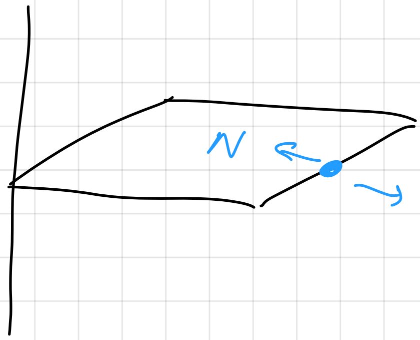
Classical Dynamics

Newton's Laws

- Good description of classical (i.e. $\hbar = 0$, $c = \infty$) physics, However, it is difficult to describe $n \gg 1$ particles, or systems with constraints, e.g. bead must stay on wire frame.



- Newton's law work best in inertial frames, else need 'ficticious forces'. Can we find a formulation that's the same form in all frames ?

These problems will overcome by Lagrangian and Hamiltonian reformation of classical dynamics.

These formulations

- make the role of symmetry much more central, e.g. invariants under spatial translations $\Leftrightarrow$ cons. of momentum.
- make connections to QM much clearier.

Hamiltonian's form	Lagrangian
$\updownarrow$	$\updownarrow$
Schrödinger picture	Feymann path integral

- The space of sol^n to Newton's eq^n itself has rich geometric structure $\Rightarrow$ symplectic geometry

Euclidean δ_{ij} $\xrightarrow[\text{Riemannian geometry}]{GR}$ $g_{ij}(x)$ "curved metric"

$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$

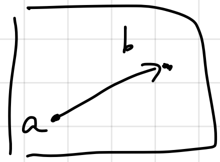
$\xrightarrow[\text{Symplectic geometry}]{}$ $\omega_{ij}(x)$ anti-sym.

$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

Galilean Space-time

An n-dimensional affine space A^n is a set together with an operation $-$ s.t.

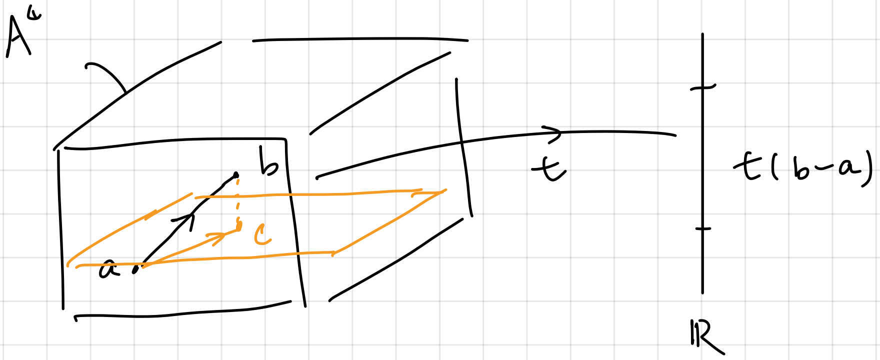
$$a - b \in \mathbb{R}^n \quad \forall\, a, b \in A^n,$$



i.e. A^n is like $\mathbb{R}^n$, but no preferred origin

Galilean space-time is

- an affine space A^4. Points $a, b \in A^4$ called events
- with a preferred linear map $t : \mathbb{R}^4 \to \mathbb{R}$ called time



the time between events b, a

- If $b - a \in \operatorname{Ker}(t)$, they are called simultaneous.

  By rank-nullity, $\operatorname{Ker}(t) \cong \mathbb{R}^3$, a linear subspace of $\mathbb{R}^4$.

- If $b, a \in A^4$ simultaneous, then the distance between them are $d(b,a) = \sqrt{(b-a,\, b-a)}$, where $(\ ,\ )$ is Euclidean inner product on $\mathbb{R}^3$.

Inertial Frame

An inertial frame is a vector space $\mathbb{R} \times \mathbb{R}^3$ with the usual inner product on $\mathbb{R}^3$. These are naturally Galilean spaces with a choice of origin. However, they're not canonically Galilean.

Consider the transformations $\mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R} \times \mathbb{R}^3$

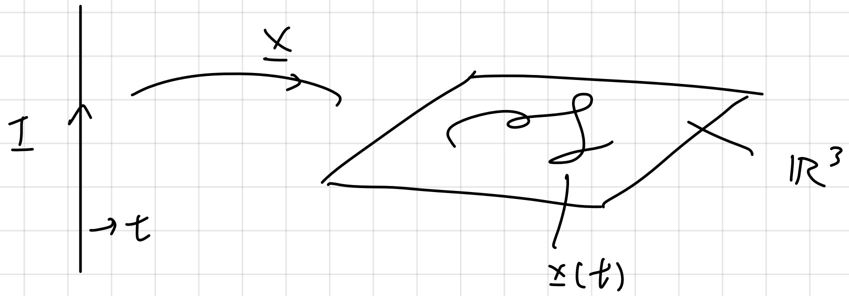
$g_1 : \quad (t, \underline{x}) \longmapsto (t+s, \underline{x} + \underline{c})$ translations

$g_2 : \quad (t, \underline{x}) \longmapsto (t, R\underline{x})$ with $R \in O(3)$ rotations + reflections

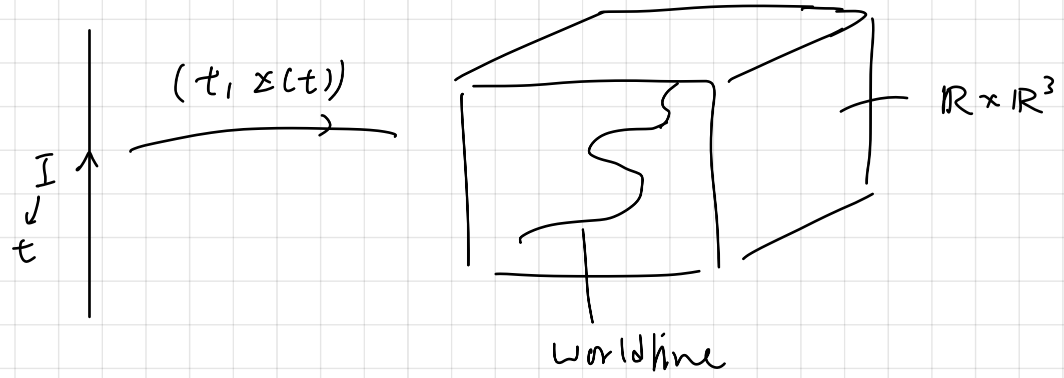
$g_3 : \quad (t, \underline{x}) \longmapsto (t, \underline{x} + \underline{v}t)$ Galilean boosts.

These translations generate the Galilean group G.
$(\dim G = \underbrace{4}_{\text{trans}} + \underbrace{3}_{\text{rot}^{\text{ns}}} + \underbrace{3}_{\text{boost}} = 10)$. Any 2 inertial frames are related by some $g \in G$.

Def^{n} A motion (of a particle) is a (smooth) map $\underline{x} : I \to \mathbb{R}^3$ where $I \subset \mathbb{R}$ interval.



The graph $(t, \underline{x}(t)) \in \mathbb{R} \times \mathbb{R}^3$ of the trajectory is the particle's worldline



The velocity of a particle is $\dot{x} = \frac{dx}{dt}$

acceleration $\ddot{x} = \frac{d^2x}{dt^2}$.

For a system of n particles, must specify n motions

$$\underset{\mathbb{R}^{3n}}{\underline{X}(t)} = \Big(\underset{\mathbb{R}^3}{\underline{x}_1(t)}, \ldots, \underset{\mathbb{R}^3}{\underline{x}_n(t)}\Big), \quad \underline{x}_i : I \to \mathbb{R}^3$$

$$\mathbb{R}^{3n} \cong \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{n \text{ copies}}$$

Here $\mathbb{R}^{3n}$ is the configuration space of our n-particle system.

Dynamics

In classical mech, a system's dynamics is determined by

$$\dot{p}_i = F_i(\underline{x}_1, \ldots, \underline{x}_n, \dot{\underline{x}}_1, \ldots, \dot{\underline{x}}_n, t), \qquad (N2)$$

where $p_i = m_i \dot{\underline{x}}_i$, m_i = mass of ith particle.

The principle of Galilean invariance says if $\{\underline{x}_i(t)\}$ is a sol^n of N2, so is $\{g \circ \underline{x}_i(t)\}$ for every $g \in \mathrm{Gal}$.

e.g. If $\{\underline{x}_i(t)\}$ solves N2, so does $\{\underline{x}_i(t+s)\}$ $\forall s \in \mathbb{R}$

$$\Rightarrow \quad m_i \ddot{\underline{x}}_i(t+s) = \underline{F}_i(\{\underline{x}_j(t+s), \dot{\underline{x}}_j(t+s)\}, t)$$

$$\Rightarrow \quad m_i \ddot{\underline{x}}_i(t') = \underline{F}_i(\{\underline{x}_j(t'), \dot{\underline{x}}_j(t')\}, t'-s)$$

$$\Rightarrow \underline{F}_i(\{\underline{x}_j(t), \dot{\underline{x}}_j(t)\}, t-s) = \underline{F}_i(\{\underline{x}_j(t), \dot{\underline{x}}_j(t)\}, t)$$

So $\underline{F}_i$ has no explicit time-dependence.

e.g. If $\{\underline{x}_i(t)\}$ obey N2, so does $\{\underline{x}_i(t) + \underline{c}\}$ $\forall \underline{c} \in \mathbb{R}^3$.

$\Rightarrow$ $\underline{F}_i(\{\underline{x}_j - \underline{x}_k\}, \{\dot{\underline{x}}_j\})$ can only depend on relative locations.

e.g. If $\{\underline{x}_i(t)\}$ solves N2, so do $\{\underline{x}_i(t) + \underline{v}t\}$

$$\Rightarrow \quad \underline{F}_i = \underline{F}_i(\{\underline{x}_j - \underline{x}_k, \dot{\underline{x}}_j - \dot{\underline{x}}_k\})$$

e.g. If $\{\underline{x}_i(t)\}$ obeys N2, so do $\{R\underline{x}_i(t)\}$.

$$m_i (R\ddot{\underline{x}}_i) = \underline{F}_i(\{R(\underline{x}_j - \underline{x}_k), R(\dot{\underline{x}}_j - \dot{\underline{x}}_k)\})$$

$$\parallel$$

$$R(m_i \ddot{\underline{x}}_i) = R\,\underline{F}_i(\{\underline{x}_j - \underline{x}_k, \dot{\underline{x}}_j - \dot{\underline{x}}_k\})$$

Hence, $\underline{F}_i(\{R(\underline{x}_j - \underline{x}_k), R(\dot{\underline{x}}_j - \dot{\underline{x}}_k)\}) = R\,\underline{F}_i(\{\underline{x}_j - \underline{x}_k, \dot{\underline{x}}_j - \dot{\underline{x}}_k\})$

Let's consider a Galilean invariant system that consist of just one particle. Then Gal inv. $\Rightarrow$ $\underline{F} = 0$. Hence,

$\dot{\underline{p}}_1 = 0 \Rightarrow \underline{x}_1(t) = \underline{x}_0 + \underline{u}t$ rectilinear motion

Now consider a Gal inv. system of two particles.

$$m_1 \ddot{\underline{x}}_1 = \underline{F}_1(\underline{x}_1 - \underline{x}_2, \dot{\underline{x}}_1 - \dot{\underline{x}}_2)$$
$$m_2 \ddot{\underline{x}}_2 = \underline{F}_2(\underline{x}_1 - \underline{x}_2, \dot{\underline{x}}_1 - \dot{\underline{x}}_2)$$

By rot. invariance,

$$\underline{F}_1(\underline{x}_1 - \underline{x}_2, \dot{\underline{x}}_1 - \dot{\underline{x}}_2) = (\underline{x}_1 - \underline{x}_2) f(\underline{x}_1 - \underline{x}_2, \dot{\underline{x}}_1 - \dot{\underline{x}}_2)$$
$$+ (\dot{\underline{x}}_1 - \dot{\underline{x}}_2) g(\underline{x}_1 - \underline{x}_2, \dot{\underline{x}}_1 - \dot{\underline{x}}_2)$$

for scaler f^n f, g. In particular,

$$m_1 \ddot{\underline{x}}_1 \cdot \left((\underline{x}_1 - \underline{x}_2) \times (\dot{\underline{x}}_1 - \dot{\underline{x}}_2)\right) = 0$$

So motion is confined to a plane.

Closed systems

A closed system is one in which the forces are forces of interaction, ie $\underline{F}_i = \sum \underline{F}_{ij}$, where $\underline{F}_{ij}$ is the force on i-th particle due to j-th particle.

$$\underline{F}_{ij} = -\underline{F}_{ji} \qquad (N3)$$

$$\underline{F}_{ij} = -\underline{F}_{ji} = (\underline{x}_i - \underline{x}_j) f(|\underline{x}_i - \underline{x}_j|, |\dot{\underline{x}}_i - \dot{\underline{x}}_j|) \qquad (N3')$$

Define $M = \sum_i m_i$ (total mass)

$$\underline{x}_{com} = \sum_i \frac{m_i \underline{x}_i}{M} \quad \text{(centre of mass location)}$$

$$\Rightarrow \quad M \ddot{\underline{x}}_{com} = \sum_i m_i \ddot{\underline{x}}_i = \sum_i \left(\sum_{j \neq i} \underline{F}_{ij} \right) = \sum_{i > j} (\underline{F}_{ij} + \underline{F}_{ji}) = 0$$

$$\Rightarrow \quad \ddot{\underline{x}}_{com} = \underline{x}_0 + \underline{u}t$$

It's often convenient to do a boost + translation $\in$ Gal s.t. $\underline{x}_{com}(t) = 0$.

Define angular momentum

$$\underline{L} = \sum_i \underline{x}_i \times \underline{p}_i = \sum_i \underline{x}_i \times (m_i \dot{\underline{x}}_i)$$

$$\Rightarrow \dot{\underline{L}} = \sum_i \underline{x}_i \times \underline{F}_i = \sum_i \underline{x}_i \times \Big(\sum_{j \neq i} \underline{F}_{ij} \Big)$$

We have

$$\underline{x}_1 \times \underline{F}_{12} + \underline{x}_2 \times \underline{F}_{21} \overset{(N3)}{=} (\underline{x}_1 - \underline{x}_2) \times \underline{F}_{12}$$

$$\overset{(N3')}{=} 0$$

$\Rightarrow$ $\underline{L}$ conserved if (N3') holds.

Non-closed system

These often arise from considering only relative motion, or "neglecting back-reaction".

e.g. For a closed system of 2 particles,

$$m_1 \ddot{\underline{x}}_1 = \underline{F}_{12} \quad , \quad m_2 \ddot{\underline{x}}_2 = -\underline{F}_{12}$$

Define $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ reduced mass

$\underline{x}_r = \underline{x}_1 - \underline{x}_2$ relative location

Then $\ddot{\underline{x}}_r = \ddot{\underline{x}}_1 - \ddot{\underline{x}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \underline{F}_{12} \Rightarrow \mu \ddot{\underline{x}}_r = \underline{F}(\underline{x}_r, \dot{\underline{x}}_r)$

So we get a closed system for the relative motion.

NB. this relative motion force does not need to obey Gal inv.

In particular, if $m_2 \gg m_1$, $\mu \approx m_1$ and we can approximate the motion $\underline{x}_1(t) \approx \underline{x}_r(t)$ in the frame which $\underline{x}_{com}(t) = 0$.

Energy

Work done along a path from $\underline{x}_0$ at t_0 to $\underline{x}_1$ to t_1 is

$$W = \int_{\underline{x}_0}^{\underline{x}_1} \underline{F} \cdot d\underline{x} = \int_{t_0}^{t_1} \underline{F} \cdot \frac{d\underline{x}}{dt}\, dt$$

$$\overset{(N2)}{=} \int_{t_0}^{t_1} m \dot{\underline{x}} \cdot \ddot{\underline{x}}\, dt$$

$$= \frac{1}{2} \int_{t_0}^{t_1} \frac{d}{dt} (m \dot{\underline{x}})^2\, dt$$

$$= T(t_1) - T(t_0), \qquad (*)$$

where we define the kinetic energy $T(t) = \frac{1}{2} m \dot{\underline{x}}^2(t)$.

A force is conservative if W is indpt of the path taken $\Rightarrow$ $\underline{F} = -\nabla V$ for a f^n $V: \mathbb{R}^3 \to \mathbb{R}$ called the potential

$$W = -\int_{\underline{x}_0}^{\underline{x}_1} \nabla V \cdot d\underline{x} = V(\underline{x}_1) - V(\underline{x}_0) \qquad (+)$$

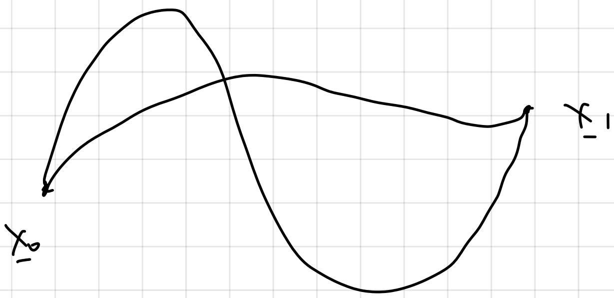
Comparing (*) and (+), we see

$$T(t_1) + V(\underline{x}(t_1)) = T(t_0) + V(\underline{x}(t_0))$$

Hence the total energy $E = T + V$ is conserved.

The Principle of Least Action

Consider a (smooth) motion / curve $\underline{x}(t)$ with $\underline{x}(t_0) = \underline{x}_0$ and $\underline{x}(t_1) = \underline{x}_1$ fixed.



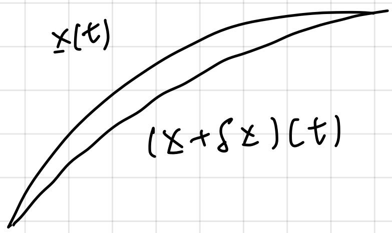
We define the action of such a curve to be

$$S[\underline{x}] = \int_{t_0}^{t_1} L(\underline{x}(t), \dot{\underline{x}}(t), t)\, dt$$

where the f^n $L : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is called the Lagrangian. (We usually choose $L = T - V$).

S is a f^n on the ∞-dim space of curves, sometimes called a functional.

Let's consider the critical points of this action. A curve $\underline{x}(t)$ will be the extramum of $S[\underline{x}]$ if



$$S[\underline{x} + \delta\underline{x}] - S[\underline{x}] = O(\delta x^2)$$

$$S[\underline{x} + \delta\underline{x}] = \int_{t_0}^{t_1} L(\underline{x} + \delta\underline{x}, \dot{\underline{x}} + \delta\dot{\underline{x}}, t)\, dt$$

$$= S[\underline{x}] + \int_{t_0}^{t_1} \delta\underline{x} \cdot \frac{\partial L}{\partial \underline{x}} + \delta\dot{\underline{x}} \cdot \frac{\partial L}{\partial \dot{\underline{x}}}\, dt + O(\delta\underline{x}^2)$$

$$\delta S = \int_{t_0}^{t_1} \delta \underline{x} \cdot \left(\frac{\partial L}{\partial \underline{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}}\right)\right) + \underbrace{\delta \underline{x} \cdot \frac{\partial L}{\partial \dot{\underline{x}}}\Big|_{t_0}^{t_1}}_{=0} + O(\delta \underline{x}^2)$$

So $\delta S = 0$ to first order for <u>every</u> variation $\delta \underline{x}$ iff

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}}\right) - \frac{\partial L}{\partial \underline{x}} = 0$$

These are called the Euler-Lagrange eqns.

E.g. the length l of a curve $(x(t), y(t))$ from $\underline{x}_0$ to $\underline{x}_1$ in $\mathbb{R}^2$ is

$$l = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

The shortest curve extremise this length, so need obey E-L for Lagrangian $\sqrt{\dot{x}^2+\dot{y}^2}$, or of $\sqrt{1+(y')^2}$

$$\Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0 \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = C$$

$$\Rightarrow y' = C$$

$$\Rightarrow y = Cx + D.$$

Hamilton's principle of least action says that the true motion of a particle extremises

$$S[\underline{x}] = \int_{t_0}^{t_1} L(\underline{x}, \dot{\underline{x}}, t)\, dt$$

for $L = T - V = \frac{1}{2} m \dot{\underline{x}}^2 - V(\underline{x}, t)$

For this L, EL become

$$\frac{d}{dt}(m\dot{\underline{x}}) + \nabla V = 0 \qquad (\text{or } \dot{\underline{p}} = \underline{F})$$

This is equivalent to N2, but gives us a different perspectives.

- Focus is on curves / trajectories, rather than forces
- This is more 'global' picture.

In QM, <u>all</u> paths make a contribution.

$$\mathbb{P}(x_1, t_1 \mid x_0, t_0) = \left| \langle x_1(t_1) \mid x_0(t_0) \rangle \right|^2$$

where $\langle x_1(t_1) \mid x_0(t_0) \rangle = \displaystyle\int_{\substack{\text{all paths} \\ \text{s.t. } x(t_0)=x_0, \\ x(t_1)=t_1}} dx \; e^{iS[x]/\hbar}$ (Feymann's path integral)

We can also easily generalise field theory rather than mechanics, where

$$S = \int \Big[\underbrace{R}_{\text{GR}} + \tfrac{1}{2} \overbrace{F^2}^{\text{EM}} + \underbrace{\bar{\psi} \not{D} \psi}_{\text{matter}} + \text{Higgs} \Big] \sqrt{-g} \, d^4x$$

<u>Generalised Coords</u>

Suppose we introduce a new coord system q^a where $q^a = q^a(\underline{x}, t)$ are smooth f^n's of $(\underline{x}, t)$. For this to be a good system of coords, need

$$\det\left(\frac{\partial q^a}{\partial \underline{x}}\right) \neq 0$$

So invertible. $\underline{x} = \underline{x}(q^a, t)$.

Then $\dot{\underline{x}} = \dfrac{d\underline{x}}{dt} = \dfrac{\partial \underline{x}}{\partial q^a} \dot{q}^a + \dfrac{\partial \underline{x}}{\partial t}$.

Let's now consider Lagrangian

$$L = L(\underline{x}(q^a, t), \dot{\underline{x}}(q^a, \dot{q}^a, t), t)$$

We have

$$\frac{\partial L}{\partial q^a} = \frac{\partial L}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial q^a} + \frac{\partial L}{\partial \dot{\underline{x}}} \cdot \frac{\partial \dot{\underline{x}}}{\partial q^a}$$

$$= \frac{\partial L}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial q^a} + \frac{\partial L}{\partial \dot{\underline{x}}} \cdot \left(\frac{\partial^2 \underline{x}}{\partial q^a \partial q^b} \dot{q}^b + \frac{\partial^2 \underline{x}}{\partial q^a \partial t} \right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}} \cdot \frac{\partial \dot{\underline{x}}}{\partial \dot{q}^a} \right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}} \cdot \frac{\partial \underline{x}}{\partial q^a} \right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}}\right) \cdot \frac{\partial \underline{x}}{\partial q^a} + \frac{\partial L}{\partial \dot{\underline{x}}} \cdot \left(\frac{\partial^2 \underline{x}}{\partial q^a \partial q^b} \dot{q}^b + \frac{\partial^2 \underline{x}}{\partial t \partial q^a} \right)$$

Comparing these, we see that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}}\right) - \frac{\partial L}{\partial \underline{x}} \right) \cdot \frac{\partial \underline{x}}{\partial q^a}$$

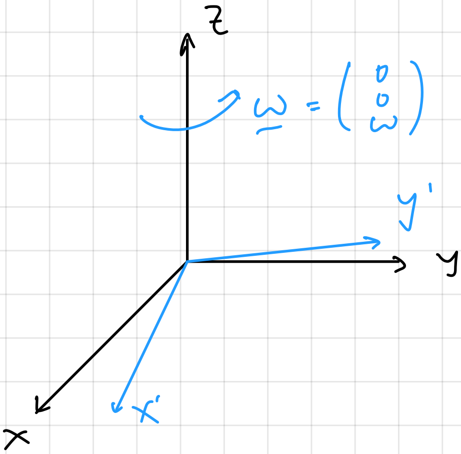
Since matrix $\frac{\partial \underline{x}}{\partial q^a}$ was non-degenerate, we see

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = 0 \iff \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}} - \frac{\partial L}{\partial \underline{x}}\right) = 0 ,$$

so E-L for q's hold iff they hold for $\underline{x}$'s.

This is a huge advantage. We only ever need to work out the form of 1 f^n $L(q^a, \dot{q}^a, t)$ to obtain motion in our generalised coord system.

E.g. Rotating coord systems



$$x' = x\cos\omega t + y\sin\omega t$$
$$y' = y\cos\omega t - x\sin\omega t$$
$$z' = z$$

The free particle in inertial frame (x, y, z) has

$$L = \frac{1}{2} m \dot{\underline{x}}^2 = \frac{1}{2} m \left(\dot{\underline{x}}' + \underline{\omega} \times \underline{x}' \right)^2 .$$

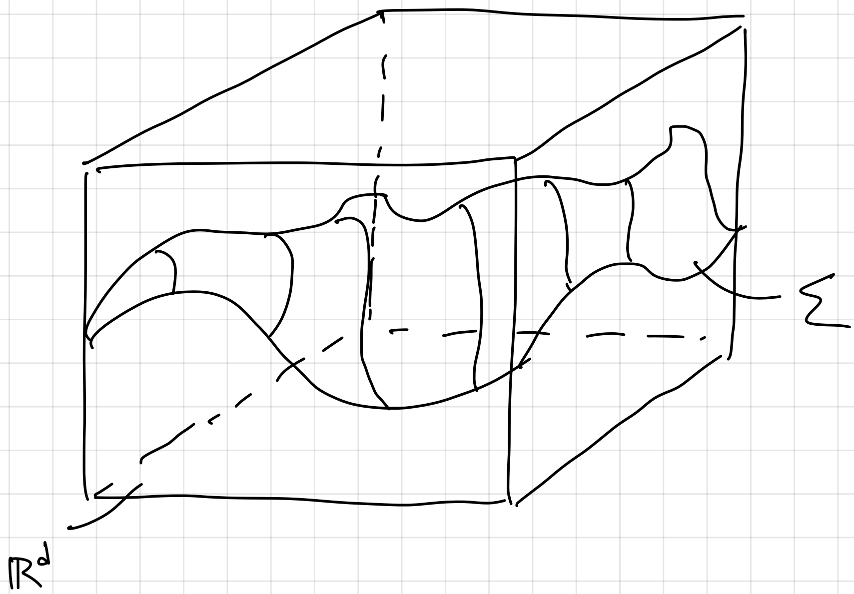
Hence in the rotating frame, the E-L become

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}'} \right) - \frac{\partial L}{\partial \underline{x}'} = m \frac{d}{dt}\left(\dot{\underline{x}}' + \underline{\omega} \times \underline{x}' \right) - m \left(\dot{\underline{x}}' \times \underline{\omega} - \underline{\omega} \times (\underline{\omega} \times \underline{x}') \right),$$

i.e. $$\ddot{\underline{x}}' + \underbrace{2\underline{\omega} \times \dot{\underline{x}}'}_{\text{Coriolis}} + \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{x}')}_{\text{Centripetal}} = 0$$

Constraints

Sometimes we want to force our particle to lie on some surfa $\Sigma \subset \mathbb{R}^d$. e.g. Particle forced to stay in a hoop.



The constraints are called holonomic if we can write

$$\Sigma = \{ f_r(\underline{x}, t) = 0, \quad r = 1, \dots, d-s \} \subset \mathbb{R}^d$$

for some fn's f_r that are indpt of $\dot{\underline{x}}$.

(E.g. For motion on $S^2_a \subset \mathbb{R}^3$, $f = x^2 + y^2 + z^2 - a^2$)

For each $r \in \{1, \dots, d-s\}$, the d-component vector $\frac{\partial f_r}{\partial \underline{x}}$ is orthogonal to Σ. If the $(d-s) \times d$ dim matrix

$$\frac{\partial f_r}{\partial \underline{x}} = \underbrace{\begin{pmatrix} \text{---} & \partial f_1 / \partial \underline{x} & \text{---} \\ & \vdots & \\ \text{---} & \partial f_{d-s} / \partial \underline{x} & \text{---} \end{pmatrix}}_{d} \Big\} {\scriptstyle d-s}$$

has maximal rank (i.e. $\dim\left(\mathrm{Im} \frac{\partial f_r}{\partial \underline{x}} \right) = d-s$), then the rows form a basis of space orthogonal to Σ at each point on Σ.

(E.g. $f = x^2 + y^2 + z^2 - a^2$, $\frac{\partial f}{\partial \underline{x}} = (x, y, z)$ points radially $\left.\frac{\partial f}{\partial \underline{x}}\right|_\Sigma = a\hat{\underline{x}}$.)

Then, by IFT, we can find generalised coords that parameterise the constraint surface Σ.

(e.g. $q^a = (q^1, q^2) = (\theta, \phi)$

$x = a \sin\theta \cos\phi$, $y = a \sin\theta \sin\phi$, $z = a\cos\theta$)

The constrained motion will then be $\underline{x}(q^a, t)$. To see this, consider a new Lagrangian

$$L'(\underline{x}, \dot{\underline{x}}, \lambda_r, t) = L(\underline{x}, \dot{\underline{x}}, t) + \sum_{r=1}^{d-s} \lambda_r f_r(\underline{x}, t).$$

The λ_r are Lagrange multipliers. They appear as non-dynamical variables $\frac{\partial L'}{\partial \dot{\lambda}_r} = 0$.

The E-L eqn for L' give

$$\frac{\partial L}{\partial \lambda_r} = 0 \Rightarrow f_r(\underline{x}, t) = 0$$, ie. motion is constrained

and

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\underline{x}}}\right) - \frac{\partial L'}{\partial \underline{x}} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}}\right) - \frac{\partial L}{\partial \underline{x}} - \sum_r \lambda_r \frac{\partial f_r}{\partial \underline{x}} = 0.$$

The new term $\underline{R} = \sum_r \lambda_r \frac{\partial f_r}{\partial \underline{x}}$ are the constraint forces.

They do no work because motion obeys $d\underline{x} \cdot \underline{R} = 0$.

Let's instead use generalised coords (q^a, f_r)

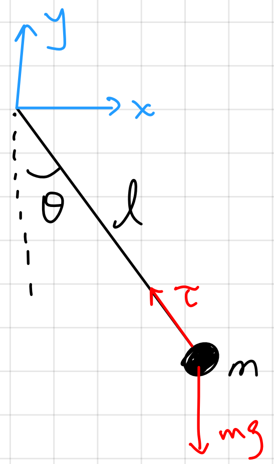
parameterise Σ $\quad$ $f_r = 0$ defines Σ

E-L say : $\quad f_r = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = \sum_r \lambda_r \cancelto{0}{\frac{\partial f_r}{\partial q^a}} = 0$$

So the motion for q^a's doesn't care about constraints

E.g. A simple pendulum



Can describe this system using the Lagrangian

$$L' = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy + \lambda(x^2 + y^2 - \ell^2)$$

Then the E-L give

$$x^2 + y^2 = \ell^2$$
$$m\ddot{x} = 2\lambda x$$
$$m\ddot{y} = -mg + 2\lambda y$$

We can solve the constraint $x^2+y^2 = \ell^2$ by setting $x = \ell \sin\theta$, $y = -\ell\cos\theta$, where upon the remaining eqn become

$$m\ell\,(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) = 2\lambda\ell\sin\theta \qquad (*)$$
$$m\ell\,(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) = -mg - 2\lambda\ell\cos\theta \qquad (+)$$

$\cos\theta \times (*) + \sin\theta \times (+)$ gives

$$m\ell\ddot{\theta} = -mg\sin\theta \quad \Rightarrow \quad \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

which is eqn of motion (eom) along the constraint surface.

We also see

$$\lambda\frac{\partial f}{\partial \underline{x}} = 2\lambda\begin{pmatrix} x \\ y \end{pmatrix} = 2\lambda\ell\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} = \underline{\tau}$$

So is just the tension in the rod.

It's much better to use generaliced coords adopted to the constraint(s). Here, these are $(q^a, f_r) \rightsquigarrow (\theta, f)$.

So our Lagrangian becomes

$$L'(\theta, f) = \frac{1}{2} m l \dot{\theta}^2 + mgl\cos\theta + \lambda f$$

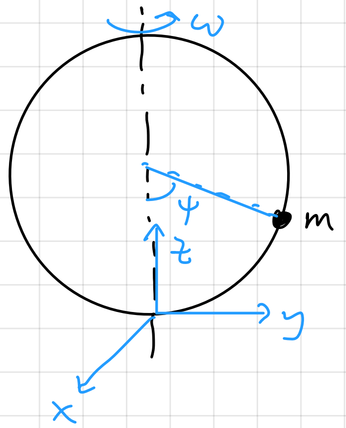
$$\begin{pmatrix} \dot{x} = -l\cos\theta\ \dot{\theta} \\ \dot{y} = l\sin\theta\ \dot{\theta} \end{pmatrix}$$

$$\frac{\partial L'}{\partial \lambda} = 0 \Rightarrow f = 0$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow ml^2\ddot{\theta} = -mgl\sin\theta$$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

E.g. Beed on a Rotating Hoop.



Beed is constrained to lie on a steel hoop of radius a, which rotates around the z-axis with frequency ω.

We can solve the constraints by setting

$$x = a\sin\psi\cos\omega t$$

$$y = a\sin\psi\sin\omega t$$

$$z = a - a\cos\psi$$

Using these in our Lagrangian,

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}ma^2(\dot{\psi}^2 + \omega^2\sin^2\psi)$$

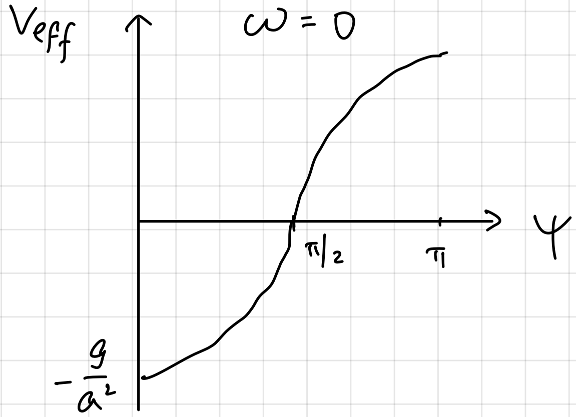
$$V = mgz = mga(1 - \cos\psi)$$

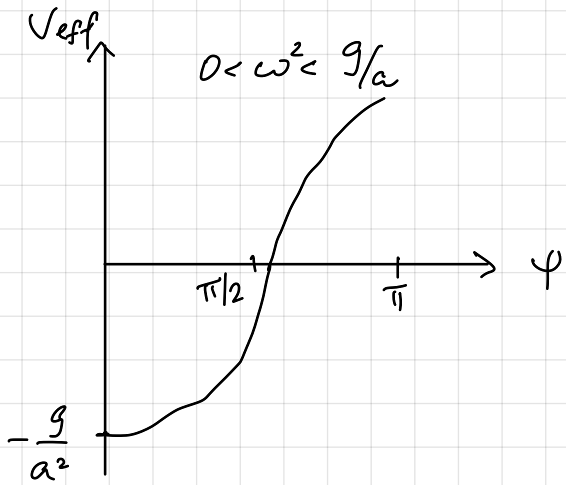
Hence the E-L for generalised coord ψ become

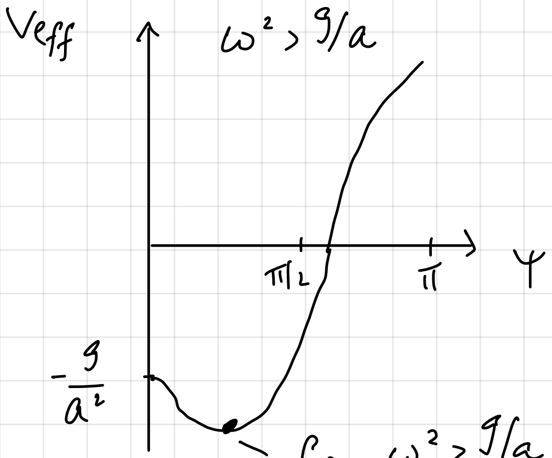
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = \frac{\partial L}{\partial \psi} \Rightarrow \ddot{\psi} = -\frac{\partial V_{eff}}{\partial \psi}.$$

where $V_{eff} = \frac{1}{ma^2}\left(-mga\cos\psi - \frac{1}{2}ma^2\omega^2\sin^2\psi\right)$.

In equilibrium, the bead must sit at an extrema of $V_{eff} = \frac{1}{ma^2}\left(-mga\cos\psi - \frac{1}{2}ma^2\omega^2\sin^2\psi\right)$.







for $\omega^2 > g/a$, the stable point is at $\psi_0 > 0$.

Noether's Theorem

There's a very beautiful relation between symmetries of a Lagrangian and concerned quantities.

We define the generalised momentum p_a associated to a generalised coord q^a by $p_a = \frac{\partial L}{\partial \dot{q}^a}$. (cf. $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$)

We also define a quantity $f(q^a, \dot{q}^a, t)$ is conserved along the motion if

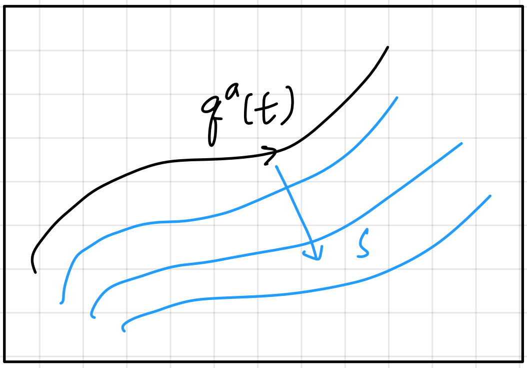
$$0 = \frac{df}{dt} = \frac{\partial f}{\partial q^a} \dot{q}^a + \frac{\partial f}{\partial \dot{q}^a} \ddot{q}^a + \frac{\partial f}{\partial t}$$

for the motion $q^a(t)$ that solves the E-L.

Consider a 1-parameter family of motions $Q^a(s,t)$ with $Q^a(0,t) = q^a(t)$, $s \in \mathbb{R}$.

This transformation is said to be a symmetry of L if

$$\frac{\partial}{\partial s} L(Q^a(s,t), \dot{Q}^a(s,t), t) = 0 .$$



Thm (Noether's thm) $\exists$ conserved quantity for each such symmetry, when the E-L holds.

Pf : $$0 = \frac{\partial L}{\partial s}\Big|_{s=0} = \frac{\partial L}{\partial q^a} \frac{\partial Q^a}{\partial s}\Big|_{s=0} + \frac{\partial L}{\partial \dot{q}^a} \frac{\partial \dot{Q}^a}{\partial s}\Big|_{s=0}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) \frac{\partial Q^a}{\partial s}\Big|_{s=0} + \frac{\partial L}{\partial \dot{q}^a} \frac{d}{dt}\left(\frac{\partial Q^a}{\partial s}\Big|_{s=0}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \frac{\partial Q^a}{\partial s}\Big|_{s=0}\right)$$

Hence, $\frac{\partial L}{\partial \dot{q}^a} \frac{\partial Q^a}{\partial s}\Big|_{s=0}$ is conserved if $Q^a(0,t) = q^a(t)$ satisfies E-L. $\square$

For example, homogenity of space implies our L should be invariant under constant translations $\underline{x}_i(t) \mapsto \underline{x}_i(t) + c$

e.g. $$L = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\underline{x}}_i^2 - V(\underline{x}_i - \underline{x}_j)$$

Setting $Q_i(s,t) = \underline{x}_i(t) + s\underline{c}$, we have a conserved quantity

$$\sum_i \frac{\partial L}{\partial \underline{x}_i} \cdot \underline{c} = \left(\sum p_i\right) \cdot \underline{c}.$$

Since true for all $\underline{c}$, the total momentum is conserved.

N.B. This is the special case of the following:

suppose $L(q^a, \dot{q}^a, t)$ is indpt of a particular generalised coord q^1, the E-L for q^1 say

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^1}\right) = \frac{\partial L}{\partial q^1} = 0$$

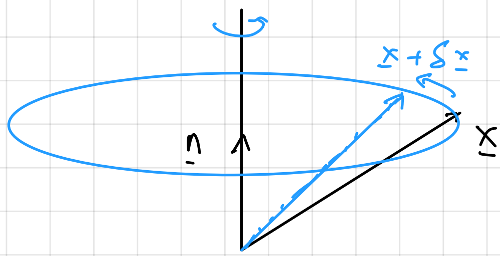
So the corresponding generaliced momentum $p_1 = \frac{\partial L}{\partial \dot{q}^1}$ is conserved.

Such coordinates are called *ignorable*

E.g. Isotropy of space $\Rightarrow$ L should be invariant under rotations of all our particles' trajectories. An infinitestimal rotation act as

$$\underline{x}_i(t) \mapsto \underline{x}_i(t) + \delta\theta\, \underline{n} \times \underline{x}_i(t)$$

if rotate by $\delta\theta$ around $\underline{n}$-axis $(\underline{n}\cdot\underline{n} = 1)$.



Hence,

$$\dot{\underline{x}}_i(t) \mapsto \dot{\underline{x}}_i(t) + \delta\theta\, \underline{n} \times \dot{\underline{x}}_i(t)$$

So a Lagrangian that is rotation invariant obeys

$$\frac{\partial}{\partial\theta} L(\underline{x}_i + \theta\, \underline{n} \times \underline{x}_i,\ \dot{\underline{x}}_i + \theta\, \underline{n} \times \dot{\underline{x}}_i,\ t)\Big|_{\theta=0} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial L}{\partial \underline{x}_i} \cdot (\underline{n} \times \underline{x}_i) + \frac{\partial L}{\partial \dot{\underline{x}}_i} \cdot (\underline{n} \times \dot{\underline{x}}_i) = 0$$

$$\Rightarrow 0 = \underline{n} \cdot \sum_i \left(\underline{x}_i \times \frac{\partial L}{\partial \underline{x}_i} + \dot{\underline{x}}_i \times \frac{\partial L}{\partial \dot{\underline{x}}_i} \right)$$

$$\overset{(E-L)}{=} \underline{n} \cdot \frac{d}{dt} \left(\sum_i \underline{x}_i \times \frac{\partial L}{\partial \dot{\underline{x}}_i} \right)$$

$$= \underline{n} \cdot \frac{d}{dt} \sum_i \underline{x}_i \times \underline{p}_i$$

Since true for any axis $\underline{n}$, the total angular momentum

$$\underline{L} = \sum_i \underline{x}_i \times \underline{p}_i$$

is conserved.

There's a more general form for Noether's thm.

Notice that L and $L + \frac{d}{dt} f(\underline{x}, t)$ lead to the same E-L.

$$S = \int \left(L + \frac{df}{dt} \right) t = \int L \, dt + f \Big|_{\underline{x}_0, t_0}^{\underline{x}_1, t_1}$$

and the boundary term is the same for all trajectories obeying $\delta \underline{x} \big|_{t_0, t_1} = 0$. Consequently, it's enough for our symmetry to preserve L up to a total time-derivative $\frac{df(\underline{x}, t)}{dt}$.

In fact, Noether's thm allows more generally L to be preserved up to $\frac{d}{dt} f(\underline{x}, \dot{\underline{x}}, t)$

Thm (Noether's thm I) Suppose we have an infinitestimal transformation

$$q^a(t) \mapsto q^a(t) + s\,u^a(t) + O(s^2)$$

such that

$$\frac{\partial}{\partial s} L(q^a + s u^a, \dot{q}^a + s\dot{u}^a, t)\Big|_{s=0} = \frac{d}{dt} f(q^a, \dot{q}^a, t)$$

then $\frac{\partial L}{\partial \dot{q}^a} u^a - f$ is conserved when E-L hold.

Pf: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} u^a - f\right)$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) u^a + \frac{\partial L}{\partial \dot{q}^a} \dot{u}^a - \frac{df}{dt}$$

$$\underset{(E\text{-}L)}{=} \frac{\partial L}{\partial q^a} u^a + \frac{\partial L}{\partial \dot{q}^a} \dot{u}^a - \frac{df}{dt}$$

$$= \frac{\partial}{\partial s} L(q + su, \dot{q} + s\dot{u}, t)\Big|_{s=0} - \frac{df}{dt} = 0 \qquad \square$$

E.g. Homogenity in time implies $\frac{\partial L}{\partial t} = 0$.

For such L, consider translating all the trajectories in time, so

$$q^a(t) \mapsto q^a(t+s) \approx q^a(t) + s\dot{q}^a(t) + O(s^2)$$

$$\dot{q}^a(t) \mapsto \dot{q}^a(t+s) \approx \dot{q}^a(t) + s\ddot{q}^a(t) + O(s^2)$$

Hence, $\frac{\partial}{\partial s} L(q^a + s\dot{q}^a, \dot{q}^a + s\ddot{q}^a)\Big|_{s=0} = \frac{dL}{dt}$.

According to Noether I, the conserved quantity is

$$H = \frac{\partial L}{\partial \dot{q}^a} \dot{q}^a - L$$

This is known as the Hamiltonian, and represents the energy.

e.g. $L = \frac{1}{2} m \dot{\underline{x}}^2 - V(\underline{x})$

$$H = m \dot{\underline{x}} \cdot \dot{\underline{x}} - \left(\frac{1}{2} m \dot{\underline{x}}^2 - V(\underline{x})\right) = \frac{1}{2} m \dot{\underline{x}}^2 + V(\underline{x}) = E.$$

If $\underline{x} = \underline{x}(q)$ with some generalised coord $q^a(t)$, then

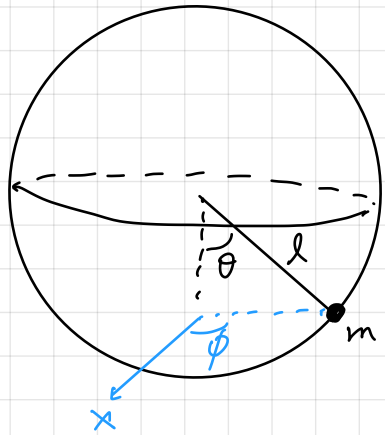
$\dot{\underline{x}} = \frac{\partial \underline{x}}{\partial q^a} \dot{q}^a$ so

$$L = T_{ab}(q)\, \dot{q}^a \dot{q}^b - V(q),$$

where $T_{ab} = \frac{1}{2} m \frac{\partial \underline{x}}{\partial q^a} \cdot \frac{\partial \underline{x}}{\partial q^b}(q)$. Then

$$H = \frac{\partial L}{\partial \dot{q}^a} \dot{q}^a - L = T_{ab} \dot{q}^a \dot{q}^b + V(q)$$

E.g. <u>Spherical pendulum.</u> : A mass m is attached to an inextensible rod of length l, but is free to swing in <u>any</u> direction.



$$L = \frac{1}{2} m \dot{\underline{x}}^2 - mgz + \lambda(x^2 + y^2 + z^2 - l^2)$$

As usual, solve the constraint by setting

$$x = l \sin\theta \cos\phi, \quad y = l \sin\theta \sin\phi, \quad z = -l \cos\theta$$

In terms of these,

$$L = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + mgl\cos\theta.$$

We see $\frac{\partial L}{\partial t} = 0$ and $\frac{\partial L}{\partial \phi} = 0$

$$\Rightarrow H = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) - mgl\cos\theta \quad \text{(Energy)}$$

$$J = \frac{1}{2} m l^2 \sin^2\theta\, \dot{\phi} \quad \text{(ang. momentum in } \phi \text{ direction)}$$

Use conserved quantity J to simplify

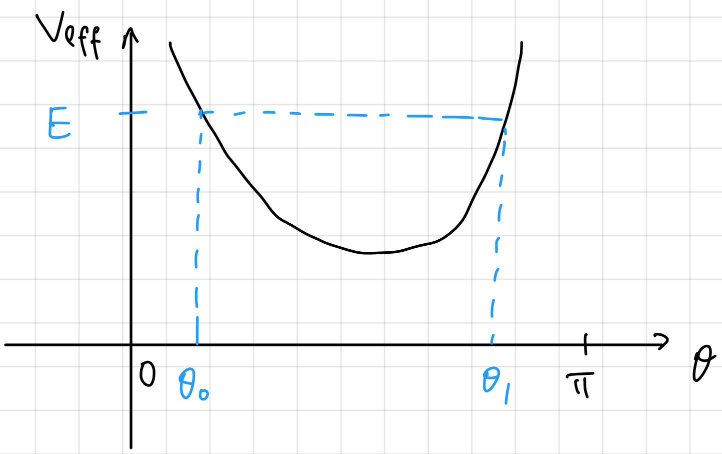
$$ml^2 \ddot{\theta} = ml^2 \frac{J^2}{(ml^2)^2} \frac{\cos\theta}{\sin^3\theta} - mgl\sin\theta$$

$$\Rightarrow \quad \ddot{\theta} = \frac{J}{(ml^2)^2} \frac{\cos\theta}{\sin^3\theta} - \frac{g}{l}\sin\theta$$

$$= -\frac{d V_{eff}}{d\theta},$$

where $V_{eff}(\theta) = -\frac{g}{l}\cos\theta + \frac{J^2}{2(ml^2)^2\sin^2\theta}$.

<u>N.B.</u> Can only substitute J into e.o.m., <u>not</u> into the Lagrangian.



For $J \neq 0$, V_{eff} grows as $\theta \to 0$.

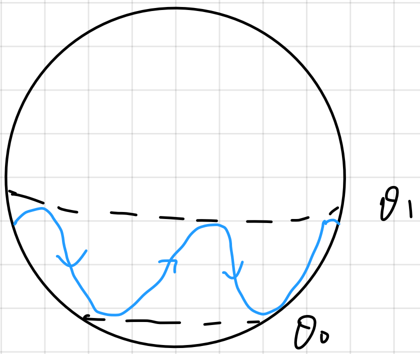
Since $\frac{\partial L}{\partial t} = 0$, $H = \dot{\theta}\frac{\partial L}{\partial \theta} + \dot{\phi}\frac{\partial L}{\partial \phi} - L$

$$= ml^2\left(\frac{1}{2}\dot{\theta}^2 + V_{eff}(\theta)\right) = E \text{ const.}$$

Hence, for fixed (E, J), the motion in θ is bounded $\theta \in [\theta_0, \theta_1]$. Hence, the motion will oscillate in θ as it precesses in ϕ.

To go further, we can write

$$\frac{d\theta}{dt} = \sqrt{\frac{2E}{ml^2} - V_{eff}(\theta)}.$$



So

$$t_1 - t_0 = \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{\frac{2E}{ml^2} - V_{eff}(\theta)}} .$$

If we could, do this integral. we'd know $\theta(t)$ and hence $\phi(t)$, but we can't (except numerically).

The Lagrangian for a Relativistic Particle

In SR, a free particle of rest mass m has action

$$S[\underline{x}] = -mc^2 \int \sqrt{1 - \dot{\underline{x}}^2/c^2} \; dt$$

Note: if $|\dot{\underline{x}}| \ll c$, $L \approx -mc^2 + \frac{1}{2} m \dot{\underline{x}}^2 + O\left(\left(\frac{\dot{\underline{x}}}{\underline{x}}\right)^2\right)$.

For this L, have

$$\underline{p} = \frac{\partial L}{\partial \dot{\underline{x}}} = \frac{m\dot{\underline{x}}}{\sqrt{1-\dot{\underline{x}}^2/c^2}} = m\gamma \dot{\underline{x}}$$

$$H = \dot{\underline{x}} \cdot \frac{\partial L}{\partial \dot{\underline{x}}} - L = \frac{m\dot{\underline{x}}^2}{\sqrt{1-\dot{\underline{x}}^2/c^2}} + mc^2\sqrt{1-\dot{\underline{x}}^2/c^2}$$

$$= \frac{mc^2}{\sqrt{1-\dot{\underline{x}}^2/c^2}}$$

$$= m\gamma c^2 .$$

However, this form of L isn't obviously Lorentz invariant.

Instead, consider the alternative action

$$S[x^\mu] = -mc \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \, d\tau .$$

where $x^\mu = (ct, \underline{x})$, τ is arbitrary parameter. and $\eta_{\mu\nu} = \operatorname{diag}(-1, 1, 1, \dots, 1)$.

This action is manifestly Lorentz invariant

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu, \qquad \frac{dx^\mu}{d\tau} \mapsto \Lambda^\mu{}_\nu \frac{dx^\nu}{d\tau}.$$

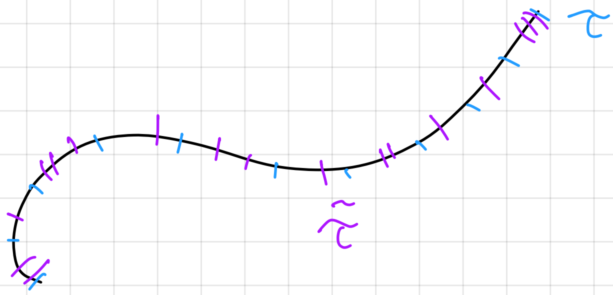
However, seem to have an extra d.o.f. $(t(\tau), \underline{x}(\tau))$.
This is illusory: the particle <u>has</u> to 'move through time'. This is reflected in a redundancy of $S[x^\mu]$.

Consider reparameterisations

$$\tau \mapsto \tilde{\tau}(\tau),$$

$$d\tau \mapsto \frac{d\tau}{d\tilde{\tau}} d\tilde{\tau}$$

$$\frac{dx^\mu}{d\tau} \mapsto \frac{d\tilde{\tau}}{d\tau} \cdot \frac{dx^\mu}{d\tilde{\tau}}.$$



Hence,

$$S[x^\mu] = -mc \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau$$

$$= -mc \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tilde{\tau}} \frac{dx^\nu}{d\tilde{\tau}}}\, d\tilde{\tau}$$

So the action is reparameterisation invariant.

We can use this to fix $\frac{dt}{d\tau} = 1$ since $\frac{dt}{d\tau} \mapsto \frac{d\tilde{\tau}}{d\tau} \frac{dt}{d\tilde{\tau}}$
so we can always set $t = \tau$.

In this parameterisation,

$$S[x^\mu]_{t=\tau} = -mc \int \sqrt{c^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{dx^i}{d\tau} \frac{dx^i}{d\tau}}\, d\tau$$

$$= -mc^2 \int \sqrt{1 - \dot{\underline{x}}^2/c^2}\, dt \qquad \left(\dot{\underline{x}} = \frac{d\underline{x}}{dt}\right)$$

and our action reduces to the previous one.

Also notice that using covariant action,

$$p_\mu = c\frac{\partial L}{\partial\left(\frac{\partial x^\mu}{\partial \tau}\right)} = mc\,\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\cdot\frac{1}{\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}}$$

In fact that we had a redundancy appears here because not all p_μ are indpt:

$$p_\mu p^\mu = m^2c^4.$$

ie. $E^2 - p^2c^2 = m^2c^4$.

Charged particle in electromagnetic field

Recall we can describe an EM field in terms of a scaler potential ϕ and vector potential $\underline{A}$.

$$\underline{E} = -\nabla\phi - \frac{\partial \underline{A}}{\partial t}, \qquad \underline{B} = \nabla\times\underline{A}$$

For a charged particle of charge e, mass m moving in this background EM field, we take

$$L = \frac{1}{2}m\dot{\underline{x}}^2 - e\left(\phi - \dot{\underline{x}}\cdot\underline{A}\right)$$

in terms of $(\phi, \underline{A})$, where $\phi(\underline{x},t)$ and $\underline{A}(\underline{x},t)$ determine the EM field.

Notice this L has a term linear in velocities, so now

$$\underline{p} = \frac{\partial L}{\partial\dot{\underline{x}}} = m\dot{\underline{x}} + e\underline{A}$$

is the momentum in the presence of EM field.

This E-L give

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}$$

$$= \frac{d}{dt}\left(m\dot{x}^i + eA^i\right) + e\frac{\partial \phi}{\partial x^i} - e\dot{x}^j \partial^i x^j$$

$$= m\ddot{x}^i + e\frac{\partial A^i}{\partial t} + e\dot{x}^j\left(\partial^j A^i - \partial^i A^j\right) + e\partial^i \phi$$

$$\Rightarrow \quad m\ddot{x}^i = -e\left(\partial^i \phi + \partial_t A^i\right) + e\dot{x}^j\left(\partial^i A^j - \partial^j A^i\right)$$

$$= eE^i + e(\underline{\dot{x}} \times \underline{B})^i$$

since $B^i = \varepsilon^{ijk}\partial^j A^k$ so $\partial^i A^j - \partial^j A^i = \varepsilon^{ijk} B^k$.

This is just Lorentz force law, justifying our choice of L.

Notice that $\underline{E}, \underline{B}$ are unchanged if we change

$$\underline{A} \mapsto \underline{A} + \nabla \chi \quad , \quad \phi \mapsto \phi - \partial_t \phi.$$

for any f^n $\chi(\underline{x}, t)$ These are gauge transformations.

For our Lagrangian, gauge transformation changes

$$L \mapsto L + e(\partial_t \chi + \underline{\dot{x}} \cdot \nabla \chi)$$

$$= L + e\frac{d\chi}{dt}$$

So this is just a boundary term in the action $S = \int L\, dt$.

which will not affect the e.o.m.

For a relativistic particle, it's natural to write

$$A_\mu = (-\phi/c, \underline{A}).$$

Then the coupled term is S are

$x^\mu = (ct, \underline{x})$

$$\int (-e\phi + e\,\dot{\underline{x}}\cdot\underline{A})\,dt = \int e A_\mu \frac{dx^\mu}{dt}\,dt$$

Again, we can write this in terms of an arbitrary parameter τ on coordinate as

$$e\int A_\mu \frac{dx^\mu}{d\tau}\,d\tau.$$

Hence, for a relativistic particle, we use

$$S = -mc\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\,d\tau + e\int A_\mu\frac{dx^\mu}{d\tau}\,d\tau.$$

This is manifestly Lorentz invariant, as well as redundant under reparameterisation $\tau \mapsto \tilde{\tau}(s)$ and gauge transformations $A_\mu \mapsto A_\mu + \partial_\mu \chi$.

Small Oscillations

Consider a generic L of the form

$$L = \frac{1}{2}T_{ab}(q)\,\dot{q}^a\dot{q}^b - V(q^a)$$

for some generalised coords q^a. The E-L say

$$\frac{d}{dt}\left(T_{ab}(q)\,\dot{q}^b\right) = -\frac{\partial V}{\partial q}.$$

So critical points of V give points of equilibrium.

If we start at some critical point $q^a = q^a_*$ with velocity $\dot{q}^a = 0$, we stay there.

We often interested in finding out what happens if we perturb around q^a_* slightly.

WLOG $T_{ab}(q)$ is real symmetric, and we'll assume it's +ve def (so KE increases if $\dot{q}^a$ increases in any direction). WLOG we'll also assume our coords are chosen s.t. the critical point of V is at $q^a_* = 0$

Then expanding L to quadratic order in $(q^a, \dot{q}^a)$ gives

$$L \approx \frac{1}{2} T_{ab}(0)\, \dot{q}^a \dot{q}^b - V(0) - \frac{1}{2} \left.\frac{\partial^2 V}{\partial q^a \partial q^b}\right|_0 q^a q^b + \dots$$

$$= \frac{1}{2}\left(T_{ab}(0)\, \dot{q}^a \dot{q}^b - V_{ab}(0)\, q^a q^b \right) + \dots$$

For this quadratic L, the E-L give

$$T_{ab}(0)\, \ddot{q}^{(b)} = -V_{ab}(0)\, q^b \quad , \text{ or } \quad T\ddot{\underline{q}} = -V\underline{q}$$

Since T is real, sym, +ve def, we have

$$\ddot{\underline{q}} = -\left(T^{-1} V \right) \underline{q}$$

So if we can diagonalise $T^{-1}V$, the motion will decompose into

$$\underline{q}(t) = \underline{\alpha}(t),$$

where $\underline{\alpha}^a(t) = \sum_a \underline{\alpha}_a e^{i\lambda_a t}$, where $(T^{-1}V)\underline{\alpha}_a = \underline{\alpha}_a$

Plugging this into eqn of motion

$$\ddot{f}(t)\, T\underline{\alpha} = -f(t)\, V\underline{\alpha}$$

Thus we solve

$$\begin{cases} \ddot{f}(t) = -\lambda f(t) \\ \lambda T\underline{\alpha} = V\underline{\alpha} \end{cases}$$

for some const. λ. To find λ, note $(\lambda T - V)\underline{\alpha} = 0$. Hence,

$$\det(\lambda T - V) = 0.$$

This char poly in λ has m (complex) sol^n, the evals of $T^{-1}V$.

Call the sol^n λ_a for $a = 1, \dots, m$. The evecs are

$$(\lambda_a T - V)\underline{\alpha}_a = 0.$$

Each sol^n has time dependence

$$\ddot{f}_a(t) = \lambda_a f_a(t).$$

The $\underline{\alpha}_a$ are called normal modes.

Claim: $\lambda_a \in \mathbb{R}$.

Pf: T and V are real and sym. T +ve def, then

$$\underline{\alpha}_a^T(\lambda_a T - V)\underline{\alpha}_a = 0$$

Since $\underline{\alpha}_a^T T \underline{\alpha}_a$ and $\underline{\alpha}_a^T V \underline{\alpha}_a$ are real and $\neq 0$, so

$\lambda_a - \lambda_a^* = 0 \Rightarrow \lambda_a \in \mathbb{R}$. $\square$

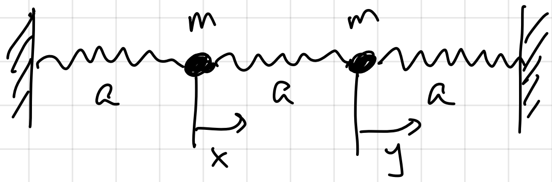
If $\lambda > 0$, then
$$f(t) = A\cos(\omega t + \beta),$$
with $\omega^2 = \lambda$ (and WLOG $\omega > 0$) and 2 integration const A (amplitude) and β (phase) fixed by initial conditions.

The ω_a are called normal frequencies

If $\lambda_a > 0$, then we have stable equilibrium.

Otherwise, $\exists\, a$ s.t. $\lambda_a \leq 0$ unstable.

<u>Example</u> (3 springs and 2 masses)



KE: $\quad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

PE: $\quad V = \frac{1}{2}k(x^2 + y^2 + (y-x)^2)$, with k spring const.

$x = y = 0$ is eqm, and L is already quadratic. To solve,

$$T = \frac{1}{2}(\dot{x}\ \ \dot{y})\underbrace{\begin{pmatrix} m & \\ & m \end{pmatrix}}_{T}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

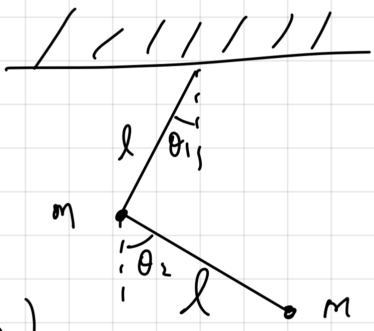
$$V = \frac{1}{2}(x\ \ y)\underbrace{\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}}_{V}\begin{pmatrix} x \\ y \end{pmatrix}$$

Char eqn is $\quad 0 = \det(\lambda T - V) = (m\lambda - k)(m\lambda - 3k)$.

So sol^n are $\lambda_1 = k/m$ and $\lambda_2 = 3k/m$, with normal frequencies $\omega_1 = \sqrt{\lambda_1} = \sqrt{k/m}$ and $\omega_2 = \sqrt{\lambda_2} = \sqrt{3k/m}$, and normal modes $\underline{\alpha}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{\alpha}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

So oscillations are in phase (α_1) and out of phase.

Example (Double pendulum)



Under force of gravity,

$$L = \frac{1}{2} m l^2 \dot{\theta}_1^2 + \frac{1}{2} m l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1 - \theta_2)\, \dot{\theta}_1 \dot{\theta}_2 \right)$$
$$+ mgl\left(2\cos\theta_1 + \cos\theta_2\right).$$

We expect eqm point at $\theta_1 = \theta_2 = 0$. Then we expand to $O(\theta_1^2, \theta_2^2, \theta_1\theta_2)$

$$V = -mlg\left(2\left(1 - \frac{1}{2}\theta_1^2\right) + \left(1 - \frac{1}{2}\theta_2^2\right) + \dots\right)$$
$$= \frac{1}{2}(\theta_1 \quad \theta_2) \begin{pmatrix} 2mlg & \\ & 2mlg \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

and

$$T = \frac{1}{2}(\dot{\theta}_1 \quad \dot{\theta}_2) \begin{pmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

Char eqn:

$$0 = \det(\lambda T - V) = m^2 l^4 \left(2(\lambda - g/l)^2 - \lambda^2\right)$$

Solved by $\lambda_1 = (2 - \sqrt{2})\, g/l$. $\lambda_2 = (2 + \sqrt{2})\, g/l$.

Since $\lambda_1, \lambda_2 > 0$, this is stable eqm.

The normal modes and frequencies are

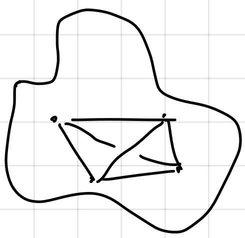
$$\omega_1 = \sqrt{\lambda_1}, \quad \underline{\alpha}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \qquad \omega_2 = \sqrt{\lambda_2}, \quad \underline{\alpha}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}.$$

Again, the low-frequency (ω_1) swings in phase and high-frequency (ω_2) swings out of phase.

Motion of Rigid Body

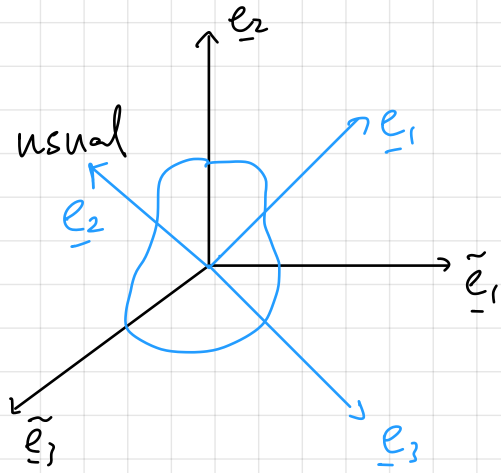
A rigid body is a collection of n (≥ 1) particles, whose motion is constrained s.t. $|\underline{r}_i - \underline{r}_j| = c_{ij}$ is const.

e.g.



We can move the body's centre of mass, and rotate the body, but not deform it.

It's useful to introduce 2 coordinate system. Suppose the c.o.m. of a rigid body is fixed.



The space frame $\{\tilde{\underline{e}}_a\}$ are our usual fixed Cartesian basis of $\mathbb{R}^3$.

The body frame $\{\underline{e}_a\}$ is fixed relative to the body, i.e. $\underline{e}_a(t)$ rotates along with the body s.t. each particle in the body has fixed coords wrt the body frame.

$$\underline{r}(t) = r_a(t)\,\tilde{\underline{e}}_a = r_a\,\underline{e}_a(t).$$

We'll choose both frames to be right-handed orthonormal. So

$$\underline{e}_a(t) = R_{ab}(t)\,\tilde{\underline{e}}_b$$

for some $R_{ab}(t)$.

$$\delta_{ab} = \underline{e}_a(t)\cdot\underline{e}_b(t) = R_{ac}R_{bd}\underbrace{\tilde{\underline{e}}_c\cdot\tilde{\underline{e}}_d}^{=\delta_{cd}}$$

$$= R_{ac}R_{bc}$$

$$= (RR^T)_{ab}$$

$\Rightarrow \quad R \in O(3)$ and in fact $R \in SO(3)$, since both frames right-handed.

We have

$$\frac{d\underline{e}_a(t)}{dt} = \frac{d}{dt}\left(R_{ab}(t)\right)\tilde{\underline{e}}_b = \left(\frac{dR_{ab}}{dt}\right) R^T_{bc}\, \underline{e}_c(t)$$

Differentiating the orthogonality condition $RR^T = I$, we have

$$\dot{R}R^T + R\dot{R}^T = 0$$

So

$$\Omega \equiv \dot{R}R^T = -R\dot{R}^T = -(\dot{R}R)^T = -\Omega.$$

And hence $\Omega_{ab} = -\Omega_{ba} =: -\varepsilon_{abc}\,\omega_c$.

Therefore,

$$\frac{d\underline{e}_a(t)}{dt} = \Omega_{ac}\,\underline{e}_c(t) = -\varepsilon_{acd}\,\omega_d\,\underline{e}_c = (\underline{\omega} \times \underline{e}_a),$$

where $\underline{\omega} = \omega_b\,\underline{e}_b(t)$ is the angular velocity (in the body frame).

For a general point $\underline{r}(t)$ in a body, we have

$$\frac{d\underline{r}(t)}{dt} = \frac{d}{dt}\left(r_a\,\underline{e}_a(t)\right) = r_a \frac{d\underline{e}_a}{dt} = r_a(\underline{\omega} \times \underline{e}_a)$$

$$\Rightarrow \quad \dot{\underline{r}} = \underline{\omega} \times \underline{r} \qquad \text{(in body frame)}.$$

We can use this to express the kinetic energy of our rotating body (with fixed c.o.m. location) as

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i \, \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i \, (\underline{\omega} \times \underline{r}_i) \cdot (\underline{\omega} \times \underline{r}_i)$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i \left(\underline{\omega}^2 \, \underline{r}_i^2 - (\underline{\omega} \cdot \underline{r}_i)^2 \right)$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i \, \omega_a \left(\delta_{ab} \, \underline{r}_i^2 - (\underline{r}_i)_a (\underline{r}_i)_b \right) \omega_b$$

$$= \frac{1}{2} \, \omega_a I_{ab} \, \omega_b ,$$

where we defined

$$I_{ab} = \sum_{i=1}^{n} m_i \left(\delta_{ab} \, \underline{r}_i^2 - (\underline{r}_i)_a (\underline{r}_i)_b \right)$$

is the inertia tensor of the body. For cts body, we instead use

$$I_{ab} = \int_{\text{body}} \rho(\underline{r}) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} d^3\underline{r} .$$

We note that I is real, sym matrix, so it can be diagonalised. If the evecs of I are called the principle axes of the body and we often choose to align our body frame with the principle axes.

We also see, for any vector $\underline{c}$,

$$I_{ab} c_a c_b = \sum_i m_i \left(\underline{r}_i^2 \underline{c}^2 - (\underline{r}_i \cdot \underline{c}) \right)$$

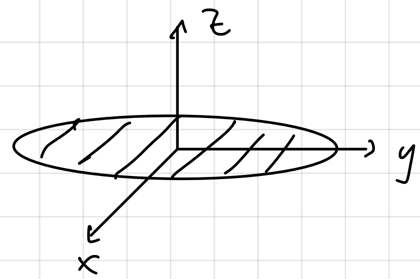
$$= \sum_i m_i \underline{r}_i^2 \underline{c}^2 \sin^2 \theta_i \geq 0 .$$

In particular, choosing $\underline{c}$ to be an evec of I,

$$I_{ab} c_a c_b = \underline{c} \cdot I \underline{c} = \lambda \underline{c}^2 \geq 0 .$$

So the evals I_1, I_2, I_3 of I are non-neg.

<u>Example</u> Consider a uniform disc of radius r and mass M. In this frame, $I_{ab} = \mathrm{diag}(I_1, I_2, I_3)$ and



$$I_1 = I_2 = \int_{\text{disc}} \frac{M}{\pi R^2} y^2 \, d^2 x$$

$$I_3 = \int_{\text{disc}} \frac{M}{\pi R^2} (x^2 + y^2) \, d^2 x \qquad (\text{so } I_3 = I_1 + I_2)$$

$$= \int_{\text{disc}} \frac{M}{\pi R^2} r^2 \, d^2 x$$

$$= \frac{2M}{R^2} \int_0^R r^3 \, dr$$

$$= \frac{MR^2}{2} , \qquad \text{and} \qquad I_1 = I_2 = \frac{MR^2}{4} .$$

<u>Parallel Axis Thm.</u>

If our body is instead fixed at some point p lying a vector $\underline{c}$ away from c.o.m. The moment of inertia around p is

$$I_{ab} = \sum_i m_i \left((\underline{r}_i - \underline{c})^2 \delta_{ab} - (\underline{r}_i - \underline{c})_a (\underline{r}_i - \underline{c})_b \right) ,$$

where the locations $\underline{r}_i$ are measured w.r.t. the point p.

Hence,

since $\sum_i m_i r_i = 0$

$$I_{ab} = I_{ab}^{\text{c.o.m.}} + \cancel{\sum_{i=1}^{n} m_i \left(-2\underline{c}\cdot\underline{r}_i\,\delta_{ab} + (\underline{r}_i)_a\,\underline{c}_b + (\underline{r}_i)_b\,\underline{c}_a \right)}$$

$$+ \underbrace{\sum_{i=1}^{n} m_i}_{=M} \left(\underline{c}^2\,\delta_{ab} - (\underline{c})_a(\underline{c})_b \right)$$

So if the pivot is displaced by $\underline{c}$ from c.o.m.,

$$I_{ab}^{\underline{c}} = I_{ab}^{\text{c.o.m.}} + M\left(\underline{c}^2\,\delta_{ab} - (\underline{c})_a(\underline{c})_b \right).$$

We can also compute the angular momentum

$$\underline{L} = \sum_i \underline{r}_i \times (m_i\,\dot{\underline{r}}_i)$$

$$= \sum_i m_i\,\underline{r}_i \times (\underline{\omega} \times \underline{r}_i)$$

$$= \sum_i m_i \left(r_i^2\,\underline{\omega} - (\underline{\omega}\cdot\underline{r}_i)\,\underline{r}_i \right) = I\,\underline{\omega}$$

In particular, if we align the body frame $\{\underline{e}_a\}$ with the principle axes, then
$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_1\,\omega_1 \\ I_2\,\omega_2 \\ I_3\,\omega_3 \end{pmatrix}.$$

Notice that in general, $\underline{L} \not\propto \underline{\omega}$, so angular momentum is typically <u>not</u> aligned with angular velocity.

Euler's equation

In the abscence of torque, $\underline{L} = L_a(t)\, \underline{e}_a(t)$ is conserved.

Therefore,

$$0 = \frac{d\underline{L}}{dt} = \frac{dL_a}{dt}\underline{e}_a + L_a \frac{d\underline{e}_a}{dt}$$

$$= \frac{dL_a}{dt}\underline{e}_a + L_a(\underline{\omega} \times \underline{e}_a)$$

So contracting with $\underline{e}_b$ gives

$$0 = \frac{dL_b}{dt} + L_a(\underline{\omega} \times \underline{e}_a)\cdot \underline{e}_b$$

$$= \frac{dL_b}{dt} + L_a\, \underline{\omega} \cdot (\underline{e}_a \times \underline{e}_b)$$

or in components, using $L_a = I_a \omega_a$,

$$\left.\begin{aligned} I_1 \dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) &= 0 \\ I_2 \dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) &= 0 \\ I_3 \dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) &= 0. \end{aligned}\right\} \text{Euler's eqn.}$$

The symmetric Top.

If our object is sph. sym, $I_{ab} = I\,\delta_{ab}$, with $I_1 = I_2 = I_3 = I$. Then Euler's eqn say $\dot{\omega}_a = 0$, or $\underline{\dot{\omega}} = 0$. In this case, the body keeps rotating around the initial axis of rotation.

A symmetric top has $I_1 = I_2 \neq I_3$ (generically).

Then Euler's eqn become

$$I_3 \dot{\omega}_3 = 0 \Rightarrow \dot{\omega}_3 = 0 \Rightarrow \omega_3(t) = \Omega \quad (\text{const.})$$

$$I_2 \dot{\omega}_1 = \omega_2 \Omega (I_2 - I_3)$$

$$I_2 \dot{\omega}_2 = \omega_1 \Omega (I_3 - I_2)$$

Let $\alpha = \Omega(I_2 - I_3)/I_2$, then

$$\dot{\omega}_1 = \alpha \omega_2 , \quad \dot{\omega}_2 = \alpha \omega_1$$

$$\Rightarrow (\omega_1(t), \omega_2(t)) = \omega_0 (\sin\alpha t, \cos\alpha t).$$

Consequently, the body's rotation "wobbles" (or processes) around the $\underline{e}_3$ axis – i.e. the normal to plane of symmetry.

The Asymmetric Top

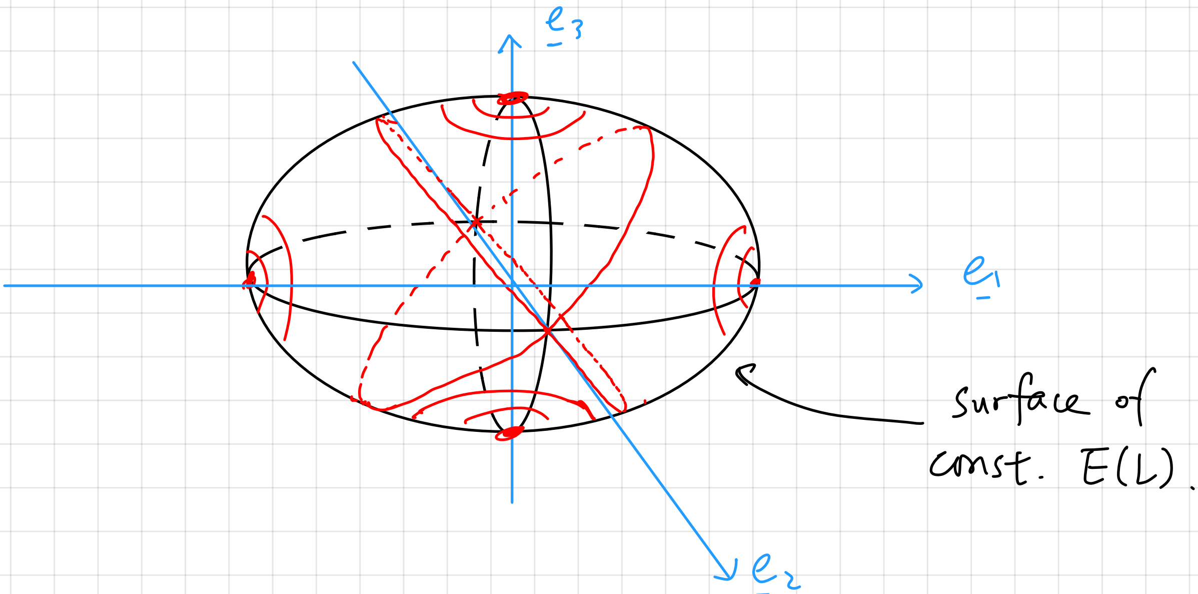
The general case $I_1 > I_2 > I_3$ cannot be solved in terms of elementary f^{n}s. However, there's a simple construction (due to Poinsot) that allows us to visualise the sol^n.

<u>Poinsot's construction</u>: We have 2 quantities

$$\underline{L}^2 = L_1^2 + L_2^2 + L_3^2 \qquad (L_a = I_a \omega_a)$$

$$2E = \underline{\omega} \, I \, \underline{\omega} = \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}$$

Surfaces of const. E are ellipsoids in the body axes



In this space, surfaces of const. $\underline{L}^2$ are spheres so motion lies on intersection.

When $L^2 = 2EI_2$, we have

$$2EI_2 = L^2 = L_1^2 + L_2^2 + L_3^2 \qquad \text{(angular mom.)}$$

$$\parallel$$

$$L_1^2 \frac{I_2}{I_1} + L_2^2 + L_3^2 \frac{I_2}{I_3} \qquad \text{(energy)}$$

$$\Rightarrow \quad 0 = L_1^2 \left(\frac{I_1 - I_2}{I_1} \right) + L_3^2 \left(\frac{I_3 - I_2}{I_3} \right)$$

$$\Rightarrow \quad L_3 = \pm L_1 \left(\frac{I_1 - I_2}{I_2 - I_3} \cdot \frac{I_3}{I_1} \right)^{1/2}$$

defines two planes.

We see:

(i) If we begin with $\underline{\omega}(0)$ perfectly aligned with any principle axes, it remains const.

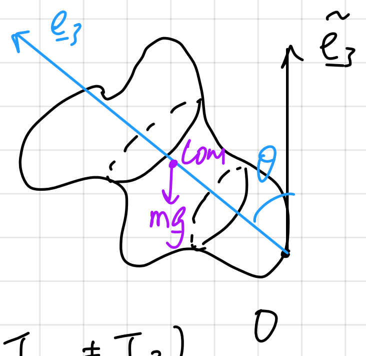
(ii) If we perturb $\underline{\omega}$ slightly around either $\underline{e}_1$ or $\underline{e}_3$, the motion is stable: the orthogonal ω's precess in small circles.

(iii) The orbits are always closed, but perturbation around intermediate axis ($\underline{e}_2$) are unstable.

Motion of Spinning Tops under Gravity

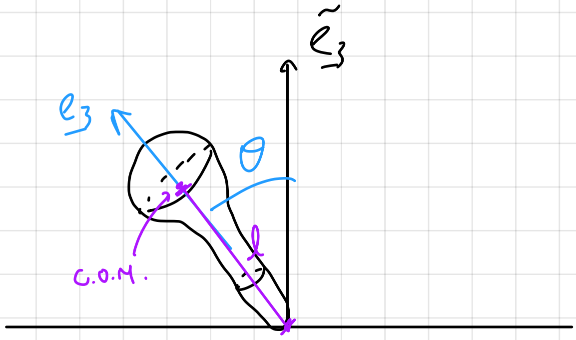
A top is called *heavy* if we include the effect of gravity. Consider a spinning top, pivoted at a fixed point O.

For an asymmetric top, E is conserved, and $\tilde{\underline{e}}_3 \cdot \underline{L}$ is conserved, but there are no further conserved quantity.



However, if the top is symmetric ($I_1 = I_2 \neq I_3$), there'll be a third conserved quantity.

The general case is unsolved, but the axisymmetric case is solved. It's called the *Lagrange Top*.



It's useful to parameterise the motion in terms of ...

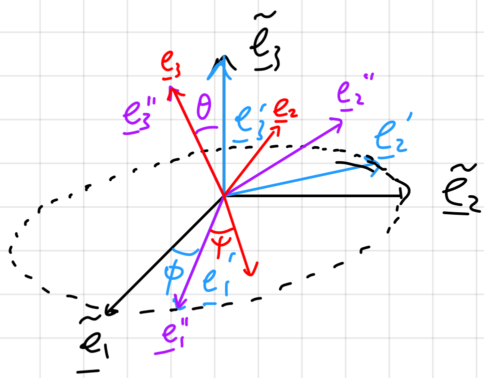
Euler Analysis

These describe a succession of 3 rotations that take us from the space frame to body frame.

$$\tilde{\underline{e}}_a \xrightarrow{R(\phi \tilde{\underline{e}}_3)} \underline{e}'_a$$

$$\underline{e}'_a \xrightarrow{R(\theta \underline{e}'_1)} \underline{e}''_a$$

$$\underline{e}''_a \xrightarrow{R(\psi \underline{e}''_3)} \underline{e}_a$$



In full, this is

$$\underline{e}_a = R_{ab}(\psi\, \underline{e}_3'')\, R_{bc}(\theta \underline{e}_1')\, R_{cd}(\phi\, \tilde{\underline{e}}_3)\, \tilde{\underline{e}}_d$$

$$= \underbrace{\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R(\psi,\theta,\phi)} \tilde{\underline{e}}_d$$

$$R(\psi,\theta,\phi) = \begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\sin\psi\sin\phi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Note $\underline{e}_1' = \cos\psi\, \underline{e}_1 - \sin\psi\, \underline{e}_2$.

In terms of Euler angles, $\underline{\omega} = \dot{R}R^T$, but it's much simpler to note $\underline{\omega} = \dot{\phi}\, \tilde{\underline{e}}_3 + \dot{\theta}\, \underline{e}_1' + \dot{\psi}\, \underline{e}_3$.

$$\underline{\omega} = \dot{\phi}\left(\sin\theta\sin\psi\, \underline{e}_1 + \sin\theta\cos\psi\, \underline{e}_2 + \cos\theta\, \underline{e}_3\right) + \dot{\theta}\left(\cos\psi\, \underline{e}_1 - \sin\psi\, \underline{e}_2\right) + \dot{\psi}\, \underline{e}_3$$

$$= (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\, \underline{e}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\, \underline{e}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\, \underline{e}_3$$

Note $\omega_1^2 + \omega_2^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta$.

Motion of Spinning Tops under Gravity

The Lagrangian of our axisymmetric heavy top is

$$L = \frac{1}{2}\omega_a I_{ab}\,\omega_b - mgl\cos\theta .$$

$$= \frac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3\,\omega_3^2 - mgl\cos\theta$$

$$= \frac{1}{2} I_1\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \frac{1}{2} I_3\left(\dot\psi + \dot\phi\cos\theta\right)^2 - mgl\cos\theta$$

Notice: since $I_1 = I_2$, ψ is ignorable. while ϕ is ignorable always. Consequently, (p_ψ, p_ϕ, E) are conserved for Lagrange Top.

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = I_3\left(\dot\psi + \dot\phi\cos\theta\right)$$

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = I_1\,\dot\phi\sin^2\theta + p_\psi\cos\theta \Rightarrow \dot\phi = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$

$$E = \dot q^a\frac{\partial L}{\partial\dot q^a} - L$$

$$= \frac{1}{2} I_1\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \frac{1}{2} I_3\left(\dot\psi + \dot\phi\cos\theta\right)^2 + mgl\cos\theta .$$

$$= \frac{1}{2} I_1\dot\theta^2 + \underbrace{\frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \underset{\text{const.}}{\frac{p_\psi^2}{2I_3}} + mgl\cos\theta}_{V_{eff}(\theta).}$$

$$= \frac{1}{2} I_1\dot\theta^2 + V_{eff}(\theta)$$

$$E' = \frac{1}{2} I_1\dot\theta^2 + I_1\frac{(b - a\cos\theta)^2}{2\sin^2\theta} + mgl\cos\theta \qquad (*)$$

where $p_\phi = I_1 b$. $p_\psi = I_1 a$.

To analyse the remaining θ eqn, let $u = \cos\theta$.

Then, $(*)$ becomes

$$\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b-au)^2 .$$

Also $\dot{\phi} = \dfrac{b-au}{1-u^2}$, $\dot{\psi} = \dfrac{I_1 a}{I_3} - \dfrac{u(b-au)}{1-u^2}$,

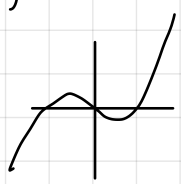
$$\alpha = \frac{2E'}{I_1}, \quad \beta = \frac{2mgl}{I_1}$$

Motion in θ is called nutation. (bobbing up + down)

Motion in ϕ is called precession.

One can solve the remaining u eqn in terms of elliptic curves, but we can understand qualitatively the motion by considering

$$f(u) = (1-u^2)(\alpha - \beta u) - (b-au)^2$$

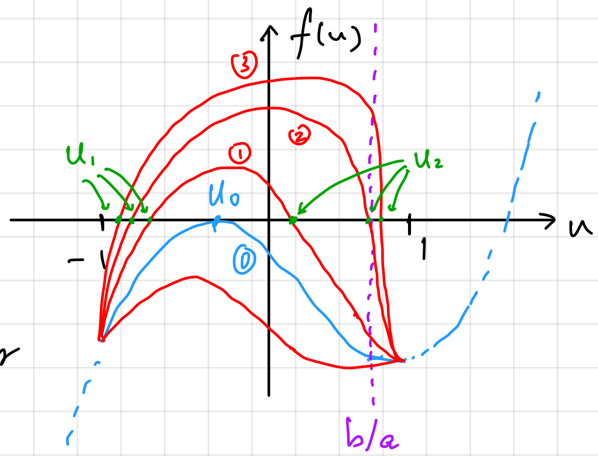
Since $f(u) \sim \beta u^3$ for $u \gg 1$, so $f(u) \sim$ .

We're only interested physically in the region $u \in [-1,1]$ and $f(u) \geq 0$.

$$f(\pm 1) = -(b-a)^2 \leq 0.$$

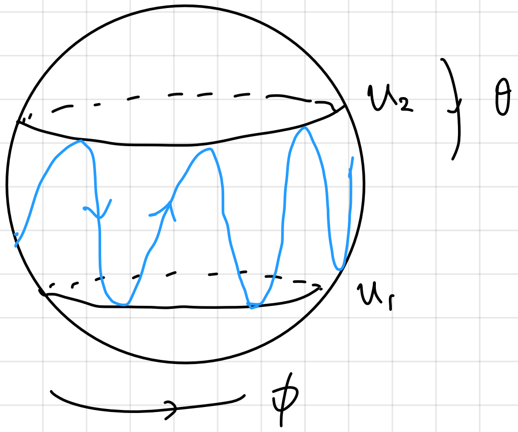
Also, we see that

$$\dot{\phi} \begin{cases} > 0 & \text{if } u < u_c \equiv b/a \\ < 0 & \text{if } u > u_c . \end{cases}$$

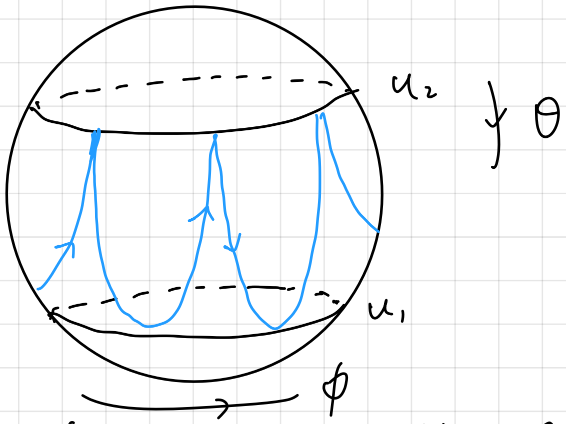


Let u_1, u_2 be the smaller / larger roots of $f(u) = 0$ in $u \in [-1,1]$

① If $u_1, u_2 < u_c = b/a$, $\dot\phi > 0$ throughout motion



② If $u_2 = u_c$, $\dot\phi = 0$ at $u = u_2$, and the motion has cusps.

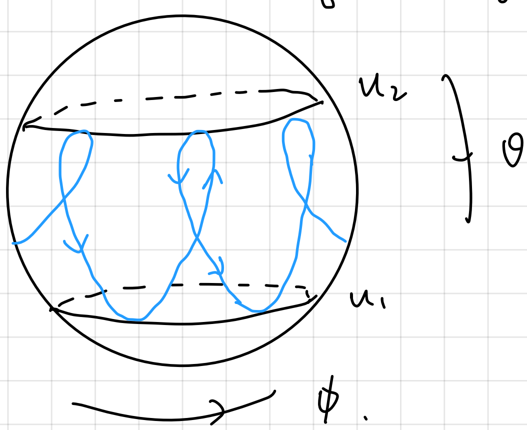


This appears fine tuned but isn't. Suppose we release the Top with $\dot\psi > 0$ but $\dot\theta = \dot\phi = 0$, i.e. we spin the top, but doesn't push it. Then initially $\dot\phi = 0 = \dfrac{b-au}{1-u^2} \Rightarrow \dfrac{b}{a} = u = u_c$.

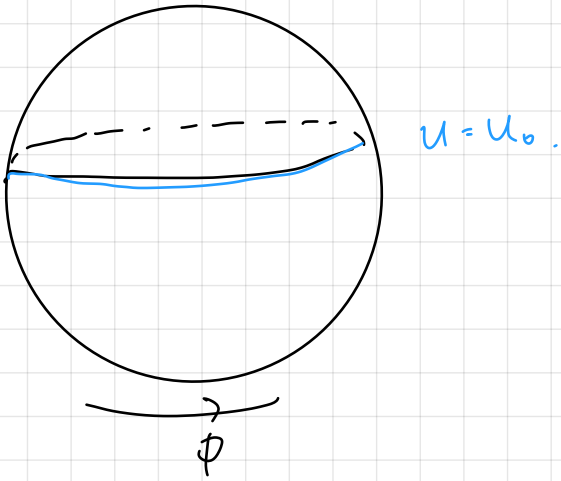
so initially u_c, and $\dot u(0) = 0$.

Thereafter, the top falls under gravity, causing u to decrease. This causes $\dot\phi$ to increase, so the top begins to precess.

③ If $u_1 < u_c$ but $u_2 > u_c$. then sign changes as the top nutates.



(0) Finally, if $f(u)$ has a repeated root in $[-1,1]$, we get precession without nutation.



This is called uniform precession. We must have

$$f(u_0) = (1-u_0^2)(\alpha - \beta u_0) - (b - a u_0)^2 = 0$$

$$f'(u_0) = -2u_0(\alpha - \beta u_0) - \beta(1-u_0^2) + 2a(b-u_0) = 0$$

from which

$$\alpha - \beta u_0 = \frac{(b-au_0)^2}{1-u_0^2} = \frac{a(b-au_0)}{u_0} - \frac{\beta(1-u_0^2)}{2u_0}$$

$$\Rightarrow \dot\phi^2 \cancel{(1-u_0^2)} = \frac{a}{u_0}\dot\phi \cancel{(1-u_0^2)} - \frac{\beta}{2u_0}\cancel{(1-u_0^2)}$$

That is, we need to give the top initial precession $\dot\phi_0$ s.t.

$$I_1 u_0 \dot\phi_0^2 - I_3 \omega_3 \dot\phi_0 + mgl = 0$$

This quadratic has 2 real roots for given $u_0(\theta_0)$ if

$$\Delta = (I_3\omega_3)^2 - 4I_1 \cos\theta_0\, mgl > 0,$$

i.e.

$$\omega_3 > \frac{2}{I_3}\sqrt{I_1 mgl \cos\theta_0}$$

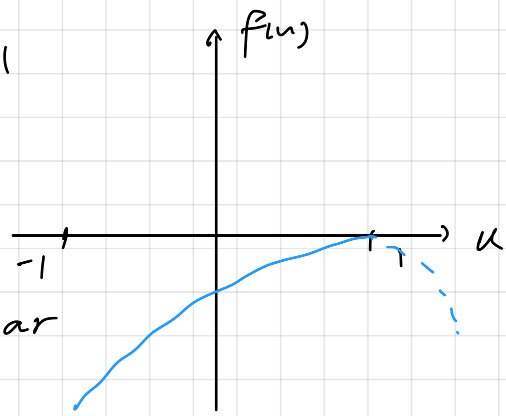
So uniform precession is only possible if the top spins quickly.

Sleeping Tops

If we start the top at $\theta = 0$ ($\underline{e}_3 = \tilde{\underline{e}}_3$ upright), then $a = b$, $\alpha = \beta$, so we get a double root at

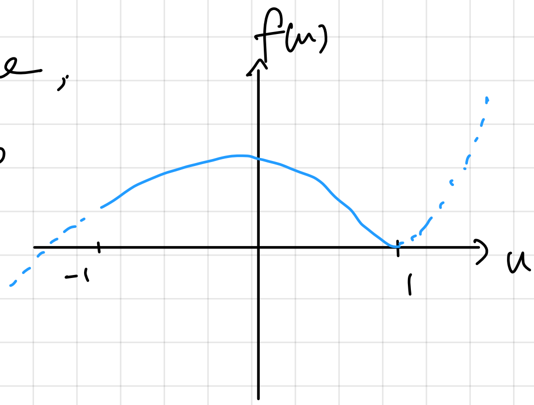
$$f(u_0) = (1-u_0^2)\left(\alpha(1+u_0) - a^2\right) = 0$$

The remaining roots is at $u_0 = \frac{a^2}{\alpha} - 1$



(i) If $u_0 > 1$, the motion is stable

$I_3\omega_3 > 2\sqrt{I_1 mgl}$. $\nexists$ any region near $u = 1$ where $f(u) > 0$.



(ii) If $u_0 < 1$, the motion is unstable, perturbing slightly cause the top to mutate

Hamiltonian mechanics

This is further reformulation of Newtonian mechanics.

It allows us to

- have still more freedom in our choice of coords / parameterisation of the motion
- get very close to the formalism of QM.
- motivates sympletic geometry.

We've seen that E-L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \frac{\partial L}{\partial q^a}$$

take the same form under general holonomic coord transformations

$$(q^a, t) \longmapsto (\tilde{q}^a(q, t), t)$$

$$\Rightarrow \dot{q}^a \longmapsto \dot{\tilde{q}}^a = \frac{\partial \tilde{q}^a}{\partial q^b}\dot{q}^b + \frac{\partial \tilde{q}^a}{\partial t}$$

However, E-L are <u>not</u> preserved under more general transformations

$$q^a \longmapsto \tilde{q}^a(q, \dot{q}, t).$$

Occasionally, such transformations are useful and it's useful to put $q^a, \dot{q}^a$ on a more symmetric footing. We could define

$$p_a = p_a(q, \dot{q}, t) \equiv \left.\frac{\partial L}{\partial \dot{q}^a}\right|_{q,t}$$

and write E-L

$$\frac{dp_a}{dt} = \frac{\partial L}{\partial q^a}, \quad p_a = \frac{\partial L}{\partial \dot{q}^a}, \quad \dot{q}^a = \frac{dq^a}{dt}.$$

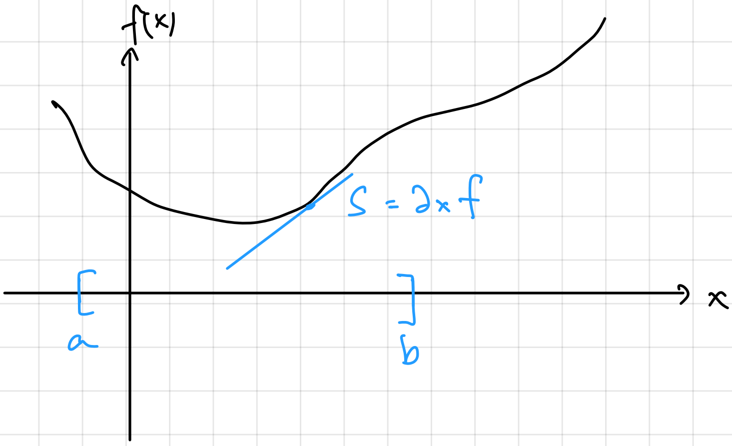
but to read Hamilton's eqn, we need to eliminate $\dot{q}^a$ in favour of p_a.

Legendre transforms

Consider f^n $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$ and let $S = \partial_x f$.

If $\partial_x S > 0$ in some interval $[a,b] \subset \mathbb{R}$ then S is monotonic on $[a,b]$, so we can invert $S(x)$ to find $x(S)$.

e.g.



We define the Legendre transform of $f(x)$ to be

$$g(s) = s\,x(s) - f(x(s))$$

viewed as a f^n of s.

Note $\partial_s g = x(s) + s\frac{dx}{ds} - \frac{\partial f}{\partial x}\frac{dx}{ds} = x(s)$, so we have $\partial_s g = x(s)$ and $\partial_x f = s(x)$, giving symmetry between $(f,x) \leftrightsquigarrow (g,s)$. This in particular implies that the Legendre transform of $g(s)$ is again $f(x)$. (Exercise)

E.g. If $f(x) = \frac{x^\alpha}{\alpha}$ with $\alpha > 1$, then $s = f' = x^{\alpha-1}$,

so $x = s^{1/\alpha-1}$. So our Legendre transform

$$g(s) = s \cdot s^{1/\alpha-1} - \frac{s^{\alpha/\alpha-1}}{\alpha}$$

$$= s^{\frac{\alpha}{\alpha-1}}\left(1 - \frac{1}{\alpha}\right)$$

$$= s^{\frac{\alpha}{\alpha-1}}\left(\frac{\alpha-1}{\alpha}\right),$$

or $g(s) = \frac{s^\beta}{\beta}$, where $\beta = \frac{\alpha}{\alpha-1}$, then $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\beta > 1$.

It's clear that taking a 2^{nd} Legendre transform will give back our original f^n.

In the case of several variables $f: \mathbb{R}^n \to \mathbb{R}$, we define $s_a = \frac{\partial f}{\partial x^a}$ (or $\underset{\sim}{s} = \nabla f$) The inverse f^n thm guarantees that the relations $s_a = s_a(\underset{\sim}{x})$ are invertible to give $x^a = x^a(\underset{\sim}{s})$ in a nbd of $\underset{\sim}{x} \in \mathbb{R}^n$ provided the Jacobian $\det\left(\frac{\partial^2 f}{\partial x^a \partial x^b}\right) > 0$.

We define the Legendre transform of $f(\underset{\sim}{x})$ to be

$$g(\underset{\sim}{s}) = \left(\sum_{a=1}^{n} s_a x^a(\underset{\sim}{s})\right) - f(\underset{\sim}{x})$$

Again, we have $\frac{\partial g}{\partial s^b} = x^b(s) + s_a \frac{\partial x^a}{\partial s^b} - \frac{\partial f}{\partial x^a}\frac{\partial x^a}{\partial s^b} = x^b(s)$.

Hamilton's Equations

We define the Hamiltonian $H(p, q, t)$ to be the Legendre transform of $L(q, \dot{q}, t)$ w.r.t. $\dot{q} \to p$ at fixed (q, t). That is

$$H(q, p, t) = p_a \dot{q}^a - L(q, \dot{q}(p, q, t), t),$$

where $p_a = \dfrac{\partial L}{\partial \dot{q}^a}$ are the usual generalised momenta.

Treating (q, p, t) as indpt, we have

$$dH = \frac{\partial H}{\partial q^a} dq^a + \frac{\partial H}{\partial p_a} dp_a + \frac{\partial H}{\partial t} dt$$

$$= dp_a \dot{q}^a + \cancel{p_a d\dot{q}^a} - \left(\frac{\partial L}{\partial q^a} dq^a + \cancel{\underbrace{\frac{\partial L}{\partial \dot{q}^a}}_{p_a} d\dot{q}^a} + \frac{\partial L}{\partial t} dt \right)$$

$$= \dot{q}^a dp_a - \frac{\partial L}{\partial q^a} dq^a - \frac{\partial L}{\partial t} dt .$$

Comparing coeff of (dp_a, dq^a, dt) gives

$$-\frac{\partial H}{\partial q^a} = \frac{\partial L}{\partial q^a} \underset{\text{by E-L}}{=} \dot{p}_a , \quad \dot{q}^a = \frac{\partial H}{\partial p_a} , \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} ,$$

i.e.

$$\dot{p}_a = -\frac{\partial H}{\partial q^a} , \quad \dot{q}^a = \frac{\partial H}{\partial p_a} . \quad \left(\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \right)$$

These are Hamilton's eqn.

E.g. $$L = \frac{1}{2} m \dot{q}^a \dot{q}^b \delta_{ab} - V(q)$$

$$p_a = \frac{\partial L}{\partial \dot{q}^a} = m \dot{q}^b \delta_{ab} \quad \Rightarrow \quad \dot{q}^a = p^a / m$$

$$H(p,q) = p_a \dot{q}^a - L$$
$$= \frac{p_a p^a}{m} - \frac{p_a p^a}{2m} + V(q)$$
$$= \frac{(p_a)^2}{2m} + V(q) \quad \text{as is familiar.}$$

E.g. $$L = \frac{1}{2} T_{ab}(q)\, \dot{q}^a \dot{q}^b + C_a(q) \dot{q}^a - V(q)$$

$$p_a = \frac{\partial L}{\partial \dot{q}^a} = T_{ab} \dot{q}^b + C_a(q)$$

$$\Rightarrow \dot{q}^b = (T^{-1})^{ab} (p_b - C_b) \quad \text{provided } T^{-1} \text{ exists.}$$

$$H(p,q) = p_a \dot{q}^a - L$$
$$= p_a (T^{-1})^{ab} (p_b - C_b) - \frac{1}{2} (p_a - C_a) (T^{-1})^{ab} (p_b - C_b)$$
$$- C_a (T^{-1})^{ab} (p_b - C_b) + V(q)$$

Altogether, this is

$$H(p,q) = \frac{1}{2} (p_a - C_a(q)) (T^{-1})^{ab} (p_b - C_b(q)) + V(q).$$

For a charged particle in EM field,

$$L = \frac{1}{2} m \dot{\underline{x}}^2 - e(\phi - \dot{\underline{x}} \cdot \underline{A}),$$

So we have momentum

$$\underline{p} = \frac{\partial L}{\partial \dot{\underline{x}}} = m \dot{\underline{x}} + e \underline{A}$$

$$\Rightarrow \dot{\underline{x}} = \frac{1}{m}(\underline{p} - e\underline{A})$$

So the Hamiltonian for a charged particle is

$$H = \underline{p} \cdot \dot{\underline{x}} - L$$

$$= \frac{1}{m}\underline{p} \cdot (\underline{p} - e\underline{A}) - \frac{1}{2m} \cdot (\underline{p} - e\underline{A})^2 + e\phi - \frac{1}{m}(\underline{p} - e\underline{A}) \cdot (e\underline{A})$$

$$= \frac{1}{2m}(\underline{p} - e\underline{A}) \cdot (\underline{p} - e\underline{A}) + e\phi$$

Hamiltonian's eqn become

$$\dot{\underline{x}} = \frac{1}{m}(\underline{p} - e\underline{A})$$

$$\dot{p}_a = -e\frac{\partial \phi}{\partial x^a} + \frac{e}{m}(\underline{p} - e\underline{A})_b \frac{\partial A_b}{\partial x^a}.$$

For example, suppose $\underline{E} = 0$, $\underline{B} = B\hat{\underline{z}}$ uniform, then we can choose $\phi = 0$ and $\underline{A} = (-By, 0, 0)$ so that $\nabla \times \underline{A} = \underline{B}$.

Then,

$$H = \frac{(p_x + eBy)^2 + p_y^2 + p_z^2}{2m}$$

$$\Rightarrow \dot{p}_x = 0 = \dot{p}_z, \quad \dot{p}_y = -\frac{eB}{m}(p_x + eBy)$$

$$m\dot{x} = p_x + eBy, \quad m\dot{y} = p_y, \quad m\dot{z} = p_z$$

This is uniform motion in direction of $\hat{\underline{B}}$, together with

$$p_y + eBx = a \text{ const. (since } eB\dot{x} = \frac{eB}{m}(p_x + eBy))$$

$$p_x = m\dot{x} - eBy = b \quad \text{const.}$$

$$\Rightarrow \quad x = \frac{a}{eB} + R\sin(\omega(t - t_0)), \quad y = -\frac{b}{eB} + R\cos(\omega(t - t_0)),$$

R, t_0 const.

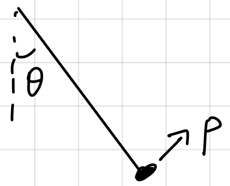
Phase Space

The (q^a, p_a) are coords on a $2n$-dimentional space called phase space $\mathcal{P}$. E.g. for a particle moving in $\mathbb{R}^n$,

$$\mathcal{P} \cong \underset{(q^a}{\overset{\mathbb{R}^n}{\Psi}} \times \underset{p_a)}{\overset{\mathbb{R}^n}{\Psi}} \cong \mathbb{R}^{2n}.$$

whilst for a simple pendulum, instead

$$\mathcal{P} \cong \underset{(\theta,}{\overset{S^1}{\Psi}} \times \underset{p)}{\overset{\mathbb{R}}{\Psi}}$$

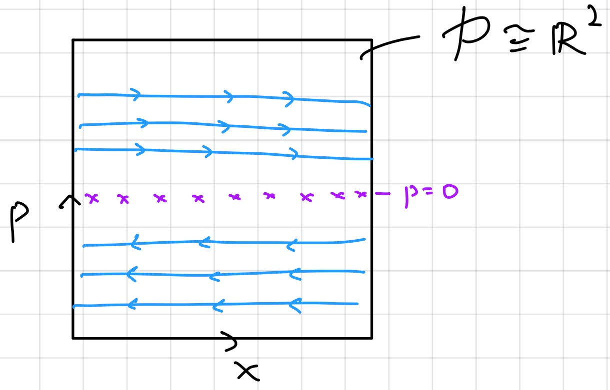


We can think of Hamilton's eqn as defining a trajectory $\gamma : \mathbb{R} \to \mathcal{P}$ in phase space, by

$$\gamma : t \mapsto (q^a(t), p_a(t)).$$

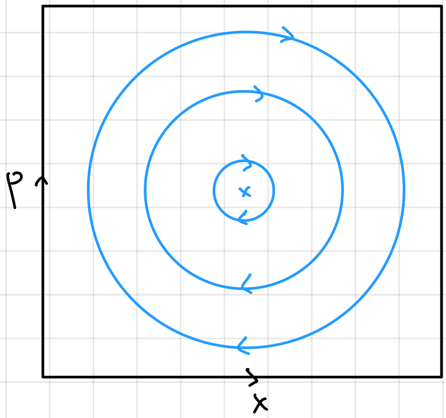
The set $\{\gamma(t)\}$ of all possible trajectories is called the Hamiltonian flow.

E.g. Free particle moving in $\mathbb{R}$.



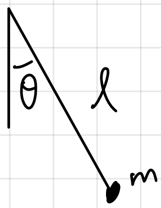
$$H = \frac{p^2}{2m}, \quad \dot{p} = 0, \quad m\dot{x} = p.$$

E.g. 1D SHO, $m = k = 1$



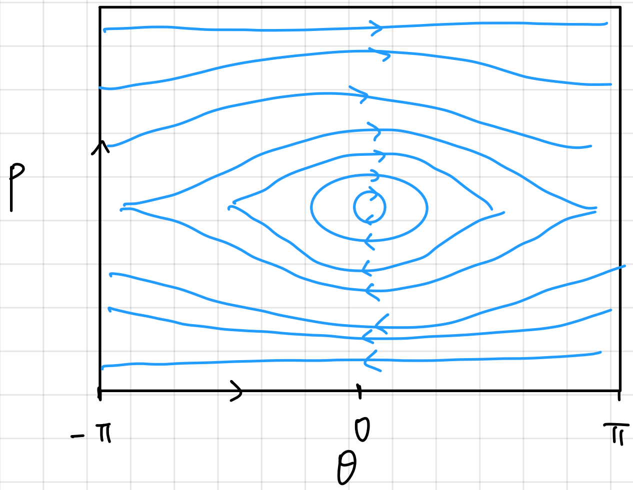
$$H = \frac{1}{2}(p^2 + x^2), \quad \dot{x} = p, \quad \dot{p} = -x.$$

E.g. Simple pendulum

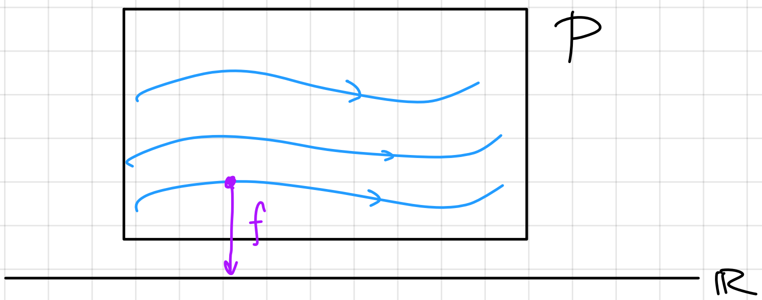


$$H = \frac{p^2}{2m} - mgl\cos\theta$$

$$\dot{\theta} = \frac{p}{m}, \qquad \dot{p} = -mgl\sin\theta$$



Arbitrary physical observables (e.g. H, $\underline{L}$, ...) can be represented by (smooth) f^ns $f: \mathcal{P} \to \mathbb{R}$. The particular value $f(q(t), p(t))$ of this observable at time t taken by a given particle is just this f^n evaluated on γ.



As time evolves,

if f has explicit t dependence.

$$\frac{df}{dt} = \frac{\partial f}{\partial q^a}\frac{\partial q^a}{\partial t} + \frac{\partial f}{\partial p_a}\frac{\partial p_a}{\partial t} + \frac{\partial f}{\partial t}.$$

$$= \frac{\partial f}{\partial q^a}\frac{\partial H}{\partial p^a} - \frac{\partial f}{\partial p_a}\frac{\partial H}{\partial q^a} + \frac{\partial f}{\partial t}$$

This motivates us to define Poisson brackets.

For any pair of smooth f^n $f, g : P \to \mathbb{R}$,

$$\{f, g\} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}$$

Therefore, $\dfrac{df}{dt} = \{f, H\} + \dfrac{\partial f}{\partial t}$.

The Poisson bracket obeys:

- $\{f, g\} = -\{g, f\}$ (antisym)
- $\{f, \alpha g + \beta h\} = \alpha\{f, g\} + \beta\{f, h\}$, $\alpha, \beta \in \mathbb{R}$ (linear)
- $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (Leibniz rule)
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity)

E.g. $\{q^a, p_b\} = \delta^a{}_b$

c.f. Dirac's quantisation took $\{\,,\,\} \to \frac{1}{i\hbar}[\,,\,]$

E.g.

$$\frac{dq^a}{dt} = \{q^a, H\} = \frac{\partial H}{\partial p_a}$$

$$\frac{dp^a}{dt} = \{p^a, H\} = -\frac{\partial H}{\partial q^a}$$

Hamilton's eqn.

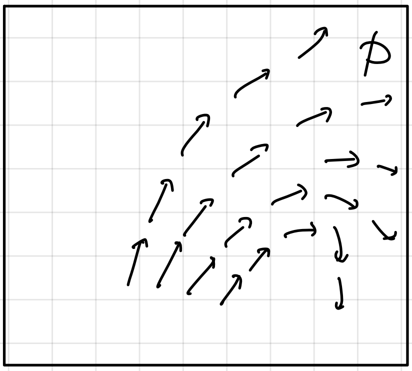
Thm (Poisson's thm) If f, g are both constants of the motion, so is $\{f, g\}$.

Pf: $$\frac{d}{dt}(\{f,g\}) = \{\{f,g\}, H\} + \frac{\partial}{\partial t}\{f,g\}$$
$$= \{\{f,H\},g\} + \{f,\{g,H\}\} + \{\tfrac{\partial f}{\partial t}, g\} + \{f, \tfrac{\partial g}{\partial t}\}$$
$$= \{\tfrac{df}{dt}, g\} + \{f, \tfrac{dg}{dt}\} = 0 \qquad \square$$

Often $\{f,g\}$ is not algebraically indpt of f, g themselves.

Suppose H has no explicit time dependence, then
$$-\{H, \cdot\} = \frac{\partial H}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial H}{\partial q^a}\frac{\partial}{\partial p^a}$$
is a vector field on $\mathcal{P}$ that generates time evolution.

Since $\frac{df}{dt} = \{f, H\} = -\{H, f\}$ (for $\frac{\partial f}{\partial t} = 0$).



We say the Hamiltonian is the **generator** of time evolution.

Suppose we translate our system in configuration space, so $(q^a, p_b) \mapsto (q^a + c^a, p_b)$ for some constant vector $\underline{c}$, then
$$f(\underline{q}, \underline{p}) \mapsto f(\underline{q} + \underline{c}, \underline{p}) = f(\underline{q}, \underline{p}) + c^a \frac{\partial f}{\partial q^a} + O(c^2).$$

Hence, for infinitestimal translations in direction $\underline{c}$, we have
$$\delta f = c^a \frac{\partial f}{\partial q^a} = \{f, c^a p_a\} = c^a\{f, p_a\}.$$

This shows that $-\{p_a, \cdot\} = \frac{\partial}{\partial q^a}$ generates translations in config space. We call $-\{p_a, \cdot\}$ the **Hamiltonian vector field associated with p_a**.

Again, p_a itself is the generator of these translations.

Under an infinitesimal rotation of our system,

$$\underline{q} \longmapsto \underline{q} + \delta\alpha \; \underline{n} \times \underline{q} + O(\delta\alpha^2)$$

angle $|\delta\alpha| \ll 1$.

$$\underline{p} \longmapsto \underline{p} + \delta\alpha \; \underline{n} \times \underline{p} + O(\delta\alpha^2),$$

axis $\underline{n}$

$$f(\underline{q}, \underline{p}) \mapsto f(\underline{q} + \delta\alpha \, \underline{n} \times \underline{q}, \; \underline{p} + \delta\alpha \, \underline{n} \times \underline{p})$$

So

$$\begin{aligned} \delta f &= \delta\alpha (\underline{n} \times \underline{q}) \cdot \frac{\partial f}{\partial \underline{q}} + \delta\alpha (\underline{n} \times \underline{p}) \cdot \frac{\partial f}{\partial \underline{p}} + \ldots \\ &= \delta\alpha \; \underline{n} \cdot \left(\underline{q} \times \frac{\partial f}{\partial \underline{q}} + \underline{p} \times \frac{\partial f}{\partial \underline{p}} \right) \\ &= \delta\alpha \; \{ f, \underline{n} \cdot \underline{L} \}, \quad \text{where } \underline{L} = \underline{q} \times \underline{p}. \end{aligned}$$

Thus $\underline{n} \cdot \underline{L}$ is the generator of rotations around the $\underline{n}$ axis.

To any (smooth) f^n $Q(\underline{q}, \underline{p})$ on phase space, we can associate a Hamiltonian vector field $-\{Q, \cdot\}$ which tells us how the system changes under a transformation generated by Q.

Thm (Noether's thm (Hamiltonian version)) Suppose $\frac{\partial Q}{\partial t} = \frac{\partial H}{\partial t} = 0$, then Q is conserved iff

$$0 = \frac{dQ}{dt} = \{Q, H\} = \frac{dH}{d\alpha}$$

where α is a parameter along the integral curve of $-\{Q, \cdot\}$

In other words, symmetries of H $\Leftrightarrow$ conserved quantity Q.

E.g. If H invariant under rotations,

$$\{H, \underline{L}\} = 0 \Rightarrow \underline{L} \text{ conserved}$$

E.g. If H invariant under translations of configuration space,

$$\{H, \underline{p}\} = 0 \Rightarrow \underline{p} \text{ conserved}$$

E.g. $\{H, H\} = 0$ trivally, so energy is always conserved.
if $\frac{\partial H}{\partial t} = 0$.

Dirac's quantisation

Suppose $f, g, h \in C^\infty(\mathcal{P})$ s.t. $\{f, g\} = h$. Dirac thought quantising this classical system means finding self-adjoint operators $\hat{f}, \hat{g}, \hat{h}$ on $L^2(\mathcal{C}, d^n q)$ s.t. $[\hat{f}, \hat{g}] = i\hbar \hat{h}$.

However, ...

Thm (Greenwald - van Hove) This is impossible in general.

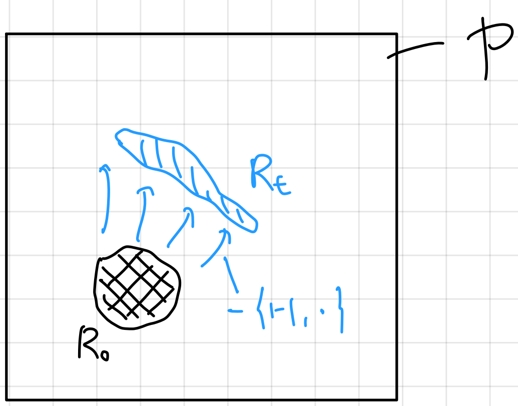
Liouville's Thm

Suppose a system evolves in time via Hamilton's eqn, then after a short time δt,

$$(q^a(t), p_b(t)) \longmapsto (\tilde{q}^a(t), \tilde{p}_b(t)) = \Big(q^a + \delta t \underbrace{\frac{\partial H}{\partial p_a}}_{\dot{q}^a}, p_b - \delta t \overbrace{\frac{\partial H}{\partial q^b}}^{-\dot{p}_b}\Big) + O(\delta t^2)$$

Suppose initially our system occupies some region $R_0 \subset \mathcal{P}$, that evolves to a region R_t under Hamiltonian flow.

Let V be the volume of R_0, i.e.



$$V = \mathrm{Vol}(R_0) = \delta^n q \, \delta^n p$$
$$= \delta q^1 \ldots \delta q^n \, \delta p^1 \ldots \delta p^n$$

After a short time δt, the region $R_{\delta t}$ has volume

$$\tilde{V} = \delta^n \tilde{q} \; \delta^n \tilde{p} = |J| \, \delta^n p \, \delta^n q \, ,$$

where $J = \begin{pmatrix} \partial \tilde{q}^a / \partial q^b & \partial \tilde{q}^a / \partial p_b \\ \partial \tilde{p}_a / \partial q^b & \partial \tilde{p}_a / \partial p_b \end{pmatrix}$.

We have

$$J = \begin{pmatrix} \delta^a{}_b + \delta t \dfrac{\partial^2 H}{\partial q^b \partial p_a} & \delta t \dfrac{\partial^2 H}{\partial p_b \partial p_a} \\ -\dfrac{\partial^2 H}{\partial q^b \partial q^a} & \delta_a{}^b - \delta t \dfrac{\partial^2 H}{\partial q^a \partial p_b} \end{pmatrix} + O(\delta t^2)$$

$$= I_{2n} + \delta t \, \delta \underline{\underline{J}} + O(\delta t^2)$$

Therefore, since $|J| = 1 + \delta t \, \mathrm{tr}(\delta J) + O(\delta t^2)$, we have

$$\tilde{V} = V\left(1 + \delta t \left(\frac{\partial^2 H}{\partial q^a \partial q_a} - \frac{\partial^2 H}{\partial q^a \partial p_a}\right)\right) = V.$$

This is Liouville's thm — the volume of any region $R \subset P$ is preserved by time evolution.

For example, we could have a phase space prob. dist. $\rho(q, p, t)$ normalised s.t. $\int_P \rho(q, p, t) \, d^n q \, d^n p = 1$, where ρ represents the prob. density to find a particle at $(q, p) \in P$ at time t.

Then Liouville's thm gives

$$0 = \frac{d}{dt}\int_{\mathcal{P}} \rho(q,p,t)\, d^n q\, d^n p$$

$$= \int_{\mathcal{P}} \left(\frac{\partial \rho}{\partial q^a}\dot{q}^a + \frac{\partial \rho}{\partial p_a}\dot{p}_a + \frac{\partial \rho}{\partial t} \right) d^n q\, d^n p$$

$$= \int_{\mathcal{P}} \left(\{\rho, H\} + \frac{\partial \rho}{\partial t} \right) d^n q\, d^n p$$

If particles are neither created nor destroyed, this holds locally, so we have Liouville's eqn

$$\frac{\partial \rho}{\partial t} = -\{\rho, H\}.$$

This can be viewed as cty eqn ∇ on $\mathcal{P}$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.$$

where j is the prob. current on phase space, i.e.

$$j = (\rho \dot{q}^a, \rho \dot{p}_a)$$

Then

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q^a}(\rho \dot{q}^a) + \frac{\partial}{\partial p_a}(\rho \dot{p}_a)$$

$$= \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q^a}\left(\rho \frac{\partial H}{\partial p_a}\right) - \frac{\partial}{\partial p_a}\left(\rho \frac{\partial H}{\partial q^a}\right)$$

$$= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q^a}\frac{\partial H}{\partial p} - \frac{\partial \rho}{\partial p_a}\frac{\partial H}{\partial q^a} + \rho\left(\frac{\partial^2 H}{\partial q^a \partial p_a} - \frac{\partial^2 H}{\partial p_a \partial q^a}\right)$$

$$= \frac{\partial \rho}{\partial t} + \{\rho, H\} = 0.$$

* The Bohr - van Leeuwen Thm *

Often, we're interested in time-indpt ($\frac{\partial \rho}{\partial t} = 0$) dist.
For example, (see stat. mech.) if a system of particles is constrained to have total energy E, i.e.

$$\int_P H(\underline{q}, \underline{p}) \, \rho(\underline{q}, \underline{p}) \, d^{3N}\underline{q} \, d^{3N}\underline{p} = E,$$

but is otherwise free, then it's governed by the Boltzmann dist.

$$\rho(\underline{q}, \underline{p}) = \frac{e^{-H(p, q)/kT}}{Z(E)},$$

where T is temp, and k, $Z(E)$ const.

E.g. If the particles are free within a box, then

$$H(\underline{q}, \underline{p}) = \sum_{i=1}^{N} \frac{\underline{p}_i^2}{2m}.$$

If the particles are electrically charged, and the system is subject to a magnetic field, then

$$H(\underline{q}, \underline{p}) = \sum_i \frac{(\underline{p}_i - e\underline{A}(\underline{q}))^2}{2m}$$

The magnetic moment $\underline{\mu}_i$ due to a charged particle is

$$\underline{\mu}_i = \underline{q}_i \times \underline{j}_i = e(\underline{q} \times \dot{\underline{q}})_i,$$

so for the whole system, the total magnetic moment in some direction $\underline{n}$ is

$$\underline{n} \cdot \underline{\mu} = e \sum_i \underline{n} \cdot (\underline{q}_i \times \dot{\underline{q}}_i) = e \sum_i (\underline{n} \times \underline{q}_i) \cdot \underbrace{\left(\frac{\underline{p}_i - e\underline{A}(\underline{q}_i)}{m} \right)}_{= \dot{\underline{q}}_i}$$

Hence, on average, we have

$$\langle \underline{n} \cdot \underline{\mu} \rangle = \int (\underline{n} \cdot \underline{\mu})\, \rho(\underline{q}, \underline{p})\, d^{3N}q\, d^{3N}p$$

$$= \frac{e}{m Z_E} \int \sum_i (\underline{n} \times \underline{q}_i) \cdot (\underline{p}_i - e\underline{A}(\underline{q}))\, e^{-\sum_i (\underline{p}_i - e\underline{A}(\underline{q}))^2 / 2mkT}\, d^N q\, d^N p$$

$$= 0 \, ,$$

Since integrand is an odd f^n of $\dot{\underline{q}}_i = \underline{p}_i - e\underline{A}_i$. Hence, there can be no net magnetic moment in classical dynamics.

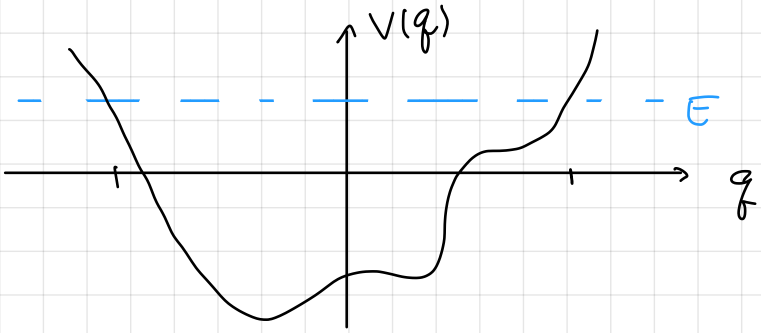
This is the Bohr-van Leeuwen thm.

Magnetism is a quantum phenomenon.

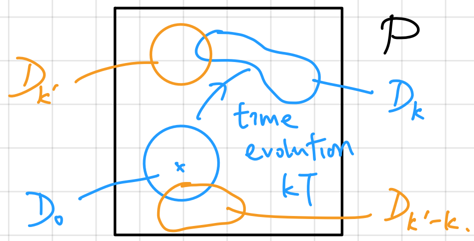
* Poincaré Recurrence *

Suppose our system is confined to a bounded region of $\mathcal{P}$.

(E.g. if the system has fixed energy E, with $T > 0$, V bounded from below, then it cannot escape to regions where $V(q) > E$.)



Suppose our system is at some point $(\underline{q}_0, \underline{p}_0) \in \mathcal{P}$ at $t = 0$, then for any open region $D_0 \in \mathcal{P}$ containing $(\underline{q}_0, \underline{p}_0)$, $\exists$ some point $(\underline{q}_0', \underline{p}_0') \in D_0$ which eventually returns to D_0.



Pf: Let D_k be the time evolution of D_0 after time kT for $k \in \mathbb{N}_0$. By Liouville thm, $\mathrm{vol}(D_k) = \mathrm{vol}(D_0)$. So if $D_k \cap D_{k'} = \phi$, $\forall k, k'$,

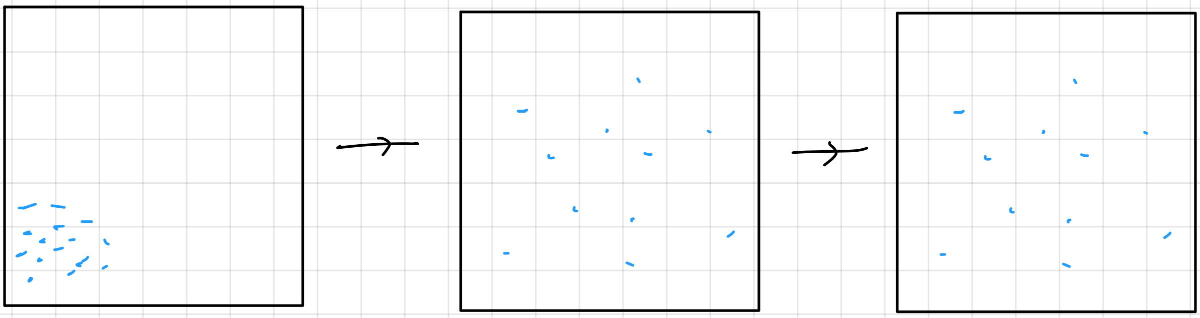
$$\mathrm{vol}\left(\bigcup_{k=0}^{n} D_k\right) = \sum_{k=0}^{n} \mathrm{vol}(D_k) = (n+1)\,\mathrm{vol}(D_0)$$

As $n \to \infty$, this diverges, contradicting our assumption that the system could only explore a bounded region of P.

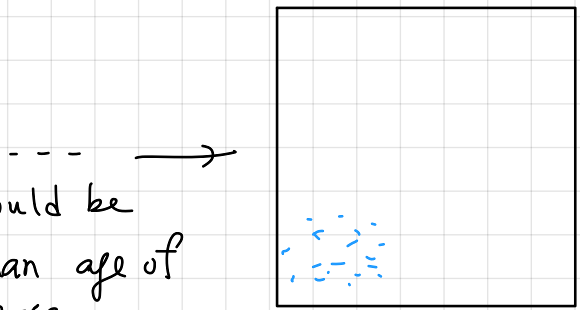
Hence, $\exists$ some $D_k, D_{k'}$ with $D_k \cap D_{k'} \neq \phi$. $\square$

Hamiltonian evolution is reversible, so we can trace these regions back to find $D_0 \cap D_{(k'-k)} \neq 0$.

This is Poincaré recurrence thm.



$t=0$ $\qquad$ kT



this could be longer than age of universe.

$(k'-k)T$ for $k'-k \gg 1$.

Canonical transformations

Because (q, p) are on a more symmetric footing in H's eqn, we can do more general coordinate transformations than the holonomic transformations $q^a \mapsto \tilde{q}^a(q, t)$ that preserve the E-L eqn.

Let $y^\alpha = (q^a, p_b)$ be $2n$-coords on $\mathcal{P}$ $(\alpha = 1, \dots, 2n)$.
A transformation $y^\alpha \mapsto Y^\alpha = Y^\alpha(y, t)$ (and $t \mapsto t$) is called canonical if it leaves Poisson brackets unchanged, in the sense that

$$\{f, g\}_y = \{f, g\}_Y$$

for all $f, g \in C^\infty(\mathcal{P})$.

To understand this, note that

$$\{f, g\} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a} = \frac{\partial f}{\partial y^\alpha}\Omega^{\alpha\beta}\frac{\partial g}{\partial y^\beta},$$

where $\Omega^{\alpha\beta} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Note that $\Omega^2 = -I_{2n}$, $\Omega^T = -\Omega$.

Viewing $f(y(Y))$, we have $\dfrac{\partial f}{\partial y^\alpha} = \dfrac{\partial f}{\partial Y^\gamma}\dfrac{\partial Y^\gamma}{\partial y^\alpha}$, so the Poisson bracket is

$$\{f, g\}_y = \frac{\partial f}{\partial Y^\gamma}\frac{\partial Y^\gamma}{\partial y^\alpha}\Omega^{\alpha\beta}\frac{\partial Y^\delta}{\partial y^\beta}\frac{\partial g}{\partial Y^\delta}.$$

This is equal to $\{f, g\}_Y$ iff

$$\frac{\partial Y^\gamma}{\partial y^\alpha}\Omega^{\alpha\beta}\frac{\partial Y^\delta}{\partial y^\beta} = \Omega^{\gamma\delta},$$

or equivalently, $J\Omega J^T = \Omega$, with $J^\gamma{}_\alpha = \dfrac{\partial Y^\gamma}{\partial y^\alpha}$.

$(\Rightarrow)$
$$\{f,g\}_y = \frac{\partial f}{\partial Y^\gamma}\left(\frac{\partial Y^\gamma}{\partial y^\alpha}\Omega^{\alpha\beta}\frac{\partial Y^\delta}{\partial y^\beta}\right)\frac{\partial g}{\partial Y^\delta}$$
$$= \frac{\partial f}{\partial Y^\gamma}\Omega^{\gamma\delta}\frac{\partial g}{\partial Y^\delta} = \{f,g\}_Y.$$

$(\Leftarrow)$ Choose $f = Y^\gamma(y,t)$, $g = Y^\delta(y,t)$, then
$$\{Y^\gamma, Y^\delta\}_y = \frac{\partial Y^\gamma}{\partial y^\alpha}\Omega^{\alpha\beta}\frac{\partial Y^\delta}{\partial y^\beta}$$
$$= \{Y^\gamma, Y^\delta\}_Y = \Omega^{\gamma\delta}.$$

At each point $y \in P$, the matrix $J(y) \in GL_{2n}(\mathbb{R})$.
For canonical transformations $J\Omega J^T = \Omega$, so
$$J \in Sp_{2n}(\mathbb{R}) \subset GL_{2n}(\mathbb{R}),$$
where $Sp_{2n}(\mathbb{R})$ is the symplectic group.

In the original coords,
$$J^\alpha{}_\beta = \frac{\partial Y^\alpha}{\partial y^\beta} = \begin{pmatrix} \dfrac{\partial Q^a}{\partial q^b} & \dfrac{\partial Q^a}{\partial p_b} \\[2ex] \dfrac{\partial P_a}{\partial q^b} & \dfrac{\partial P_a}{\partial p_b} \end{pmatrix}$$

So the condition $J\Omega J^T = \Omega$ becomes
$$\begin{pmatrix} \{Q^a, Q^b\}_{(q,p)} & \{Q^a, P_b\}_{(q,p)} \\ \{P_a, P_b\}_{(q,p)} & \{P_a, P_b\}_{(q,p)} \end{pmatrix} = \begin{pmatrix} 0 & \delta^a{}_b \\ -\delta_a{}^b & 0 \end{pmatrix}$$

E.g. Suppose we 'swap' positions and momenta by defining

$$\underline{Q} = \underline{p}, \qquad \underline{P} = -\underline{q},$$

Then $J = \begin{pmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{pmatrix} = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} = \Omega$ in this case.

Since $J\Omega J^T = \underbrace{\Omega\Omega}_{=-I}\underbrace{\Omega^T}_{=-\Omega} = \Omega$, this transformation is canonical. Hence, it's really just a matter of convention what we think of as a 'position' coordinate and what is a 'momentum' coordinate.

We say coords (q^a, p_b) obeying usual Poisson bracket relations form a canonically conjugate pair.

Hamilton's eqn and canonical transformation

H's eqn say $\frac{dy^\alpha}{dt} = \Omega^{\alpha\beta}\frac{\partial H}{\partial y^\beta}$. Under a canonical transformation $y^\alpha \mapsto Y^\alpha(y,t)$, we have

$$\frac{dY^\alpha}{dt} = \frac{\partial Y^\alpha}{\partial y^\beta}\frac{dy^\beta}{dt} + \frac{\partial Y^\alpha}{\partial t}$$

$$= \frac{\partial Y^\alpha}{\partial y^\beta}\Omega^{\alpha\beta}\frac{\partial H}{\partial y^\beta} + \frac{\partial Y^\alpha}{\partial t}$$

Since $J\Omega J^T = \Omega$, we have $J\Omega = \Omega(J^T)^{-1}$, so

$$\frac{dY^\alpha}{dt} = \Omega^{\alpha\beta}\frac{\partial y^\gamma}{\partial Y^\beta}\frac{\partial H}{\partial y^\gamma} + \frac{\partial Y}{\partial t} = \Omega^{\alpha\beta}\frac{\partial H}{\partial Y^\beta} + \frac{\partial Y}{\partial t}$$

Hence, for canonical transformations with no explicit t dependence, we have

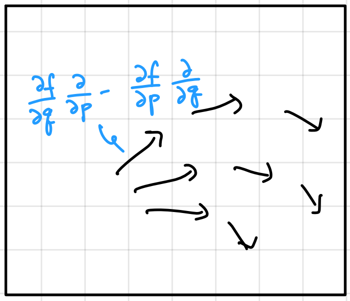
$$\dot{y}^\alpha = \Omega^{\alpha\beta} \frac{\partial H}{\partial y^\beta} \quad \Leftrightarrow \quad \dot{Y}^\alpha = \Omega^{\alpha\beta} \frac{\partial H}{\partial Y^\beta}$$

Generating Functions

There are several ways to construct coord transformations that are guaranteed to be canonical.

Recall we defined a Hamiltonian vector field associated to $f(q,p)$ as

$$\{f, \cdot\} =: D_f$$



This obeys properties following from properties of the Poisson bracket:

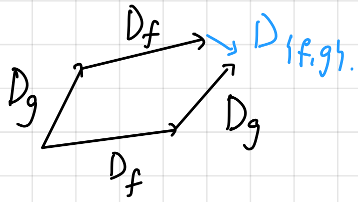
- $D_f(g) = \{f, g\} = -D_g(f)$
- $D_{(f_1+f_2)}(g) = D_{f_1}(g) + D_{f_2}(g)$ (linear in f)
- $D_f(g_1+g_2) = D_f(g_1) + D_f(g_2)$ (linear in g)
- $D_f(gh) = (D_f g)h + g D_f(h)$ (Leibniz rule).
- $[D_f, D_g]h = D_f(D_g h) - D_g(D_f h)$

$$= \{f, \{g,h\}\} - \{g, \{f,h\}\}$$

$$= \{\{f,g\}, h\} \quad \text{(by Jacobi identity)}$$

$$= D_{\{f,g\}}(h).$$

$$\Rightarrow [D_f, D_g] = D_{\{f,g\}}.$$



To construct a canonical transformation, pick any $f \in C^\infty(P)$.
Then define $Y^\alpha(y,s)$ by

$$Y^\alpha(y,s) = e^{-sD_f}(y_\alpha) = \sum_{n\geq 0} \frac{(-s)^n}{n!} \underbrace{D_f \cdots D_f}_{n \text{ times}}(y_\alpha)$$

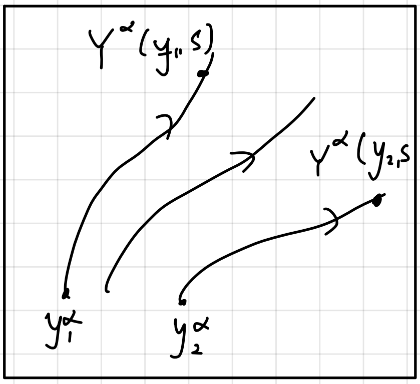
This will be a canonical coord transformation for any $s \in \mathbb{R}$.

$$\frac{\partial}{\partial s} Y^\alpha(y,s) = -\sum_{n=1}^{\infty} \frac{(-s)^{n-1}}{(n-1)!} \underbrace{D_f \cdots D_f}_{n \text{ times}}(y^\alpha)$$

$$= -D_f Y^\alpha(y,s)$$

$$= -\{f, Y^\alpha(y,s)\} = \{Y^\alpha(y,s), f\}. \quad (*)$$

Also, $Y^\alpha(y,0) = y^\alpha$.



These transformation $y^\alpha \longmapsto Y^\alpha(y,s)$ is canonical for every choice of (f,s).

To see this, note

$$\frac{\partial}{\partial s}\{Y^\alpha, Y^\beta\} = \left\{\frac{\partial Y^\alpha}{\partial s}, Y^\beta\right\} + \left\{Y^\alpha, \frac{\partial Y^\beta}{\partial s}\right\}$$

$$= \{\{Y^\alpha(y,s), f\}, Y^\beta\} + \{Y^\alpha, \{Y^\beta(y,s), f\}\}$$

$$= \{\{Y^\alpha(y,s), Y^\beta(y,s)\}, f\}$$

has the same form as $(*)$ now for the f^n $\{Y^\alpha, Y^\beta\}(y,s)$. Hence, we can solve this 1^{st} order PDE as

$$\{Y^\alpha(y,s), Y^\beta(y,s)\} = e^{-sD_f}\{Y^\alpha(y,0), Y^\beta(y,0)\}$$

$$= e^{-sD_f}\{y^\alpha, y^\beta\} = e^{-sD_f}(\Omega^{\alpha\beta}) = \Omega^{\alpha\beta}$$

Hence, $\{Y^\alpha, Y^\beta\} = \Omega^{\alpha\beta}$ if $Y^\alpha = e^{-sD_f}(y^\alpha)$, so the transformation is canonical.

E.g. Choose $P = \mathbb{R}^2 \ni (q,p)$ and pick $f = ap$ for some const. a. Then

$$D_f = \{f, \cdot\} = -a\frac{\partial}{\partial q}$$

So we have

$$Q(q,p) = e^{-sD_f}(q) = e^{sa\frac{\partial}{\partial q}}(q) = q + sa.$$

$$P(q,p) = e^{-sD_f}(p) = e^{-sa\frac{\partial}{\partial q}}(p) = p.$$

So $f = ap$ generates translations which are canonical coord transformations.

In particular, if we choose $f(q,p) = H(q,p)$ to be the system's Hamiltonian, then (relabelling $s \mapsto t$), the transformation

$$Q(q,p,t) = e^{-tD_H}(q) \quad , \quad P(q,p,t) = e^{-tD_H}(p)$$

is canonical, since $\frac{\partial Y^\alpha}{\partial t} = \{Y^\alpha(y,t), H\}$. This is just time evolution by Hamilton's eqn.

E.g. Consider SHO with $m = 1 = \omega$, so $H = \frac{1}{2}(p^2 + q^2)$. Then $D_H = q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}$ and under time evolution / Hamiltonian flow.

$$Q(q,p,t) = e^{-tD_H}q$$

$$= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right)^n q$$

$$= \sum_{n=0}^{\infty} \frac{(-t)^{2n}}{(2n)!} (-1)^n q + \sum_{n=0}^{\infty} \frac{(-t)^{2n+1}}{(2n+1)!} (-1)^{n+1} (-p)$$

So

$$Q(q,p,t) = q \cos t - p \sin t ,$$

while

$$P(q,p,t) = p \cos t - q \sin t ,$$

which are indeed new pair of canonically conjugate coords on $\mathcal{P}$.

Hamilton's Principle of Least Action

Another way to generate canonical coord transformation uses the Hamiltonian version of the principle of least action.

We view the action as a functional $S = S[q,p]$ and write

$$S = \int_{t_0}^{t_1} L(q, \dot{q}(q,p), t)\, dt = \int_{t_0}^{t_1} \left(p\dot{q} - H(q,p) \right) dt$$

Extremising w.r.t. changes of the curve in phase space, we have

$$0 = \delta S = \int_{t_0}^{t_1} \left(\delta p_a \dot{q}^a + p_a \delta \dot{q}^a - \frac{\partial H}{\partial q^a} \delta q^a - \frac{\partial H}{\partial p_a} \delta p_a \right) dt$$

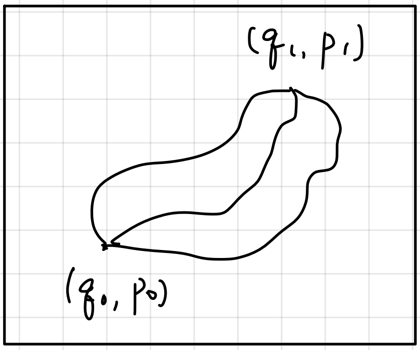
$$= \int_{t_0}^{t_1} \left(\dot{q}^a - \frac{\partial H}{\partial p_a} \right) \delta p_a - \left(\dot{p}_a + \frac{\partial H}{\partial q^a} \right) \delta q^a \, dt$$

$$+ \; p_a \delta q^a \Big|_{t_0}^{t_1}$$

So if $\delta q^a(t_0) = \delta q^a(t_1) = 0$, we extremise by setting

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q^a} \qquad (\text{H's eqn})$$

However, there's more freedom, because now we're extremising over curves in phase space. Since all our curves begin and end at the same two points in phase space, we also have $\delta p_a(t_0) = \delta p_a(t_1) = 0$.



Thus, we can add the total time derivative of any $F \in C^\infty(\mathcal{P})$

$$S \mapsto S + \int_{t_0}^{t_1} \frac{d}{dt} F(q, p, t)\, dt.$$

This will not change the form of the eqns of motion.

Suppose we have two coord systems (q^a, p_b) and (Q^a, P_b) on $\mathcal{P}$, with Hamiltonians $H(q,p)$ and $K(Q,P) = H(q(Q,P), p(Q,P))$. These lead to the same e.o.m. if

$$P_a\, dQ^a - K\, dt = p_a\, dq^a - H\, dt - dF_1(q, Q)$$

Here, F_1 is called the generating f^n of the first kind. (Also $dQ^a = \frac{dQ^a}{dt} dt$ etc). We have

$$dF_1 = \frac{\partial F_1}{\partial q^a} dq^a + \frac{\partial F_1}{\partial Q^a} dQ^a,$$

so comparing gives

$$\left(P_a + \frac{\partial F_1}{\partial Q^a}\right) dQ^a + (H - K)\, dt = \left(p_a - \frac{\partial F_1}{\partial q^a}\right) dq^a.$$

For these to hold, we should define

$$P_a = -\frac{\partial F_1}{\partial Q^a}, \qquad p_a = \frac{\partial F_1}{\partial q^a} \qquad (\text{and } K = H)$$

$p_a = \frac{\partial F_1}{\partial q^a} = p_a(q, Q)$, so invert to get $Q = Q(q, p)$.

E.g. Choose $F_1 = qQ$, then

$$P_a = -\frac{\partial F_1}{\partial Q^a} = -q^a, \qquad p_a = \frac{\partial F_1}{\partial q^a} = Q^a,$$

i.e. $(Q^a, P_b) = (p_a, q^b)$ so this switches momenta and coords

Instead of viewing our generating f^n F as depending on both sets of coords, it's sometimes helpful to think of $\tilde{F}_2 = \tilde{F}_2(q, P)$ in which case it's called a generating f^n of the second kind. Here we have

$$d\tilde{F}_2 = \frac{\partial \tilde{F}_2}{\partial q^a} dq^a + \frac{\partial \tilde{F}_2}{\partial P_b} dP_b$$

and collecting terms now gives

$$P_a dQ^a - K dt - \frac{\partial \tilde{F}_2}{\partial P_a} dP_a = \left(p_a - \frac{\partial \tilde{F}_2}{\partial q^a}\right) dq^a - H dt$$

$$\Rightarrow d(P_a Q^a) - \left(Q_a + \frac{\partial \tilde{F}_2}{\partial P_a}\right) dP_a = \left(p_a - \frac{\partial \tilde{F}_2}{\partial q^a}\right) dq^a + (K - H) dt$$

We now define $F_2(q, P) = \tilde{F}_2 + P_a Q^a$

$$\Rightarrow p_a = \frac{\partial F_2}{\partial q^a}, \qquad Q^a = \frac{\partial F_2}{\partial P_a}, \qquad K(Q, P) = H(q, p)$$

E.g. Consider the generating f^n

$$F_2(q,P) = \int^q \sqrt{2P-u^2}\, du$$

$$p = \frac{\partial F_2}{\partial q} = \sqrt{2P-q^2} \, . \quad Q = \frac{\partial F_2}{\partial P} = \int^q \frac{du}{\sqrt{2P-u^2}}$$

$$= \tan^{-1}\left(\frac{q}{\sqrt{2P-q^2}}\right),$$

i.e. $q = \sqrt{2P} \sin Q$, $p = \sqrt{2P} \cos Q$.

In particular, this transformation is useful for SHO, since, with $m = 1 = \omega$,

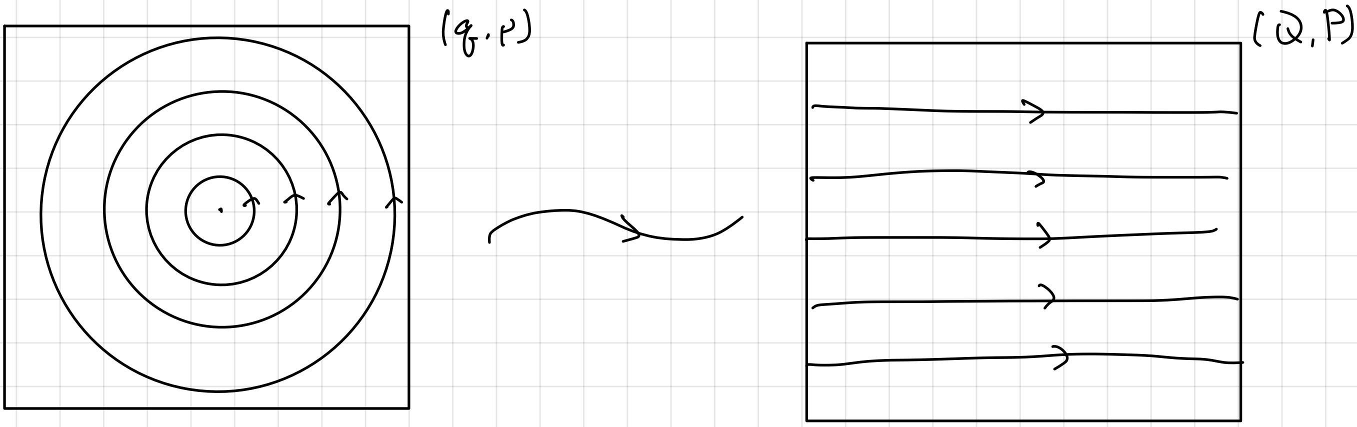
$$H_{SHO}(q,p) = \frac{1}{2}(p^2+q^2) = P,$$

so that in the new coords, $K(Q,P) = P$

Hence, in the (Q,P) coords. H's eqn become

$$\dot{Q} = \frac{\partial K}{\partial P} = 1 \, . \quad \dot{P} = -\frac{\partial K}{\partial Q} = 0$$

So $P(t) = \text{const}$, $\dot{Q}(t) = t - t_0$.



Integrable Systems

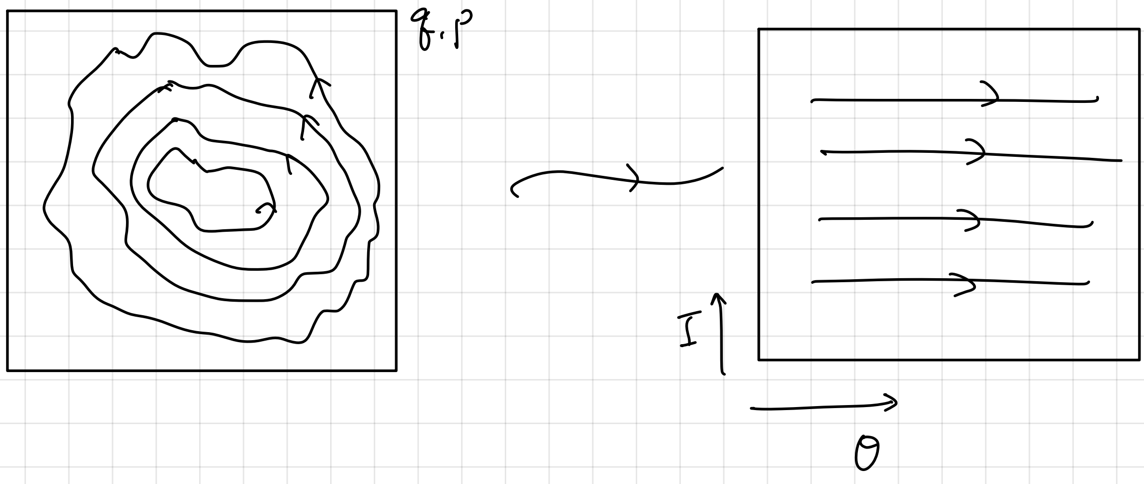
In the previous example, we found a coord transformation $(q^a, p_b) \mapsto (Q^a, I_b)$ s.t. in the new coords, $H = H(I)$. These are called "action-angle" coords. Any system where such a transformation is possible is called integrable.

A thm of Liouville states that such a transformation is possible iff $\exists$ n algebraicly indpt conserved quantities $I_a : \mathcal{P}^{2n} \to \mathbb{R}$ that obey $\{I_a, I_b\} = 0$. These constants are said to be in involution.

In action-angle coords, $H = H(I)$, we'll trivially have $\dot{I}_a = \frac{\partial H}{\partial \theta^a} = 0$, while $\dot{\theta}^a = \frac{\partial H}{\partial I_a} = \omega_a$, which is also const.

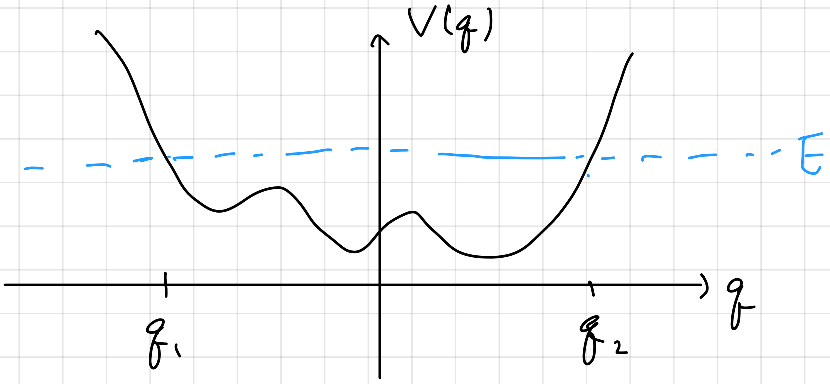
$$\Rightarrow \quad \theta^a(t) = \omega^a(t - t_0)$$

That is, we'll have "straightened out the flow in $\mathcal{P}$".



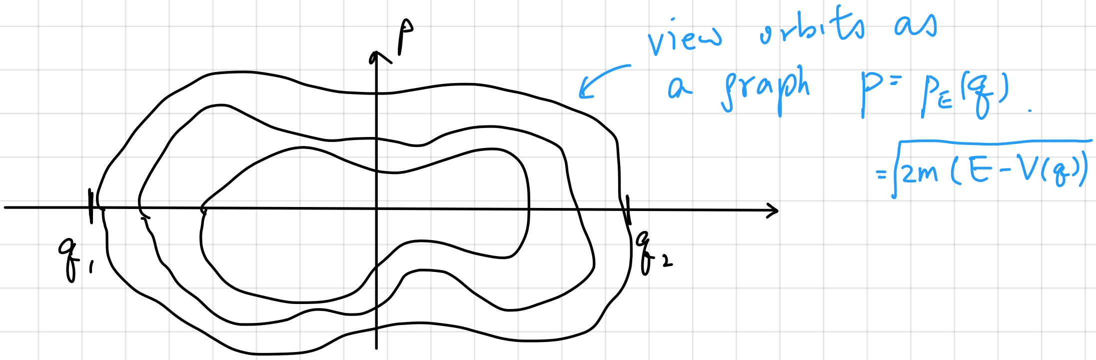
Integrable systems are very special. However, every 1D system whose H obeys $\frac{\partial H}{\partial t} = 0$ is integrable (trivially).

E.g. Suppose $H(q,p) = \frac{p^2}{2m} + V(q)$ and we'll suppose the motion is bounded (ie we have $H = E$ with $E < \lim_{|q|\to\infty} V(q)$)



The motion will oscillate back and forth between (q_1, q_2).

On phase space, the orbits will look like



Claim: $I = \frac{1}{2\pi} \oint p\, dq = \frac{1}{2\pi}$ (Area enclosed by orbit)

Pf: We know for this $H(q,p)$, that $p = m\dot{q}$, so

$$dt = \sqrt{\frac{m}{2}} \frac{dq}{\sqrt{E - V(q)}} \quad \text{ie.} \quad dt = m\frac{dq}{p}$$

$$\Rightarrow \oint_{\text{orbit}} dt = \sqrt{\frac{m}{2}} \oint_{\text{orbit}} \frac{dq}{\sqrt{E - V(q)}}$$

$$\Rightarrow \frac{2\pi}{\omega} = \sqrt{\frac{m}{2}} \oint \frac{dq}{\sqrt{E - V(q)}}$$

$$= \sqrt{2m} \oint \frac{d}{dE} \sqrt{E - V(q)}\, dq$$

$$\stackrel{!}{=} \frac{d}{dE} \oint \sqrt{2m(E - V(q))}\, dq$$

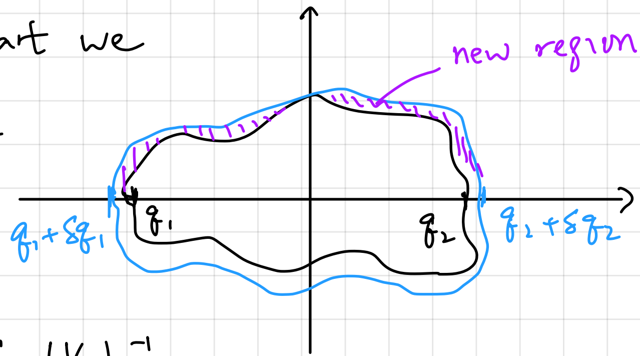
$$= \frac{d}{dE} \oint p \, dq$$

$$= 2\pi \frac{dI}{dE} ,$$

and therefore, $\frac{dE}{dI} = \omega$, so $\frac{\partial H}{\partial I} = \omega$.

To justify exchanging the derivative and integral, note that as we change E, the curve $p(q, E)$ changes and the endpoints q_1, q_2 of the orbit change.

$\delta q_i = \left(\frac{dV}{dq} \right)^{-1} \delta E$ so the part we neglected (i.e. area of the "new region") is



$$\sim \int_{q_1 + \delta q_1}^{q_1} \sqrt{2m(E - V(q))} \left(\frac{dV}{dq} \right)^{-1} dE$$

$$\approx \sqrt{2m\left(E - V\left(q_1 + \frac{\delta q_1}{2}\right)\right)} \left(\frac{dV}{dq} \right)^{-1} \Bigg|_{q_1 + \frac{\delta q_1}{2}} \delta E \sim O(\delta E^2)$$

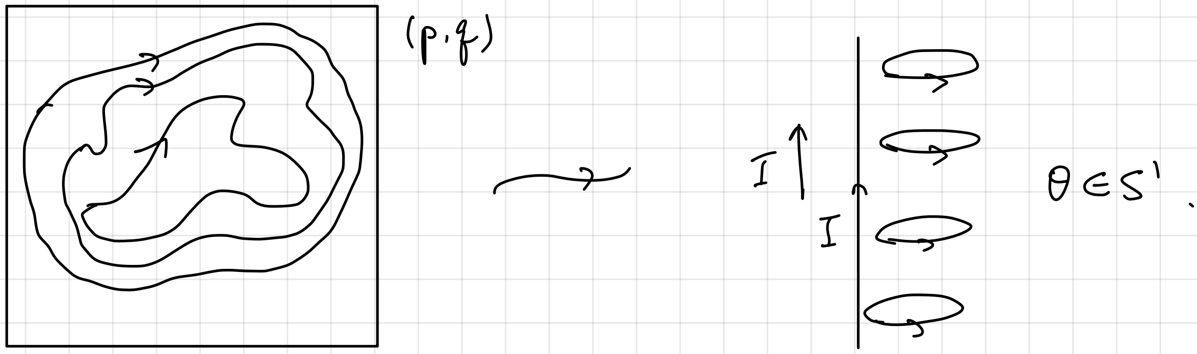
since by def^n $E - V(q) \to 0$ at the endpoints.

Hence, we are justified in exchanging

$$\oint \frac{d}{dE} p \, dq = \frac{d}{dE} \oint p \, dq.$$

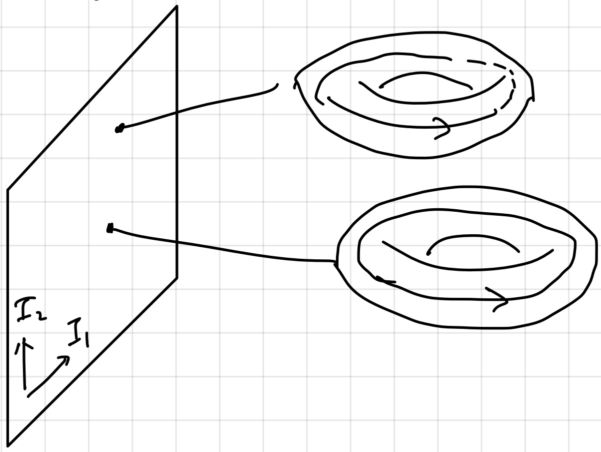
This def^n of action variables also extends to higher dimensional phase spaces.

If we've found n constants of the motion (alg. indpt, in involution), then the orbits must look like $(S^1)^n$ and we've foliated the phase space with these tori.



In 4d phase space of an integrable system,

$$\dot{I}_a = 0 \quad , \quad \theta^a = \omega^a t$$



We define the action variables to be

$$I_a = \frac{1}{2\pi} \oint_{\gamma_a} \underline{p} \cdot d\underline{q} = \frac{1}{2\pi} \oint_{\gamma_a} p_b \, dq^b .$$

where γ_a are the non-contractible curves on our tori.

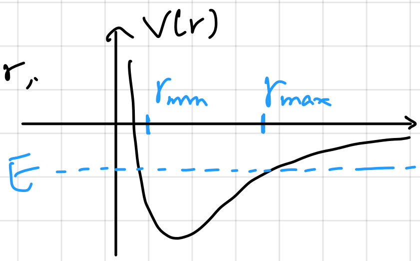
E.g. Action-angle variables for the Kepler problem.

If $V(\underline{r}) = -\frac{k}{r}$, then angular momentum is conserved and the motion $\underline{r}(t) \in \mathbb{R}^3$ is confined to the plane $\underline{r}(t) \cdot \underline{L} = 0$.

Let (r, ϕ) be 2D polar coords on this plane

The Hamiltonian in the plane is

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{k}{r} .$$



The motion is periodic in both ϕ and r, and these two angles give us our two action variables.

$$I_\phi = \frac{1}{2\pi} \oint_{\phi \text{ cycle}} p_r\, dr + p_\phi\, d\phi = \frac{1}{2\pi} \oint_0^{2\pi} p_\phi \, d\phi = p\phi$$

$$I_r = \frac{1}{2\pi} \oint_{r \text{ cycle}} p \cdot dq = \frac{2}{2\pi} \int_{r_{min}}^{r_{max}} p_r \, dr$$

We have $p_a^2 = 2m\left(E + \frac{k}{r}\right) - \frac{I_\phi^2}{r^2}$

$$= 2m|E| \left(-1 + \frac{k}{|E| r} - \frac{I_\phi^2}{2m|E| r^2} \right)$$

$$= 2m|E| \left(1 - \frac{r_{min}}{r} \right) \left(\frac{r_{max}}{r} - 1 \right),$$

where $r_{min} + r_{max} = -\frac{k}{|E|}$, $r_{min} r_{max} = \frac{I_\phi^2}{2m|E|}$.

Hence,

$$I_r = \frac{\sqrt{2m|E|}}{\pi} \int_{r_{min}}^{r_{max}} \sqrt{\left(1 - \frac{r_{min}}{r}\right)\left(\frac{r_{max}}{r} - 1\right)} \, dr$$

$$= \sqrt{2m|E|} \left(\frac{r_{min} + r_{max}}{2} - \sqrt{r_{min} r_{max}} \right)$$

$$= \sqrt{\frac{m}{2|E|}}\, k - I_\phi .$$

Rearranging, this gives $I_r + I_\phi = k\sqrt{\frac{m}{2|E|}}$, or in other words,

$$H(I) = -\frac{mk^2}{2(I_r + I_\phi)^2}.$$

for the Kepler problem.

We see the corresponding angle variables obey

$$\dot{\phi} = \frac{\partial H}{\partial I_\phi} = \omega_\phi, \quad \dot{\theta} = \frac{\partial H}{\partial I_r} = \omega_r$$

for freqs in the angular and radial directions.

Since this Kepler $H = H(I_r + I_\phi)$, we see $\omega_\phi = \omega_r$. This is the fact that the orbits are elliptical in the Kepler problem.

Adiabatic Invariants

Suppose that the Hamiltonian depends on some parameter $\lambda(t)$ which is very slowly varied $\left(\frac{|\dot\lambda|}{|\lambda|} \ll |\omega|\right)$. A quantity $I(p, q, \lambda)$ is called an adiabatic invariant if for every $\kappa > 0$, $\exists\ \varepsilon_0 > 0$ s.t.

$$\left| I(p(t), q(t); \lambda(t)) - I(p(0), q(0); \lambda(0)) \right| < \kappa.$$

$\forall t \in (0, 1/\varepsilon)$ and all $\varepsilon \in (0, \varepsilon_0)$.

Claim: The action variable $I = \frac{1}{2\pi} \oint p\, dq$ in a 2D phase space is an adiabatic invariant.

To see this, note $\dot{E} = \frac{\partial H}{\partial \lambda} \dot\lambda$ and

$$I = \frac{1}{2\pi} \oint \sqrt{2m\left(E(\lambda(t)) - V(q, \lambda(t))\right)}\, dq.$$

So

$$\dot{I} = \left.\frac{\partial I}{\partial E}\right|_V \dot{E} + \left.\frac{\partial I}{\partial \lambda}\right|_E \dot\lambda$$

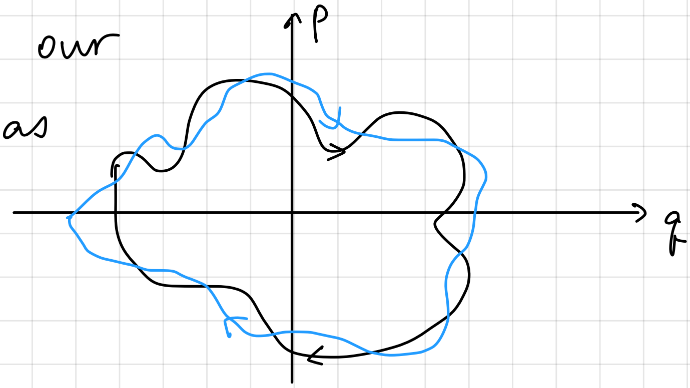
$$= \left(\left.\frac{\partial I}{\partial E}\right|_V \frac{\partial H}{\partial \lambda} + \left.\frac{\partial I}{\partial \lambda}\right|_E \right) \dot\lambda.$$

We know that, at fixed parameter λ,

$$\left.\frac{\partial I}{\partial E}\right|_\lambda = \frac{1}{\omega(\lambda)} = \frac{T(\lambda)}{2\pi}$$

for orbit period $T(\lambda)$.

For the remaining term, note our orbit in phase space changes as we vary the parameter λ



We have

$$\left.\frac{\partial I}{\partial \lambda}\right|_E = \frac{1}{2\pi} \left.\frac{\partial}{\partial \lambda}\right|_E \oint p\, dq$$

$$= \frac{1}{2\pi} \oint \left.\frac{\partial p}{\partial \lambda}\right|_E dq$$

$$= \frac{1}{2\pi} \int_0^{T(\lambda)} \left.\frac{\partial p}{\partial \lambda}\right|_E \dot{q}\, dt$$

$$= \frac{1}{2\pi} \int_0^{T(\lambda)} \left.\frac{\partial p}{\partial \lambda}\right|_E \left.\frac{\partial H}{\partial p}\right|_\lambda dt \qquad \text{(by H's eqn)}$$

Also, on our orbit, $H(p, q, \lambda) = E$, so taking $\frac{d}{dE}$ gives

$$0 = \frac{\partial H}{\partial \lambda} + \left.\frac{\partial H}{\partial p}\right|_{\lambda, q, E} \left.\frac{\partial p}{\partial \lambda}\right|_E .$$

where $p = \sqrt{2m(E - V(\lambda, q))}$. Hence,

$$\left.\frac{\partial I}{\partial \lambda}\right|_E = -\frac{1}{2\pi} \int_0^{T(\lambda)} \left.\frac{\partial H}{\partial \lambda}\right|_E dt$$

Combining both terms, we have

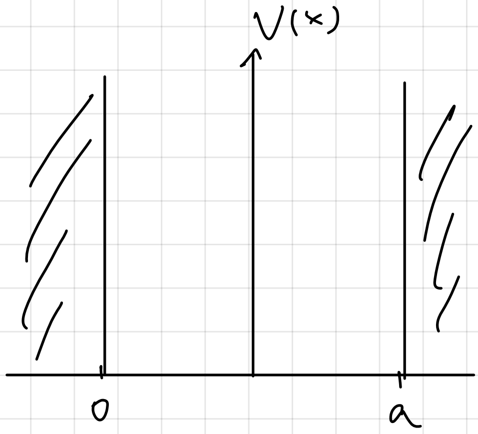
$$\dot{I} = \left(\frac{\partial H}{\partial \lambda} - \frac{1}{T(\lambda)} \int_0^{T(\lambda)} \frac{\partial H}{\partial \lambda} dt \right) \frac{T(\lambda)}{2\pi} \dot{\lambda}$$

$$= \left(\frac{\partial H}{\partial \lambda} - \left\langle \frac{\partial H}{\partial \lambda} \right\rangle \right) \frac{T(\lambda)}{2\pi} \dot{\lambda}$$

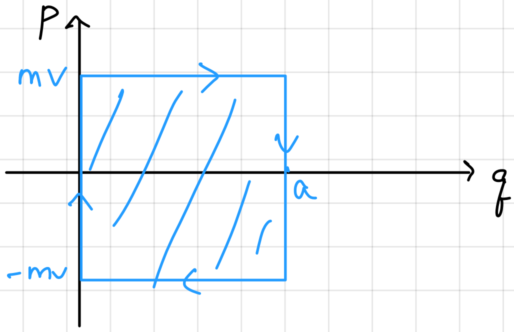
Consequently, so long as λ varies sufficiently slowly that it's approximately const. over a period of oscillation, we'll have $\dot{I} \simeq 0$.

E.g. suppose a particle bounces back and forth between the walls of a rigid box.

$$V(x) = \begin{cases} 0 & 0<x<a \\ \infty & o/w \end{cases}$$



If the collisions with the walls are perfectly elastic, then the phase space orbit's are



$$I = \frac{1}{2\pi} \oint p\, dq = \frac{mva}{\pi}$$

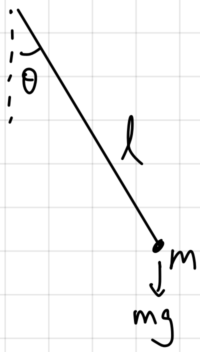
and this quantity will be an adiabatic invariant. In particular, if the separation between the walls is slowly varied, the velocity of the particle must change as

$$v(t) = v(0) \frac{a(0)}{a(t)}$$

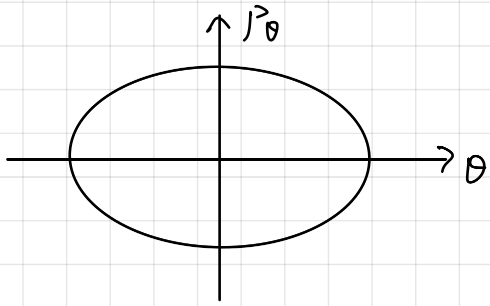
E.g. Suppose we slowly the length l of a simple pendulum.

$$H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

$$\approx \frac{p_\theta^2}{2ml^2} + \frac{mgl}{2}\theta^2$$



$$I = \frac{1}{2\pi} \oint p\, dq = \frac{\pi a b}{2\pi},$$



where $a = \sqrt{\frac{2E}{mgl}}$. $b = \sqrt{2mEl^2}$

$$\Rightarrow \quad I = E\sqrt{\frac{l}{g}} = \frac{E}{\omega} \quad , \text{ where } \quad \omega = \sqrt{\frac{g}{l}}.$$

We can write the energy in terms of the amplitude

$$E = \frac{mgl}{2}\, \theta_{max}^2 .$$

Hence, as length of pendulum is altered, the amplitude will vary as $E/\omega = m g^{1/2} l^{3/2} \theta_{max}^2$ stays const.

$$\Rightarrow \quad \theta_{max}(t) = \theta_{max}(0) \cdot \left(\frac{l(0)}{l(t)} \right)^{3/4}$$

* Field Theory *

A co-vector (or a 1-form) on a space $\mathcal{M}$ is a collection $\omega_a(x)\, dx^a$ where $\omega_a(x)$ for $a=1,\dots,n=\dim(\mathcal{M})$ and $\{dx^a\}$ are a basis of the cotangent space to $\mathcal{M}$ at x.

A vector field on $\mathcal{M}$ is likewise a collection $V^a(x)\frac{\partial}{\partial x^a}$.

Vectors and Covectors are naturally dual by the pairing

$$(V,\omega) \longmapsto V^a(x)\,\omega_a(x) \; : \; \mathcal{M} \to \mathbb{R}.$$

Similarly, a 2-form is a collection $\omega_{ab}(x)\, dx^a \wedge dx^b$, where $dx^a \wedge dx^b = -dx^b \wedge dx^a$, so $\omega_{ab}(x) = \omega_{[ab]}(x)$ and hence consists of $\binom{n}{2}$ indpt. f^{n}s. We can generalise to p-forms

$$\phi_{abc\dots d}(x)\, dx^a \wedge \dots \wedge dx^d$$

consisting of $\binom{n}{p}$ f^{n}s.

The space of p-forms is denoted $\Omega^p(\mathcal{M})$, and is ∞-dim vector space. There's a natural map, called the exterior derivative

$$d : \Omega^p \to \Omega^{p+1}.$$

E.g. For a 1-form $A = A_a(x)\, dx^a$, we have

$$d : A \longmapsto dA = \partial_a A_b \, dx^a \wedge dx^b.$$
$$= \tfrac{1}{2}(\partial_a A_b - \partial_b A_a)\, dx^a \wedge dx^b,$$

E.g. in E-M, $F = dA$ is shorthand for

$$F_{ab} = \partial_a A_b - \partial_b A_a.$$

Importantly, $d^2 = 0$, since

$$d^2\omega = \partial_a \partial_b \omega_{cd\ldots e} \, dx^a \wedge \ldots \wedge dx^e = 0,$$

since $\partial_a \partial_b = \partial_b \partial_a$.

In EM, this is gauge freedom $A \sim A + d\lambda$, which leaves $F = d(A + d\lambda) = dA + d^2\lambda = dA$.

Now let's consider an action $S[x] = \int L(x, \dot{x})\, dt$.

This is a f^n on the space M of curves in "our" space $\mathbb{R}^3$. Let δ be the exterior derivatives on this space of curves. When the e.o.m. hold,

$$\delta S = \int (\text{bulk e.o.m.})\, \delta x \, dt + \left(\frac{\partial L}{\partial \dot{x}}\right) \delta x \Big|_{t_i}^{t_f}$$

$$= \frac{\partial L}{\partial \dot{x}} \delta x \Big|_{t_i}^{t_f} = p \, \delta x \Big|_{t_i}^{t_f}$$

It follows that

$$0 = \delta^2 S = \delta p \wedge \delta x \Big|_{t_i}^{t_f}$$

In other words, $\delta p \wedge \delta x \, (t_i) = \delta p \wedge \delta x \, (t_f)$ so long as e.o.m. hold in between (t_i, t_f). This is

$$\omega = \delta p \wedge \delta x$$

the sympletic form on space of sol^n.

- $\delta\omega = 0$
- $\omega = \frac{1}{2} \omega_{\alpha\beta} \, dy^\alpha \wedge dy^\beta$, $\omega_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

The inverse of ω is the Poisson bracket

$$\omega^{-1} = -\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$$

$$\omega^{-1}(df, dg) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}.$$

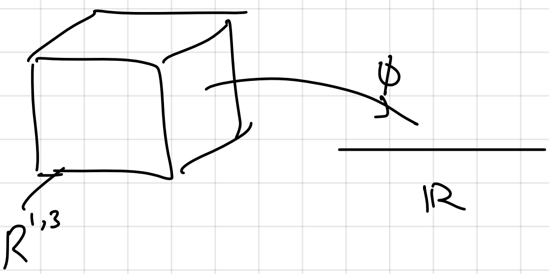
In mechanics, we've considered Lagrangian / actions defined along the 1D worldline of the particle. In field theory, we have further integrals.

world line

mech : $x : I \longrightarrow \mathbb{R}^3$

field : $\phi : \mathbb{R}^{1,3} \longrightarrow \mathbb{R}$

" $A_\mu : \mathbb{R}^{1,3} \longrightarrow \mathbb{R}^4$ "



$$S[\phi] = \int_{\mathbb{R}^{1,3}} \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - V(\phi) \, d^4x$$

So the action is a f^n on the space of all field configurations. The E-L eqns say $\delta S = 0$.

$$\delta S = \int \partial_\mu \delta\phi \, \partial^\mu \phi - \delta\phi \frac{\partial V}{\partial \phi} \, d^4 x$$

$$= \int -\left(\partial^\mu \partial_\mu \phi + \frac{\partial V}{\partial \phi}\right)\delta\phi \, d^4x + \int n_\mu \partial^\mu \phi \, \delta\phi \, d^3x \Big|_{\Sigma_i}^{\Sigma_f}$$

$= (\partial_0 \phi)^2 + (\nabla \phi)^2$

Hence, the e.o.m. (for $\delta\phi|_{\Sigma_{i,f}} = 0$) give

$$\partial^\mu \partial_\mu \phi = -\frac{\partial V}{\partial \phi}$$

at each $x^\mu \in \mathbb{R}^{1,3}$.

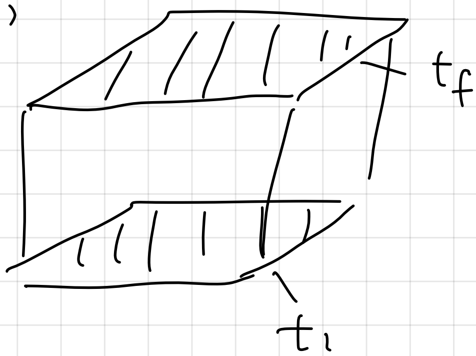
E.g. • $V(\phi) = 0 \Rightarrow \partial^\mu \partial_\mu \phi = 0$ wave eqn.

• $V(\phi) = \frac{1}{2}m^2\phi^2 \Rightarrow \partial^\mu \partial_\mu \phi = -m^2 \phi$ Klein-Gordon eqn.

If we express $\phi(x^\mu) = \int \frac{d^4 k}{(2\pi)^2} e^{ik\cdot x} \tilde{\phi}(k_\mu)$, then these eqns require

$$k^\mu k_\mu = E^2 - \underline{k}\cdot\underline{k} = m^2.$$

The field momentum density is $\pi = \frac{\partial L}{\partial \dot\phi} = \dot\phi$. Again, if our boundaries are slices of constant time, then the boundary term from VP is



$$\int \partial_0 \phi \Big|_{\varepsilon_i}^{\varepsilon_f} d^3x = \int \pi(\underline{x},t)\, \delta\phi(\underline{x},t)\Big|_{t_i}^{t_f} d^3x$$

We get a corresponding symplectic form and Poisson bracket

$$\{\phi(\underline{x},t), \pi(\underline{y},t)\} = \delta^3(\underline{x}-\underline{y})$$

In QFT, we can canonically quantise the system to give

$$[\hat\phi(\underline{x},t), \hat\pi(\underline{y},t)] = i\hbar\, \delta^3(\underline{x}-\underline{y})$$

QM	QFT
$q(t)$	$\phi(\underline{x},t)$
$p(t) = \frac{\partial L}{\partial \dot q}$	$\pi(\underline{x},t) = \frac{\partial L}{\partial \dot\phi}$
$[\hat q, \hat p] = i\hbar$	$[\hat\phi(\underline{x},t), \hat\pi(\underline{x},t)] = i\hbar\, \delta^3(\underline{x}-\underline{y})$
$\psi(q)$	$\Psi[\phi]$.