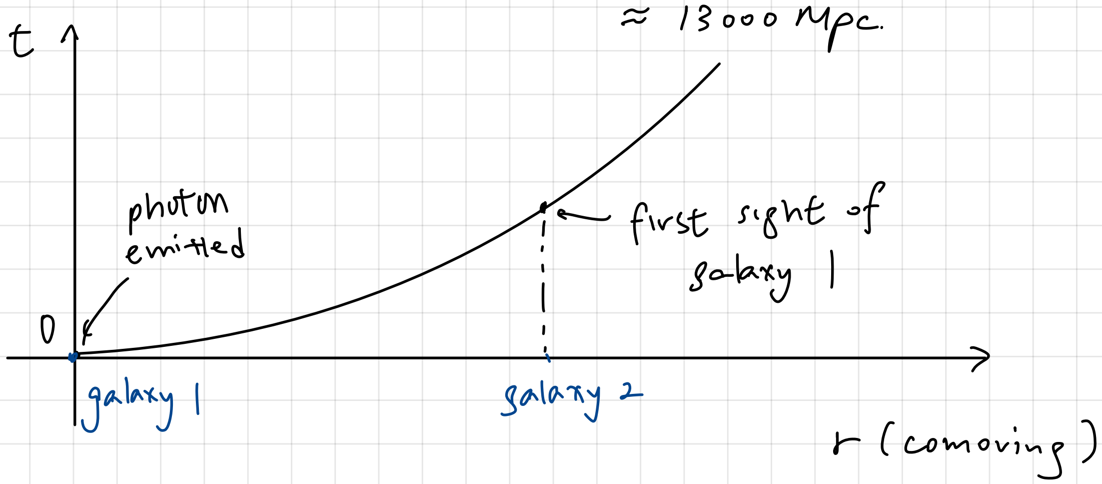
For EdS (1.30) $a(t) = (t/t_0)^{2/3}$, gives the radius of the observable universe

$$d_H(t) = c\left(\frac{t}{t_0}\right)^{2/3}\cdot 3t^{1/3}t_0^{2/3} = 3ct.$$

$$= 3\times(3\times10^8)\times(4.4\times10^7)$$

$$= 4\times10^{26}\,\text{m}$$

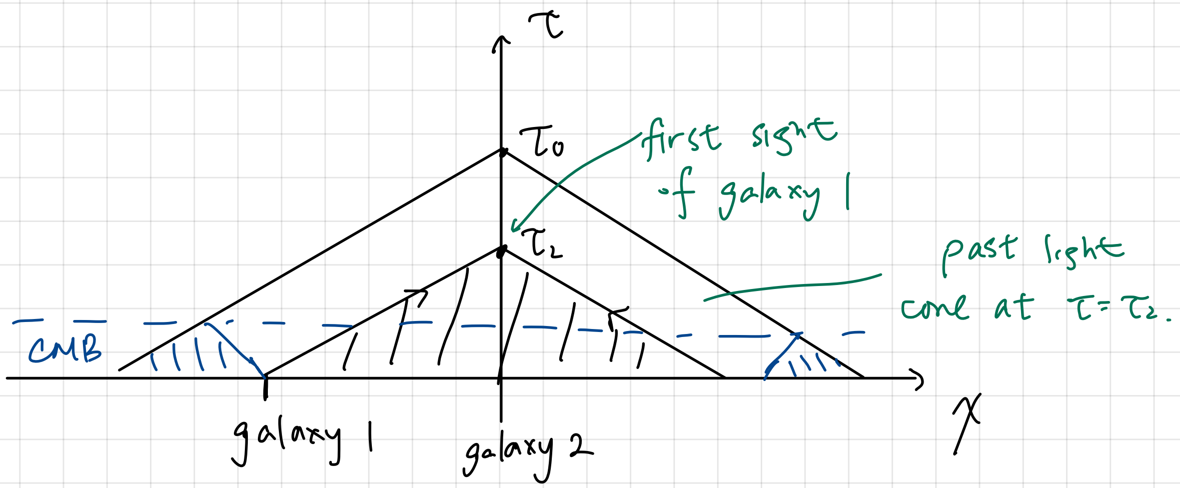
$$\approx 13000\ \text{Mpc}.$$



Better in conformal metric (2.20)

$$d_H(\tau) = c\,a(\tau)\,\tau$$

with $c\tau$ is the comoving horizon.



__Horizon Problem__ There are 10^5 casually disconnected regions $d_H(t_{dec})$ which have same temp. T to one part in 100000. why?

2.4 FLRW Spacetime Dynamics

__Matter content:__ The energy + momentum tensor $T_{\mu\nu}$ of a perfect fluid is $T^\mu{}_\nu = \mathrm{diag}(\rho, P, P, P)$

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} \tag{2.23}$$

where the fluid 4-vel: $u^\mu = (1, 0, 0, 0)$ in the comoving (rest) frame. The conservation law

$$\nabla^\mu T_{\mu\nu} = 0 \tag{$*$}$$

and (1.29) $\dot\rho = -3\frac{\dot a}{a}(\rho + P)$.

We envisage several components $\rho = \sum_i \rho_i$ with different EOs.

(1.27) $P_i = w_i \rho_i$ with each ρ_i satisfying (1.28) $\rho_i = \rho_{i0} a^{-3(1+w_i)}$.

__Exercise__ Using the density parameter (1.22)

$$\Omega_i = \rho_i / \rho_{crit} = \frac{3\pi G}{3c^2} \cdot \frac{1}{H^2} \rho_i .$$

show

$$\frac{8\pi G}{3c^2} \sum_i \rho_i = H_0^2 \sum_i \frac{\Omega_{i0}}{a^{3(1+w_i)}} \tag{2.24}$$

where $\Omega_{i0} = \Omega_i(t_0)$ today.

Eistein Equation : Non-zero curvature tensor components for FLRW metric (2.15) are

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{ij} = g_{ij}\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right) + \frac{2k^2c^2}{a^2}\right) \qquad (\dagger).$$

Exercise Evaluate $(\dagger)$ and $(*)$. from (2.15). Equating with (2.23) in Eistein eqns (2.2), yields (1.32)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P)$$

and (1.20)

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho$$

from $(ij) - \frac{(00)}{3}$

Singularity Theorem

In an expanding universe $(H>0)$ with matter satisfying the strong energy condition

$$\rho + 3P \geq 0 \qquad (2.25)$$

there must have been a singularity $(a \to 0, \rho \to \infty)$ at a finite time t in the past, with $|t_0 - t| \leq H_0^{-1}$.

Pf: We can rewrite accel eqn (1.32) as (1.33)

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3c^2}(\rho + 3P) \leq 0.$$

by (2.25), then

$$\frac{\dot{H}}{H^2} \leq -1, \quad \text{or} \quad -\frac{d}{dt}\left(\frac{1}{H}\right) \leq 1$$

Integrate from t to t_0

$$-\frac{1}{H_0} + \frac{1}{H} \leq -t_0 + t \Rightarrow H^{-1} \leq H_0^{-1} + t - t_0$$

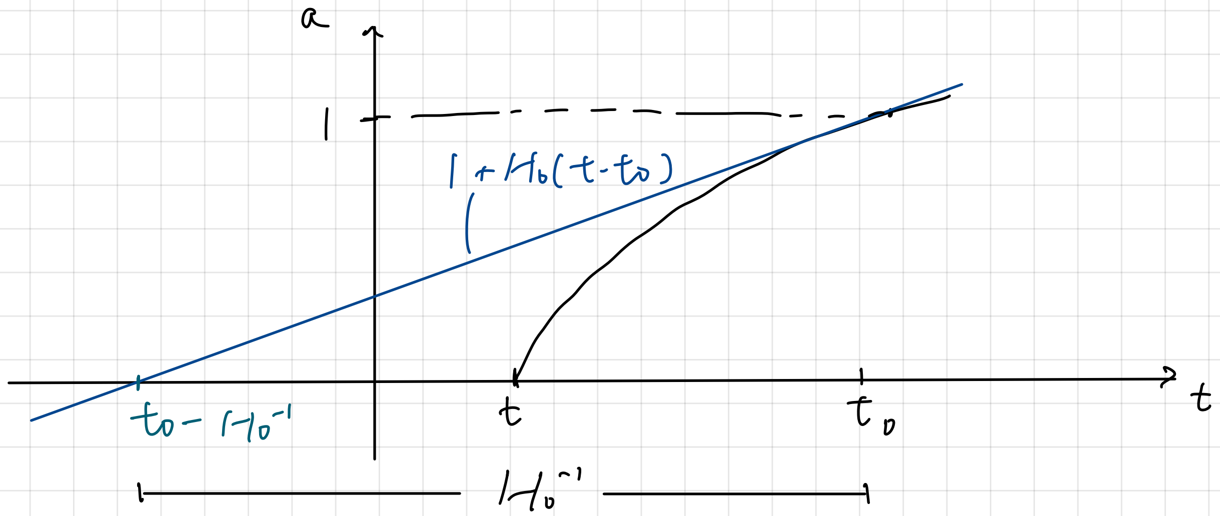
Inverting $H = \frac{\dot{a}}{a} = \frac{1}{H_0^{-1} + t - t_0}$.

Integrate again, $a_0 = 1$

$$\log(a_0) - \log(a) = \ln(H_0^{-1}) - \ln(H_0^{-1} + t - t_0)$$

Hence

$$a(t) \leq 1 + H_0(t - t_0) \qquad (2.26)$$



Since a is bounded above, must have $a \to 0$ for $t_0 - H_0^{-1} \leq t \leq t_0$. This is a physical singularity in both space and time, where $\rho \to \infty$ and physical laws break down — the singularity problems.

2.5 Cosmological solutions

- NR matter model ($P = 0$, $k = 0$)

See EdS (1.29) $\rho_m = \rho_{m_0}/a^3$, (1.30) $a(t) = (t/t_0)^{2/3}$

- Radiation model ($P = \rho/3$, $k = 0$)

Hot real gas $E = c^2\sqrt{m^2c^2 + p^2} \approx pc$ for $p \gg m$

(or $m = 0$, $E = h\nu_{photons}$)

Continuity (1.25)

$$\Rightarrow \rho_R = \rho_{R_0}/a^4 \qquad (2.27)$$

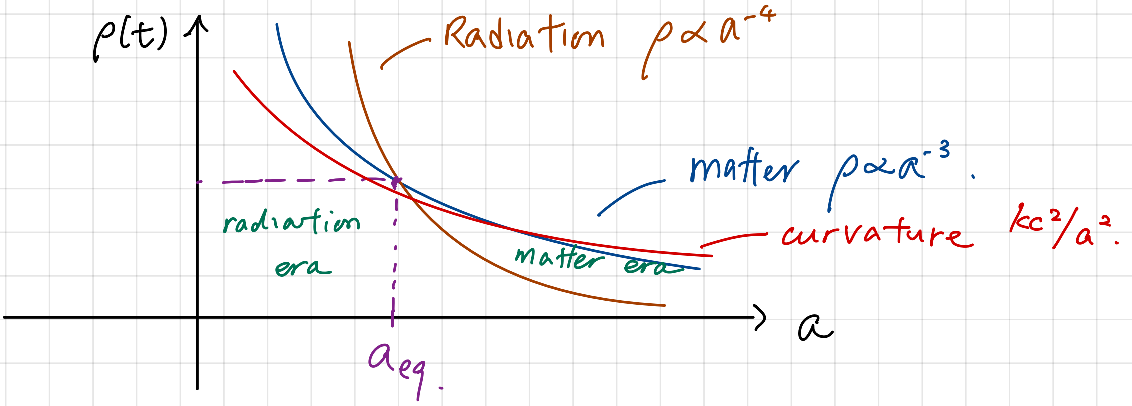
Friedmann (1.20)

$$\Rightarrow \quad a\dot{a} = \sqrt{\frac{8\pi G}{3c^2}\rho_{R_0}}$$

$$\Rightarrow \quad a(t) = (t/t_0)^{1/2} \tag{2.28}$$

with $H = \frac{1}{2t}$ and

$$\rho_R = \frac{3c^2}{32\pi G t^2} \tag{2.29}$$



· Matter - radiation Transition

Our universe has both NR matter ρ_{M_0} and radiation ρ_{R_0}.
today with $\rho_{M_0} \gg \rho_{R_0}$ ($\sim 3000\times$).

$$\rho = \frac{\rho_{R_0}}{a^4} + \frac{\rho_{M_0}}{a^3} \tag{2.30}$$

Equal density at $a_{eq} = \rho_{R_0}/\rho_{M_0} = \Omega_{R_0}/\Omega_{M_0}$, where

$$\Omega_i = \frac{\rho_i}{\rho_{crit}} = \frac{8\pi G}{3c^2}\frac{\rho_i}{H}.$$

Friedmann (1.20)

$$\Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = H_0\left(\frac{\Omega_{R_0}}{a^4} + \frac{\Omega_{M_0}}{a^3}\right) \tag{2.31}$$

with conformal time $\frac{dt}{d\tau} = a$, $\left(\frac{\dot{a}}{a}\right)^2 = \frac{a'^2}{a^4}$

$$\Rightarrow \quad a' = \alpha\sqrt{a_{eq} + a}$$

with $\alpha = H_0\sqrt{\Omega_{M_0}}$

Integrate $(0,0) \rightarrow (\alpha, \tau)$

$$\Rightarrow \quad 2\sqrt{a_{eq} + a} - \sqrt{a_{eq}} = \alpha\tau$$

$$\Rightarrow \quad a(\tau) = \frac{\alpha^2}{4}\tau^2 + \alpha\, a_{eq}\,\tau \qquad (2.32)$$

Integrating,
$$t(\tau) = \frac{\alpha^2}{12}\tau^3 + \frac{\alpha}{2} a_{eq}\,\tau^2 \qquad (2.33)$$

Exercise As $t \rightarrow 0$, show rad sol^n (2.28), and conversely $t \gg t_{eq}$ matter sol^n (1.30).

Possible Worlds : open, closed and flat universes

With NR matter $(p \approx 0)$ and curvature $(k \neq 0)$, (1.20) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\frac{\rho_M}{a^3} - \frac{kc^2}{a^2} = \frac{H_0^2\,\Omega_{M0}}{a^3} - \frac{kc^2}{a^2}$$

There is an equal contribution from k when

$$a_k = a(t_k) = \frac{H_0^2\,\Omega_{M0}}{kc^2} = \frac{\Omega_{M0}}{\Omega_{k0}}$$

with eff. curvature parameter $\Omega_{k0} = -kc^2/H_0^2$ and curvature dominates when $t \gtrsim t_k$.

Key ratio $\beta = \dfrac{1}{|\Omega_k|^{1/2}} = \dfrac{H_0 R_0}{c} = \dfrac{R_0}{cH_0^{-1}}$ ← curvature radius, ← Hubble radius

In conformal time (2.18) $\dot{a} = \dfrac{d\tau}{dt} a' = \dfrac{a'}{a} \equiv \mathcal{H}$

$$\ddot{a} = \frac{d\tau}{dt}\frac{d}{d\tau}\left(\frac{a'}{a}\right) = \frac{1}{a}\mathcal{H}' \qquad (2.34)$$

and accel eqn (1.32),

$$\mathcal{H}' = -\frac{4\pi G}{3c^2}\rho a^2 = \frac{H_0^2\,\Omega_{M0}}{2a} \qquad (2.35)$$

and Friedmann (1.20),

$$\mathcal{H}^2 + kc^2 = \frac{8\pi G}{3c^2}\rho a^2 \tag{2.36}$$

$$\Rightarrow 2\mathcal{H}' + \mathcal{H}^2 + kc^2 = 0. \tag{2.37}$$

with sol^n

$$\mathcal{H} = \begin{cases} \frac{c}{R_0} \cot\left(\frac{c\tau}{2R_0}\right) & k>0 \\ \frac{c}{R_0} \coth\left(\frac{c\tau}{2R_0}\right) & k<0 \end{cases}$$

Now find $\mathcal{H}'$, then $a(\tau)$ from (2.35)

$$a(\tau) = \begin{cases} \frac{1}{2}(\beta^2+1)\left(1-\cos(c\tau/R_0)\right) & k>0 \\ \frac{1}{2}(\beta^2-1)\left(\cosh(c\tau/R_0)-1\right) & k<0 \end{cases} \tag{2.38}$$

Integrating $dt = a\, d\tau$,

$$t(\tau) = \begin{cases} \frac{1}{2}(\beta^2+1)\left(\tau - \frac{R_0}{c}\sin(c\tau/R_0)\right) & k>0 \\ \frac{1}{2}(\beta^2-1)\left(\frac{R_0}{c}\sinh(c\tau/R_0) - \tau\right) & k<0 \end{cases} \tag{2.39}$$

Exercise Check as $\tau \to 0$, $t \sim \tau^3$, $a \sim \tau^2 \sim t^{2/3}$, i.e. EdS (1.30).

Fate of Universe

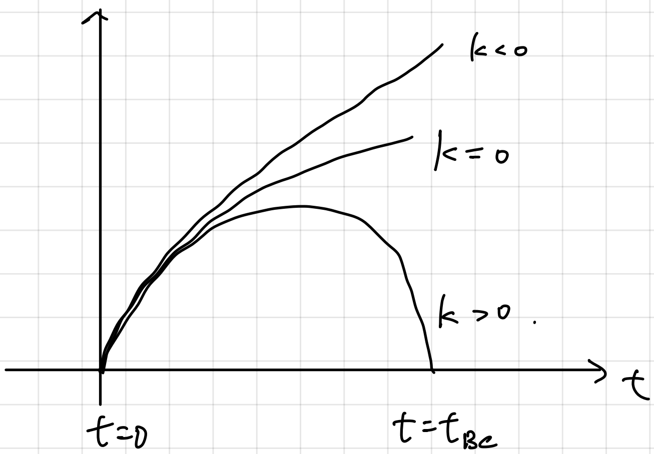
Depends on k with

- S^3 ($k>0$) then big Crunch at $\tau = 2\pi R_0/c$. with

  $$t_{BC} = \pi R_0(\beta^2+1) = \pi R_0\left(\frac{H_0^2 R_0^2}{c^2}+1\right).$$

  If $R_0 \gtrsim 2H_0^{-1}$, then $t_{BC} \gtrsim 40$ billion years.

- H^3 ($k<0$), faster expansion in a cold empty universe ($a \propto t$).
- R^3 ($k=0$) EdS (critical) sol^n lying in between.



<u>Flatness problem</u> ($\Omega = \Omega_M$)

Rewriting conformal Fried eqn (2.36),

$$\mathcal{H}^2 + kc^2 = \frac{8\pi G}{3c^2}\rho a^2$$

and compare with

$$\mathcal{H}^2 = \frac{8\pi G}{3c^2}\rho_{crit}\, a^2 \qquad (\ddagger)$$

$$\Rightarrow \Omega = \rho/\rho_{crit} = \frac{\mathcal{H}^2 + kc^2}{\mathcal{H}^2} \Rightarrow \Omega - 1 = \frac{kc^2}{\mathcal{H}^2} \quad (*).$$

Diff to get

$$\Omega' = \frac{-2kc^2\mathcal{H}'}{\mathcal{H}^3} = \mathcal{H}\underbrace{\left(-\frac{2\mathcal{H}'}{\mathcal{H}^2}\right)}_{\substack{= \Omega \text{ by } (\ddagger) \\ \text{and } (2.35)}}\underbrace{\left(\frac{kc^2}{\mathcal{H}^2}\right)}_{= \Omega - 1 \text{ by } (*)}$$

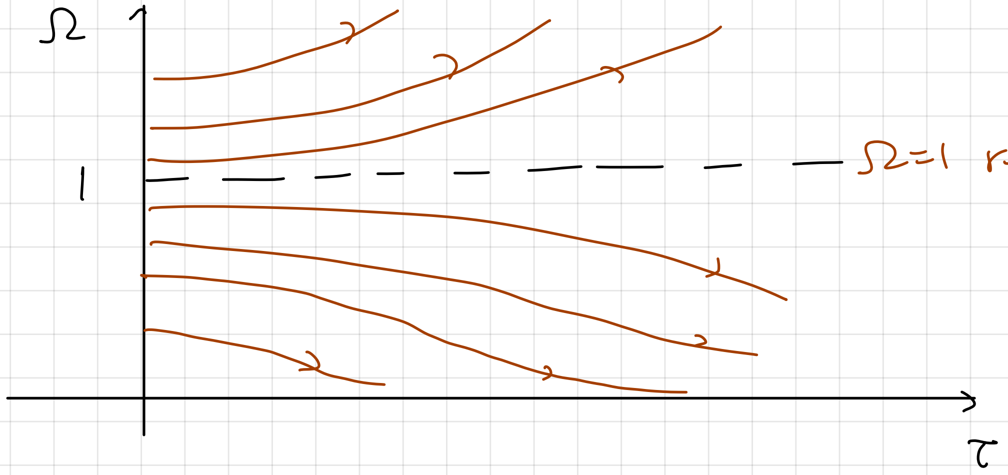
So

$$\Omega' = \mathcal{H}\,\Omega\,(\Omega - 1) \qquad (2.40)$$

In expanding universe ($\mathcal{H} > 0$), we are always driven away from $\Omega = 1$ ($k=0$, flat universe)

E.g. $\Omega > 1 \Rightarrow \Omega' > 0$, so Ω increases

$\Omega < 1 \Rightarrow \Omega' < 0$, so Ω decreases.



Exercise Solve (2.40) to find

$$\Omega - 1 = (\Omega_0 - 1)\frac{1}{1-\Omega_0+\Omega_0(1+z)}$$

$$\approx \frac{\Omega_0 - 1}{\Omega_0(1+z)} \quad \text{as } t \to 0$$

Radiation $(\Omega - 1) \propto \frac{1}{(1+z)^2}$

Project back to $t = \frac{1}{100}s$ and thus represents huge fine-tuning $|\Omega_{eq} - 1| = |\Omega_0 - 1| / (1 + z_{eq}\Omega_0) \approx 10^{-5}$

Radiation $|\Omega(1/100) - 1| \approx \left(\frac{1+z}{1+z_{eq}}\right)^2 |\Omega_{eq} - 1| \lesssim 10^{-18}$.

This is called the flatness problem.

Models with a Cosmological constant

Consider the isotropic force laws (1.35) with Λ (also arises naturally in GR), which acts as a uniform field with EoS $P = -\rho$, or $w = -1$ from (1.27) and the energy density $\rho_\Lambda = \frac{\Lambda c^4}{8\pi G}$ const. Friedmann eqn becomes (1.36)

$$\left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho + \frac{1}{3}\Lambda c^2. \qquad (2.42)$$

and accel. eqn

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P) + \frac{1}{3}\Lambda c^2 \qquad (2.43)$$

where $[\Lambda] = L^{-2}$.

<u>Vacuum sol^n (de Sitter)</u>

Empty universe $\rho = 0$, with $\Lambda > 0$, then (2.42) has sol^n

$$a(t) = \begin{cases} \alpha \cosh\left[\sqrt{\frac{\Lambda}{3}}\, c(t-t_0) + \beta\right] & k>0 \\ \exp\left(\sqrt{\frac{\Lambda}{3}}\, c\,(t-t_0)\right) & k=0 \\ \alpha \sinh\left[\sqrt{\frac{\Lambda}{3}}\, c(t-t_0) + \beta\right] & k<0 \end{cases} \qquad (2.43)$$

where $\alpha^2 = \frac{3|k|}{\Lambda} \approx 3\left(\frac{R_\Lambda}{R_0}\right)^2$ and $\beta = \coth(1-\alpha^2)$

curvature radius

" Λ scale "

Simplest case $(k=0)$ has $a(t_0) = 1$ and

$$H = \sqrt{\frac{\Lambda}{3}}\, c \equiv H_0 \quad , \text{ const.}$$

As $t \to \infty$, for all k,

$$a(t) \approx e^{H_0 t}.$$

These exhibit a <u>comoving event horizon</u>, i.e. how far a photon can move in the future).

Recall (2.33),

$$d_E(t) = \int_{t_0}^{\infty} \frac{dt'}{a(t')} = c\int_t^{\infty} e^{-H_0(t'-t_0)}\, dt' = cH_0^{-1}. \qquad (2.45)$$

comoving

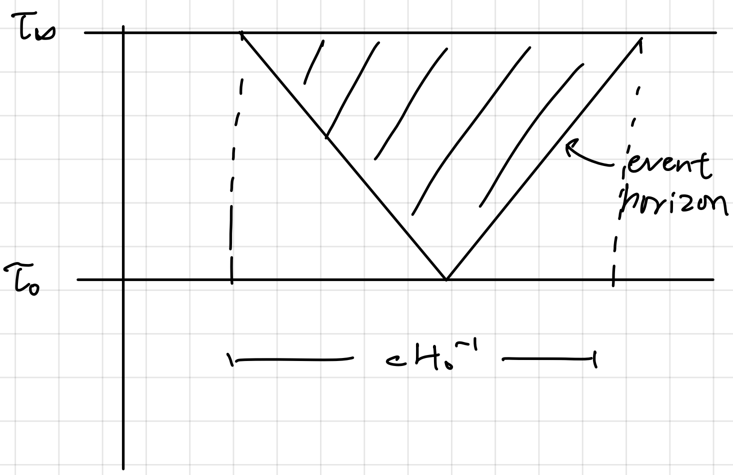
i.e. a finite comoving distance, so we can only have causal contact with galaxies within Hubble radius cH_0^{-1}.

To the past, we "avoid" a big bang evading (2.26) by violating SEC (2.25). Conformal time gives

$$\tau = \int_{t_0}^{t} \frac{dt'}{a(t')} = H_0^{-1}\left(1 - e^{-H_0(t'-t_0)}\right) \qquad (2.46)$$

so $t = t_0 \Rightarrow \tau = 0$ and $t \to \infty \Rightarrow \tau \to H_0^{-1}$.

Reveals casual structure with light rays $r = c\tau$

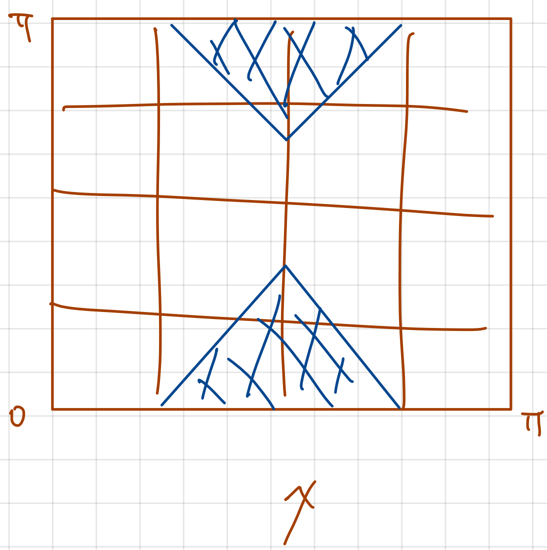


Exercise For $k>0$ in (2.44), show that

$$\tau = 2H_0^{-1} \tan^{-1}\left(\tanh\left(\tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}\, ct\right)\right) \qquad \text{(ignore } \alpha, \beta)$$

with $t \to -\infty \Rightarrow \tau = 0$, $t \to +\infty$, $\tau = \pi$

Hence show both cosmological and event horizon (in Penrose diag.)



Einstein - static Universe

Einstein introduced $\Lambda > 0$ (anti-gravity) counteract the collapse of NR matter ρ_M due to gravity. Balancing forces in (1.35) or the accel. eqn (2.43), to make static $(H=0)$ sol^n. requires

$$\Lambda = \frac{4\pi G}{c^4} \rho_{M0} > 0.$$

Sub into Fried. eqn (2.42) implies at $t = t_0$, $a_0 = 1$,

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \rho + \frac{1}{3} \Lambda c^2 - kc^2 = (\Lambda - k) c^2 = 0.$$

If $\Lambda = k > 0$, (closed universe)

Lemaitre showed this only achieves <u>unstable equilibrium</u>.

3. The Accelerating Universe

3.1 The standard Concordance Model (ΛCDM)

The effect of Λ in (2.42), (2.43) completes the present picture of our universe. Using the def^n of density parameter,

$$\Omega_{i0} = \frac{8\pi G}{3c^2} \frac{\rho_{i0}}{H_0^2}$$

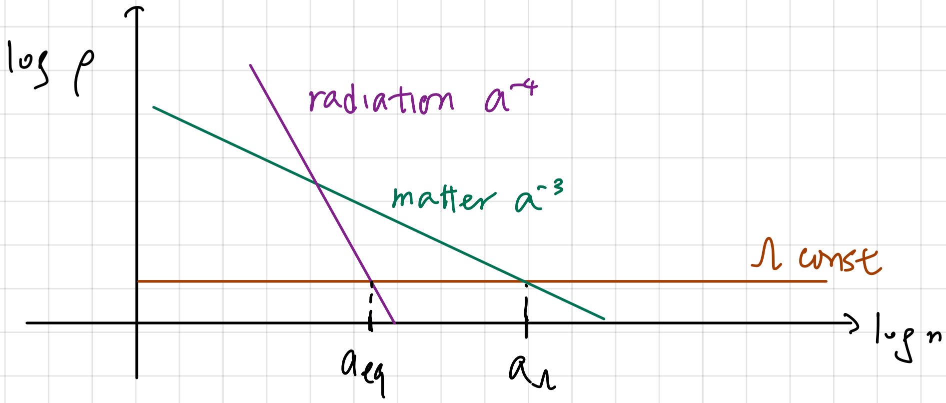
with $\Omega_{\Lambda 0} \equiv \dfrac{\Lambda c^2}{3H_0^2}$, $\Omega_{k0} = \dfrac{-kc^2}{H_0^2}$.

Friedmann (2.42) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = H_0^2 \left(\frac{\Omega_{M0}}{a^3} + \frac{\Omega_{R0}}{a^4} + \Omega_{\Lambda_0} + \frac{\Omega_{k0}}{a^2} \right) \qquad (3.1)$$

Accel (2.43) become

$$\frac{\ddot{a}}{a} = -\frac{1}{2} H_0^2 \left(\frac{\Omega_{M0}}{a^3} + \frac{2\Omega_{R0}}{a^4} - 2\Omega_{\Lambda 0} \right) \qquad (3.2)$$



Continuity (1.25) unchanged.

Energy budget of our universe

Simplest FLRW model consistent with observations

$H_0 = 674\ (\pm 0.5)$ km s^{-1} Mpc^{-1}

$\Omega_{\Lambda_0} = 0.689\ (\pm 0.006)$ const. (dark energy)

$\Omega_{c0} = 0.264\ (\pm 0.006)$ cold dark matter.

$\Omega_{B0} = 0.049\ (\pm 0.001)$ Baryons (ordinary matter).

$\Rightarrow$ NR total : $\Omega_{M0} = 0.315\ (\pm 0.007)$ (3.3)

$\Omega_{\gamma 0} = 5.0 \times 10^{-5}$ photons

$\Omega_{\nu_0} = 3.4 \times 10^{-5}$ neutrinos

$\Rightarrow$ Radiation total: $\Omega_{R_0} = 8.4 \times 10^{-5}$ (negligible today).

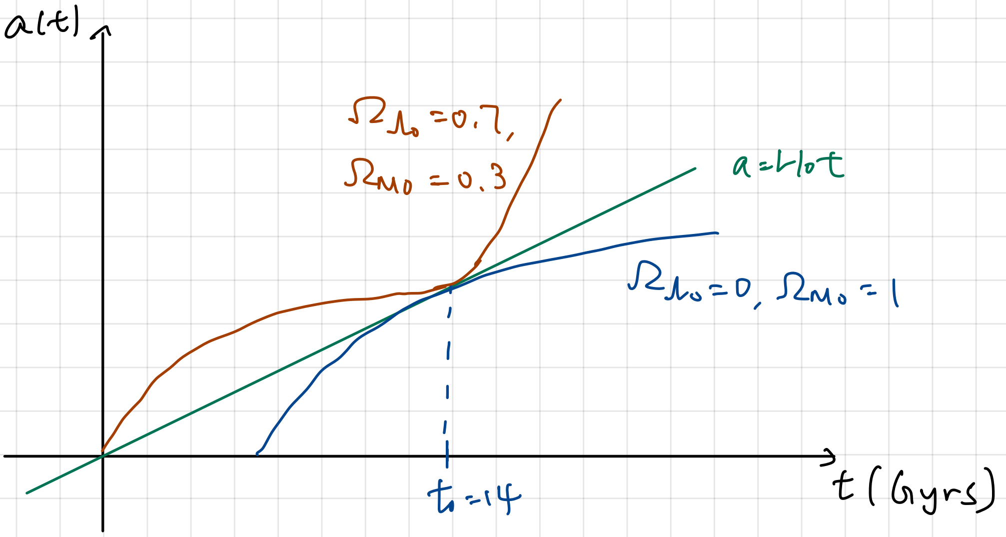
$\Omega_{k0} = 0.001\ (\pm 0.002)$ Curvature (spatially flat to 0.02%)

Matter - dark energy sol^n

With $\Omega_{R0} \ll \Omega_{\Lambda 0}, \Omega_{M0}$. Consider a flat $k \approx 0$ model filled with Λ and NR matter, i.e. $\Omega_{\Lambda_0} + \Omega_{M0} = 1$. In Ex 2 #3, find the sol^n to (3.1), (3.2) (use $b = a^{3/2}$ unnormalised)

$$a(t) = \left(\frac{\Omega_{M0}}{\Omega_{\Lambda_0}}\right)^{1/3} \sinh^{2/3}\left(t/t_\Lambda\right) \qquad (3.4)$$

As $t \to 0$, EdS (1.30) $a \propto t^{2/3}$. As $t \to \infty$, dS (2.44) $a \propto e^{Ht}$, H const



Age of universe

$$H_0 = \frac{\dot{a}}{a}\bigg|_{t=t_0} = \frac{2}{3t_\Lambda} \frac{1}{\tanh(t_0/t_{\Lambda_0})}$$

$$\Rightarrow \quad t_0 = t_\Lambda \tanh^{-1}\left(\sqrt{\Omega_{\Lambda_0}}\right)$$

$$\approx 0.955\, H_0^{-1} \approx 13.79\ (\pm 0.02) \text{ billion years}$$

So acceleration today solves time <u>timescale problem</u>. Universe transitions to accel $\dot{a} \approx 0$ at $t = 7.6$ Gyrs

<u>3.2 Evidence of Dark Energy</u>

<u>Luminosity distance d_L</u>

Measured flux or apparent luminosity

$$F = \frac{L}{4\pi d_L^2} \tag{3.5}$$

where L is the absolute luminosity (e.g. Sun 3.8×10^{26} W) and d_L is the luminosity distance. This is modified by expansion at redshift $1 + z = 1/a(t)$ $(a_0 = 1)$

$$F = \begin{pmatrix}\text{abs. lum} \\ L\end{pmatrix} \cdot \begin{pmatrix}\text{SA of sphere} \\ 4\pi a_0^2 r^2\end{pmatrix} \cdot \underbrace{\begin{pmatrix}\text{freq red shift} \\ \nu \mapsto \frac{\nu}{1+z}\end{pmatrix}\begin{pmatrix}\text{time dilation} \\ \Delta t \mapsto (1+z)\Delta t\end{pmatrix}^{-1}}_{\text{Power} = \frac{\Delta E}{\Delta t}}$$

$$\Rightarrow \qquad F = \frac{L}{4\pi r^2 (1+z)^2}$$

Hence,

$$d_L = \left(\frac{L}{4\pi F}\right)^2 = (1+z) r \qquad (3.6)$$

Deceleration parameter

Derivations from Hubble's law (1.1) can distinguish expansion rates using measurable quantities z and d_L.

Taylor expand $a(t)$ today $(t = t_0)$

$$a(t) = \underset{1}{\underset{\shortparallel}{a(t_0)}} + \underset{H_0}{\underset{\sim}{\dot a(t_0)}}\,(t - t_0) + \frac{1}{2}\ddot a(t_0)(t-t_0)^2 + \dots$$

So

$$\frac{1}{1+z} = 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 \qquad (3.7)$$

where

$$q = \frac{-\ddot a(t)\, a(t)}{\dot a(t)^2}, \qquad (3.8)$$

with $q_0 = \ddot a(t_0)/H_0^2$ is the deceleration parameter.

When matter obeys SEC (2.25) by (3.2), $q_0 \geq 0$.

EdS $\Omega_{M0} = 1$ has $q_0 = 0.5$.

Exercise Show ΛCDM (3.3) with $\Omega_{M0} = 0.3$, $\Omega_{\Lambda 0} = 0.7$ by (3.2), $q_0 \approx -0.54$.

Distance - redshift relation

We can invert (3.7) to find

$$1+z = 1 - H_0(t-t_0) + \left(1+\tfrac{1}{2}q_0\right) H_0^2 (t-t_0)^2 + \dots \qquad (3.9)$$

(Refer to notes)

In a time $t \to t_0$, light travels a distance given by (2.22)

$(k \simeq 0)$

$$r = c\int_t^{t_0} \frac{dt'}{a(t')} = c\int dt' \left[1 - H_0(t-t_0) + \dots\right]$$

$$= c(t_0 - t) + \frac{cH_0^2}{2}(t-t_0)^2 + \dots \qquad (3.10)$$

Now revert (3.9) to find

$$t_0 - t = \frac{z}{H_0}\left(1 - \left(1+\tfrac{1}{2}q_0 z\right) + \dots\right)$$

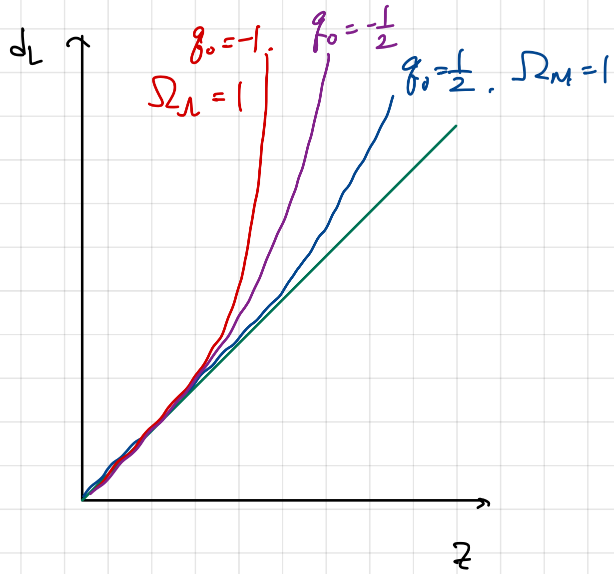
and substituting yields

$$r = \frac{cz}{H_0}\left(1 - \tfrac{1}{2}(1+q_0)z + \dots\right)$$

Hence, with lum dist. (3.6) we obtain the measurable Hubble law

$$d_L = (1+z)r = \frac{cz}{H_0}\left(1 + (1-q_0)z + \dots\right) \qquad (3.10)$$

In 1998, astronomers using SNIa measurements found $q_0 \approx -0.6 \pm 0.2 < 0$ in the Hubble diagram, i.e. the universe is accelerating



3.3 Evidence of dark matter

Estimating the matter density Ω_M is challenging since not all matter are visible (in stars)

Two step process:

(i) Estimate the galaxy mass with velocities and relate to luminosity of the galaxy (mass-to-light ratio M/L)

(ii) Integrate M/L over the observed galaxy luminosity f^n $\phi(L)$ to find mass density $\rho_M c^2$.

The Virial Theorem

A bound grav. system of massive particles in equilibrium obeys the relation

$$2\langle T\rangle = -\langle V\rangle \qquad (3.11)$$

where $\langle T\rangle$ is the time-avg total KE and $\langle V\rangle$ of the grav. PE.

Example Circular orbital motion. Taking $m \ll M$, $F_{grav} = \frac{GMm}{r^2} = \frac{mv^2}{r}$

$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r}$, i.e. $T = -\frac{1}{2}V$. Estimate mass from $M = v^2 r/G$.

Pf: Consider $\mathcal{G} = \sum_{i=1}^{N} \underline{p}_i \cdot \underline{r}_i$, $\underline{p}_i = m\frac{d\underline{r}_i}{dt}$. (Large N, equal mass m)

$$\frac{d\mathcal{G}}{dt} = \sum_{i=1}^{N} m\left(\dot{\underline{r}}_i \cdot \dot{\underline{r}}_i + \ddot{\underline{r}}_i \cdot \underline{r}_i\right)$$

$$= 2T + \sum_{i=1}^{N} \underline{F}_i \cdot \underline{r}_i$$

where $\underline{F}_i = \sum_{i \neq j}^{N} \underline{f}_{ij} = \sum_{i\neq j}^{N} \frac{Gm(\underline{r}_j - \underline{r}_i)}{|\underline{r}_j - \underline{r}_i|^3}$ $(*)$, then

$$\sum_{i=1}^{N} \underline{F}_i \cdot \underline{r}_i = \sum_{i=1}^{N}\sum_{j<i}^{N} \underline{f}_{ij}\cdot\underline{r}_i + \sum_{i=1}^{N}\sum_{j>i}^{N} \underline{f}_{ij}\cdot\underline{r}_i$$

$$= \sum_{i=1}^{N} \sum_{j<i}^{N} \underline{f}_{ij} \cdot \underline{r}_i + \sum_{j} \sum_{i<j} \underline{f}_{ij} \cdot \underline{r}_i \qquad (\text{swap } i \leftrightarrow j)$$

$$= \sum_{i=1}^{N} \sum_{j<i}^{N} \underline{f}_{ij} \cdot \underline{r}_i - \sum_{i} \sum_{j<i} \underline{f}_{ij} \cdot \underline{r}_j \qquad (N3)$$

$$= \sum_{i} \sum_{j<i} \underline{f}_{ij} \cdot (\underline{r}_i - \underline{r}_j)$$

$$= \sum_{i=1}^{N} \sum_{j<i} \frac{Gm^2}{|\underline{r}_j - \underline{r}_i|} \qquad (\text{grav energy})$$

So $\frac{dG}{dt} = 2T + V$. Take time-avg.

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{dG}{dt}\, dt = \lim_{t\to\infty} \frac{G(t) - G(0)}{t} = 0$$

as G bounded. Hence, $2\langle T \rangle_t = -\langle V \rangle_t$. $\square$

For large particle no., ergodic hypothesis allows time as $\langle T \rangle_t$ to be replaced by instantaneous spatial as $\langle T \rangle_r$:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{dG}{dt}\, dt \dots \longrightarrow \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \dots \qquad (3.13)$$

Coma cluster mass

Zwicky estimated Coma mass $M = Nm$ using $N = 1000$ galaxies and their radial velocities $\langle V_r^2 \rangle \approx 1000\,\mathrm{km\,s^{-1}}$ from redshift.

i.e. $\langle V^2 \rangle = 3 \langle V_r^2 \rangle$ (include transverse components), also $\langle R \rangle \approx 1.5\,\mathrm{Mpc} \approx 5 \times 10^{22}\,\mathrm{m}$. From (3.11),

$$M_{coma} = \frac{2 \langle V^2 \rangle \langle R \rangle}{G} \approx 2 \times 10^{15} M_\odot$$

But Coma Luminosity is $L \sim 8 \times 10^{12} L_\odot$, so mass to light ratio $M/L \approx 250$. What is the unseen ("dunkel") matter?

Hot gas in Coma (from X-ray): $M_{gas} = 2 \times 10^{14} M_\odot$, so M_{coma} is 90% DM.

Galaxy Rotation curves

With mass density profile $\rho(r)/c^2$, we have

$$M(r) = \int_0^r \rho(r') 4\pi r'^2 dr'$$

Measure $\langle v^2(r) \rangle$ for a mass shell m_s at r. From (3.11), implies

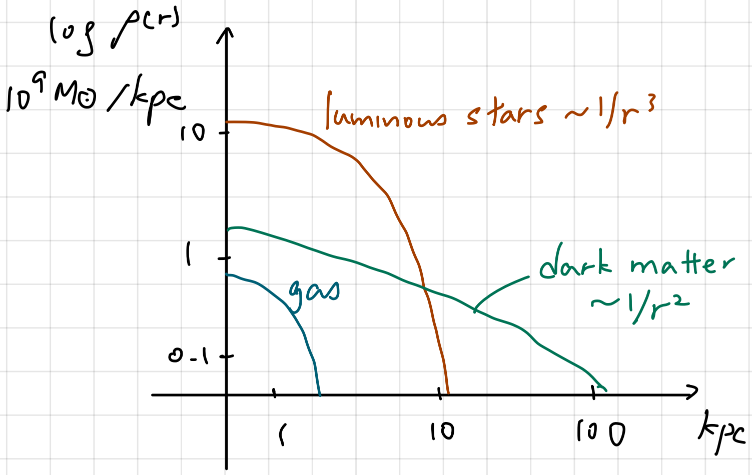
$$m_s \langle v^2 \rangle = \frac{GMm_s}{r} .$$

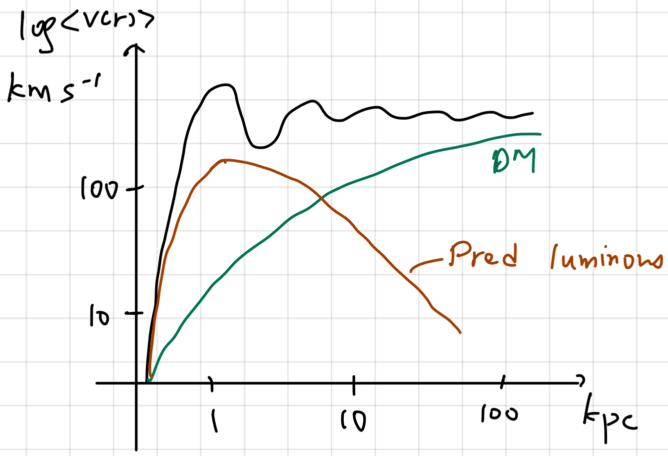
So $M(r) = r\langle v^2(r)\rangle / G$, or profile

$$\rho(r) = \frac{1}{4\pi r^2}\frac{dM}{dr} = \frac{1}{4\pi r^2}\frac{d}{dr}\left(r\langle v^2\rangle\right) \qquad (3.14)$$

E.g. $\rho(r) \sim \frac{1}{r^3} \iff \langle v^2 \rangle \sim \ln r / r$

$\rho(r) \sim \frac{1}{r^2} \iff \langle v^2\rangle \sim \text{const.}$





Typically, galaxies have $M_{DM} = 20\, M_{stars}$.

Accurate measures of $\Omega_B + \Omega_{DM} = \Omega_N$ come from modelling the CMB (refer §4-5).

3.4 Inflationary Universe

Cosmic inflation is a brief period of accelerated (exponential) expansion in the very early universe $a(t) \approx e^{Ht}$, H const, with de Sitter sol^n (2.44). Inflation alleviates the horizon and flatness problems, while quantum fluctuations provide the seeds for structure formation.

Scaler field inflation

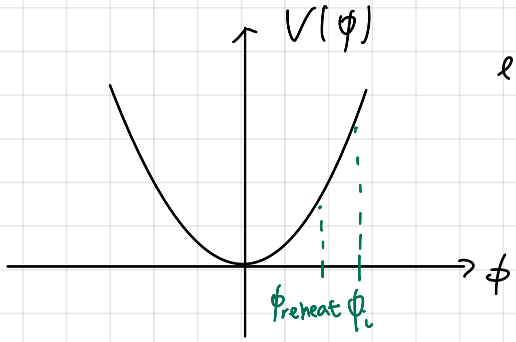
Inflation is usually driven by the (vacuum) potential energy $V(\phi)$ of a scaler field ϕ, called the inflaton. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2c^2}\dot{\phi}^2 - \frac{1}{2}\nabla\phi\cdot\nabla\phi - V(\phi)$$

In an expanding FLRW $(k=0)$ universe, we have matter action

$$S_\phi = \int dt \int d^3\underline{x}\ a^3(t)\left[\frac{1}{2c^2}\dot{\phi}^2 - \frac{1}{2}\nabla_{\underline{x}}\phi\cdot\nabla_{\underline{x}}\phi - V(\phi)\right] \quad (3.16)$$

using comoving coords $d^3\underline{r} = a^3(t)\,d^3\underline{x}$.



e.g $V = \frac{1}{2}m^2\phi^2$

For a uniform field $\phi(\underline{x},t) \equiv \phi(t)$, variation $\frac{\delta S_\phi}{\delta\phi}$ of the action (3.16) yields EoM

$$\ddot{\phi} + 3H\dot{\phi} + c^2\frac{dV}{d\phi} = 0 \quad (3.17)$$

Exercise Perform $\delta\phi$-variation on (3.16). IBP and ignore boundary terms to find (3.17).

The scaler field acts as a uniform fluid with energy density and pressure given by

$$\rho = \underbrace{\frac{1}{2c^2}\dot{\phi}^2}_{\text{field KE}} + \underbrace{V(\phi)}_{\text{field PE}} \quad (3.18)$$

and

$$P = \frac{1}{2c^2}\dot{\phi}^2 - V(\phi) \tag{3.19}$$

Exercise Substitute (3.18) into energy conservation eqn (1.25) to validate pressure in (3.19)

From (2.42) Friedmann eqn

$$H^2 = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{3.20}$$

and accel eqn

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \tag{3.21}$$

Inflation occurs when vacuum energy dominates $V(\phi) \gg \frac{1}{2c^2}\dot{\phi}^2$

So we have $P = -\rho$ in (3.18), (3.19) (ie. $w \approx -1$ in (1.27) approximately const), ie the condition for accel $\ddot{a} > 0$ in (3.21). The effective cosmological const. is

$$\Lambda_{eff} = \frac{8\pi G}{c^4} V(\phi)$$

and from (3.20),

$$H = \sqrt{\frac{8\pi G}{3c^2} V(\phi)}$$

In most models, inflation occurs $t_{inf} \approx 10^{-35}$ s, ending with "reheating" at $t = t_{reheat}$ when $\frac{1}{2c^2}\dot{\phi}^2 \approx V(\phi)$. The vacuum energy is converted into relativistic particles, restarting the Hot Big Bang.

Canonical example : Slow-roll inflation

During inflation with $V(\phi) \gg \frac{1}{2c^2}\dot{\phi}^2$, and we have overdamped evolution

$$\ddot{\phi} \ll 3H\dot{\phi}.$$

Then

$$3H\dot{\phi} = -c^2 \frac{dV}{d\phi} \tag{3.22}$$

and

$$H^2 = \frac{8\pi G}{3c^2} V(\phi) \tag{3.23}$$

Eqns (3.22), (3.23) are known as the <u>slow-roll approximation</u>.

Now consider a massive scalar field (mass m) with potential

$$V(\phi) = \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi^2 \equiv \frac{1}{2}M^2\phi^2 \tag{3.24}$$

Eqn (3.22) becomes

$$H\dot{\phi} = -\frac{c^2}{3}M^2\phi^2 \tag{3.25}$$

and (3.23) becomes

$$H^2 = \frac{4\pi G}{3c^2}M^2\phi^2 \equiv c^2\frac{M^2}{M_{pl}^2}\phi^2 \tag{3.26}$$

where $M_{pl} = \sqrt{3c^2/4\pi G}$.

Note: Natural units more convenient $c = G = \hbar = k_B = 1$.

Here $[\rho] = [E/V] = kg\, m^{-1}s^{-2} \Rightarrow [\phi^2] = kg\, m s^{-2}$.

Eliminate H, then

$$\dot{\phi} = -\frac{c}{3}M M_{pl}$$

$$\Rightarrow \phi(t) = \phi_i - \frac{1}{3}M M_{pl}\, c(t - t_i) \tag{3.27}$$

With initial $\phi(t_i) = \phi_i$

Now substitute into (3.26)

$$\frac{\dot{a}}{a} = \frac{c\mathcal{M}\phi}{M_{pl}}$$

to find

$$a(t) = \exp\left(\frac{3}{2M_{pl}^2}\left(\phi_i^2 - \phi^2(t)\right)\right) \qquad (3.28)$$

This is quasi-exponential expansion with N e-folds of expansion

$$N = \log\left(\frac{a}{a_i}\right) \simeq \frac{3}{2}\frac{\phi_i^2}{M_{pl}^2} \qquad (3.29)$$

e.g. $N \geq 60$, $\phi_i \geq 6 M_{pl}$.

Reheating

Inflation ends when KE $\frac{1}{2c^2}\dot{\phi}^2 \gtrsim \frac{1}{2}\mathcal{M}^2\phi^2$ PE, i.e.

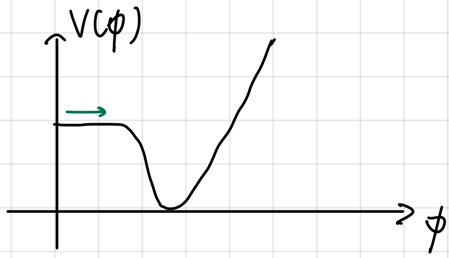
$$\phi_{reh} \approx \frac{M_{pl}}{3}$$

using $\dot{\phi}$ from (3.27). Oscillation about $\phi \approx 0$ begin. (damped harmonic oscillator) and scalar field decays into thermal particles.

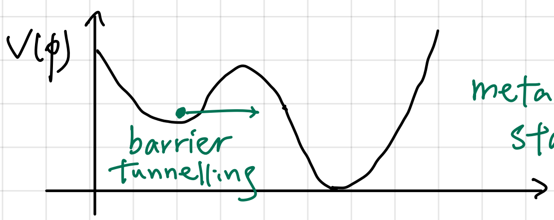
Classes of inflation

- Large field inflation (or chaotic inflation): see earlier.
- Small field inflation

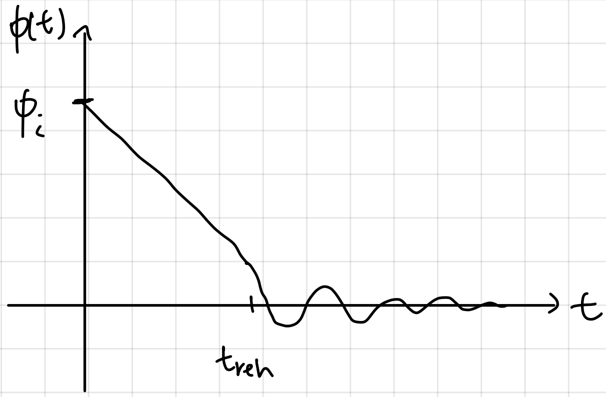
flat potential
$\Rightarrow \dot{\phi}^2 \ll V(\phi)$

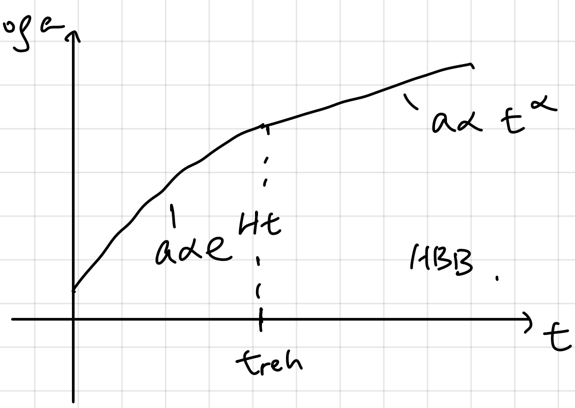


- False vacuum. (old inflation)



metastable state





Motivation for Inflation

- **Flatness problem**: Rewrite conformal Fried (2.40) as

$$\Omega - 1 = \frac{kc^2}{a^2H^2}$$

Then during inflation, $\Omega - 1 \to 0$ is an "attractor" because with $\ddot{a} > 0$, $a^2H^2 = \dot{a}^2$ increases with t.

During inflation,

$$|\Omega_{reh} - 1| \approx \left(\frac{a_{reh}}{a_i}\right)^2 |\Omega_i - 1| \qquad (3.30)$$

where $a_{reh}/a_i \approx e^{H(t_{reh} - t_i)}$. Require 60 e-folds

$$N = \log\left(\frac{a_{reh}}{a_i}\right) \approx H(t_{reh} - t_i) \geq 60.$$

- **Horizon problem**: Consider the evolution of comoving scale corresponding to the Hubble radius Today (cH_0^{-1}) at $t = t_0$, i.e. $a(t)\,(cH_0^{-1})$.

At time $t = t_{reh}$, this comoving scale had a physical scale size

$$\frac{a(t_{reh})}{a(t_0)}(cH_0^{-1}) \gg cH_{reh}^{-1}.$$

But, during inflation, $H_i \approx \text{const} \approx H(t_{reh})$, so this comoving length scale becomes comparable to the Hubble radius $cH_I^{-1} \approx cH_{reh}^{-1}$ when

$$\frac{a(t_I)}{a(t_{reh})} \frac{a(t_{reh})}{a(t_0)} cH_0^{-1} = cH_I^{-1}$$

ie. at $t = t_I$ defined by

$$a(t_I) H_I = a(t_0) H_0 \qquad (3.31)$$

inflation

Hot big bang

So the observable universe could have begun inside one Hubble radius (ie. in casual contact).

<u>Exercise</u> Show that a comoving scale x today ($a_0 = 1$) will exit the horizon (Hubble radius) at time t_I (during inflation) and re-enter at t_H during HBB given by

$$\boxed{a(t_I) H_I = a(t_H)\, H(t_H) = c x^{-1}} \qquad (3.32)$$

<u>Origin of quantum fluctuations</u>

Heisenberg's uncertainty principle $\Delta x\, \Delta p \geq \hbar/2$, or $\Delta E\, \Delta t \geq \hbar/2$ is relavant during inflation because of the event horizon $\Delta x \approx cH_I^{-1}$ or $\Delta t = H_I^{-1}$.

$$\Rightarrow \quad \Delta E \sim \hbar/\Delta t \sim \hbar H_I .$$

But energy in a Hubble volume

$$E = \rho (cH_I^{-1})^3 = \left(\frac{3c^2}{8\pi G}\right) \cdot \left(\frac{c^3}{H_I^3}\right) \sim \frac{(m_{pl} c^2)^2}{\hbar H_I}$$

with $m_{pl} = \left(\frac{\hbar c}{G}\right)^{1/2}$ Planck mass.

So

$$\frac{\delta\rho}{\rho} = \frac{\delta G}{G} = \left(\frac{\hbar H_I}{m_{pl} c^2}\right)^2 \tag{3.33}$$

These quantum fluctuations (const. amplitude) exit during inflation and re-enter the universe during HBB at $t = t_H$ in (3.32). After $t > t_{eq}$, these inhomogenities can grow to form structures like galaxies etc. (§ 5).

4. Thermal History of the Universe

4.1 Matter Content of the Universe

*The standard model

- There are 12 building blocks — quarks, leptons, out of which baryons (protons and neutrons) are made.

- Forces between particles mediate by gauge bosons — gluons, vector bosons $W^{\pm}$, Z, γ photons — plus Higgs scaler Φ and gravitons $h_{\mu\nu}$.

Bosons and Fermions

Two indistinguishable particles in states $\phi_k(1)$ and $\phi_\ell(2)$.

Wavefn $\psi(1,2)$, could be $\phi_k(1)\,\phi_\ell(2)$, or $\phi_k(2)\,\phi_\ell(1)$, but

$$|\psi(1,2)|^2 = |\psi(2,1)|^2$$

- Bosons are symmetric under $1 \leftrightarrow 2$

$$\psi_B(1,2) = \phi_k(1)\,\phi_\ell(2) + \phi_k(2)\,\phi_\ell(1) = \psi_B(2,1)$$

If $k=\ell$, any no. of particles possible. $n_k = 0, 1, 2, 3, \ldots \quad \forall k$ (4.1)

- Fermions are antisymmetric $1 \leftrightarrow 2$

$$\psi_F(1,2) = \phi_k(1)\,\phi_\ell(2) - \phi_k(2)\,\phi_\ell(1) = -\psi_F(2,1)$$

So occupations is never above unity.

$$n_k = 0, 1 \quad \forall k \tag{4.2}$$

This is Pauli exclusion principle.

4.2 Laws of Thermodynamics

An isolated system is in equilibrium if its state variables do not change in time, e.g. energy E, volume V, particle no. N (extensive change with system size), temperature T, pressure p, chemical potential μ (intensive).

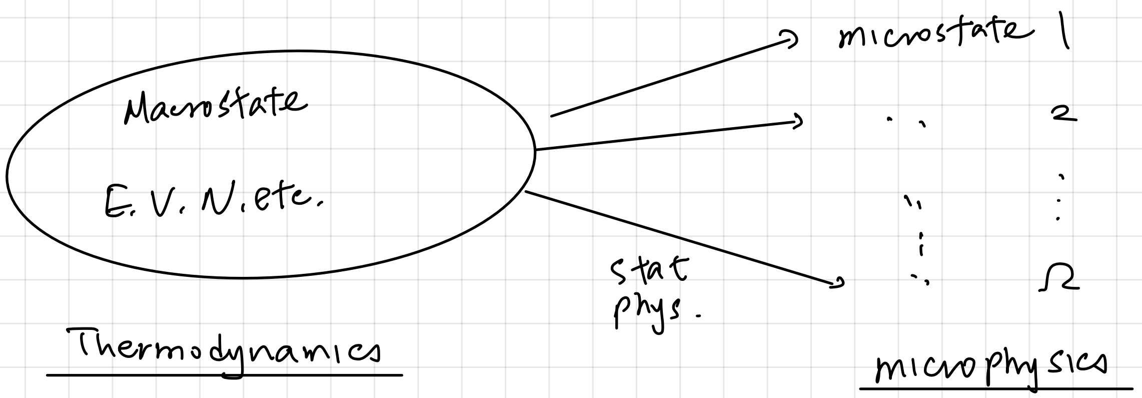
For every macrostate of a system (E, V, N, etc) there are many possible microstates, a very large number Ω.

We assume that each of these microstates is equally likely (principle III).

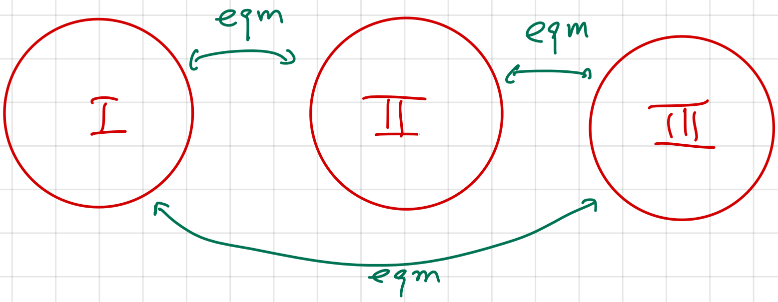
The entropy S of the system is defined as

$$S = k \log \Omega \tag{4.3}$$

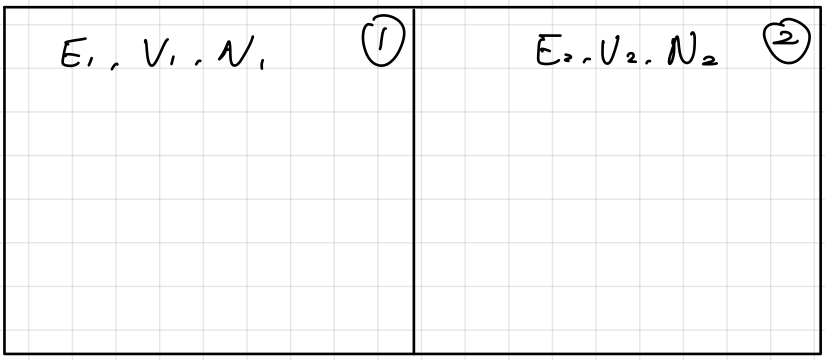
where $k = k_B = 1.38 \times 10^{-23}\ J\,K^{-1}$ is the Boltzmann's constant.



Zeroth law: Two systems in equilibrium with each other have the same temp. $T_1 = T_2 = T$, pressure $P_1 = P_2 = P$, and chemical potential $\mu_1 = \mu_2 = \mu$.



Consider two systems with fixed E, V, N.



The total no. of microstates of combined system of combined system

$$\Omega = \Omega_1 \Omega_2 \qquad (4.4)$$

So entropy is additive.

$$S = S_1 + S_2 \qquad (4.5)$$

Allow exchange of energy E_i (at fixed V_i, N_i), so given E_1 for system 1, we have

$$E_2 = E - E_1 \qquad (*)$$

This implies $S = S_1(E_1, V_1, N_1) + S_2(E - E_1, V_2, N_2)$.

Most probable partition maximises Ω (and thus S) w.r.t. changing E, i.e. $\dfrac{\partial S}{\partial E_1} = 0$ (+).

But $\frac{\partial S}{\partial E_1} = \frac{\partial S_1}{\partial E_1}(E_1, V_1, N_1) + \frac{\partial S_2}{\partial E_1}(E-E_1, V_2, N_2)$

$$= \frac{\partial S_1}{\partial E_1} - \frac{\partial S_2}{\partial E_2} = 0.$$

So we have

$$\frac{\partial S_1(E_1, V_1, N_1)}{\partial E_1} = \frac{\partial S_2(E_2, V_2, N_2)}{\partial E_2} \tag{4.6}$$

Define temperature by

$$\frac{\partial S}{\partial E} = \frac{1}{T} \tag{4.7}$$

So eqm requires

$$T_1 = T_2 \tag{4.8}$$

Similarly, for volume V exchange, define

$$\frac{\partial S}{\partial V} = P \tag{4.9}$$

implies

$$P_1 = P_2 \tag{4.10}$$

Exchange of particles N, define

$$\frac{\partial S}{\partial N} = -\mu/T \tag{4.11}$$

implies

$$\mu_1 = \mu_2 \tag{4.12}$$

First law: Energy is conserved

$$dE = \underbrace{T\,dS}_{\text{"heat energy" } dQ} - \underbrace{P\,dV}_{\text{mech work "}dW\text{"}} + \underbrace{\mu\,dN}_{\text{chem energy}} \tag{4.13}$$

This can be obtainded from $S(E, V, N)$

$$dS = \frac{\partial S}{\partial E}dE + \frac{\partial S}{\partial V}dV + \frac{\partial S}{\partial N}dN$$

$$= \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$$

and rearrange.

Second law: For an isolated system, entropy change is always non-negative.

$$\Delta S \geq 0 \qquad (4.14)$$

Chemical potentials: Affinity ν_i for a general interaction

$$\sum \underset{\text{no. of particles}}{\nu_i} \underset{\text{species label}}{A_i} = 0 \qquad (4.15)$$

is chemical eqm (for fixed E, V) from (4.13) implies

$$\sum \mu_i dN_i = 0 \qquad (4.16)$$

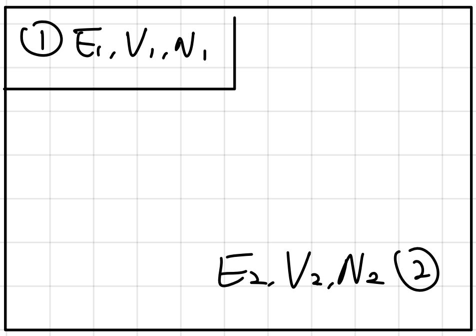
E.g. nuclear reaction $\quad p + n \longleftrightarrow D, \; \nu_p = 1, \; \nu_n = 1, \; \nu_D = -1$

then at eqm,

$$\mu_p + \mu_n = \mu_D.$$

4.3 Equilibrium distribution

Boltzmann distribution



Two systems $E_1 \ll E_2$ etc, but fixed total E, V, N (Fix V_1, T, μ)

Label microstates for system 1.

Taking $N_1 = n$, non-degenerate eigenstates $r^{(n)}$ with evals $E_r^{(n)}$ with

$$E_1^{(n)} < E_2^{(n)} < \ldots < E_r^{(n)} < \ldots \qquad (4.17)$$

Recall (4.4) $\Omega = \Omega_1 \Omega_2$, so probability $p_r^{(n)}$ of microstate $r^{(n)}$ is

$$p_r^{(n)} = \frac{\text{no. of microstates of system 2}}{\text{total microstates}} = \frac{\Omega_2(E_2, V_2, N_2)}{\Omega(E, V, N)}$$

where $E = E_r^{(n)} + E_2$, $N = n + N_2$ and ignore fixed V.

From entropy def^n,

$$\Omega_2(E_2, N_2) = \exp\left(\frac{1}{k} S_2(E_2, N_2)\right)$$
$$= \exp\left(\frac{1}{k} S_2(E - E_r^{(n)}, N - n)\right)$$

So

$$S_2(E - E_r^{(n)}, N - n) = S_2(E, N) - \frac{\partial S_2}{\partial E_2} E_r^{(n)} - \frac{\partial S_2}{\partial N_2} n + \dots$$
$$= \text{const.} - E_r^{(n)}/T - \mu n/T + \dots$$

Hence in eqm, we have

$$p_r^{(n)} = \text{const.} \times \exp\left(\frac{1}{kT}(\mu n - E_r^{(n)})\right)$$
$$= \frac{\exp(\beta(\mu n - E_r^{(n)}))}{Z} \qquad (4.19)$$

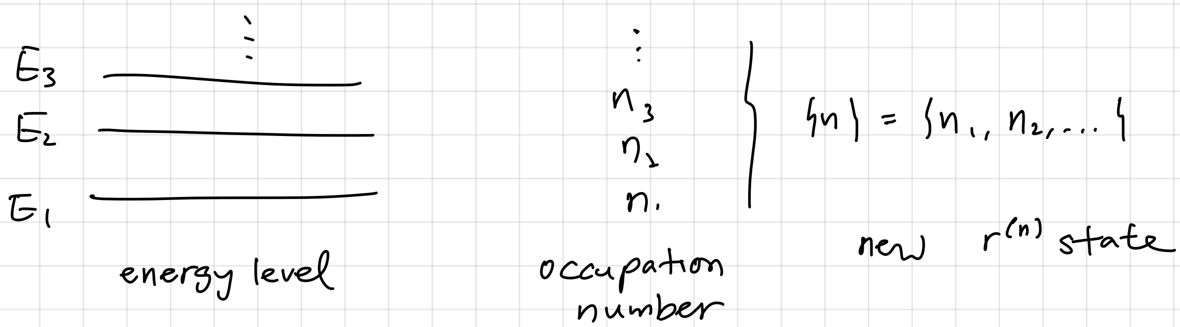
where $\beta = \frac{1}{kT}$ and normalisation factor $\left(\sum_{n,r} p_r^{(n)} = 1\right)$

$$Z = \sum_{n,r} \exp(\beta(\mu n - E_r^{(n)})) \qquad (4.20)$$

known as the <u>partition function</u>.

<u>Ideal gas</u> : Assume identical partition in one of the discrete set of energy levels

$$E_1 < E_2 < E_3 < \dots \qquad (4.21)$$



n-particles $r^{(n)} \to \{n_1, n_2, n_3\}$

$$n = \sum_k n_k \, , \quad E_r^{(n)} = \sum_k n_k E_k \qquad (4.22)$$

Boltzmann-Gibbs distribution factorises

$$p_r^{(n)} \longrightarrow p\{n\} = \text{const.} \ \exp\left(\beta \sum_k (\mu - E_k)\right) \tag{4.23}$$

$$= \prod_k p_k(n_k)$$

where

$$p_k(n_k) = \exp\left(\beta n_k (\mu - E_k)\right) / Z_k \ . \tag{4.24}$$

and

$$Z_k = \sum_n \exp\left(\beta n (\mu - E_k)\right) . \tag{4.25}$$

This probability yields the avg value of n_k.

$$\bar{n}_k = \sum_{n_k} n_k \, p_k(n_k) = \frac{\sum_{n_k} n_k \exp\left(\beta n_k (\mu - E_k)\right)}{\sum_{n_k} \exp\left(\beta n_k (\mu - E_k)\right)}$$

$$\Rightarrow \bar{n}_k = kT \frac{\partial \log Z_k}{\partial \mu} \tag{4.26}$$

4.4 Statistics of Bosons and Fermions

Bose-Einstein distribution

Bosons can have $n_k = 0, 1, 2, 3, \ldots \ \forall k$. From (4.25),

$$Z_k = \sum_{n=0}^{\infty} \left(e^{\beta(\mu - E_k)}\right)^n = \sum_{n=0}^{\infty} A_k^n = \frac{1}{1 - A_k} = \frac{1}{1 - e^{\beta(\mu - E_k)}} \tag{4.27}$$

if $|A_k| < 1$.

Diff using (4.26)

$$\bar{n}_k(E_k) = \frac{g_k}{e^{\beta(E_k - \mu)} - 1} \qquad \text{(bosons)} \tag{4.28}$$

where g_k is the degeneracy (e.g. photons $g_k = 2$ for two polarisations.)

Fermi-Dirac distribution

Pauli exclusion $\Rightarrow$ $n_k = 0, 1$ only.

$$Z_k = \sum_{n=0}^{1} e^{\beta n(\mu - E_k)} = 1 + e^{\beta(\mu - E_k)}$$

Thus (4.26)

$$\Rightarrow \bar{n}_k(E_k) = \frac{g_k}{e^{\beta(E_k - \mu)} + 1} \quad \text{(fermions)} \qquad (4.29)$$

E.g. electron $g_k = 2$, spin "up" and spin "down".

Maxwell-Boltzmann distribution

If ground state $E_1 - \mu \gg kT = \frac{1}{\beta}$ $\Rightarrow$ $\forall k$, $E_k - \mu \gg kT$, so $e^{\beta} \gg 1$, implies

$$\bar{n}_k(E_k) \approx g_k e^{-\beta(E_k - \mu)} \quad \binom{\text{bosons}}{\text{fermions}} \qquad (4.30)$$

Density of states

Consider momentum estates of particles in a box ($V = L^3$)

$$p = \frac{h}{L}(n_1, n_2, n_3), \quad n_i \in \mathbb{Z}$$

In the continuum limit on a shell, $p \to p + dp$. no. of e-states is

$$4\pi p^2 \, dp \, \frac{L^3}{h^3} \equiv f(p)\, dp.$$

sph. shell total states

where

$$f(p) = \frac{4\pi V}{h^3} p^2 \qquad (4.31)$$

In 3D, (4.28 - 29) become

$$\bar{n}(p)\, dp \equiv \frac{4\pi g_s}{h^3} \frac{p^2 \, dp}{e^{\beta(E(p) - \mu)} \mp 1} \quad \begin{cases} - & \text{bosons} \\ + & \text{fermions} \end{cases} \qquad (4.32)$$

where g_s is degeneracy and

$$E(p) = \sqrt{(pc)^2 + (mc^2)^2} \tag{4.33}$$

Particle no. (average total)

$$N = \int \bar{n}(p)\, dp \tag{4.34}$$

or number density

$$n = N/V$$

Energy E is

$$E = \int E(p)\, \bar{n}(p)\, dp \tag{4.35}$$

or energy density

$$\rho = E/V$$

Pressure P : Consider a slowly changing volume $V = L^3$, then the momentum state $p \propto h/L \propto V^{-1/3}$. So

$$\frac{dp}{dV} = -p/3V \tag{4.36}$$

If eqm maintained (entropy const. $dS = 0$), then occupation no. does not change, so $\bar{n}(p)\, dp$ is same, and N remains fixed. (Ehrenfest principle).

So overall change in energy is

$$\begin{aligned} dE &= \int_0^\infty dE(p)\, \bar{n}(p)\, dp \\ &= dV \int_0^\infty E'(p) \frac{dp}{dV} \bar{n}(p)\, dp \\ &= -\frac{dV}{3V} \int_0^\infty p\, E'(p)\, \bar{n}(p)\, dp. \end{aligned}$$

But $dS = dN = 0$. we have $\left(\frac{\partial E}{\partial V}\right)_{N,S} = -P$ by 1st law (4.13),

Hence,

$$P = \frac{1}{3V}\int_0^\infty p\,E'(p)\,\bar{n}(p)\,dp \tag{4.37}$$

<u>Ultrarelativistic limit</u> $(kT \gg mc^2, \mu)$

$$E(p) = \sqrt{(pc)^2 + (mc^2)^2} \approx pc$$

$$\Rightarrow p\,E'(p) = E_p$$

So (4. 37) becomes

$$P = \frac{1}{3V}\int_0^\infty E(p)\,\bar{n}(p)\,dp = E/3V$$

$$\Rightarrow \qquad P = \frac{1}{3}\rho \tag{4.38}$$

satisfying (1.27) and (2.27)

<u>Non-relativistic limit</u> $(kT \ll mc^2)$

$$E(p) = mc^2 + p^2/2m$$

with $KE = U(p) = p^2/2m$, so $p\,E'(p) = 2U(p)$, so

$$P = \frac{2}{3}\int_0^\infty U(p)\,\bar{n}(p)\,dp$$

$$\Rightarrow P = \frac{2}{3}\frac{U}{V} \tag{4.39}$$

where U is the internal energy (excluding rest mass).

<u>4.5 Classical ideal gas</u>

NR gas with Maxwell-Boltzman distribution (4.31), $E(p) = mc^2 + p^2/2m$

$$N = \frac{4\pi V g_s}{h^3} e^{\beta(\mu - mc^2)} \int_0^\infty e^{-\beta p^2/2m} p^2\,dp$$

Gaussian

$$= \frac{4\pi V g_s}{h^3} e^{\beta(\mu - mc^2)} \sqrt{\frac{\pi}{2}} \left(\frac{m}{\beta}\right)^{3/2}$$

by IBP $\quad p^2 e^{-bp^2} = -\frac{1}{2b} p \frac{d}{dp}\left(e^{-bp^2}\right)$. Hence

$$n = \frac{N}{V} = g_s \left(\frac{2\pi m kT}{h^2}\right)^{3/2} \exp\left((\mu - mc^2)/kT\right) \qquad (4.40)$$

where $n_Q = \left(\frac{2\pi m kT}{h^2}\right)^{3/2}$ is the quantum concentration.

Exercise Evaluate internal energy $U(p) = p^2/2m$

$$U = \frac{4\pi g_s V}{h^3} e^{\beta(\mu - mc^2)} \int_0^\infty dp \left(\frac{p^2}{2m}\right) p^2 \, e^{-\beta p^2/2m}$$

$$= \left(\qquad \right) \left(-\frac{\partial}{\partial\beta} \sqrt{\frac{\pi}{2}} \left(\frac{m}{\beta}\right)^{3/2}\right)$$

$$= \frac{3}{2\beta} N$$

So

$$U = \frac{3}{2} NkT \qquad (4.41)$$

Note equipartition of energy $\frac{1}{2}kT$ for energy degree of freedom.

Combine (4.39), we get Boyle's law

$$pV = NkT \qquad (4.42)$$

4.6 Ultrarelativistic ideal gas

With $kT \gg mc^2 \gg \mu$, using (4.32) for bosons with $E(p) \simeq pc$,

$$n_b = \frac{N_b}{V} = \frac{4\pi g_s}{h^3} \int \frac{p^2}{e^{\beta pc} - 1} dp$$

$$= \frac{4\pi g_s}{(hc)^3} (kT)^3 \int_0^\infty \frac{y^2 dy}{e^y - 1}, \quad y = \frac{pc}{kT}.$$

Integral

$$I_n = \int_0^\infty \frac{y^n}{e^y - 1} dy = \zeta(n+1)\,\Gamma(n+1) \qquad (*)$$

Given $\zeta(3) = 1.202\ldots$, we have

$$n_b = \frac{8\pi \zeta(3)}{(hc)^3} g_s (kT)^3 \qquad \text{(boson)} \qquad (4.43)$$

For energy density, integrate with $E = pc$,

$$\rho_b = \frac{4\pi g_s}{h^3} \int \frac{c\,p^3}{e^{\beta pc}-1}\,dp$$

$$\overset{(*)}{=} \frac{4\pi g_s}{(hc)^3}\, g_s\, (kT)^4\, I_3$$

$$= \frac{4\pi^5}{15(hc)^3}\, g_s\, (kT)^4 \qquad \text{(bosons)} \qquad (4.44)$$

since $\zeta(4) = \pi^4/90$.

For fermions, require

$$\int \frac{y^n}{e^y+1}\,dy = \int \frac{y^n(e^y-1)}{e^{2y}-1}\,dy$$

$$= \int \left(\frac{y^n}{e^y-1} - \frac{2y^n}{e^{2y}-1} \right) dy$$

$$= \left(1 - \frac{1}{2^n}\right) I_n$$

So using (4.32),

$$n_f = \frac{3}{4}\, n_b \qquad (4.45)$$

and

$$\rho_f = \frac{7}{8}\, \rho_b \qquad (4.46)$$

Effective no. of relativistic degrees of freedom

$$g_* = \sum_{\text{bosons}} g_i + \frac{7}{8} \sum_{\text{fermions}} g_i \qquad (4.47)$$

with total energy

$$\rho_r = \frac{4\pi^5}{15(hc)^3}\, g_*(T)\, (kT)^4 = \frac{2\sigma}{c}\, g_*(T)\, T^4 \qquad (4.48)$$

where $\sigma = \dfrac{2\pi^5 k^4}{15 h^3 c^2} = 5.67 \times 10^{-8}$.

4.7 Photon gas and Planck spectrum

Consider photons in equilibrium for $t < 400.000$ yrs.

- Massless, so $E(p) = pc = h\nu$. $\Rightarrow p = \frac{h\nu}{c}$.
- Photon no. not conserved (e.g. $e^- + \gamma \to e^- + \gamma + \gamma$), so from (4.6), we have $\mu_\gamma = 0$
- Bosons with 2 polarisation states, so $g_\gamma = 2$.

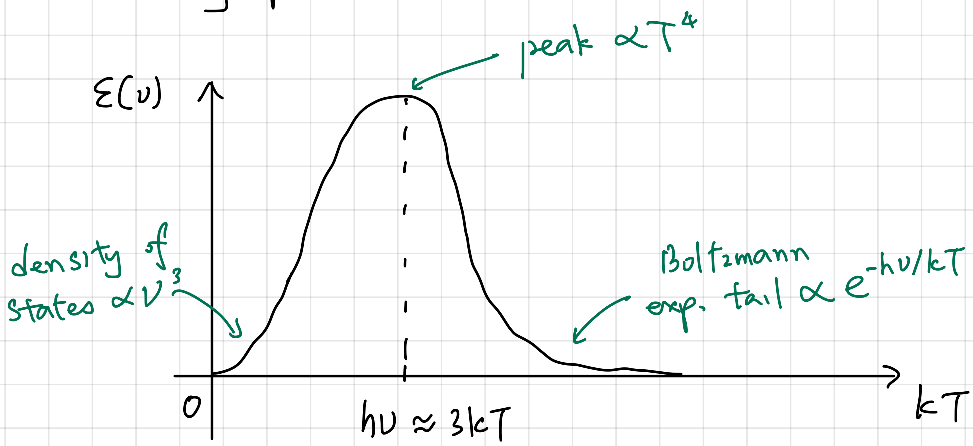
Bose-Eistein distribution (4.32) yields Planck spectrum: photons number density is

$$\bar{n}(p)\,dp \longrightarrow \bar{n}(\nu)\,d\nu = \frac{16\pi}{c^3}\frac{\nu^2 d\nu}{\exp(h\nu/kT)-1} \tag{4.49}$$

Energy density

$$\mathcal{E}(\nu)\,d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{\exp(h\nu/kT)-1} \tag{4.50}$$

which is black body spectrum



Peak of spectrum ($\mathcal{E}'(\nu) = 0$) about

$$h\nu_{peak} \approx 3kT \tag{4.51}$$

The shape is independent of temp.

The most accurately known Planck spectrum is the cosmic microwave with temp.

$$T_0 = 2.7255 \pm 0.0006 \tag{4.52}$$

by COBE satellite.

Integrate (4.50) over all frequencies ν to obtain Stefan-Boltzmann law

$$\rho_\gamma = \frac{4\sigma}{c} T^4 \tag{4.53}$$

From (4.43), today we have

$$n_\gamma = \frac{16\pi}{hc^3} \zeta(3) (kT)^3 = 4.11 \times 10^8 \ /m^3 . \tag{4.54}$$

Comparing to baryons $n_{B0} = \frac{\Omega_{B0} \rho_{crit}}{m_{proton}} = 0.24 \ /m^3$

Why is the CMB Planck spectrum so perfect?

Decoupling at $T \approx 4000K$, so it is not in eqm now! Red shifts of photons

$$\nu(t) = \frac{a(t_{dec})}{a(t)} \nu_{dec} \tag{4.55}$$

and dilute $\rho_\gamma \propto \left(\frac{a(t_{dec})}{a(t)}\right)^4$, means Planck spectrum is preserved after decoupling with temp.

$$T(t) = \frac{a(t_{dec})}{a(t)} T_{dec} \tag{4.56}$$

4.8 Photon Energy

First law for photons $(\mu_\gamma = 0)$ is

$$dE = TdS - PdV.$$

So

$$\begin{aligned} TdS &= dE + PdV \qquad \text{(4.38) } P = \tfrac{1}{3}\rho \\ &= d(\rho V) + \tfrac{1}{3}\rho dV \\ &= \tfrac{4}{3}\rho dV + V d\rho \\ &= \frac{4\sigma}{c}\left(\frac{4}{3}T^4 dV + 4T^3 V dT\right) \\ &= \frac{4\sigma}{c}\frac{4}{3}\left[T^4 dV + 3T^3 V dT\right] \end{aligned}$$

$$\Rightarrow dS = \frac{16\sigma}{3c}\left(T^3 dV + 3T^2 V dT\right) = \frac{16\sigma}{3c} d(T^3 V)$$

Integrate.
$$S = \frac{16\sigma}{3c} T^3 V + S_0 ,$$

but third law $S \to 0$ as $T \to 0 \Rightarrow S_0 = 0$, thus entropy density
$$s = \frac{S}{V} = \frac{16\sigma}{3c} T^3 \qquad (4.57)$$

Now, if interaction rate Γ is much faster than the expansion rate H,
$$\Gamma \gg H \qquad (4.58)$$

The expansion is quasi-static and universe passes slowly through a cts series of equilibrium states, with the occupation no. $N \propto VT^3 = \text{const}$, so likewise entropy $S \propto VT^3 = \text{const}$.

But with $V \propto a^3$, we have $T \propto 1/a$, so photons undergo adiabatic cooling.

In radiation era (2.28), $a \propto t^{1/2}$, so cosmic time $t \propto 1/T^2$ and $H = \frac{1}{2t} \propto T^2$.

effective rel. deg. of freedom
$$H = \sqrt{\frac{8\pi G \rho_\varepsilon}{3c^2}} = \sqrt{\frac{32\pi G\sigma}{3c^3}}\, g_*^{1/2} T^2 .$$
$$\approx 2.17 \times 10^{-21}\, g_*^{1/2} T^2 \qquad (4.60)$$

Particle no. N_s

Photon no. N_γ conserved, cooling with $T \propto 1/a$. Compare to other species N_s with conserved particle no.
$$\frac{N_s}{N_\gamma} = \frac{n_s}{n_\gamma} = \text{const}. \qquad (4.61)$$

key example: baryon-to-photon ratio (4.55)
$$\eta \equiv \frac{n_B}{n_\gamma} = 5.9 \times 10^{-10} \quad (\text{for } t < 10\,s)$$

4.9 Recombination

When $T \lesssim 5000\,K$ (about 300k years), protons and electrons begin to combine to form hydrogen.

$$p + e^- \longrightarrow H + \gamma$$

Chemical eqm:

$$\mu_p + \mu_{e^-} = \mu_H + \cancel{\mu_\gamma}^{=0} \qquad (4.62)$$

Charge neutrality:

$$n_e = n_p \qquad (4.63)$$

From (4.40), number density for NR particles is

$$n_i = g_i \left(\frac{2\pi m_i kT}{h^2}\right)^{3/2} e^{(\mu - mc^2)/kT}$$

Divide these to eliminate μ's in (4.62)

$$\frac{n_e n_p}{n_H} = \frac{g_e g_p}{g_H}\left(\frac{2\pi m_e m_p kT}{m_H h^2}\right)^{3/2} \exp\left(\underbrace{-(m_p + m_e - m_H)}_{\text{ionisation energy}} c^2/kT\right)$$

Here, $g_H = 4$, $m_H \simeq m_p$, so using ionisation energy $I = 13.6\,eV$, we obtain Saha's eqn

$$\frac{n_e^2}{n_H} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-I/kT} \qquad (4.64)$$

Usually expressed with free electrons.

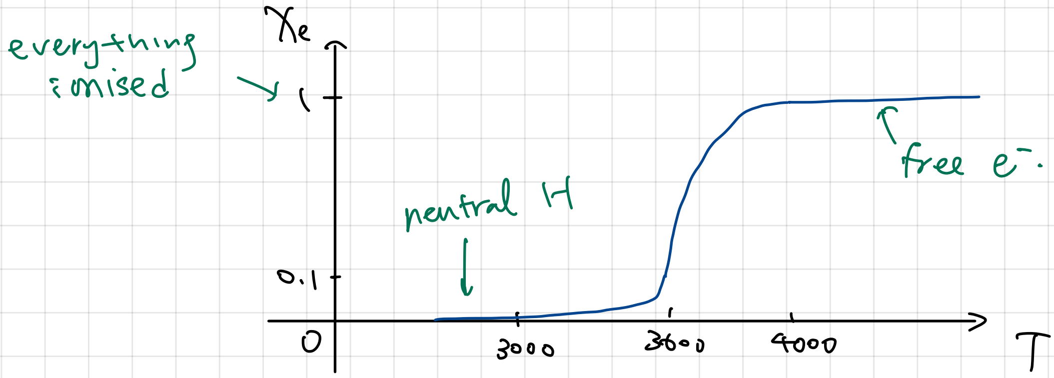
Ratio:

$$X_e = \frac{n_e}{n_B} = \frac{n_e}{n_p + n_H} \qquad (4.65)$$

$$\Rightarrow n_e = X_e n_B = X_e \eta n_\gamma$$

Exercise Show Saha becomes

$$\frac{1 - X_e}{X_e} = \eta \zeta(3) \sqrt{\frac{32}{\pi}} \left(\frac{kT}{c^2}\right)^{3/2} e^{I/kT} \qquad (4.66)$$



For $X_e = 0.1$, can solve to show that $T \simeq 3600\ K$ ($kT = 0.31\,eV$)

4.10 Photon decoupling

Photons and free e^- interact via (elastic) Thomson scattering ($h\nu \ll m_e c^2$)

$$e^- + \gamma \leftrightarrow e^- + \gamma$$

with the interaction rate

$$\Gamma_\gamma = n_e \sigma_T c \qquad (4.68)$$

with the cross-section $\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2 = 6.7\times 10^{-29} m^2$.

Photons fall out of eqm when

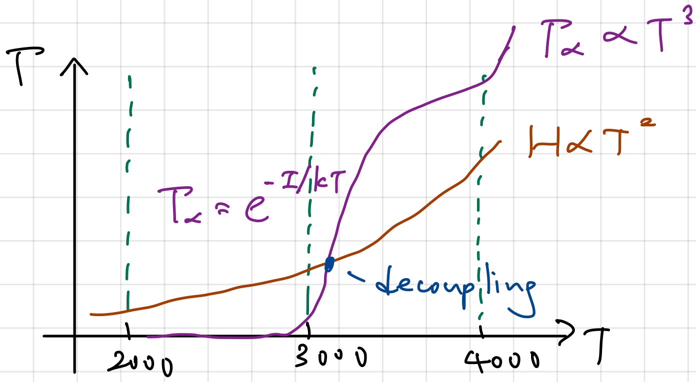
$$\Gamma_\gamma \lesssim H, \qquad (4.69)$$

ie. recombination $n_e \to 0$.

We know $H \simeq 4.3\times 10^{-21}\, T^2$, $g^*_{eff} = 3.9$ (γ, ν's).

We compare with

$$\Gamma_\gamma = n_e \sigma_T c = X_e \eta n_\gamma \sigma_T c = \begin{cases} 2.4\times 10^{-22} T^3 & T > 4000K \\ 4\times 10^{-11}\, T^{-3/2}\, e^{-8\times 10^4/T} & T \lesssim 3500K \end{cases} \qquad (4.70)$$



Setting $\Gamma = H$ above yields

$$T_{dec} \approx 3000\,K \qquad (kT \approx 0.26\,eV)$$

with $X_{dec} \simeq 0.004$ when $t_{dec} \simeq 380{,}000$ yrs. $z_{dec} \approx 1100$.

Decoupled photons maintain their eqm. dist. (4.50) as shown in (4.56) with $T \propto 1/a$.

Small <u>residual ionisation</u> left $X_e \approx 5 \times 10^{-4}$.

4.10 Precursor Lepton Decoupling

About $t \simeq 1s$ with $kT = 1$ MeV ($T = 10^9 K$), the universe is dominated by rel. lepton species

$$g_* = 2[\gamma] + \frac{7}{8}\left(4[e^-, e^+] + 6[\nu_i, \bar{\nu}_i]\right) = 10.75 \qquad (4.73)$$

Small no. of NR protons and neutrons

$$n_p + n_n = n_B \ll n_\gamma$$

Total entropy is

$$S = \frac{8\sigma}{3c} g_* T^3 \qquad (4.74)$$

(N.B. $1\,MeV = 10^6\,eV = 1.6 \times 10^{-13}\,J$)

<u>Neutrino decoupling</u> (massless or UR species)

Maintain eqm through weak interactions

$$e^- + \nu_e \longleftrightarrow e^- + \nu_e \;,\; e^- + e^+ \longleftrightarrow \nu_e + \bar{\nu}_e \text{, etc.}$$

With Fermi cross-section

$$\sigma_W = G_F^2 T^2,$$

where $\frac{G_F}{(\hbar c)^3} \sim \frac{10^{-5}}{(m_p c^2)^3} \sim 10^{-5}\,GeV^{-2}$, with $m_p c^2 = 938\,MeV$.

With interactions

$$\Gamma_{\nu_e} \approx n_e \sigma_W \sim 0.3\, G_F^2\, T^5 \tag{4.75}$$

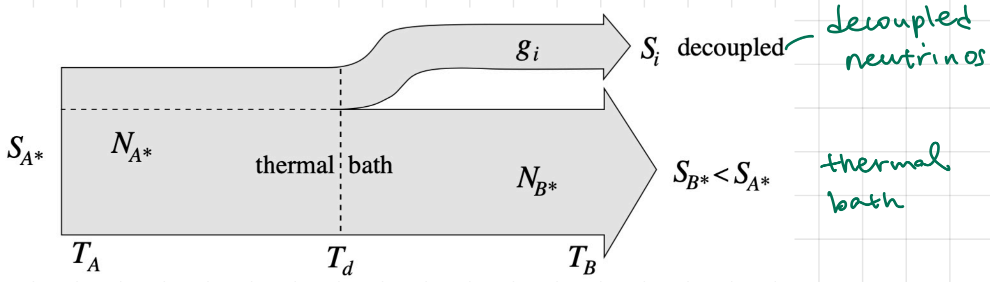
Comparing with H (4.60),

$$\left(\frac{\Gamma_\nu}{H}\right) \approx \left(\frac{kT}{1\,\mathrm{MeV}}\right)^3 \tag{4.76}$$

So neutrin decoupling at $T_{\nu_{dec}} \approx 1\,\mathrm{MeV}$.

Neutrinos still UR $g_\nu = \frac{7}{8} \times 6 = 5.25$, so they take entropy out of thermal bath

$$\frac{S_\nu}{S_*} = \frac{5.25}{10.75} = 49\,\%$$



$T_A > T_{dec} > T_B$

For $T_B < T_{\nu_{dec}}$, neutrinos temp

$$T_\nu \propto \frac{1}{a} \tag{4.77}$$

just like γ, but later T_ν differ from $T \equiv T_\gamma$.

So add all rel. dof.

$$\sum g_*^{eff} = \underbrace{\sum_b g_b + \frac{7}{8}\sum_f g_f}_{\text{thermal bath temp } T} + \underbrace{\sum_{b'} g_{b'}\left(\frac{T_{b'}}{T}\right)^3 + \frac{7}{8}\sum_{f'} g_{f'}\left(\frac{T_{f'}}{T}\right)^3}_{\text{decoupled massless/UR species}} \tag{4.78}$$

Electron-positron annilation (massive decoupling)

At temp. T below mass threshould $m_e c^2 = 0.511$ MeV (i.e. $kT \lesssim 0.5$ MeV)

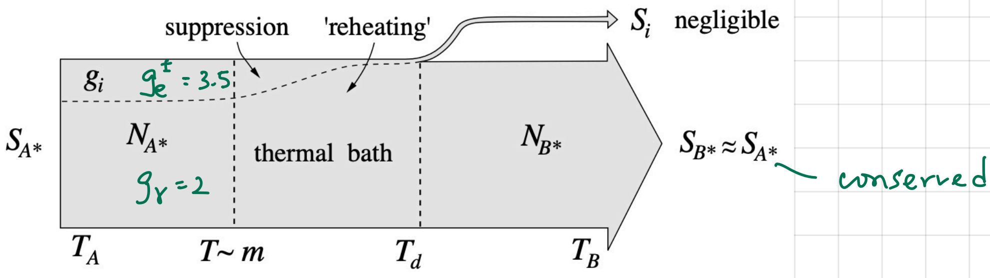
and e^+ are exponentially surpressed (by (4.40))

$$n = g_s \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} \exp\left((\mu - mc^2)/kT \right)$$

This is due to annilation

$$e^- + e^+ \longrightarrow \gamma + \gamma$$

This "reheats" photons. As entropy is conserved, temp. T of thermal bath must fall more slowly ("delayed cooling")



So equil. temp. falls slowly $T_{m_e} < T < T_{d_e}$. $S(T_A) = S(T_B)$

$$\Rightarrow \quad s(T_A)\, a^3(t_A) = s(T_B)\, a^3(t_B).$$

entropy density

$$\Rightarrow \quad \frac{T_B}{T_A} = \left(\frac{g_*(T_A)}{g_*(T_B)} \right)^{1/3} \frac{a(t_A)}{a(t_B)} = \left(\frac{5.5}{2} \right)^{1/3} \frac{a(t_A)}{a(t_B)} \qquad (4.79)$$

i.e. not simply $T \propto 1/a$.

Neutrino temperature

Assume $T_A > T_{m_e} > T_{d_e} > T_B$, neutrino temp falls as

$$\frac{T_B^\nu}{T_A^\nu} = \frac{T_B^\nu}{T_A} = \frac{a(t_A)}{a(t_B)} \qquad (4.80)$$

Combining with (4.79), when $T < T_{d_e}$, we have

$$\frac{T_\nu}{T} = \left(\frac{4}{11}\right)^{1/3} \qquad (4.81)$$

So today $T_0 = 2.73\,K \Rightarrow T_0^{\nu} = 1.95\,K$.

4.11 Light element synthesis

Neutron-proton ratio

At $t = 1s$ ($kT = 1\,MeV$), neutrons and protons are in eqm through weak interactions

$$n + \nu_e \longleftrightarrow p + e^-$$

$$n \longrightarrow p + e^- + \bar{\nu}_e \quad \text{(decay)}$$

with interaction rate

$$\frac{\Gamma_W}{H} \approx \left(\frac{kT}{0.7\,MeV}\right)^3 \qquad (4.82)$$

ie "effective" decoupling temp. $T \approx 0.7\,MeV$.

Divide NR eqm distributions

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left[\left((\mu_n - \mu_p) - (m_n - m_p)c^2\right)/kT\right] \approx \exp\left(-\frac{Q}{kT}\right) \qquad (4.83)$$

where $Q = (m_n - m_p)c^2 = 1.29\,MeV$ and assume

$$\frac{\mu_n - \mu_p}{kT} = \frac{\mu_e - \mu_{\nu_e}}{kT} \approx 0$$

(since $\frac{\mu_e}{kT} \approx \frac{\mu_{\nu_e}}{kT} \approx \eta \sim 10^{-9}$). Using $kT_D \approx 0.7\,MeV$, the neutrons "freeze out" with

$$\frac{n_n}{n_p} = \frac{1}{6} \qquad (4.84)$$

The neutrons decay continues with mean lifetime $\tau_n \approx 879\,s$ until $t = 100\,s$ ($kT \approx 0.1\,MeV$) when

$$\frac{n_n}{n_p} e^{-t/\tau_n} \approx \frac{1}{7}$$

Deuterium distribution

$$p + n \leftrightarrow D + \gamma$$

So (4.40) yields

$$\frac{X_D}{X_n X_p} = \underbrace{16.3\,\eta \left(\frac{kT}{m_p c^2}\right)^{3/2}}_{10^{-16}} \exp\left(B_D / kT\right) \qquad (4.86)$$

where $B_D = 2.2$ MeV.

For $X_D \sim \mathcal{O}(1)$, need $B_D / kT \approx 35$, i.e. $kT \ll B_D$.

By $kT \approx 0.07$ MeV, deuterium forms, and past this threshold, reactions cascade to stable the $^4He = 28.3$ MeV into which almost all neutrons are captured.

Helium-4 abundance

: Relative mass density in ^4He.

$$Y_{He} = \frac{\rho_{4He}}{\rho_B} = \frac{m_{4He}}{m_p} \times \frac{n_{4He}}{n_B} = 4 \times \frac{n_n/2}{n_n + n_p}$$

$$= \frac{2 n_n / n_p}{n_n/n_p + 1}$$

$$= \frac{2 \cdot \frac{1}{7}}{\frac{1}{7} + 1} = \frac{1}{4} \qquad (4.87)$$

The prediction that 25% of ordinary matter is ^4He is confirmed by observation.

$$Y_{4He} = 0.245 \pm 0.002,$$

There are small residual abundances of deuterium D ($D/H = (2.62 \pm 0.05) \times 10^{-5}$) and ^3He, thus trace "proof" of Li and Be — heavier elements are made in stars.

Nuclear predictions match for (4.35) baryon-to-photon.

$$\eta = 6.14(\pm 0.19) \times 10^{-10}$$

This is linked to the origin of dark matter and baryon asymmetry.

5. Large-Scale Structure Formation

5.1 Traces of the matter density

Definition and Statistical assumptions

We describe inhomogeneities with the density perturbation $\delta(\underline{r},t)$ by comparing mass density $\rho(\underline{r},t)/c^2$ as

$$\delta(\underline{r},t) = \frac{\delta\rho}{\rho}(\underline{r},t) = \frac{\rho(\underline{r},t) - \bar{\rho}(t)}{\bar{\rho}(t)} \tag{5.1}$$

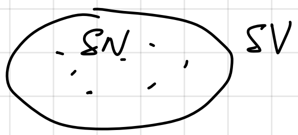
where $$\bar{\rho}(t) = \langle \rho(\underline{r},t) \rangle = \frac{1}{V}\int_V \rho(\underline{r},t)\, d^3\underline{r}. \tag{5.2}$$

So around a hom. FLRW background, we have

$$\rho(\underline{r},t) = \bar{\rho}(t)\left(1 + \delta(\underline{r},t)\right) \tag{5.3}$$

Linear if $\delta \lesssim 1$, but can be non-linear.

For galaxies with $n(\underline{r},t) = \frac{\delta N}{\delta V}$. assume $\rho/c^2 = \bar{m}\, n$ with $\rho/\bar{\rho} = m/\bar{m}$ (caveat is "bias" for different galaxy types)

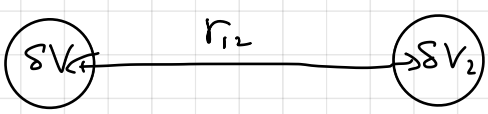


Galaxy clustering described by two-point correlation f^n.

$$\xi(r) = \langle \delta(\underline{r}',t)\, \delta(\underline{r}'+\underline{r},t)\rangle \tag{5.4}$$

$$= \frac{1}{\bar{\rho}(t)} \langle \rho(\underline{r}',t)\, \rho(\underline{r}'+\underline{r},t)\rangle - 1$$

where $r = |\underline{r}|$



This represents the probability

$$P(1,2) = \bar{n}^2 \left(1 + \xi(r_{12})\right) \delta V_1 \, \delta V_2$$

in excess of random of finding one galaxy in δV_1 and another in δV_2 separated by a distance $r_{12} = |\underline{r}_1 - \underline{r}_2|$

Note that $\xi(\underline{r}) \equiv \xi(|\underline{r}|)$ assumes that

- indpt of $\underline{r}'$ by statistical homogeneity
- indpt of $\underline{r}'$ by statistical isotropy.

We also use ergodic hypothesis

$$\langle \quad \rangle_{\substack{\text{ensemble of} \\ \text{realisation}}} \longrightarrow \langle \quad \rangle \quad \begin{matrix} \text{spatial average} \\ \text{(large vol. realisation)} \end{matrix}$$

Variance : $\sigma^2 = \langle \delta^2(\underline{r},t) \rangle = \xi(r=0)$. Usually defined as

$$\sigma_R^2 = \langle \delta_R^2 \rangle \qquad (5.5)$$

after smoothing with a window f^n on a lengthscale R.

$$\delta_R(\underline{r},t) = \int d^3\underline{r}' \; W_R(|\underline{r}' - \underline{r}|) \; \delta(\underline{r}',t) \qquad (5.6)$$

e.g. with a top hat

$$W_R(\underline{r}) = \begin{cases} 3/4\pi R^3 \quad (\text{const}) & . \; r < R \\ 0 & \quad\; r \geq R \end{cases} \qquad (5.7)$$

Measurement of σ_R^2 on a non-linear scale $R \approx 8h^{-1}\,\text{Mpc} \approx 12\,\text{Mpc}$, (with $h = H_0 / 100\,\text{km s}^{-1}\text{Mpc}^{-1} \approx 0.67$), yield

$$\sigma_8^2 = \begin{cases} 0.83 \pm 0.02 & (\text{Planck}) \\ 0.78 \pm 0.02 & (\text{Weak lensing}) \end{cases} \qquad (5.8)$$

Fourier space

$$\delta(\underline{k}) = \int d^3\underline{x}\; e^{-i\underline{k}\cdot\underline{x}} \qquad (5.9)$$

$$\delta(\underline{x}) = \int \frac{d^3\underline{k}}{(2\pi)^3}\; e^{i\underline{k}\cdot\underline{x}} \qquad (5.10)$$

Since $\delta(\underline{x})$ real, we have $\delta^*(\underline{k}) = \delta(-\underline{k})$

Derivative: $\nabla \rightarrow i\underline{k}$

Spatial translation: $\delta(\underline{r}+\underline{r}') \longrightarrow \delta(\underline{k})\, e^{i\underline{k}\cdot\underline{r}'}$

Dirac δ f^n: $\delta^{(3)}(\underline{r}-\underline{r}') \longrightarrow \int \frac{d^3\underline{k}}{(2\pi)^3} e^{i\underline{k}\cdot(\underline{r}-\underline{r}')}$

Convolution thm:

$$f(\underline{r}) = \int d^3\underline{r}'\; g(\underline{r}')\, h(\underline{r}-\underline{r}') \iff f(\underline{k}) = g(\underline{k})\, h(\underline{k}) \qquad (5.11)$$

Sph. sym. $\quad f(r) = \int \frac{d^3\underline{k}}{(2\pi)^3} f(k)\, e^{i\underline{x}\cdot\underline{r}}$.

Take $\mu = \hat{\underline{k}}\cdot\hat{\underline{r}} = \cos\theta \Rightarrow i\underline{k}\cdot\underline{r} = i\mu kr$, $k = |\underline{k}|$, $r = |\underline{r}|$.

$$f(r) = 2\pi \int \frac{k^2\, dk}{(2\pi)^3} f(k) \int_{-1}^{1} d\mu\, e^{i\mu kr}$$

$$= \frac{1}{2\pi^2} \int k^2 \frac{\sin kr}{kr} f(k)\, dk \qquad (5.12)$$

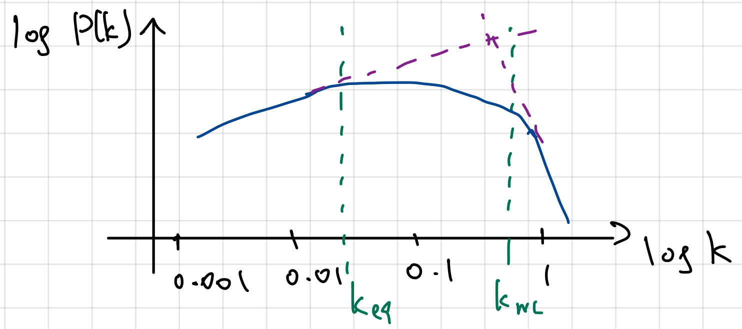
Power spectrum

This is FT of two-point correlation $\xi(r)$

$$P(k) = \int d^3\underline{r}\; \xi(r)\, e^{i\underline{k}\cdot\underline{r}}$$

$$= 4\pi \int r^2 \frac{\sin kr}{kr} \xi(r)\, dr \qquad (5.13)$$

ie. how much power there is on a given lengthscale $\lambda \sim k^{-1}$.



Dimensionless power spectrum

$$\Delta^2(k) := \frac{k^3}{2\pi^2} P(k) \tag{5.14}$$

which is power per log interval

$$\int P(k) \frac{d^3\underline{k}}{(2\pi)^3} = \frac{1}{(2\pi)^3} \int P(k)\, k^2 dk \underbrace{d\mu\, d\phi}_{4\pi}$$

$$= \frac{1}{2\pi^2} \int P(k)\, k^3\, d(\log k)$$

ie. effectively we are using a window f^n $\Delta k = k$.

This is <u>scale-invariant</u> if

$$\Delta^2(k) := \frac{A}{2\pi^2} k^{n-1} \approx \text{const.} \tag{5.15}$$

ie. we have <u>spectral index</u> $n=1$ (equal power on each lengthscale) known as the <u>Harrison-Zel'dovich spectrum</u>.

This is a prediction of inflation for premordial perturbation.

$$P_i(k) \approx A k^{n-4} \approx A/k^3 \tag{5.16}$$

and that $\delta(k,t)$ is a <u>Gaussian random field</u>.

$$\langle \delta(\underline{k})\, \delta(\underline{k}') \rangle = (2\pi)^3 \, \delta^{(3)}(\underline{k}+\underline{k}')\, P(\underline{k}) \tag{5.17}$$

↖ Dirac delta

5.2 Newtonian Dynamics

Equation of motion

Consider a flat ($k=0$) FLRW model on subhorizon scales $r \ll H^{-1}$ ($v \ll c$) with comoving coords

$$\underline{r} = a(t)\, \underline{x} \tag{5.18}$$

physical ↗ ↖ comoving

In a grav. potential Φ,

$$\ddot{\underline{r}} = -\nabla_{\underline{r}} \Phi = -\frac{1}{a} \nabla_{\underline{x}} \Phi \tag{5.19}$$

• Homogeneous background $\underline{x} = \text{const.}$

$$\dot{\underline{r}} = H\underline{r} = \dot{a}\,\underline{x} = \frac{1}{a}\frac{da}{d\tau}\,\underline{x} = \frac{a'}{a}\,\underline{x} = \mathcal{H}\,\underline{x} \qquad (5.20)$$

where τ conformal time $d\tau = a\,dt$.

$$\ddot{\underline{r}} = \frac{1}{a}\mathcal{H}'\,\underline{x} = -\nabla_{\underline{r}}\bar{\Phi} = -\frac{\nabla_x}{a}\bar{\Phi} \qquad (5.21)$$

The background potential $\bar{\Phi}$ is

$$\bar{\Phi} = -\frac{1}{2}\mathcal{H}'\,x^2 \qquad (5.22)$$

with $x = |\underline{x}|$.

In a matter-dominated universe $(\Omega_M \simeq 1$ for $\tau_{eq} < \tau < \tau_\Lambda)$.

Poisson eqn (1.14) implies

$$\nabla_{\underline{x}}^2\,\bar{\Phi} = \frac{4\pi G}{c^2}\,a^2\,\bar{\rho}_M \qquad (5.23)$$

• Inhomogeneous universe $\underline{x} \neq \text{const.}$

$$\dot{\underline{r}} = H\underline{r} + a\dot{\underline{x}} = \mathcal{H}\underline{x} + \underline{x}'$$

Hubble flow   pec vel

$$\ddot{\underline{r}} = \frac{1}{a}\mathcal{H}'\underline{x} + \frac{1}{a}\mathcal{H}\underline{x}' + \frac{1}{a}\underline{x}''$$

$$= -\frac{1}{a}\nabla_{\underline{x}}\Phi$$

$$= -\frac{1}{a}\nabla_{\underline{x}}\bar{\Phi} - \frac{1}{2}\nabla_{\underline{x}}\phi .$$

where $\Phi = \underbrace{\bar{\Phi}}_{\text{background}} + \underbrace{\phi}_{\text{peculiar}}$

Hence,

$$\underline{x}'' + \mathcal{H}\underline{x}' = -\nabla_{\underline{x}}\phi . \qquad (5.24)$$

With Friedmann eqn $\mathcal{H}^2 = \frac{8\pi G}{3c^2}\bar{\rho}_M a^2$, the perturbed Poisson eqn (1.14)

with (5.3) $\rho = \bar{\rho}_m(1+\delta)$ becomes (matter era)

$$\nabla_x^2 \phi = \frac{4\pi G}{c^2} a^2 \bar{\rho}_m \delta = \frac{3}{2} \mathcal{H}^2 \Omega_m \delta = \frac{3}{2} \Omega_{M_0} H_0^2 \frac{\delta}{a} \qquad (5.25)$$

where $\Omega_{M_0} = \left.\frac{\rho_m}{\rho_{crit}}\right|_{t=t_0}$. Here, we assume $\delta\rho_m$ is the most important component, not $\delta\rho_R$, $\delta\rho_\Lambda$.

<u>Canonical momentum</u> for "comoving $\underline{x}$":

$$\underline{p} = a m \underline{u} \qquad (5.26)$$

where $\underline{u} = \underline{x}'$.

$$\underline{p}' = m(a'\underline{x}' + a\underline{x}'') = am(\mathcal{H}\underline{x}' + \underline{x}'') = -am \nabla_x \phi \qquad (5.27)$$

* Fluid equations in an expanding universe *

Consider phase space dist^n f^n $f(\underline{x}, \underline{p}, \tau)$ where

$$dN = f(\underline{x}, \underline{p}, \tau)\, d^3\underline{p}\, d^3\underline{x} \qquad (5.28)$$

is the particle no. in phase space vol. $d^3\underline{p}$, $d^3\underline{x}$ (see e.g. eqm. dist^n (4.32) w/o spatial dependence). NR dark matter obeys the <u>collisionless Boltzmann eqn</u>

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{d\underline{x}}{d\tau}\cdot\frac{\partial f}{\partial \underline{x}} + \frac{d\underline{p}}{d\tau}\cdot\frac{\partial f}{\partial \underline{p}} \qquad (5.29)$$

$$= \frac{\partial f}{\partial \tau} + \frac{\underline{p}}{am}\cdot\nabla_{\underline{x}} f - am\nabla_{\underline{x}}\phi\cdot\nabla_{\underline{p}} f = 0$$

vanishes by Liouville's thm that phase space is conserved along trajectories (no interaction), i.e. $\frac{df}{d\tau} = 0$.

- <u>Density</u>

$$\rho(\underline{x}, \tau) = \frac{m}{a^3}\int d^3\underline{p}\, f(\underline{x}, \underline{p}, \tau) \qquad (5.30)$$

- <u>Velocity</u>

$$v_i(\underline{x}, \tau) = \frac{\int d^3\underline{x}\, \frac{p_i}{am} f(\underline{x}, \underline{p}, \tau)}{\int d^3\underline{p}\, f(\underline{x}, \underline{p}, \tau)} \qquad (5.31)$$

Moments of Boltzmann eqn (5.29) yield e.o.m. for ρ and $\underline{v}$.

(zeroth)
$$\int d^3p \, \frac{\partial f}{\partial \tau} + \int d^3p \, \frac{p_i}{am} \frac{\partial f}{\partial x_i} + am \nabla_i \phi \underbrace{\int d^3p \, \frac{\partial f}{\partial p_i}}_{\text{total derivative vanishes } p=0,\infty}$$
$$= \frac{\partial}{\partial \tau}\left(\frac{a^3}{m}\rho\right) + \nabla_i\left(\frac{a^3}{m}\rho \underline{v}\right)$$

Now substitute $\rho = \frac{\bar{\rho}_m}{a^3}(1+\delta)$ yields the cty (using $\bar{\rho}_m$ eqn)

$$\boxed{\delta' + \nabla_x \cdot (\underline{v}(1+\delta)) = 0} \qquad (5.32)$$

Now take the first moment $\int d^3p \, \frac{p_i}{am}$ (5.29) to find the

<u>Euler eqn</u> or momentum conservation eqn

$$\boxed{\underline{v}' + \mathcal{H}\underline{v} + (\underline{v}\cdot\nabla_x)\underline{v} = -\nabla_x \phi - \frac{1}{\rho}\nabla_x P.} \qquad (5.33)$$

where a pressure term has been added and anistropic stress is ignored.

velocity decomposes into scaler (compressional) and vector (rotational) parts
$$\underline{V} = \underline{V}_{\parallel} + \underline{V}_{\perp}$$

with $\nabla \times \underline{V}_{\parallel} = 0$, $\nabla \cdot \underline{V}_{\perp} = 0$

Take divergence
$$\theta = \nabla_x \cdot \underline{v} = \nabla_x \cdot \underline{V}_{\parallel} \qquad (5.34)$$

and curl
$$\underline{\omega} = \nabla_x \times \underline{V} = \nabla_x \times \underline{V}_{\perp} \qquad (5.35)$$

<u>Linear perturbation eqn</u>

Assume δ, $\underline{v}$ small and discard 2nd-order terms ($\underline{v}\delta$, $\underline{v}\cdot\nabla\underline{v}$) to find

$$(5.32) \Rightarrow \quad \delta' + \nabla_x \cdot \underline{v} = \delta' + \theta = 0 \qquad (*)$$
$$(5.33) \Rightarrow \quad \underline{v}' + \mathcal{H}\underline{v} = -\nabla_x \phi$$

Decompose $\underline{v} \to \theta, \underline{\omega}$.

Scaler : $\theta' + \mathcal{H}\theta = -\nabla_x^2 \phi \overset{(5.25)}{=} -\frac{3}{2}\mathcal{H}^2 \Omega_m \delta$ (†)

vector : $\underline{\omega}' + \mathcal{H}\underline{\omega} = 0 \Rightarrow \underline{\omega} \propto a^{-1}$ (5.36)

as rotational modes are surpressed as the universe expands.

Comparing (*) and (†), we find

$$\boxed{\delta''(\underline{x},\tau) + \mathcal{H}(\tau)\,\delta'(\underline{x},\tau) - \frac{3}{2}\Omega_m(\tau)\,\mathcal{H}^2(\tau)\,\delta(\underline{x},\tau) = 0} \quad (5.37)$$

Hubble damping   grav. attraction

We need to solve this to find $\delta(\underline{x},\tau)$, or equivalently, $\delta(k,\tau)$ evolves and grows.

With cosmic time t,

$$\boxed{\ddot{\delta}(\underline{x},t) + 2H(t)\,\dot{\delta}(\underline{x},t) - \frac{4\pi G}{c^2}\bar{\rho}\,\delta(\underline{x},t) = 0} \quad (5.38)$$

<u>Matter dominated era</u>: between $\tau_{eq} < \tau < \tau_\Lambda$, NR matter is dominant with $\Omega_m \approx 1$. Recall EdS (1.29),

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}, \quad t_0 = \frac{2}{3}H_0^{-1}, \quad \bar{\rho}_m = \frac{\bar{\rho}_{m0}}{a^3} = \frac{1}{6\pi G t^2}.$$

In conformal time,

$$a = (\tau/\tau_0)^2, \quad \tau_0 = 2\mathcal{H}_0^{-1} = 2H_0^{-1}, \quad t = t_0(\tau/\tau_0)^3.$$

We take

$$a(\tau) = (\tau/\tau_i)^2, \quad \mathcal{H} = 2/\tau$$

So perturbation eqn (5.37) becomes

$$\boxed{\delta'' + \frac{2}{\tau}\delta' - \frac{6}{\tau^2}\delta'' = 0} \quad (5.39)$$

Seek power sol^n $\delta = \tau^\beta$

$$\beta(\beta-1) + 2\beta - 6 = 0 \Rightarrow \beta = 2, -3$$

So general sol^n is

$$\delta(\underline{x}, \tau) = \underbrace{A(\underline{x}) \left(\tau/\tau_i\right)^2}_{\substack{\text{growing mode} \\ (\propto a)}} + \underbrace{B(\underline{x}) \left(\tau/\tau_i\right)^{-3}}_{\substack{\text{decaying mode} \\ \text{(neglect)}}} \qquad (5.40)$$

where A, B arbitrary.

So we have growth at time τ given by (equiv. as FT).

$$\delta(\underline{x}, \tau) \approx A(\underline{k}) \frac{a(\tau)}{a(\tau_i)} = A(\underline{k}) \frac{1+z_i}{1+z} \qquad (5.41)$$

where $\tau_{eq} < \tau < \tau_\Lambda$.

The maximum linear growth is

$$\frac{1+z_{eq}}{1+z_\Lambda} = \left(\frac{\Omega_{m0}}{\Omega_{R0}}\right)\left(\frac{\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{-1/3} \approx 2800 \qquad (5.42)$$

$\uparrow$ (2.31) $\quad$ $\uparrow$ (3.4)

After which non-linear effects take hold if $\delta = 1.69$, we expect the true virialised inhomogenity to be about $\delta_{NL} \approx 178$ (spherical collapse model)

5.3 The Jeans length

The Euler eqn (5.33) includes the effects of non-zero pressure $P_m \ll \rho_m$ with sound speed

$$c_s^2 = \frac{dP}{d\rho} \qquad (5.43)$$

with EoS $P_m = \omega_m \rho_m$, we have $c_s^2 = \omega_m$. In Fourier space, we have

$$\frac{1}{\bar\rho} \nabla_{\underline{x}} P(\underline{x}, t) = \frac{1}{\bar\rho} \nabla_{\underline{x}} \left(\bar P(t) + \delta P(\underline{x}, t)\right)$$

$$= \frac{1}{\bar\rho} \frac{dP}{d\rho} \nabla_{\underline{x}} \delta\rho(\underline{x}, t) = c_s^2 \nabla_{\underline{x}} \delta \longrightarrow c_s^2 i\underline{k}\delta.$$

So Euler (5.33) linearised to

$$\theta' + \mathcal{H}\theta = -\frac{3}{2}\Omega_m \mathcal{H}^2 \delta + c_s^2 k^2 \delta$$

Combining with continuity (5.32) or (*), we find

$$\delta'' + \mathcal{H}\delta' - \left(\frac{3}{2}\Omega_m \mathcal{H}^2 - c_s^2 k^2\right)\delta = 0 \qquad (5.44)$$

We define the Jeans wavenumber (comoving)

$$k_J = \frac{a}{c_s}\sqrt{4\pi G \rho_m / c^2} \qquad (5.45)$$

and a Jeans length (physical)

$$\lambda_J = \frac{2\pi a}{k_J} = c_s \left(\frac{\pi}{G\bar{\rho}_m / c^2}\right)^{1/2} \qquad (5.46)$$

In conformal frame, $k_J = \left(\frac{3}{2}\frac{\Omega_m \mathcal{H}^2}{c_s^2}\right)^{1/2} = \frac{\sqrt{6}}{c_s \tau}$ in matter era $\Omega_m \approx 1$, $\mathcal{H} = 2/\tau$, so Jeans length becomes

$$\lambda_J = \frac{2\pi a}{k_J} = \frac{2\pi}{\sqrt{6}} \underbrace{c_s a \tau}_{c_s a\tau \text{ horizon}} \simeq \mathcal{O}(1)\, c_s t .$$

ie. sound speed horizon.

- For short wavelengths $\lambda < \lambda_J$ ($k > k_J$), the density perturbation oscillates with freq

$$\omega^2 = \frac{c_s^2 k^2}{a^2} - 4\pi G \rho_m > 0 \qquad (5.48)$$

So with $k > k_J$, we have

$$\delta(\underline{k}, t) \simeq A(\underline{k}) \cos(\omega t + \phi) \qquad (5.49)$$

with time averages

$$\langle \delta^2(\underline{k}, t)\rangle \simeq \frac{|A|^2}{2}$$

ie. there is no net growth.

- For long $\lambda > \lambda_J$ ($k < k_J$), then grav. collapse at

$$t_{grav} = 1 / \sqrt{4\pi G \bar{\rho}_m / c^2} \qquad (5.50)$$

which proceeds unimpeded outside the sound horizon.

The Jeans mass is the minimum mass that can collapse at time t.

$$M_J(t) = \frac{4\pi}{3}\bar{\rho}_m \lambda_J^3(t) \qquad (5.51)$$

Example (Baryons and photon decoupling) For $t < t_{dec}$ (4.71), baryons (protons), e^- and photons in eqm. Photon sound speed dominant $P = \frac{1}{3}\rho$, but baryon-photon fluid has $\rho = \rho_B + \rho_\gamma$, so sound speed

$$c_s^2 = \frac{dP_\gamma}{d\rho} = \frac{\partial P_\gamma}{\partial \rho_\gamma} \cdot \frac{\partial \rho_\gamma}{\partial \rho} = \frac{c^2}{3}\left(1 + \frac{\partial \rho_B}{\partial \rho_\gamma}\right)^{-1} \qquad (\ddagger)$$

with $\rho_B = \rho_{B0}/a^3$, $\rho_\gamma = \rho_{\gamma 0}/a^4$, so

$$\frac{\partial \rho_B}{\partial \rho_\gamma} = \frac{\partial \rho_B}{\partial a}\left(\frac{\partial \rho_\gamma}{\partial a}\right)^{-1} = \frac{3}{4}\frac{\rho_{B0}}{\rho_{\gamma 0}} a = \frac{3}{4}\frac{\rho_{m0}/6}{\rho_{R0}/1.67} a$$

include cold dark matter

include neutrinos

Thus $(\ddagger)$ gives

$$c_s = \frac{c}{\sqrt{3}}\left(1 + 0.2\,\frac{1+z_{eq}}{1+z}\right)^{-1/2} \qquad (5.52)$$

until recombination when baryons rapidly leave eqm

$$p + e^- \to H$$

For $t > t_{dec}$, a NR ideal gas has $P \propto \rho^\gamma$ with adiabatic index

$$\gamma = 1 + \frac{2}{f} = \frac{5}{3}$$

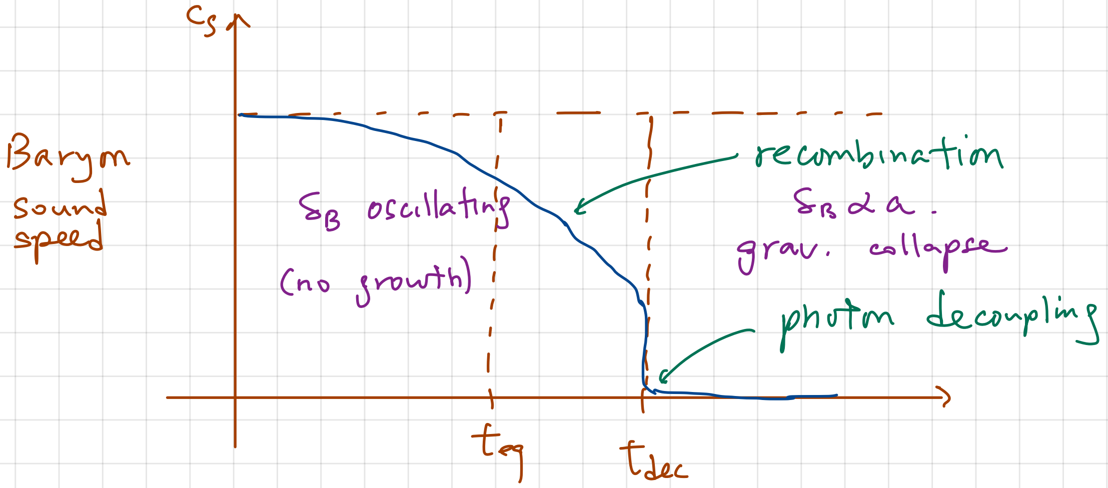
deg. of freedom

So

$$c_s = \sqrt{\gamma \frac{P}{\rho}} = \sqrt{\gamma \frac{k_B T}{m_p}} \approx 6000 \left(\frac{1+z}{1+z_{dec}}\right) \qquad (5.53)$$

since $T \propto a^{-2}$ decoupled NR gas.

So at $t = t_{dec}$, c_s dramatically falls from $c_s \approx 1.8\,c$ with $M_J \simeq 10^{18} M_\odot$ down to $c_s \lesssim 10^{-5} c$, $M_J \lesssim 10^5 M_\odot$



For $t > t_{dec}$, baryons fall into potential wells from CDM, and structures start to form.

5.4 Density perturbation across cosmic epochs

Horizon-crossing timescales

Newtonian analysis (5.37, 5.44) is valid on sub-horizon scales with comoving mode $k \gtrsim 2\pi/c\tau$. or physical scales

comoving horizon

$$\lambda(t) = \frac{2\pi a}{k} \lesssim cH^{-1}(t)$$

(sometimes $k \geqslant 2\pi aH/c$)

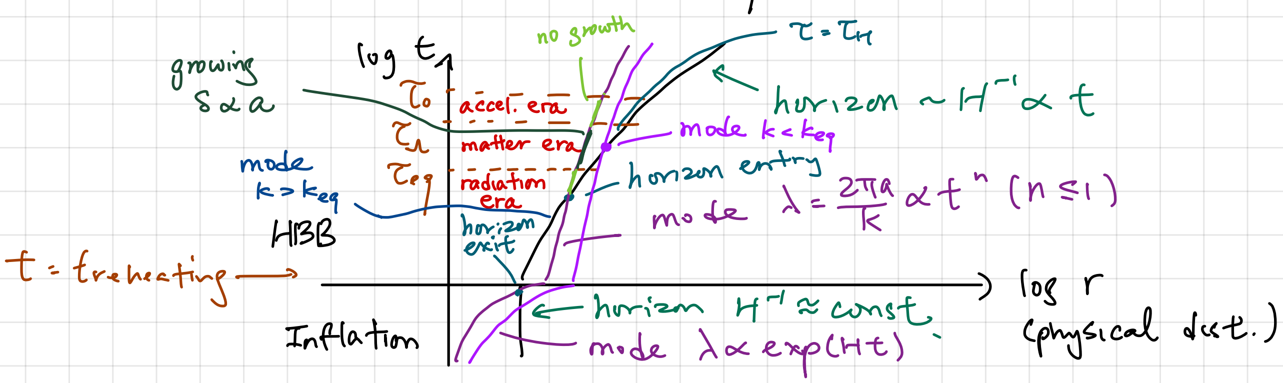
Horizon-crossing for mode k occurs at

$$\tau_H = \frac{2\pi}{kc} \qquad (5.54)$$

(or in cosmic time solve $a(t_H)\, H(t_H) = \frac{2\pi}{ck}$ for t_H).

Key other lengthscale is crossing at equal matter-radiation $\tau = \tau_{eq}$. Define

$$k_{eq} = \frac{2\pi}{c\,\tau_{eq}} \qquad (5.55)$$



Sub-horizon growth (Newtonian $k \gg 2\pi/c\tau$)

Radiation era ($\tau < \tau_{eq}$)

Use (5.37) for NR matter in the rad. era. With $a \propto \tau$, $\mathcal{H} = \frac{1}{\tau}$,

$$\delta'' + \frac{\delta'}{\tau} - \frac{3}{2}\frac{\delta}{\tau(\tau+\tau_{eq})} = 0 \qquad (5.56)$$

where we have used

$$\Omega_m(\tau) \approx \frac{\bar{\rho}_{m0}/a^3}{\bar{\rho}_{m0}/a^3 + \bar{\rho}_{R0}/a^4} = \frac{1}{1+\frac{a}{a_{eq}}}$$

with $a_{eq} = \Omega_{R0}/\Omega_{M0} = \bar{\rho}_{R0}/\bar{\rho}_{M0}$

Taking $\tau \ll \tau_{eq}$. We can find approx. sol^n

$$\delta(\underline{k},\tau) \approx \underbrace{A(\underline{k})\left(1+\frac{3}{2}\frac{\tau}{\tau_{eq}}\right)}_{\text{growing mode}} + \text{decaying mode} \qquad (5.57)$$

So deep in rad. era, $\delta \approx$ const. with no growth until $\tau = \tau_{eq}$.
(See improved analytic sol^n in Ex 4 Q5)

Matter era ($\tau_{eq} < \tau < \tau_\Lambda$) (see §5.2)

Key linear growth (5.14)

$$\delta(\underline{k},t) = A(\underline{k})\frac{a(\tau)}{a(\tau_i)} = A(\underline{k})\underbrace{\frac{1+z_i}{1+z}}_{=D(z)\ \text{growth rate}} \lesssim A(\underline{k})\frac{1+z_{eq}}{1+z_\Lambda}$$

yields grav. collapse and structural formation.

Caveat is delayed collapse of baryons $\tau_{eq} < \tau < \tau_\Lambda$.

Acceleration epoch ($\tau_\Lambda < \tau < \tau_0$)

The last term $\frac{3}{2}\Omega_m \mathcal{H}^2$ becomes negligible, so (5.37) becomes

$$\delta'' + \mathcal{H}\delta' = 0 \qquad (5.58)$$

with sol^n

$$\delta(\underline{k},\tau) = \underbrace{A(\underline{k})}_{\text{const.}} + \underbrace{B(\underline{k})/a(\tau)}_{\text{decaying}} \qquad (5.59)$$

So accel. switches off "linear perturbation" growth.

* Initial Conditions *

Assume inflationary fluctuations for there is good observed evidence (see § 3.4) with the following properties:

- Scale-invariant in metric or potential fluctuations

$$P_\phi(k) \approx A^2/k^3 \qquad (5.60)$$

(strictly Ak^{n_s-4}, so (5.15) variance $\Delta^2_\phi(k) = \frac{k^3}{2\pi^2}P(k) \simeq$ const.
$\Rightarrow n_s \approx 1$)

- Gaussian, so described by $P(k)$ only
- Adiabatic with $\delta_R = \frac{4}{3}\delta_M$ (*) All particle species are created together, e.g. no. of photons is fixed relative to DM particle, i.e. $\frac{n_R}{n_M} =$ const.

But $\rho_R \propto a^{-4} \propto n_R^{4/3} \Rightarrow \delta_{\rho_R} \propto \frac{4}{3}\delta_{n_R}$ (linearising).
Also $\rho_M \propto a^{-3} \Rightarrow \delta_{\rho_M} \propto \delta_{n_R}$. hence (*).

Superhorizon evolution ($k \ll 2\pi/c\tau$)

Require relativistic perturbation theory.

For $k \ll 2\pi/c\tau$, potential fluctuations are "frozen".

$$\frac{d\phi}{d\tau}(\underline{k}, \tau) = 0$$

So from (5.60),

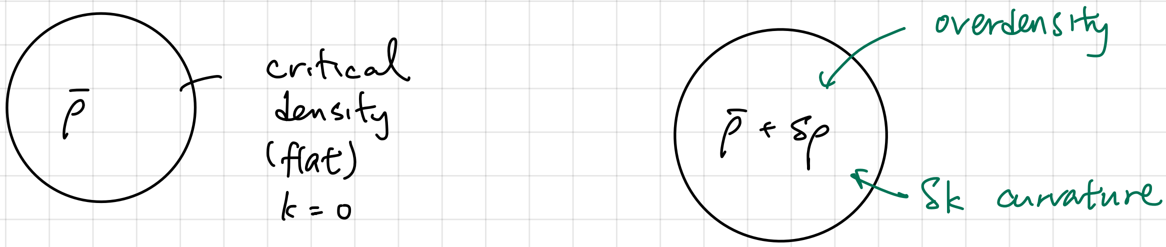
$$\phi(\underline{k}, \tau) = \phi(\underline{k}) = A / k^{3/2} \qquad (5.62)$$

up to horizon re-entry $\tau_H = 2\pi/kc$.

<u>Exercise</u> Use perturbed Poisson eqn (5.25) to show $\phi =$ const. in matter era.

* <u>Aside</u>: Superhorizon evolution ($\tau < \tau_H = 2\pi/kc$)

Consider two regions expanding at rate H:



$$H^2 = \frac{8\pi G}{3c^2}\bar{\rho} \qquad\qquad H^2 = \frac{8\pi G}{3c^2}(\bar{\rho} + \delta\rho) - \frac{\delta k\, c^2}{a^2}.$$

Subtract to find curvature

$$\delta k\, c^2 = \frac{8\pi G}{3c^2}\bar{\rho}\,\delta a^2 \qquad (*)$$

Constant on superhorizon scales ($\tau < \tau_H = 2\pi/kc$)

- Matter era $\delta_m \propto a \propto \tau^2$
- Radiation $\delta_R \propto a^2 \propto \tau^2$

Relativistic Poisson equation include pressure

$$\nabla_x^2 \bar{\Phi} = \frac{4\pi G}{c^2} a^2(\rho+3P) = \frac{4\pi G}{c^2} a^2(1+3w)\rho$$

which has a perturbed part $\Phi = \bar{\Phi} + \phi$.

$$\nabla_x^2 \phi = \frac{4\pi G}{c^2}(1+3w)\bar{\rho}\,\delta a^2 = \frac{3}{2}(1+3w)\,\delta k\, c^2 \qquad (+)$$

const.?

The potential ϕ is constant on superhorizon scales,

$$\delta_m(k,\tau_H) = \frac{k^2\phi(k)}{\frac{3}{2}\mathcal{H}^2} = \frac{\alpha}{6}\tau_H^2 A k^{1/2}.$$

where $\alpha = \begin{cases} 1 & \tau_H > \tau_{eq} \quad \text{(matter)} \\ 2 & \tau_H < \tau_{eq} \quad \text{(rad.)} \end{cases}$

So $P(k)\big|_{\tau_H} = \left(\frac{\alpha}{6}\right)^2 \tau_H^4 A^2 k = \left(\frac{\alpha}{6}\right)\left(\frac{\tau_H}{\tau_{eq}}\right)^4 (\tau_{eq})^4 A^2 k$

$$= \left(\frac{\tau_H}{\tau_{eq}}\right)^4 B_{eq}^2\, k.$$

where $B_{eq} = \frac{\alpha}{6}\tau_{eq}^2 A$ to normalise at $k = k_{eq}$.

Transfer functions

Collate these results into a sol^n of perturbation eqn called a transfer f^n

$$P(k) = D^2(z_{eq})\, T^2(k)\, k^4\, P_\phi(k) \qquad (5.66)$$

Power spectrum today

maximum growth factor $\tau_{eq} \to \tau_0$

transfer for specific k

convert $\phi \to \delta$

inflationary fluctuation

- Radiation era modes ($k > k_{eq}$, $\tau_H < \tau_{eq}$).

At horizon-crossing $\tau_H = 2\pi/kc < \tau_{eq}$ from (5.65), we have

$$P(k)\big|_{\tau_H} = \left(\frac{\tau_H}{\tau_{eq}}\right)^4 B_{eq}\, k = P(k)\big|_{\tau_{eq}} \qquad (5.67)$$

since effectively <u>no growth</u> (stagnation).

For $\tau_{eq} < \tau < \tau_{eq}$, we have full growing mode (5.41).

$\delta \propto \frac{a}{a_{eq}} = \left(\frac{\tau}{\tau_{eq}}\right)^2$.

$$P(k)|_{\tau_\Lambda} = \left(\frac{\tau_H}{\tau_{eq}}\right)^4 \left(\frac{\tau_\Lambda}{\tau_{eq}}\right)^4 B_{eq}^2 k = P(k)|_{\tau_0} \qquad (5.69)$$

because growth is surpressed $\tau_\Lambda < \tau < \tau_0$

Substituting $\tau_H = 2\pi/k$, $\tau_{eq} = 2\pi/k_{eq}$.

$$P(k)|_{\tau_0} = \left(\frac{1+z_{eq}}{1+z_\Lambda}\right)^2 \left(\frac{k_{eq}}{k}\right)^4 B_{eq} k \approx D^2 B_{eq}^2 \frac{k_{eq}^4}{k^3} \quad (k > k_{eq}) \quad (\dagger)$$

· <u>Matter era</u> $(k < k_{eq}, \tau_H > \tau_{eq})$

At $\tau = \tau_H$, we have

$$P(k)|_{\tau_H} = \left(\frac{\tau_H}{\tau_{eq}}\right)^4 B_{eq}^2 k$$

which grows from $\tau_H < \tau < \tau_\Lambda$ is $a(\tau_\Lambda)/a(\tau_H)$.

$$P(k)|_{\tau_0} = P(k)|_{\tau_\Lambda} = \left(\frac{\tau_\Lambda}{\tau_H}\right)^4 \left(\frac{\tau_H}{\tau_{eq}}\right)^4 B_{eq}^2 k$$

$$= \left(\frac{1+z_{eq}}{1+z_\Lambda}\right)^2 B_{eq}^2 k \qquad (\dagger\dagger).$$

<u>Power spectrum</u>

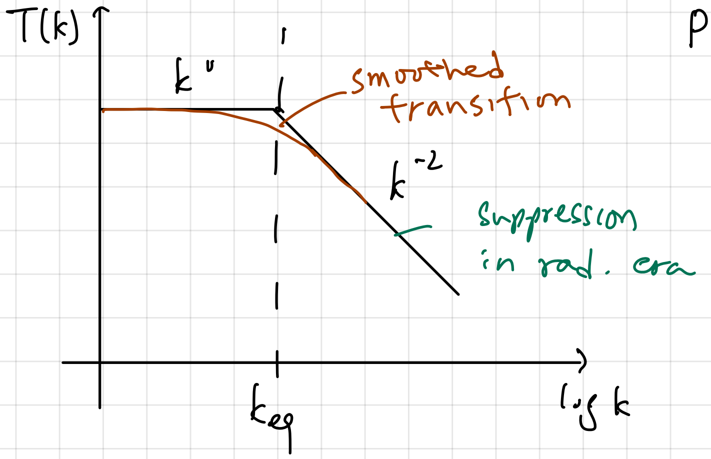
$$P(k) = D^2(z_{eq})\, T^2(k)\, B_{eq} k$$

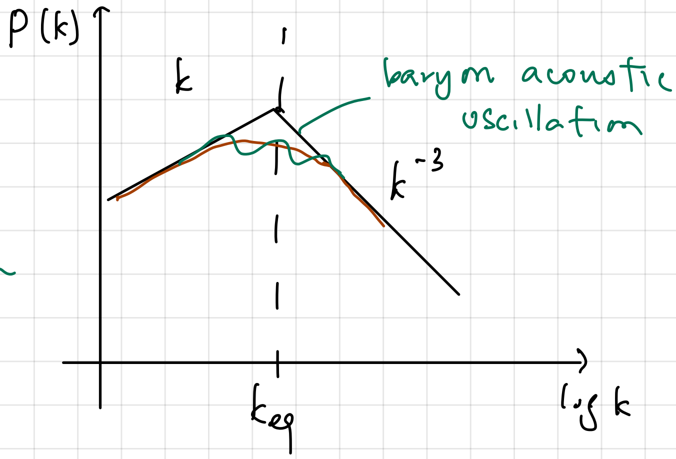
$$= \left(\frac{1+z_{eq}}{1+z_\Lambda}\right)^2 B_{eq}^2 \begin{cases} k & k < k_{eq} \\ k_{eq}^4/k^3 & k > k_{eq} \end{cases}$$

with transfer f^n

$$T(k) = \begin{cases} 1 & k < k_{eq} \\ \left(\frac{k_{eq}}{k}\right)^2 & k > k_{eq}. \end{cases}$$

with (apparant) suppression for $k > k_{eq}$, no growth in rad. era,





where $k_{eq} \approx 0.015\ h\ \mathrm{Mpc}^{-1}$. Smooth transfer at $k = k_{eq}$, obtained from Ex. 4 Q5.

Baryon acoustic effects imprinted $r_{BAO} \approx 100\ h^{-1}\ \mathrm{Mpc}$.

*5.4 Cosmic Microwave Sky

(See notes)