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Advanced Quantum Field Theory

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Abstract: Lecture Notes to accompany the Part III AQFT in Lent Term 2026. Please send any comments or corrections to the email address above.

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A Note on These Notes

- These notes are intended to supplement the Part III lectures. They are not an alternative to the lectures. In places they go beyond the what is covered in the lectures and may include topics that we do not have time to cover in the lectures. Only those topics that are covered in the lectures are examinable. Sections that will definitely not be examinable are highlighted with an *.

- The lecture notes will be updated from time to time, to correct errors or to fill in gaps. There may be some sections that are 'under construction'.

- Please send any comments or corrections to my email address on the front cover.

A Note on Problem Sheets

- There will be four classes based on problem sheets.

- For the most part, the problem sheets will cover material after that material has been covered in lectures. There are a few places where the problems sheets require you to read ahead of the lectures. You should take this as an opportunity to get ahead and not as a drawback. Indeed, in some places, this is called 'flipped learning' and is considered a virtue[1].

- Whilst it is possible to have gained some understanding of the course material without having done the problems, it is unlikely that you have understood the course material to the required depth unless you can do the problems. The lesson: do the problem sheets. Only then will you know if you have understood the material.

Other Support

- There will be weekly office hours during term time at a time and place tba during the lectures.

- There will be drop-in sessions, hosted by graduate students, where you may ask any questions you have about high energy theory, including AQFT.

- These lectures are recorded. Re-watching a lecture multiple times is probably a waste of time. I make no claim to have found the perfect way to teach this subject and, if something in the lectures appears puzzling, it may be that an alternative perspective may be more beneficial. Try one of the many excellent textbooks out there. We learn by looking at a subject from many different perspectives. And by doing problems (see above). That said, I hope these lectures are informative, challenging and fun.

[1] I tell a lie. This isn't quite what is meant by flipped learning; however, the benefits of reading ahead of lectures are apparent nonetheless.

Spacetime Conventions

Were possible, we shall use the same conventions as Peskin and Schroder.

Natural units $\hbar = c = G = 1$ will be used throughout unless otherwise specified.

We will use the mostly-minus metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

A space-time coordinate will be denoted by

$$x^\mu = (x^0, \mathbf{x}), \qquad x_\mu := \eta_{\mu\nu} x^\nu = (x^0, -\mathbf{x})$$

$$p \cdot x = \eta_{\mu\nu} p^\mu x^\nu = p^0 x^0 - \mathbf{p} \cdot \mathbf{x}$$

For a particle of mass m, the mass-shell condition is

$$p^2 := p_\mu p^\mu = E^2 - |\mathbf{p}|^2 = m^2$$

The partial derivative is written as

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \nabla \right),$$

and

$$\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial_0^2 - \nabla^2.$$

Fourier Transforms

For Fourier transforms, the factors of 2π will appear with the momentum integral

$$f(x) = \int \frac{d^4p}{(2\pi)^4} \, \tilde{f}(p) \, e^{-ip\cdot x}, \qquad \tilde{f}(x) = \int d^4x \, f(x) \, e^{ip\cdot x}$$

The delta function is

$$\int \frac{d^4x}{(2\pi)^4} \, e^{ip\cdot x} = \delta^4(p)$$

The Feynman propagator is

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}.$$

and is a Greens function for the Klein-Gordon equation[2]

$$i(\Box_x + m^2)D_F(x-y) = \delta^4(x-y)$$

Gamma Matrices

The Minkowski spacetime gamma matrices will be taken to be in the Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\lambda\sigma}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\bar{\psi}(x) := \psi^\dagger(x)\gamma^0.$$

Euclidean Space

It will often prove fruitful to Wick rotate (work with an imaginary time τ given by $it = \tau$) from four-dimensional Minkowski spacetime to four-dimensional Euclidean space. In Euclidean space ($\mathbb{R}^4$) we take

$$it = ix^0 = x_4, \qquad \int_{\mathbb{R}^{1,3}} dt d\mathbf{x} = \int_{\mathbb{R}^{1,3}} d^4x = -i\int_{\mathbb{R}^4} d^4x_E,$$

then, if we also define $ip^0 = p_4$, we see

$$p\cdot x = p_0x^0 - \mathbf{p}\cdot\mathbf{x} = -ip_0x_4 - \mathbf{p}\cdot\mathbf{x} = -p_4x_4 - \mathbf{p}\cdot\mathbf{x} := -p_E\cdot x_E$$

and so phases change sign.

The action changes as

$$S[\phi] = \int dtd^3\mathbf{x}\Big(\dot{\phi}^2 - \nabla\phi\cdot\nabla\phi - V(\phi)\Big) = -i\int d^4\mathbf{x}\left(-\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \nabla\phi\cdot\nabla\phi - V(\phi)\right) := iS_E[\phi],$$

where

$$S_E[\phi] = \int d^4\mathbf{x}\left(\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \nabla\phi\cdot\nabla\phi + V(\phi)\right)$$

Notice that the potential changes relative to the 'kinetic' term. The equation of motion is then $\Delta\phi - V'(\phi) = 0$ where the Laplacian is given by

$$\Delta := \frac{\partial^2}{\partial\tau^2} + \nabla^2.$$

[2]NB: Many authors (including Weinberg), who agree with our other conventions but define the propagator as

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m^2 + i\epsilon}e^{-ip\cdot(x-y)}.$$

as a Greens function for the Klein-Gordon equation

$$(\Box_x - m^2)D_F(x-y) = -\delta^4(x-y).$$

Functional integrals in Minkowski space become

$$\int \mathcal{D}\phi \; e^{\frac{i}{\hbar} S[\phi]} = \int \mathcal{D}\phi \; e^{-\frac{1}{\hbar} S_E[\phi]}$$

The Euclidean path integral will usually have much better convergence properties than its Minkowski space counterpart.

1 Introduction

The aim of this course is to introduce you to:

- Path integral methods of quantization and calculation
- Renormalization
- Non-abelian gauge theory

Things to keep in mind

At the end of this course you will know **more** about QFT (I hope). You will not know everything. There is a huge list of topics that are not covered here. But since QFT, as a language of nature, is ubiquitous, some of these topics will be covered in other Part III courses (e.g. OPE in string theory).

People often complain QFT is not rigorous. Many of the topics we discuss will have what many consider to be a low level of mathematical rigour. For example, we will frequently write down and manipulate integrals that don't exist. There are a (at least) three views to take on this

- Some of these ideas may provide intuition that leads us to write down statements that can then be made rigorous. The path integral is a good example of this; as noted above some of the integrals we will write down do not exist; however, few would argue that there are many examples of physical ideas that are difficult to understand in any approach other than the path integral. Perhaps the example from history to have in mind is the state of Calculus before Analysis - it was clear the formalism was correct and powerful but one had to wait until the likes of Cauchy, Riemann and Weierstrass for a rigorous understanding.
- These are effective theories. At the very least, spacetime is probably not a smooth manifold when quantum gravitational effects become important so Quantum Field Theories are only expected to hold up to some scale (maybe the Planck length or string scale?). This is a theme that will become very apparent when we study renormalization and owes much to the pioneering work of Ken Wilson and others.
- Some of the issues are well-understood and we know enough to see that the lack of rigour will not cause problems and allow us to streamline arguments (and generally see the wood for the trees). This is another way of saying that there are some situations where we feel we understand what is going on well enough to be very confident that a lack of rigour will not lead us astray (perhaps because we have other more rigorous methods at our disposal to check, as with BRST quantization). An early example of this we shall see is in perturbation theory where we consider issues of asymptotic Vs absolute convergence of a series in perturbation theory.

A good philosophy to approach is that, QFT is a work in progress. It is also the language in which some of the most successful theories in science are written (QED). We clearly don't understand QFT as well as we'd like to but we are clearly on the path to something worthwhile/useful/profound.

This pioneering spirit was nicely summed up by the mathematician Edward J. McShane (1904-1989):

"There are in this world optimists who feel that any symbol that starts off with an integral sign must necessarily denote something that will have every property that they should like an integral to possess. This is of course quite annoying to us rigorous mathematicians; what is even more annoying is that by doing so they often come up with the right answer."

In this course, we shall be optimists.

2 The Path Integral and Quantum Field Theory in One Dimension

To the best of our knowledge, the world is described by quantum theories of one sort or another and classical physics emerges in some (often difficult to make precise) limit. In practice however, we do things the other way round, with a quantum theory constructed from some well-motivated classical theory by the magic of "quantization". There are a number of ways of constructing a quantum theory from a classical one. In this course we will motivate, study and make use of the path integral approach to quantization. Much time and effort will be spent trying to learn something from path integrals that we cannot evaluate explicitly. In many ways an alternative title for the course could be

"How to write Quantum Physics in terms of integrals and how to still learn something even when you cannot evaluate those integrals"

But this title is not so catchy, so we'll stick with *"Advanced Quantum Field Theory"* - not quite so on-the-nose but a better guarantor of a good turnout in the lecture theatre.

We will start with one-dimensional QFT. What we mean by dimension is the as follows: If the action can be written in terms of a (local) Lagrangian denisty

$$S[\phi] = \int d^D x \, \mathcal{L}(\phi),$$

the theory is said to be a D-dimensional quantum field theory. Broadly speaking, this is the number of parameters the objects ϕ are functions of. In this sense, Quantum Mechanics is a $D = 1$ dimensional QFT, even though it may describe physics in three-dimensional space, plus time. Similarly, the worldsheet theory of string theory is a 2-dimensional QFT, even though the target space may be 10 or 26 dimensional[3].

2.1 The Path Integral

As we shall see, the zero-dimensional case is the simplest to discuss but we shall start with one-dimensional quantum field theory as this turns out to be familiar quantum mechanics. Once we see how to write quantum mechanics in terms of path integrals we will refine our techniques in the simplest zero-dimensional case before tackling our real subject of interest; four-dimensional quantum field theory.

We start with the Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \widehat{H} |\psi(t)\rangle,$$

For simplicity, we will take the Hamiltonian to be time-translation invariant and of the form[4]

$$\widehat{H} = \frac{\widehat{P}^2}{2m} + V(\widehat{X}),$$

[3]A key point here is that, even though the worldsheet physics is described by a QFT, the target space physics in certainly not that of a QFT (modulo statements of holography).

[4]We will usually put hats on operators.

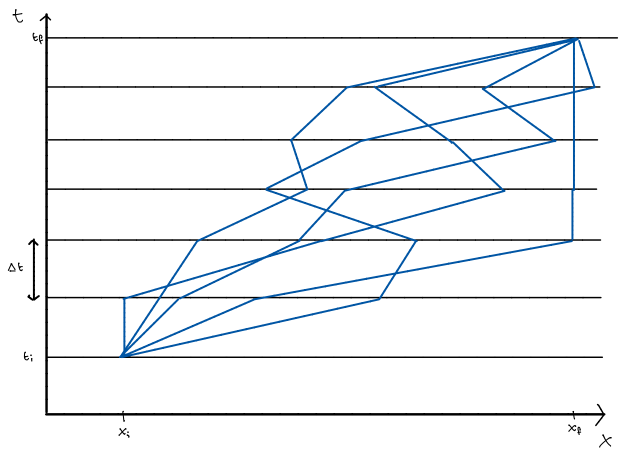


Figure 1. The path integral gives a weighted sum of all possible trajectories between the initial and final events.

but it is straightforward to generalise to more general Hamiltonions. The solution is

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\widehat{H}(t-t')}|\psi(t')\rangle.$$

This implicitly assumes $t > t'$. In terms of time-independent position eigenstates[5] $|x\rangle$, the position space wavefunction is

$$\psi(x,t) = \langle x|\psi(t)\rangle,$$

so that

$$\psi(x,t) = \langle x|e^{-\frac{i}{\hbar}\widehat{H}(t-t')}|\psi(t')\rangle.$$

Inserting a complete basis of position states

$$I = \int dx'\, |x'\rangle\langle x'|,$$

gives

$$\begin{aligned} \psi(x,t) &= \int dx'\, \langle x|e^{-\frac{i}{\hbar}\widehat{H}(t-t')}|x'\rangle\langle x'|\psi(t')\rangle \\ &= \int dx'\, U(x,t:x',t')\psi(x',t'), \end{aligned} \tag{2.1}$$

which we can view as an evolution or transition equation relating the wavefunction at one time with the wavefunction at a later time. The transition amplitude which appears as a Kernel is

$$U(x,t:x',t') := \langle x|e^{-\frac{i}{\hbar}\widehat{H}(t-t')}|x'\rangle.$$

[5] Which satisfy $\widehat{X}|x\rangle = x|x\rangle$.

For some initial and final states of some process we could then write in an obvious notation

$$\psi(x_f, t_f) = \int dx_i \, U(x_f, t_f : x_i, t_i)\psi(x_i, t_i).$$

This is an integral version of the Schrödinger equation[6] and we can see that the Kernel $U(x_f, t_f : x_i', t_i')$ contains the physics of the process that evolves the wavefunction from the initial to final state. If we want to understand the physical process involved, it is $U(x_f, t_f : x_i', t_i')$ that we need to study.

From transition amplitudes to path integrals

The transition amplitude clearly plays a central role in the dynamics of the quantum system. Our task now is to find a different way to think about the transition amplitude $U(x_f, t_f : x_i', t_i')$. We can introduce N discrete time slices and break this time evolution up into steps

$$e^{-\frac{i}{\hbar}\widehat{H}(t_f - t_i)} = e^{-\frac{i}{\hbar}\widehat{H}(t_f - t_N)} e^{-\frac{i}{\hbar}\widehat{H}(t_N - t_{N-1})} ... e^{-\frac{i}{\hbar}\widehat{H}(t_2 - t_1)} e^{-\frac{i}{\hbar}\widehat{H}(t_1 - t_i)},$$

where we will often identify $t_i := t_0$ and $t_f := t_{N+1}$. We now insert a complete basis of states into each of these steps so that we can get a spatial description of what is going on at the intermediate steps.

$$U(x_f, t_f : x_i', t_i') = \int dx_1 ... dx_N \prod_{r=0}^{N} \langle x_{r+1}| \exp\left(-\frac{i}{\hbar}\widehat{H}(t_{r+1} - t_r)\right)|x_r\rangle,$$

where $x_r := x(t_r)$. Note that we only integrate over the unconstrained positions x_r for $r = 1, ..., N$. The initial and final positions x_0 and x_{N+1} are fixed boundary conditions. Let

$$x_0 = x_i = x(t_i), \qquad x_{N+1} = x_f = x(t_f), \qquad \delta t := t_{r+1} - t_r$$

be independent of r and consider one term in the product

$$U_{r+1,r} := \langle x_{r+1}| \exp\left(-\frac{i}{\hbar}\widehat{H}(t_{r+1} - t_r)\right)|x_r\rangle.$$

Now the kinetic and potential terms in $\widehat{H}$ don't commute in general so we have to think carefully about this exponential. A useful result is:

[6] Since $t > t'$, we can write

$$U(t, t') = \Theta(t - t')e^{-\frac{i}{\hbar}H(t-t')}.$$

This satisfies the Greens function for the time-*dependent* Schrödinger equation

$$\left(i\hbar\frac{\partial}{\partial t_1} - H\right)U(t_1, t_2) = i\hbar\delta(t_1 - t_2),$$

and so $U(t, t')$ can be thought of as a Greens-function, or propagator.

Suzuki-Trotter Decomposition

In general

$$e^{A+B} \neq e^A e^B.$$

Indeed, using the Baker-Campbell-Hausdorf formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}.$$

For small ϵ,

$$e^{\epsilon A}e^{\epsilon B} = e^{\epsilon(A+B)+\frac{\epsilon^2}{2}[A,B]+\dots} = e^{\epsilon(A+B)}(1+\mathcal{O}(\epsilon)).$$

Turning this around

$$e^{\epsilon(A+B)} = e^{\epsilon A}e^{\epsilon B}(1+\mathcal{O}(\epsilon)).$$

Let $\epsilon = 1/n$ for some large n. Raising both sides to the power n and taking the large n limit gives Trotter's theorem

$$e^{A+B} = \lim_{n\to\infty}\left(e^{A/n}e^{B/n}\right)^n.$$

In the context we are using it in, we assume δt is small and take

$$\langle x_{r+1}|e^{-\frac{i}{\hbar}\widehat{H}\delta t}|x_r\rangle = \langle x_{r+1}|e^{-\frac{i}{\hbar}\frac{\widehat{P}^2}{2m}\delta t}\, e^{-\frac{i}{\hbar}V(\widehat{X})\delta t}|x_r\rangle$$

to leading order in δt. In this context, this is called the Trotter-Suzuki decomposition. Note that, since $\widehat{X}$ commutes with itself, we can construct arbitrary functions $V(\widehat{X})$ without worrying too much. When it comes to interpreting such functions, we shall usually have in mind some formal power series of the operator.

An aside: At the moment, assuming it is meaningful to take $\delta t \to 0$ seems reasonable and it is very easy to get used to using the path integral formalism without thinking about where it came from. However, if we were attempting to construct a quantum theory of gravity then the assumption that $\delta t \to 0$ is meaningful begins to feel less certain. In fact, there are quite a few times we shall implicitly rely on a continuum structure of spacetime to think about quantum field theory. The construction of quantum theories without a fundamental spacetime continuum, causal structure etc is an interesting ongoing problem.

Back to the calculation at hand. $V(\widehat{X})$ has eigenstate $|x\rangle$, hence $V(\widehat{X})|x_r\rangle = V(x_r)|x_r\rangle$, and so the potential term may be factored out to give

$$U_{r+1,r} = \langle x_{r+1}|e^{-\frac{i}{\hbar}\widehat{H}\delta t}|x_r\rangle = \langle x_{r+1}|e^{-\frac{i}{\hbar}\frac{\widehat{P}^2}{2m}\delta t}|x_r\rangle\, e^{-\frac{i}{\hbar}V(x_r)\delta t}$$

The potential contributes some phase which feels like something we should be able to understand. As such, we shall forget about the potential term for now and focus on the kinetic term. We insert a complete basis of momentum states so that

$$\langle x_{r+1}|e^{-\frac{i}{\hbar}\frac{\widehat{P}^2}{2m}\delta t}|x_r\rangle = \int\frac{dp}{2\pi\hbar}\langle x_{r+1}|e^{-\frac{i}{\hbar}\frac{\widehat{P}^2}{2m}\delta t}|p\rangle\langle p|x_r\rangle.$$

Using the fact that the overlap is just the momentum wavefunction in position space

$$\langle p|x\rangle = e^{-\frac{i}{\hbar}p\cdot x},$$

gives

$$\begin{aligned}
\langle x_{r+1}|e^{-\frac{i}{\hbar}\frac{\hat{P}^2}{2m}\delta t}|x_r\rangle &= \int \frac{dp}{2\pi\hbar}\langle x_{r+1}|e^{-\frac{i}{\hbar}\frac{p^2}{2m}\delta t}|p\rangle\, e^{-\frac{i}{\hbar}p\cdot x_r} \\
&= \int \frac{dp}{2\pi\hbar}\langle x_{r+1}|p\rangle\, e^{-\frac{i}{\hbar}\frac{p^2}{2m}\delta t}\, e^{-\frac{i}{\hbar}p\cdot x_r} \\
&= \int \frac{dp}{2\pi\hbar} e^{-\frac{i}{\hbar}\frac{p^2}{2m}\delta t}\, e^{\frac{i}{\hbar}p\cdot(x_{r+1}-x_r)} \\
&= \int \frac{dp}{2\pi\hbar}\exp\left(-\frac{i}{\hbar}\left[\frac{p^2}{2m}\delta t - p\cdot(x_{r+1}-x_r)\right]\right)
\end{aligned} \tag{2.2}$$

we then complete the square in the exponent

$$\frac{\delta t}{2m}\left(p^2 - \frac{2m}{\delta t}p\cdot(x_{r+1}-x_r)\right) = \frac{\delta t}{2m}\left(p - \frac{m}{\delta t}(x_{r+1}-x_r)\right)^2 - \frac{m}{2}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2\delta t$$

and integrate over $p - \frac{m}{\delta t}(x_{r+1}-x_r)$, to give (up to a phase which we do not care about[7])

$$\langle x_{r+1}|e^{-\frac{i}{\hbar}\frac{\hat{P}^2}{2m}\delta t}|x_r\rangle = \sqrt{\frac{m}{2\pi i\hbar\delta t}}\exp\left(\frac{i}{\hbar}\frac{m}{2}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2\delta t\right)$$

Putting all of this together, we have

$$\langle x_{r+1}|e^{-\frac{i}{\hbar}\hat{H}\delta t}|x_r\rangle = \sqrt{\frac{m}{2\pi i\hbar\delta t}}\exp\left(\frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2 - V(x_r)\right]\delta t\right)$$

Thus, if we take the limit $N\to\infty$ and $\delta t\to 0$ such that $T := N\delta t = t_f - t_i$ is held fixed, we arrive at

$$U(x_f,t_f : x_i',t_i') = \lim_{\substack{\delta t\to 0\\ N\to\infty}}\int\prod_{r=1}^{N}dx_r,\left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{N+1}{2}}\exp\left(\frac{i}{\hbar}\sum_{r=0}^{N}\left[\frac{m}{2}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2 - V(x_r)\right]\delta t\right).$$

This is interesting as we seem to have written the transition amplitude in terms of functions, rather than operators. In this limit, the exponent becomes the classical action

$$\frac{i}{\hbar}\sum_{r=0}^{N}\left[\frac{m}{2}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2 - V(x_r)\right]\delta t \to \frac{i}{\hbar}\int_{t_i}^{t_f}dt\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \frac{i}{\hbar}S[x].$$

Already this is a remarkable result. We are beginning to see, in a precise way, the role of the classical action in describing the dynamics of a quantum system. We can write our final result as follows

[7] See Osborn's notes for details.

The Transition Function as a Path Integral

The transition function may be written as

$$\langle x_f | e^{-\frac{i}{\hbar}H(t_f - t_i)} | x_i \rangle = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x \, e^{\frac{i}{\hbar}S[x]},$$

where we have defined the 'measure'

$$\mathcal{D}x := \lim_{\substack{\delta t \to 0 \\ N \to \infty}} \sqrt{\frac{m}{2\pi i \hbar \delta t}} \prod_{r=1}^{N} \left(\sqrt{\frac{m}{2\pi i \hbar \delta t}} dx_r \right).$$

We often drop the limits of the integral, where they are clear from the context. The action is

$$S[x] = \int_{t_i}^{t_f} dt \, L(x, \dot{x}).$$

Actually, what we should call a measure is something closer to the combination $\mathcal{D}x \; e^{\frac{i}{\hbar}S[x]}$. We will not really care about the normalization.

The path integral gives a new interpretation to quantum mechanics. The probability amplitude for a particle to be found at two distinct spacetime points is given by a sum over all paths connecting these two points. The sum is weighted by the classical action, which provides the physics input. In Lagrangian mechanics the configuration space we care about is the physical space $\mathbb{R}^3$. This undergoes a slight augmentation in Hamiltonian mechanics, where we work in phase space. In quantum mechanics the arena is far greater; we are working on the space of all paths.

Some further comments:

- There are no operators! All of the objects in the path integral are (commuting) functions.
- The limit of this integral probably doesn't exist in most cases. However, we shall see that in some select cases the prescription does make sense and gives the right answer. Crucially, it works unambiguously for the free particle and harmonic oscillator. Since these cases are the starting points for perturbation theory in QFT, this is a non-trivial result.
- It is generally very hard to solve, even simple quantum mechanical systems in this formalism but it is very useful for QFT.

This provides an example of an important lesson that we shall see appear in a number of different guises throughout the course. Given a classical theory with action $S[\phi]$, we do not generally have a well-defined quantum field theory simply by using this action in the path integral. A naive path integral will contain divergences and in general will not exist. The definition of a quantum field theory thus also requires a prescription to regularise the divergences. We shall see this again when we consider renormalization in quantum field theory but already we see that the divergences in the quantum mechanical path integral can be naturally regularized (rendered finite) by imposing a finite minimal time interval δt. Such regularizing features seem un-physical but, if the physics of the theory does not depend on the scale chosen then the theory is well-defined.

In some sense, one could take such lattice regularizations as a way of defining quantum field theory.

The Classical Limit and Imaginary Time

Convergence properties of the path intergral may be improved by the replacement

$$t \to \tau = it.$$

The path integral for the propagator becomes

$$\int_i^f \mathcal{D}x\, e^{-\frac{1}{\hbar}S_E[x]}, \qquad S_E[x] = \int d\tau \left[\left(\frac{dx}{d\tau} \right)^2 + V(x) \right]$$

where we note that the action is required to be real. The notation $S_E[x]$ is short for Euclidean action and denotes the usual action with t replaced by τ. Why we call this Euclidean will become clear when we look at the relativistic theory (QFT). We turn to some observations

- Written in this way, we see that, in the limit $\hbar \to 0$, the integral is dominated by the extrema of the action. This is a hint that Hamilton's principle of least action has its origins in the classical limit of quantum theory.
- If we work with real time, the phase oscillates wildly as $\hbar \to 0$. We still believe the least action principle arises in this limit. The intuition is that other phases cancel each other out (like wave superposition), leaving only the zero phase $S[x] = 0$ configuration.
- The action plays an important and perhaps unexpected role. The measure

  $$\mathcal{D}x := \lim_{\substack{\delta\tau \to 0 \\ N \to \infty}} \sqrt{\frac{m}{2\pi\hbar\delta\tau}} \prod_{r=1}^{N} \left(\sqrt{\frac{m}{2\pi\hbar\delta\tau}} dx_r \right).$$

  is still not well defined in the given limits; however, the combination

  $$\begin{aligned} &\lim_{\substack{\delta\tau \to 0 \\ N \to \infty}} \left(\sqrt{\frac{m}{2\pi\hbar\delta\tau}} \right)^{N+1} \prod_{r=1}^{N} dx_r \, \exp\left(-\frac{1}{\hbar} \sum_{r=0}^{N} \left[\frac{m}{2} \left(\frac{x_{r+1} - x_r}{\delta\tau} \right)^2 + V(x_r) \right] \delta\tau \right) \\ &= \mathcal{D}x\, e^{-S_E[x]/\hbar}, \end{aligned} \tag{2.3}$$

  is a well-defined measure (for suitable $V(x)$).

Notice that the Lagrangian has been replaced by the Hamiltonian density

$$\lim_{\delta\tau \to 0} \left[\frac{m}{2} \left(\frac{x_{r+1} - x_r}{\delta\tau} \right)^2 + V(x_r) \right] = \mathcal{H}$$

and so the path integral begins to resemble a Gibbs distribution from statistical field theory $\rho = e^{-\frac{H}{k_B T}}$. This analogy can be made more precise and is a useful tool to understand QFT at finite temperature in thermal equilibrium[8]

Example I: The Free Particle

Let us consider a simple (the simplest) example of a free particle of mass m with action

$$S[x] = \frac{1}{2}m\int_{t_i}^{t_f} dt\,\dot{x}^2.$$

In this case, the Schrodinger equation is like a heat equation with an imaginary diffusion coefficient

$$\frac{\partial\psi}{\partial t} = D\frac{\partial^2\psi}{\partial x^2}, \qquad D = -\frac{i\hbar}{2m}.$$

Wick rotating to imaginary time, we find a heat equation with real diffusion coefficient. It is a simple adaptation of what you know from finding the Green's function for the heat equation to show that

$$U(x_f, t_f : x_i, t_i) = \left(\frac{m}{2\pi i\hbar(t_f - t_i)}\right)^{\frac{1}{2}} \exp\left(\frac{i}{\hbar}\frac{m(x_f - x_i)^2}{2(t_f - t_i)}\right).$$

In the Euclidean case, this solution is a normalised probability density function with standard deviation $\sigma \propto \hbar$. In the limit $\hbar \to 0$, the distribution tends to a delta function. We can also recover this result from the path integral directly. In this case we have

$$U(x_f, t_f : x_i, t_i) = \lim_{\substack{\delta t\to 0\\ N\to\infty}} \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{N}{2}} \int dx_1 ... dx_{N-1} \exp\left(\frac{i\delta t}{\hbar}\sum_{n=1}^{N}\frac{m}{2}\left(\frac{x_n - x_{n-1}}{\delta t}\right)^2\right),$$

where

$$\delta t = \frac{t_f - t_i}{N}.$$

Changing variables to

$$y_n := \sqrt{\frac{m}{2\delta t\hbar}}x_n,$$

gives

$$U(x_f, t_f : x_i, t_i) = \lim_{\substack{\delta t\to 0\\ N\to\infty}} \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{N}{2}} \left(\frac{2\hbar\delta t}{m}\right)^{\frac{N-1}{2}} \int dy_1 ... dy_{N-1} \exp\left(i\sum_{n=1}^{N}(y_n - y_{n-1})^2\right).$$

Consider the first integral, involving y_1. By completing the square in the exponent we find that

$$\begin{aligned}\int dy_1\, e^{i[(y_1-y_0)^2+(y_2-y_1)^2]} &= \int dy_1 e^{2i\left(y_1 - \frac{1}{2}(y_0+y_2)\right)^2 + \frac{i}{2}(y_2-y_0)^2} \\ &= \left(\frac{i\pi}{2}\right)^{\frac{1}{2}} e^{\frac{i}{2}(y_2-y_0)^2}.\end{aligned} \tag{2.4}$$

[8]More sophisticated constructions are needed if we also want to include time-dependence.

Similarly one can show that

$$\int dy_1 dy_2\, e^{i[(y_1-y_0)^2+(y_2-y_1)^2+(y_3-y_2)^2]} = \left(\frac{(i\pi)^2}{3}\right)^{\frac{1}{2}} e^{\frac{i}{3}(y_3-y_0)^2}$$

Proceeding in this manner we find,

$$\begin{aligned} U(x_f,t_f:x_i,t_i) &= \lim_{\substack{\delta t\to 0\\ N\to\infty}} \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{N}{2}} \left(\frac{2\hbar\delta t}{m}\right)^{\frac{N-1}{2}} \left(\frac{(i\pi)^{N-1}}{N}\right)^{\frac{1}{2}} e^{\frac{i}{N}(y_N-y_0)^2} \\ &= \lim_{\substack{\delta t\to 0\\ N\to\infty}} \left(\frac{m}{2\pi i\hbar N\delta t}\right)^{\frac{1}{2}} e^{\frac{im}{2\pi i\hbar N\delta t}(x_f-x_i)^2} \\ &= \left(\frac{m}{2\pi i\hbar(t_f-t_i)}\right)^{\frac{1}{2}} \exp\left(\frac{i}{\hbar}\frac{m(x_f-x_i)^2}{2(t_f-t_i)}\right), \end{aligned} \tag{2.5}$$

as argued above. It is reassuring that the path integral gives the expected Greens function in this case.

Relationship to the Classical Action

Note that the classical solution $x_{\rm cl}$ is given by the stationary points of the action

$$0 = \frac{\delta S[x]}{\delta x(t)} = m\ddot{x} \qquad \Longrightarrow \dot{x}_{\rm cl} := v = \text{ constant.}$$

Evaluating the action on the classical solution gives

$$S[x_{\rm cl}] := \frac{1}{2}m\int_{t_i}^{t_f} dt\, \dot{x}_{\rm cl}^2 = \frac{1}{2}mv^2(t_f-t_i).$$

However we also know that, since v, is a constant

$$v = \frac{x_f-x_i}{t_f-t_i},$$

giving

$$S[x_{\rm cl}] = \frac{m}{2}\frac{(x_f-x_i)^2}{t_f-t_i}.$$

The transition amplitude may then be written as

$$U(x_f,t_f:x_i,t_i) = \left(\frac{m}{2\pi i\hbar(t_f-t_i)}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}S[x_{\rm cl}]}.$$

This form of the path integral, in terms of the classical action is common to many exactly-solvable models.

2.2 Functional Calculus

In this section we give the briefest of overviews of functional calculus. The introduction of the functional integral $\int \mathcal{D}x$ over the space of paths $x(t)$ motivates us to consider a generalisation of calculus to such infinite-dimensional spaces.

A function is a map from a number to a number; if $t \in \mathbb{R}$, then $x(t)$ is a map $x : \mathbb{R} \to \mathbb{R}$. By contrast, the action $S[x]$ takes a function $x(t)$ and maps it to a number. $S[x]$ is clearly not a function; it is a *functional*. Similar to differentiation of functions

$$\frac{dx(t)}{dt} = \lim_{\epsilon \to 0} \frac{x(t+\epsilon) - x(t)}{\epsilon},$$

we can consider how a functional changes as we change the function it is evaluated on. One way of doing this is as follows: Given a path $x(t)$ with endpoints $x_i = x(t_i)$ and $x_f = x(t_f)$, we can introduce an arbitrary variation as

$$x(t) \to x(t) + \epsilon\eta(t),$$

where $\eta(t)$ is an arbitrary (differentiable) function such that $\eta(t_i) = 0 = \eta(t_f)$. A sensible first guess at a derivative of the functional $S[x]$ would then be[9]

$$\frac{d}{dx(t)} S[x] = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} S[x + \epsilon\eta].$$

We generalise this slightly cumbersome approach by first defining the variation of the Lagrangian

$$\boxed{\frac{\delta L(x(t'))}{\delta x(t)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big\{ L\Big(x(t') + \epsilon\delta(t'-t)\Big) - L\Big(x(t')\Big) \Big\}.}$$

this measures the change in the Lagrangian when the function $x(t)$ is changed at a particular point in time.

To see this makes heuristic sense, consider the function $L = x^2(t)$. The functional variation is

$$\begin{aligned} \delta L(x(t')) &= \Big(x(t') - \epsilon\delta(t'-t)\Big)^2 - \Big(x(t)\Big)^2 \\ &= 2\epsilon x(t')\delta(t'-t) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.6}$$

and so the functional derivative is

$$\frac{\delta L(x(t'))}{\delta x(t)} = 2x(t')\delta(t'-t).$$

We see that the functional derivative gives what we expect from a derivative (the $2x(t)$ factor) as well as a delta function which has support where $t' = t$. This is natural when we consider functionals constructed as integrals of local functions. Since Quantum Field

[9]This is what you might have done in an introductory course on the Calculus of Variations.

Theory is local[10], all of the functionals we shall consider in this course will be of this kind. Of course, there is no reason why all functionals should be of this kind but our definition of the functional derivative given here, though limited, will be sufficient for our purposes.

From the functional variation of the Lagrangian we can now define the *functional derivative* of the action as the sum of all such changes at all such times (i.e. an integral over t)

$$\frac{\delta S[x]}{\delta x(t)} = \int dt' \frac{\delta L(x(t'))}{\delta x(t)}.$$

For fixed t, $x(t)$ is just a number and we can think of $L(x)$ as just a function of x. This is particularly clear in the time-slicing construction of the path integral, where we where we could focus on a fixed time and then treat $L(x_r)$ as a function and differentiate with respect to x_r as a standard variable. This reasoning leads us to the proposal

Functional Derivative

The derivative of the functional $S[x] = \int dt L(x(t))$ with respect to the function $x(t)$, with parameter t is

$$\frac{\delta S[x]}{\delta x(t)} = \lim_{\epsilon \to 0} \int dt' \frac{L\Big(x(t') + \epsilon\delta(t'-t)\Big) - L\Big(x(t')\Big)}{\epsilon}.$$

We see that the delta function ensures we are only varying $L(x)$ at a particular t; otherwise, the derivative takes the expected form. We will assume the integration and the limit commute. Heuristically, we might write this as

$$\frac{\delta S[x]}{\delta x(t)} = \lim_{\epsilon \to 0} \frac{S[x+\epsilon] - S[x]}{\epsilon}.$$

The derivative satisfies the properties you would hope a derivative satisfies (linear, distributive, associative). For the example above of $L = x^2(t')$, we see

$$S[x] = \int dt'\, x^2(t'), \qquad \frac{\delta S[x]}{\delta x(t)} = 2x(t).$$

The relationship between our functional derivative and the naive first guess is

$$\delta x(t) \frac{d}{dx(t)} S[x] = \eta(t) \times \lim_{\epsilon \to 0} \frac{d}{d\epsilon} S[x + \epsilon\eta] = \int dt'\, \frac{\delta L[x]}{\delta x(t)} \eta(t'),$$

and in many practical applications this latter relationship is the simplest approach to take.

The functional derivative of a function gives a very useful result

[10]There are lots of caveats which should accompany this statement of locality. The Reeh-Schlieder theorem is a useful reminder of the subtleties involved - look it up!

$$\boxed{\frac{\delta\phi(x)}{\delta\phi(y)} = \delta^D(x-y).}$$

Applying this to the functionals, in conjunction with other rules for derivatives (Linearity, Leibnitz), it is then easy to see determine most needed results. For example, if ϕ and J are independent fields[11]

$$\frac{\delta}{\delta J(y)}\int d^4x\,\phi(x)J(x) = \int d^4x\,\phi(x)\frac{\delta J(x)}{\delta J(y)} = \int d^4x\,\phi(x)\delta^4(x-y) = \phi(y).$$

Because delta-functions are best understood under an integral, most applications of functional differentiation will be on functionals (where, in our considerations, an integral is always present).

Another example is the variation of the Klein-Gordon action

$$\begin{aligned}\frac{\delta}{\delta\phi(x)}S[\phi] &= \frac{\delta}{\delta\phi(x)}\int d^4y\left(\frac{1}{2}\partial_\mu\phi(y)\partial^\mu\phi(y) - \frac{1}{2}m^2\phi(y)^2\right)\\ &= \int d^4y\Big(-\partial^\mu\partial_\mu\phi(y) - m^2\phi(y)\Big)\frac{\delta\phi(y)}{\delta\phi(x)}\\ &= -(\Box + m^2)\phi(x),\end{aligned} \tag{2.7}$$

in agreement with the Euler-Lagrange equations.

Whilst it is good to understand the first principles definition of our functional integral, we will rarely need to resort to it. Just as with the calculus of functions, one quickly gets a feel for the general idea and applies the rules without a second thought. Happily the rules for functional differentiation are directly analogous to the rules you are already familiar with so there is not much new to learn. Indeed, we shall spend a lot of time understanding quantum field theory is zero dimensions, where the action is a function and the functional derivatives reduce to simple partial derivatives. The lessons we learn in zero dimensions will carry over to general dimension without obstruction. New lessons will have to be learned in going from zero dimensions to two or more dimensions but they will not arise from functional calculus.

Example II: The Harmonic Oscillator

The harmonic oscillator is the most important problem in Quantum Mechanics. It would be hasty to progress further without at least reconsidering this model from the path integral perspective. The action for the harmonic oscillator is

$$S[x] = \int dt\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2\right).$$

It will be helpful to expand about the classical solution. Let us write $x(t) = x_{\rm cl}(t) + \eta(t)$, where $x_{\rm cl}(t)$ is the classical solution to the Euler-Lagrange equations of motion and $\eta(t)$ is

[11] Before any classical equations of motion are applied. Remember, these are quantum fields, not constrained by the classical equations of motion.

an arbitrary differentiable function that vanishes at the endpoints; i.e. $\eta(t_i) = 0 = \eta(t_f)$. One can think of $\eta(t)$ as describing quantum fluctuations about a classical solution. We will assume a functional analogue of the Taylor expansion such that

$$\begin{aligned} S[x] &= S[x_{\text{cl}} + \eta] \\ &= S[x_{\text{cl}}] + \int dt_1\, \eta(t_1)\, \frac{\delta S[x]}{\delta x(t_1)}\bigg|_{x=x_{\text{cl}}} + \frac{1}{2!}\int dt_1 dt_2\, \eta(t_1)\eta(t_2)\, \frac{\delta^2 S[x]}{\delta x(t_1)\delta x(t_2)}\bigg|_{x=x_{\text{cl}}}. \end{aligned}$$

There are no higher terms since the Lagrangian is quadratic in $x(t)$. By definition of x_{cl}

$$\frac{\delta S[x]}{\delta x(t_1)}\bigg|_{x=x_{\text{cl}}} = 0,$$

so we only need to evaluate the second variation of the action. It is not hard to show that

$$\begin{aligned} \frac{d}{d\epsilon}S[x+\epsilon\eta]\bigg|_{\epsilon=0} &= \int dt\, m\Big(\dot{x}(t)\dot{\eta}(t) - \omega^2 x(t)\eta(t)\Big) \\ &= -\int dt\, m\Big(\ddot{x}(t) + \omega^2 x(t)\Big)\eta(t), \end{aligned} \tag{2.8}$$

and so we can read off

$$S'[x] := \frac{\delta S[x]}{\delta x(t)} = -m\ddot{x}(t) - m\omega^2 x(t).$$

It is useful to write this as

$$S'[x] = -m\int dt'\, \delta(t-t')\Big(\ddot{x}(t') + \omega^2 x(t')\Big).$$

Differentiating again gives

$$\begin{aligned} \frac{d}{d\epsilon'}S'[x+\epsilon'\eta]\bigg|_{\epsilon'=0} &= -m\int dt'\, \delta(t-t')\Big(\ddot{\eta}(t') + \omega^2\eta(t')\Big) \\ &= -m\int dt'\, \delta(t-t')\Big(\frac{d^2}{dt'^2} + \omega^2\Big)\eta(t') \end{aligned} \tag{2.9}$$

and so

$$\frac{\delta^2 S[x]}{\delta x(t_1)\delta x(t_2)} = -m\delta(t_1-t_2)\Big(\frac{d^2}{dt_1^2} + \omega^2\Big),$$

so that we then have

$$\begin{aligned} S[x] &= S[x_{\text{cl}}] - \frac{m}{2!}\int dt_1 dt_2\, \eta(t_1)\delta(t_1-t_2)\Big(\ddot{\eta}(t_2) + \omega^2\eta(t_2)\Big) \\ &= S[x_{\text{cl}}] + \frac{m}{2!}\int dt\, \Big(\dot{\eta}^2(t) - \omega^2\eta(t)^2\Big), \end{aligned} \tag{2.10}$$

where we have integrated by parts to get to the second line. For fixed x_{cl}, the sum over $x(t)$ becomes a sum over $\eta(t)$ so that[12] it is clear $\mathcal{D}x = \mathcal{D}\eta$, giving and expression for the

[12] If there are many x_{cl} satisfying the boundary conditions, then we must sum over them too.

propagator

$$\mathcal{N}\int_{t_i}^{t_f}\mathcal{D}x\, e^{\frac{i}{\hbar}s[x]} = \mathcal{N}\int\mathcal{D}\eta\,\exp\left(\frac{i}{\hbar}S[x_{\rm cl}]+\frac{im}{2\hbar}\int_{t_i}^{t_f}dt\,\left(\dot{\eta}^2(t)-\omega^2\eta(t)^2\right)\right)$$
$$= \mathcal{N}e^{\frac{i}{\hbar}S[x_{\rm cl}]}\int\mathcal{D}\eta\,\exp\left(\frac{im}{2\hbar}\int_{t_i}^{t_f}dt\,\left(\dot{\eta}^2(t)-\omega^2\eta(t)^2\right)\right) \tag{2.11}$$

We can do this integral. To simplify matters, let us take $t_i = 0$ and define $T := t_f - t_i$. It is also useful to use a Fourier representation for $\eta(t)$, so that without loss of generality

$$\eta(t) = \sum_n a_n \sin\left(\frac{n\pi t}{T}\right), \qquad n = 1, 2, 3, ..., N-1$$

which clearly satisfies the boundary conditions for $\eta(t)$. Note that, since the trajectory is dividend into N intervals, the path must be specified by $N-1$ independent coefficients. e will take $N \to \infty$ a the end so this is something of a formality. It is easy to then show that

$$\int_0^T dt\,\dot{\eta}^2 tF(t) = \sum_{m,n} a_n a_m \frac{nm\pi^2}{T^2}\int_0^T dt\,\cos\left(\frac{n\pi t}{T}\right)\cos\left(\frac{m\pi t}{T}\right) = \frac{T}{2}\sum_n\left(\frac{n\pi}{T}\right)^2 a_n^2,$$

and

$$\int_0^T dt\,\eta^2(t) = \sum_{m,n} a_n a`_m \int_0^T dt\,\sin\left(\frac{n\pi t}{T}\right)\sin\left(\frac{m\pi t}{T}\right) = \frac{T}{2}\sum_n a_n^2,$$

and so we have[13]

$$U(x_i, t_i : x_f, t_f) = \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \mathcal{N}'\, e^{\frac{i}{\hbar}S[x_{\rm cl}]}\int da_1 ... da_{N-1}\,\exp\left(\frac{imT}{4\hbar}\sum_{n=1}^{N-1}\left(\left[\frac{n\pi}{T}\right]^2-\omega^2\right)a_n^2\right)$$

The integral in the exponent is Gaussian can be done

$$\int da_n\,\exp\left(\frac{imT}{4\hbar}\left(\left[\frac{n\pi}{T}\right]^2-\omega^2\right)a_n^2\right) = \sqrt{\frac{4\pi i\hbar}{mT}}\frac{T}{n\pi}\left(1-\left(\frac{\omega T}{n\pi}\right)^2\right)^{-\frac{1}{2}}$$

so we have

$$U(x_i, t_i : x_f, t_f) = \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \mathcal{N}''\, e^{\frac{i}{\hbar}S[x_{\rm cl}]}\prod_{n=1}^{N-1}\left(1-\left(\frac{\omega T}{n\pi}\right)^2\right)^{-\frac{1}{2}}$$

Using the (apparently) a well-known identity

$$\lim_{N\to\infty}\prod_{n=1}^{N-1}\left(1-\left(\frac{\omega T}{n\pi}\right)^2\right)^{-\frac{1}{2}} = \frac{\sin(\omega T)}{\omega T},$$

we have

$$U(x_i, t_i : x_f, t_f) = \mathcal{N}'''\, e^{\frac{i}{\hbar}S[x_{\rm cl}]}\left(\frac{\sin(\omega T)}{\omega T}\right)^{-\frac{1}{2}}.$$

[13] A possible Jacobian could arise in changing from $\mathcal{D}\eta$ to $\prod_{i=1}^{N-1} da_i$, which we absorb into the normalisation; hence, $\mathcal{N} \to \mathcal{N}'$ which is yet to be determined.

The normalization constant is independent of ω and can be found by taking the $\omega \to 0$ limit and comparing with the free case (2.5). We find

$$\mathcal{N}''' = \sqrt{\frac{m}{2\pi i\hbar T}},$$

and so

$$U(x_i, t_i : x_f, t_f) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega T)}} \exp\left(\frac{i}{\hbar} S[x_{\text{cl}}]\right).$$

One can show that, on the classical solutions, the action is given by

$$S[x_{\text{cl}}] = \frac{m\omega}{2\sin(\omega T)}\Big((x_i^2 + x_f^2)\cos(\omega T) - 2x_i x_f\Big).$$

It is a relief that we can solve the simple harmonic oscillator using path integrals but it is a lot of work doing it this way. The hydrogen atom is even worse! We shall see that path integrals are very useful in many applications but we shouldn't become blinkered; sometimes other approaches will be better suited to the problem at hand.

Energy Spectrum

How can we recover the energy eigenvalues in the path integral formalism? The trace over the phase $e^{-iHT/\hbar}$ can be computed in two ways

$$\text{Tr}(e^{-iHT/\hbar}) = \int_{-\infty}^{\infty} dx \langle x|e^{-iET/\hbar}|x\rangle = \sum_{n=0}^{\infty} \langle n|e^{-iHT/\hbar}|n\rangle = \sum_{n=0}^{\infty} e^{-iE_n T/\hbar},$$

where the integral involves the position basis and the sum uses the energy basis with associated eigenvalues E_n. We therefore have

$$\int_{-\infty}^{\infty} dx \langle x|e^{-iET/\hbar}|x\rangle = \int_{-\infty}^{\infty} dx\, U(x, 0; x, T) = \sum_{n=0}^{\infty} e^{-iE_n T/\hbar},$$

where we have set $x_i = x_f := x$ and have chosen, wlog, $t_i = 0$. We thus have

$$U(x, 0 : x, T) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega T)}} \exp\left(\frac{i}{\hbar} S[x_{\text{cl}}]\right).$$

where

$$S[x_{\text{cl}}] = \frac{m\omega x^2}{\sin(\omega T)}\Big(\cos(\omega T) - 1\Big).$$

The only x-dependence is in the classical action. The x-integral is Gaussian and is easily done to give

$$\int_{-\infty}^{\infty} dx\, e^{iS[x]/\hbar} = \sqrt{-\frac{i\pi\hbar\sin(\omega T)}{2m\omega\sin^2(\omega T/2)}}$$

Thus

$$\text{Tr}(e^{-iHT/\hbar}) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega T)}} \times \sqrt{-\frac{i\pi\hbar\sin(\omega T)}{2m\omega\sin^2(\omega T/2)}} = \frac{1}{2i\sin(\omega T/2)}.$$

To compare with the sum over phases expression, we can write this as

$$\frac{1}{2i\sin(\omega T/2)} = \frac{1}{e^{i\omega T/2} - e^{-i\omega T/2}} = e^{i\omega T/2}\frac{1}{1-e^{-i\omega T}} = e^{i\omega T/2}\sum_{n=0}^{\infty} e^{-in\omega T} = \sum_{n=0}^{\infty} e^{-iE_n T},$$

where we identify the energy eigenvalues as $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$.

Note that this could have been done in Euclidean signature with $\beta = iT$. The result would have been the partition function

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_n}, \qquad \beta = iT.$$

Since $x_i = x_f$ the motion of the particle is in a topological circle of radius β. Making contact with the standard partition function of statistical mechanics we would identify $\beta = 1/k_B\theta$, where k_B is the Boltzmann constant and θ the temperature. As such, in this context, one often refers to the motion of the particle as along the *thermal circle*.

Wavefunctions

If the path integral approach is to be as useful as the operator approach, we had better be able to recover the known expressions for the wavefunctions. Our transition amplitude

$$U(x_i,t_i:x_f,t_f) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega T)}}\exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left[(x_i^2+x_f^2)\cos(\omega T) - 2x_ix_f\right]\right). \tag{2.12}$$

can also be written as

$$\begin{aligned} U(x_i,t_i:x_f,t_f) &= \langle x_f|e^{-iHT/\hbar}|x_i\rangle = \sum_{n=0}^{\infty}\langle x_f|n\rangle\langle n|e^{-iHT/\hbar}|x_i\rangle \\ &= \sum_{n=0}^{\infty}\langle x_f|n\rangle e^{-iE_nT/\hbar}\langle n|x_i\rangle = \sum_{n=0}^{\infty}\phi_n(x_f)\phi_n^*(x_i)\,e^{-iE_nt/\hbar} \\ &= e^{-i\omega T}\sum_{n=0}^{\infty}\phi_n(x_f)\phi_n^*(x_i)\,e^{-in\omega t/2}. \end{aligned} \tag{2.13}$$

Thus, the wavefunction can be determined by expanding the expression (2.12) in powers of $e^{-i\omega t/2}$ and equating the coefficients with products of wavefunctions.

2.3 Time-Ordering

The time-slicing construction is important and all path integral approaches (that I know of) require some sort of ordering, at least locally[14]. Here we look at two examples of where time-ordering, coming from time-slicing plays and important role.

[14]For example, in a Euclidean CFT this might be radial ordering.

2.3.1 Time-Ordered Correlation Functions

We have talked extensively about the path integral expression for the propagator but there are other observables we may want to calculate. In general, we may wish to calculate a correlation function involving the insertion of operators such as

$$\langle x_f, t_f|\widehat{X}_H(t_1)\widehat{X}_H(t_2)|x_i, t_i\rangle, \qquad t_f > t_1 > t_2 > t_i$$

where the subscript denotes we have inserted Heisenberg picture operators. We can write this as

$$\langle x_f|e^{-\frac{i}{\hbar}\widehat{H}(t_1-t_f)}\widehat{X}_H(0)e^{-\frac{i}{\hbar}\widehat{H}(t_2-t_1)}\widehat{X}_H(0)e^{-\frac{i}{\hbar}\widehat{H}(t_i-t_2)}|x_i\rangle,$$

Inserting a complete set of states $\{|x_1\rangle\}$ and $\{|x_1\rangle\}$ to the right of the position operators gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \langle x_f|e^{-\frac{i}{\hbar}\widehat{H}(t_1-t_f)}\widehat{X}_H(0)|x_1\rangle\langle x_1|e^{-\frac{i}{\hbar}\widehat{H}(t_2-t_1)}\widehat{X}_H(0)|x_2\rangle\langle x_2|e^{-\frac{i}{\hbar}\widehat{H}(t_i-t_2)}|x_i\rangle \\
&= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\; x_1x_2\langle x_f|e^{-\frac{i}{\hbar}\widehat{H}(t_1-t_f)}|x_1\rangle\langle x_1|e^{-\frac{i}{\hbar}\widehat{H}(t_2-t_1)}|x_2\rangle\langle x_2|e^{-\frac{i}{\hbar}\widehat{H}(t_i-t_2)}|x_i\rangle \qquad (2.14)
\end{aligned}$$

where we have used the fact that $\widehat{X}(0)|x\rangle = x|x\rangle$ to replace the position operators by the appropriate eigenvalue. We may therefore write

$$\begin{aligned}
&\langle x_f, t_f|\widehat{X}_H(t_1)\widehat{X}_H(t_2)|x_i, t_i\rangle \\
&= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\; x_1x_2\, U(x_f, t_f; x_1, t_1)U(x_1, t_1; x_2, t_2)U(x_2, t_2; x_i, t_i). \qquad (2.15)
\end{aligned}$$

We have path integral expressions for each of the transition functions $U(x_b, t_b; x_a, t_a)$ and so

$$\begin{aligned}
&\langle x_f, t_f|\widehat{X}_H(t_1)\widehat{X}_H(t_2)|x_i, t_i\rangle \\
&= \mathcal{N}\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{x_1}^{x_f} \mathcal{D}x\, e^{\frac{i}{\hbar}\int_{t_1}^{t^f} dt L(x)} \int_{x_2}^{x_1} \mathcal{D}x'\, e^{\frac{i}{\hbar}\int_{t_2}^{t^1} dt' L(x')} \\
&\times \int_{x_2}^{x_1} \mathcal{D}x''\, e^{\frac{i}{\hbar}\int_{t_i}^{t^2} dt'' L(x'')}\; x_1x_2. \qquad (2.16)
\end{aligned}$$

If we recall the definition of the measure $\mathcal{D}x$, we see that the constrained end points are not integrated over. The functional integrals can be easily combined with the finite dimensional integrals as

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{x_1}^{x_f} \mathcal{D}x \int_{x_2}^{x_1} \mathcal{D}x' \int_{x_2}^{x_1} \mathcal{D}x'' = \int_{x_i}^{x_f} \mathcal{D}x, \qquad (2.17)$$

and the sum of the exponents is simply the integral of the action between times t_i and t_f, as such we have

$$\langle x_f, t_f|\widehat{X}_H(t_1)\widehat{X}_H(t_2)|x_i, t_i\rangle = \mathcal{N}\int_{x_i}^{x_f} \mathcal{D}x\, x(t_2)x(t_1)e^{\frac{i}{\hbar}\int_{t_i}^{t^f} dt\, L(x)}, \qquad (2.18)$$

where we have denoted $x(t_1) := x_1$ and $x(t_2) := x_2$.

If we had swapped the order of the operator insertions (i.e. if $t_2 > t_1$) then we would arrive at the result

$$\begin{aligned}\langle x_f, t_f|X_H(t_2)X_H(t_1)|x_i, t_i\rangle &= \mathcal{N}\int_i^f \mathcal{D}x\, x(t_1)x(t_2)\, e^{\frac{i}{\hbar}S[x]} \\ &= \mathcal{N}\int_i^f \mathcal{D}x\, x(t_2)x(t_1)\, e^{\frac{i}{\hbar}S[x]}, \qquad t_f > t_2 > t_1 > t_i,\end{aligned}$$

since $x(t_1)$ and $x(t_2)$ are classical commuting quantities. So we see, whatever order we put functions into the functional integral, the time-slicing construction implies that the path integral is always computing the time-ordered correlation function.

$$\langle x_f, t_f|T\{X_H(t_1)X_H(t_2)\}|x_i, t_i\rangle = \mathcal{N}\int_i^f \mathcal{D}x\, x(t_1)x(t_2)\, e^{\frac{i}{\hbar}S[x]}.$$

More generally, if we have composite operators $\mathcal{O}(X(t))$, then we have

$$\langle x_f, t_f|T\{\mathcal{O}_1(X(t_1))...\mathcal{O}_n(X(t_n))\}|x_i, t_i\rangle = \mathcal{N}\int_i^f \mathcal{D}x\, \mathcal{O}_1(x(t_1))...\mathcal{O}_n(x(t_n))\, e^{\frac{i}{\hbar}S[x]}.$$

as the path integral expression of the time-ordered correlation function. Normalization of the correlation function suggests we define

$$\mathcal{N}^{-1} = \int_i^f \mathcal{D}x\, e^{\frac{i}{\hbar}S[x]}.$$

Thus, we may write the normalized time-ordered correlation functions as

$$\boxed{\frac{\langle x_f, t_f|T\{\mathcal{O}_1(\widehat{X}(t_1))...\mathcal{O}_n(\widehat{X}(t_n))\}|x_i, t_i\rangle}{\langle x_f, t_f|x_i, t_i\rangle} = \frac{\int_i^f \mathcal{D}x\, \mathcal{O}_1(x(t_1))...\mathcal{O}_n(x(t_n))\, e^{\frac{i}{\hbar}S[x]}}{\int_i^f \mathcal{D}x\, e^{\frac{i}{\hbar}S[x]}}.}$$

Note that these expressions do not depend on the, possibly ill-defined normalisation constants that appeared in the path integral. This is a common feature in working with path integrals - we often work in intermediate stages with objects that are ill-defined but the physically meaningful objects will always be well-defined. The path integral seems to provide an intuitive way to get from one correct expression to another, often by a (naively) dubious route[15].

2.3.2 Canonical Commutation Relations

In the path integral formalism all objects appearing in the path integral are (commuting) functions. Given this, how do we make sense of the canonical commutation relations? For example, we expect

$$\langle x_f, t_f|[X(t), P(t)]|x_i, t_i\rangle = i\hbar\langle x_f, t_f|x_i, t_i\rangle \neq 0,$$

[15] I say emphasize ‘naive’ here as there is in all likelihood a rigorous way to state the argument, it’s just that we might not yet be smart enough to know it!

But in the path integral,

$$\langle x_f, t_f | [X(t), P(t)] | x_i, t_i \rangle = \mathcal{N} \int_i^f \mathcal{D}x \Big(x(t)p(t) - p(t)x(t) \Big) e^{\frac{i}{\hbar} S[x]},$$

seems to vanish, since $x(t)p(t) = p(t)x(t)$. The disparity is resolved one we note an important subtlety. What we mean by the composite operator $X(t)P(t)$ is really

$$\lim_{\epsilon \to 0} X(t+\epsilon)P(t),$$

as such the correct path integral description is

$$\langle x_f, t_f | T\{[X(t), P(t)]\} | x_i, t_i \rangle = \lim_{\epsilon \to 0} \mathcal{N} \int_i^f \mathcal{D}x \Big(x(t+\epsilon)p(t) - p(t+\epsilon)x(t) \Big) e^{\frac{i}{\hbar} S[x]},$$

which does not obviously vanish. In order to see how this works in detail we need the following result. Consider

$$\left\langle \frac{\delta F}{\delta x(t)} \right\rangle := \langle x_f, t_f | \frac{\delta F}{\delta x(t)} | x_i, t_i \rangle = \int_i^f \mathcal{D}x \, \frac{\delta F}{\delta x(t)} \, e^{\frac{i}{\hbar} S[x]},$$

where F is some object (function or functional) and time-ordering is assumed. Our first assumption is that there is some analogue of Stoke's theorem for functionals. Our second assumption will be that the boundary terms vanish, so that we can integrate by parts to give

$$\left\langle \frac{\delta F}{\delta x(t)} \right\rangle = -\frac{i}{\hbar} \int_i^f \mathcal{D}x \, F \frac{\delta S[x]}{\delta x(t)} \, e^{\frac{i}{\hbar} S[x]} = -\frac{i}{\hbar} \left\langle F \frac{\delta S[x]}{\delta x(t)} \right\rangle \tag{2.19}$$

Let us think about the functional derivative of the action more carefully. The action is given by

$$S[x] = \sum_r \left[\frac{m}{2} \left(\frac{x_{r+1} - x_r}{\delta t} \right)^2 - V(x_r) \right] \delta t.$$

It turns out that the potential $V(x)$ plays no role in this argument (its contribution drops out in the limit), so we will ignore it from here. In the discretisation, we take $x(t) \to x_k$, which corresponds to the value of x at time t, so that

$$\begin{aligned} \frac{\delta S[x]}{\delta x(t)} \to \frac{\partial S[x]}{\partial x_k} &= \sum_r m \frac{(\delta_{k,r+1} - \delta_{k,r})(x_{r+1} - x_r)}{\delta t} \\ &= -m \left(\frac{x_{k+1} - x_k}{\delta t} - \frac{x_k - x_{k-1}}{\delta t} \right) \end{aligned} \tag{2.20}$$

We then have

$$\left\langle \frac{\delta F}{\delta x(t)} \right\rangle = \frac{i}{\hbar} \left\langle F \, m \left(\frac{x_{k+1} - x_k}{\delta t} - \frac{x_k - x_{k-1}}{\delta t} \right) \right\rangle$$

For the commutator in question, we take $F = x_k$, so that (assuming th correlation functions are normalized)

$$\frac{i}{\hbar} \left\langle x_k \, m \left(\frac{x_{k+1} - x_k}{\delta t} - \frac{x_k - x_{k-1}}{\delta t} \right) \right\rangle = 1,$$

now taking the small δt limit and noting that

$$\lim_{\delta t \to 0} m \frac{x_{k+1} - x_k}{\delta t} = m\dot{x}(t_+) = p(t_+), \qquad \lim_{\delta t \to 0} m \frac{x_k - x_{k-1}}{\delta t} = m\dot{x}(t_-) = p(t_-)$$

where $t_\pm$ denotes the fact that we are approaching t from above or below. We find

$$1 = \frac{i}{\hbar} \left\langle x(t) \Big(p(t_+) - p(t_-) \Big) \right\rangle,$$

This should be time-ordered, so

$$1 = \frac{i}{\hbar} \left\langle \Big(p(t_+) x(t) - x(t) p(t_-) \Big) \right\rangle,$$

which corresponds to the correlation function

$$\langle x_f, t_f | T\{[\widehat{P}(t), \widehat{X}(t)]\} | x_i, t_i \rangle = -i\hbar.$$

This is certainly consistent with the commutation relation $[\widehat{P}(t), \widehat{X}(t)] = -i\hbar I$, where I is the identity operator. Again we see, as with time-ordering, that time-slicing is often the key to understanding some of the subtle issues in the quantum theory.

2.4 The Path Integral for Quantum Field Theories

What changes when we go to higher dimensions? We should be clear, what we mean by dimension is the as follows: If the action can be written in terms of a (local) Lagrangian density

$$S[\phi] = \int_{\mathcal{M}} d^D x \, \mathcal{L}(\phi),$$

the theory is said to be a D-dimensional quantum field theory defined on a manifold $\mathcal{M}$. Broadly speaking, the dimension is the number of parameters the objects ϕ are functions of. In thins sense, Quantum Mechanics is a $D = 1$ dimensional QFT, even though it may describe physics in three-dimensional space, plus time. Similarly, the worldsheet theory of string theory is a 2-dimensional QFT, even though the target space may be 10 or 26 dimensional[16].

2.4.1 *QFT using Schrodinger Functionals

In classical particle mechanics, we try to solve differential equations such Newton's second law $m\ddot{\mathbf{x}} = -\nabla V(\mathbf{x})$ to find the trajectory $\mathbf{x}(t)$ as a function of the parameter t. Newton's second law tells us how the function evolves in time. In the Schrodinger picture of quantum mechanics, the object that plays the role of the function $\mathbf{x}(t)$ is the wavefunction $\Psi(\mathbf{x})$ which may be thought of as the state $|\Psi\rangle$ in the position basis $|\mathbf{x}\rangle$ as $\Psi(\mathbf{x}) = \langle \mathbf{x} | \Psi \rangle$. Time evolution is given by solving the Schrodinger equation for a Hamiltonian

$$\widehat{H} = \frac{\widehat{P}^2}{2m} + V(\widehat{X}).$$

[16] A key point here is that, even though the worldsheet physics is described by a QFT, the target space physics in certainly not that of a QFT (modulo statements of holography).

The canonical commutation relations are between observables such as $[X, P] = i\hbar$. In classical field theory, position now becomes a parameter (like time) and the object of interest in the classical field $\phi(x)$, where we write $x = (\mathbf{x}, t)$. The time evolution is determined by solving the classical field equations, such as the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\phi(x) = 0.$$

In quantum field theory, the object of interest is the quantum field $\phi(x)$. The field itself satisfies the commutation relations

$$[\phi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = i\hbar\delta^3(\mathbf{x} - \mathbf{y}),$$

The analogue of $|\mathbf{x}, t\rangle$ are the states of the quantum field $|\phi(\mathbf{x}, t)\rangle$ with an associated operator

$$\widehat{\mathbf{X}}|\mathbf{x}, t\rangle = \mathbf{x}(t)|\mathbf{x}, t\rangle \qquad \rightarrow \qquad \widehat{\Phi}(\mathbf{x})|\phi(\mathbf{x}, t)\rangle = \phi(\mathbf{x}, t)|\phi(\mathbf{x}, t)\rangle.$$

Time-dependence can be introduced via a Hamiltonian in the usual way. The Hamiltonian can be written as

$$\widehat{H} = \int d^3\mathbf{x}\left(\frac{1}{2}\widehat{\Pi}^2(\mathbf{x}) + \mathcal{V}(\widehat{\Phi})\right),$$

where the potential is often of the form $\mathcal{V}(\Phi) = \frac{1}{2}\nabla^2\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi)$. Noting that the commutation relations may be realised by

$$\widehat{\Pi}(\mathbf{x}) = -i\hbar\frac{\delta}{\delta\widehat{\Phi}(\mathbf{x})},$$

We could then make QFT look very much like QM and introduce wavefunctionals

$$\Psi[\phi] = \langle\phi(\mathbf{x})|\Psi\rangle,$$

which solve the functional Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi[\phi] = \widehat{H}\Psi[\phi]$$

This can be solved to relate the wavefunctional at a given time to some initial wavefunctional

$$\Psi[\phi_f, t_f] = \int \mathcal{D}\phi_i\,\langle\phi_f|e^{-\frac{i}{\hbar}\widehat{H}(t_2-t_1)}|\phi_f\rangle\,\Psi[\phi_i, t_i], \tag{2.21}$$

as can be verified by plugging this into the Schrodinger equation above[17]. The route to the path integral construction is now straightforward. We can introduce the amplitude

$$\langle\Psi[\phi_f, t_f]|\Psi[\phi_i, t_i]\rangle$$

relating two field configurations ϕ_f and ϕ_i at times t_f and t_i respectively. We time-slice as before and insert the Huygens-type expression (2.21) to describe between one slice and the next. The final expression is written as a functional integral over the space of field configurations, rather than the space of paths.

[17]The notation $\Psi[\phi, t]$ is a little misleading as Ψ is a function of t but a function*al* of ϕ. What the more logical notation $\Psi[\phi](t)$ gains in precision it loses in clarity.

2.4.2 From Functions to Functionals

The most obvious change is that the fields ϕ are now functions of a universal parameter - time, so $\phi \to \phi(t)$. Our key objects, such as the action and generating function(al) are now functionals of $\phi(t)$ and $J(t)$ respectively. The objects appearing in the zero-dimensional case have corresponding functionals

$$S[\phi] = \int dt L(\phi, \dot{\phi}, ...), \qquad Z[J] = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}S[\phi] + \int dt \, J(t)\phi(t)},$$

with time-ordered correlation functions given by

$$\langle 0|T\{\phi(t_1)...\phi(t_n)\}|0\rangle_J = \frac{(-i\hbar)^n}{Z[J]} \frac{\delta^n Z[J]}{\delta J(t_1)...\delta J(t_n)}.$$

The connected correlation functions are given by the effective potential $W[J]$, which is given by

$$Z[J] = e^{\frac{i}{\hbar}W[J]},$$

so that

$$\langle 0|T\{\phi(t_1)...\phi(t_n)\}|0\rangle_{J,\mathrm{conn}} = \frac{\delta^n W[J]}{\delta J(t_1)...\delta J(t_n)}.$$

2.4.3 *States and wavefunctionals from the path integral

We have seen how to compute an operator - the time evolution operator (or transition amplitudes) U - how do we describe states in the path integral framework? The starting point is to note two facts

- The transition amplitude may be written as

$$\langle \phi_2|e^{-iHt}|\phi_1\rangle = \int_{\phi(0)=\phi_1}^{\phi(t)=\phi_2} \mathcal{D}\phi \; e^{iS[\phi]},$$

  where $S[\phi] = \int_0^t dt' \mathcal{L}(\phi)$.

- A ket is a linear map from the space of bras to $\mathbb{C}$

Te second statement can be understood from the Schrodinger wavefunction in a particular basis

$$\psi(\mathbf{x}, t) = \langle \mathbf{x}|\psi(t)\rangle,$$

where we have chosen the position basis for definiteness. This is function, not a vector in the state space. We could write the ket as

$$|\psi(t)\rangle = \psi(\bullet, t),$$

where the $\bullet$ denotes the fact that we need to input a choice of basis (a set of bras) to map to to $\mathbb{C}$. The first point above suggests a natural way to describe a state in path integral language is to leave one of the boundary conditions in the path integral unspecified.

We can make this discussion more precise by recalling the time-slicing definition of the path integral, before we take the continuum limit, along the lines of the quantum mechanics discussion. There, the procedure to define states could be summarized by the slogan "a state is given by splitting the correlation function". This was made precise for quantum mechanics as follows:

1. Split the time interval into two pieces $[t_i, t_f] = [t_i, t] \cup [t, t_f]$.
2. $e^{-\frac{i}{\hbar}(t-t_i)}|x_i\rangle$ is the path integral over configurations in $[t_i, t]$ with $x(t_i) = x_i$ and $x(t)$ unconstrained. We can write this as

$$\int_{x(t_i)=x_i}^{\bullet} \mathcal{D}x\, e^{\frac{i}{\hbar}S[x]}.$$

3. $\langle x_f|e^{-\frac{i}{\hbar}(t_f-t)}$ is the path integral over configurations in $[t, t_f]$ with $x(t_f) = x_f$ and $x(t)$ unconstrained. We can write this as

$$\int_{\bullet}^{x(t_f)=x_f} \mathcal{D}x\, e^{\frac{i}{\hbar}S[x]}.$$

4. In each case the action is defined as the integral of the Lagrangian over the appropriate interval in time.

The generalisation to $d+1$ dimensions is clear. One fixes initial and final times $t_{i,f}$ and choose an appropriate spacelike slice $\mathcal{S}_{i,f}$ - Cauchy surfaces - on which the initial and final field configurations are defined; i.e.

$$\phi(\mathbf{x})|_{\mathcal{S}_{i,f}} = \phi_{i,f}(\mathbf{x}).$$

The transition function from initial to final field configurations may be written as

$$\langle\phi_f|e^{-\frac{i}{\hbar}H(t_f-t_i)}|\phi_i\rangle = \int_{\phi|_{\mathcal{S}_i}=\phi_i}^{\phi|_{\mathcal{S}_f}=\phi_f} \mathcal{D}\phi\, \exp\left(\frac{i}{\hbar}\int_{\mathcal{I}(\mathcal{S}_i,\mathcal{S}_f)} d^{d+1}x\, \mathcal{L}(\phi,\partial\phi)\right)$$

where $\mathcal{I}(\mathcal{S}_i, \mathcal{S}_f)$ is the spacetime region bounded by $\mathcal{S}_i$ and $\mathcal{S}_f$. This can be captured in the picture [**picture**]. States can then be written as

$$|\phi_i(t)\rangle := e^{-\frac{i}{\hbar}H(t-t_i)}|\phi_i\rangle = \int_{\phi|_{\mathcal{S}_i}=\phi_i}^{\bullet} \mathcal{D}\phi\, \exp\left(\frac{i}{\hbar}\int_{\mathcal{I}(\mathcal{S}_i,\mathcal{S}_\bullet)} d^{d+1}x\, \mathcal{L}(\phi,\partial\phi)\right)$$

and

$$\langle\phi_f(t)| := \langle\phi_f|e^{-\frac{i}{\hbar}H(t_f-t)} = \int_{\bullet}^{\phi|_{\mathcal{S}_f}=\phi_f} \mathcal{D}\phi\, \exp\left(\frac{i}{\hbar}\int_{\mathcal{I}(\mathcal{S}_t,\mathcal{S}_\bullet)} d^{d+1}x\, \mathcal{L}(\phi,\partial\phi)\right)$$

where the $\bullet$ denote unconstrained limits on the integral.

Path Integral Description of the Wavefunctional

If we include a general field ϕ on $\mathcal{S}_\bullet$ we can define the wavefunctional $\Psi_i[\phi]$ as

$$\Psi_i[\phi] := \langle\phi|\phi_i(t)\rangle := \int_{\phi'|_{\mathcal{S}_i}=\phi_i}^{\phi'|_{\mathcal{S}_\bullet}=\phi} \mathcal{D}\phi' \, \exp\left(\frac{i}{\hbar}\int_{\mathcal{I}(\mathcal{S}_i,\mathcal{S}_\bullet)} d^{d+1}x\,\mathcal{L}(\phi',\partial\phi')\right),$$

where we take ϕ as arbitrary. Correlation functions can be glued together in the obvious way

$$\langle\phi_f|\phi_i\rangle = \int \mathcal{D}\phi' \; \Psi_f^*[\phi']\Psi_i[\phi']. \tag{2.22}$$

This last statement may be understood as

$$\begin{aligned}\langle\phi_f|\phi_i\rangle &= \int \mathcal{D}\phi' \; \Psi_f^*[\phi']\Psi_i[\phi'] \\ &= \int \mathcal{D}\phi' \int_{\phi'}^{\phi''|_{\mathcal{S}_f}=\phi_f} \mathcal{D}\phi'' \, \exp\left(\frac{i}{\hbar}\int_{\mathcal{I}(\mathcal{S}_{t''},\mathcal{S}_{t'})} d^{d+1}x\,\mathcal{L}(\phi'')\right) \\ &\quad\times \int_{\phi'''|_{\mathcal{S}_{t_i}}=\phi_i}^{\phi'} \mathcal{D}\phi''' \, \exp\left(\frac{i}{\hbar}\int_{\mathcal{I}(\mathcal{S}_{t_i},\mathcal{S}_{t'})} d^{d+1}x\,\mathcal{L}(\phi''')\right)\end{aligned} \tag{2.23}$$

The imposition of the boundary conditions ϕ' on each integral followed by the integral over all intermediate field configurations ϕ' is equivalent to not having the ϕ' boundary constraints at all

$$\begin{aligned}\langle\phi_f|\phi_i\rangle &= \int_{\phi|_{\mathcal{S}_i}=\phi_i}^{\phi|_{\mathcal{S}_f}=\phi_f} \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar}\int_{\mathcal{I}(\mathcal{S}_{t_f},\mathcal{S}_t)} d^{d+1}x\,\mathcal{L}(\phi) + \frac{i}{\hbar}\int_{\mathcal{I}(\mathcal{S}_{t_i},\mathcal{S}_t)} d^{d+1}x\,\mathcal{L}(\phi)\right) \\ &= \int_{\phi|_{\mathcal{S}_i}=\phi_i}^{\phi|_{\mathcal{S}_f}=\phi_f} \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar}\int_{\mathcal{I}(\mathcal{S}_{t_i},\mathcal{S}_{t_f})} d^{d+1}x\,\mathcal{L}(\phi)\right)\end{aligned} \tag{2.24}$$

3 Quantum Theory Without Space or Time

Previous sections has hopefully convinced you that quantum theory has something to do with (functional) integrals. Before looking at higher dimensional QFT, let's look at a finite-dimensional analogue: Matrix Mechanics (aka quantum theory in zero dimensions). Many of the tools we will use make sense for finite-dimensional integrals and so we will take a step back (just one) and introduce these ideas in the zero-dimensional case, where the action is just a function (there are no parameters to integrate over). This means that, when we introduce these techniques in higher dimensions, we (hopefully) will not be overwhelmed with technicalities.

3.1 The Generating Function $Z(J)$

We consider the single variable ϕ, which we assume can take values on $\mathbb{R}$. There is no notion of time, as we are in zero dimensions, so ϕ is not a function of t it is just a number. The action is just a function of ϕ. We shall be interested in polynomial actions[18]

$$S(\phi) = \sum_{n=2}^{N} \frac{g_n}{n!}\phi^n.$$

For example, we could have

$$S(\phi) = \frac{\alpha}{2}\phi^2 + \frac{\lambda}{4!}\phi^4.$$

In fact, this is the model we will work with throughout this section (note that the action is symmetric about $\phi \to -\phi$). There is no notion of space or time, so there are no derivatives. The universe is a point at an instant and the only observable is a (real) number ϕ. The 'dynamics' of the theory tell us what values functions of ϕ can take and with what probability. In some sense this is the purest quantum system you can imagine, with all other distractions removed. Once we understand this system, we will have a good framework to understand more realistic systems. We shall always assume $\alpha > 0$.

The analogue of a generating function is the finite-dimensional integral

$$Z_\lambda(J) = N \int_{\mathbb{R}} d\phi \, e^{-S(\phi)+J\phi},$$

where we choose to work with imaginary 'time'[19] and N is some normalization, independent of J. It will be helpful to denote the value of λ but we shall suppress the parameter α. The integral $Z_\lambda(0)$ is sometimes called the *partition function*. As a precursor to the calculation of correlation functions, let us try to understand the partition function.

[18]Except when dealing with gravitational theories, constant terms will not be important, so we neglect ϕ^0 terms. Linear terms can be absorbed into a redefinition of the field, so we ignore them too. Terms with negative powers of ϕ, though interesting to consider, will not have analogues in the realistic theories we consider later so will not be discussed here.

[19]Let's not pass up the chance to work with well-defined integrals!

The Partition Function

In the Euclidean description,

$$Z_\lambda(0) = N \int_{\mathbb{R}} d\phi\, e^{-S(\phi)/\hbar}.$$

has the same form as the classical partition function from statistical mechanics

$$Z = \sum_{\text{states}} e^{-H(\phi)/k_B T} = \mathrm{Tr}(e^{-\beta H}), \qquad \beta = \frac{1}{k_B T}$$

where k_B is Boltzmann's constant and T is the temperature. We note that the Euclidean action has the same form as the Hamiltonian in Quantum Mechanics. We will understand this relationship better later.

If $\lambda, J = 0$, the integral is Gaussian and we can evaluate it directly

$$Z_0(0) = N \int_{-\infty}^{+\infty} d\phi\, e^{-S(\phi)} = N\sqrt{\frac{2\pi}{\alpha}}.$$

This is the analogue of the free particle and it is often helpful to normalize $Z(J)$ by dividing out by this factor and so we shall choose

$$N = \sqrt{\frac{\alpha}{2\pi}},$$

so that $Z_0(0) = 1$. A helpful trick can be used to calculate correlation functions. Let us introduce the source J and couple it to ϕ, to give the generating function

$$Z(J) = N \int d\phi\, e^{-S(\phi)+J\phi}.$$

We see then that differentiating with respect to J and then setting $J = 0$ gives correlation functions for polynomials in ϕ; i.e. $Z(J)$ is a generating function for correlation functions

$$\boxed{\langle \phi^n \rangle = \frac{1}{Z(0)} \left[\frac{\partial^n Z(J)}{\partial J^n} \right]_{J=0} = \frac{1}{Z(0)} \int d\phi\, \phi^n\, e^{-S(\phi)}.}$$

More generally, correlation functions are given by

$$\langle f(\phi) \rangle = \frac{1}{Z(0)} \left[f\left(\frac{\partial}{\partial J} \right) Z_\lambda(J) \right]_{J=0} = \frac{1}{Z(0)} \int d\phi\, f(\phi)\, e^{-S(\phi)}$$

for some normalization we shall choose shortly. Our task then is to find $Z(J)$. We shall first try to do this for the Gaussian model.

We see that the path integral formulation gives a very clear relationship to the probability interpretation of quantum theory. For a real field, the generating function is a conventional generating function for a continuous random variable. The remarkable ability to extend this to complex functions, is due to the Strum-Liouville structure inherent in quantum theory.

The Free Theory $(\lambda = 0)$

In this case

$$Z_0(J) = \sqrt{\frac{\alpha}{2\pi}} \int d\phi \, e^{-\frac{1}{2}\alpha\phi^2 + J\phi}.$$

We can complete the square in the exponent

$$Z_0(J) = \sqrt{\frac{\alpha}{2\pi}} \int d\phi \, e^{-\frac{1}{2}\alpha\left(\phi - \frac{J}{\alpha}\right)^2 + \frac{J^2}{2\alpha}} = \exp\left(\frac{J^2}{2\alpha}\right),$$

where we have done the Gaussian integral over $\tilde{\phi} := \phi - \frac{J}{\alpha}$. We see then that the correlations for odd n vanish and for even n

$$\langle \phi^{2n} \rangle = \frac{1}{Z_0(0)} \left[\frac{\partial^{2n}}{\partial J^{2n}} Z_0(J)\right]_{J=0} = \left[\frac{\partial^{2n}}{\partial J^{2n}} e^{\frac{J^2}{2\alpha}}\right]_{J=0} = \frac{(2n-1)!!}{\alpha^n}.$$

where the 'double factorial' denotes

$$n!! = n(n-2)(n-4)...4.2$$

if n is even and

$$n!! = n(n-2)(n-4)...3.1$$

if n is odd. For example

$$\langle \phi^2 \rangle = \frac{1}{\alpha}, \qquad \langle \phi^4 \rangle = \frac{3}{\alpha^2}, \qquad \langle \phi^6 \rangle = \frac{15}{\alpha^3}.$$

These results are easy to understand combinatorially. For example; for $n = 4$ we have to connect eight nodes with propagators. Starting with one node, we have 7 nodes to choose to connect it to. Once the choice is made, only 6 nodes are left unconnected. We choose one of these nodes and look for another node to connect to. There are 5 to choose from. Once we make a connection, we have 4 unconnected nodes left. There are 3 ways to connect these together. Each propagator carries a $1/\alpha$ so we have a total weighting for the graph of

$$\frac{7.5.3.1}{\alpha^4},$$

as described above. Let us now apply this to the interacting theory.

The ϕ^4 Interacting Theory $(\lambda \neq 0)$

The generating function, with the normalization as above, is

$$Z_\lambda(J) = \sqrt{\frac{\alpha}{2\pi}} \int d\phi \, e^{-\frac{\alpha}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi}.$$

We treat the interaction perturbatively, assuming $\lambda << 1$, and expand the interaction term as a power series

$$\begin{aligned}
Z_\lambda(J) &= \sqrt{\frac{\alpha}{2\pi}} \int d\phi \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\lambda\phi^4}{4!}\right)^k e^{-\frac{\alpha}{2}\phi^2 + J\phi} \\
&= \sqrt{\frac{\alpha}{2\pi}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\lambda}{4!}\right)^k \int d\phi\, \phi^{4k}\, e^{-\frac{\alpha}{2}\phi^2 + J\phi} \qquad \text{Integral and sum exchanged!} \\
&= \sqrt{\frac{\alpha}{2\pi}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\lambda}{4!}\right)^k \frac{\partial^{4k}}{\partial J^{4k}} \int d\phi\, e^{-\frac{\alpha}{2}\phi^2 + J\phi} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\lambda}{4!}\right)^k \frac{\partial^{4k}}{\partial J^{4k}} Z_0(J). \qquad (3.1)
\end{aligned}$$

In summary

$$\boxed{Z_\lambda(J) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\lambda}{4!}\right)^k \frac{\partial^{4k}}{\partial J^{4k}} Z_0(J)} \qquad (3.2)$$

where the 'free' partition function is

$$\boxed{Z_0(J) = \exp\left(\frac{J^2}{2\alpha}\right).} \qquad (3.3)$$

This basic structure is for perturbative calculations will appear again and again in the course.

There are a couple of dodgy steps in the above calculation which we will discuss in the following section. For now, let's just hope for the best and plough on! We know $Z_0(J)$ so we can, in principle, evaluate this expression. The correlations are then given by

$$\langle \phi^n \rangle = \frac{1}{Z_\lambda(0)} \left[\frac{\partial^n Z_\lambda(J)}{\partial J^n}\right]_{J=0}$$

As an indicative example, let's look at the quadratic correlator, up to order λ^2.

$$\langle \phi^2 \rangle = \frac{1}{Z_\lambda(0)} \left[\frac{\partial^2 Z_\lambda(J)}{\partial J^2}\right]_{J=0}$$

We need then to expand $Z_\lambda(J)$ to order λ^2

$$\begin{aligned}
Z_\lambda(J) &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{4!^k k!} \frac{\partial^{4k}}{\partial J^{4k}} Z_0(J) \\
&= Z_0(J) - \frac{\lambda}{4!} \frac{\partial^4}{\partial J^4} Z_0(J) + \frac{1}{2}\left(-\frac{\lambda}{4!}\right)^2 \frac{\partial^8}{\partial J^8} Z_0(J) + \ldots \qquad (3.4)
\end{aligned}$$

This is starting to look a bit daunting and we are only in zero dimensions! The free theory correlation functions could be understood simply in terms of diagrams. Can we find a similar organising principle for our calculations here in the interacting theory? The answer of course is yes (we have all done QFT and leaned about Feynman diagrams last term). We now turn our thoughts to understanding how Feynman diagrams arise from the path integral. Before we deal with correlation functions, let us consider a slightly simpler object.

3.2 The Partition Function and its diagrammatic interpretation

We can think of the J as sources and so if $J = 0$ we have the vacuum diagrams only. The generating function for the vacuum diagrams is the partition function. With a bit of work we can show that

$$\frac{\partial^4}{\partial J^4} Z_0(J) = \left(\frac{J^4}{\alpha^4} + \frac{6J^2}{\alpha^3} + \frac{3}{\alpha^2}\right) Z_0(J), \tag{3.5}$$

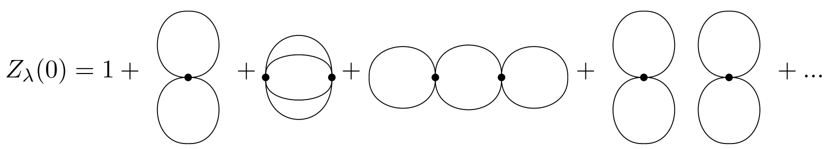
and

$$\frac{\partial^8}{\partial J^8} Z_0(J) = \left(\frac{J^8}{\alpha^8} + \frac{28J^6}{\alpha^7} + \frac{210J^4}{\alpha^6} + \frac{420J^2}{\alpha^5} + \frac{105}{\alpha^4}\right) Z_0(J).$$

Putting these results into (3.4) and setting $J = 0$ gives the partition function to order λ^2

$$Z_\lambda(0) = 1 - \frac{\lambda}{8\alpha^2} + \frac{\lambda^2 105}{2(4!)^2\alpha^4} + ... = 1 - \frac{\lambda}{8\alpha^2} + \frac{\lambda^2 35}{384\alpha^4} + ... \tag{3.6}$$

which, we claim, can be expressed diagramatically as a series of 'vacuum bubbles'



What have we done here? Firstly, we note that we can write this as

$$Z_\lambda(0) = 1 + (-\lambda)\left(\frac{1}{\alpha}\right)^2\left(\frac{1}{8}\right) + (-\lambda)^2\left(\frac{1}{\alpha}\right)^4\left(\frac{1}{48} + \frac{1}{16} + \frac{1}{128}\right) + ...$$

There is a factor of α^{-1} for each propagator and a factor of $-\lambda$ for each vertex V, giving

$$Z_\lambda(0) = \sum_{V=0}^{\infty}(-\lambda)^V\left(\frac{1}{\alpha}\right)^{2V}\sum_{\mathcal{G}_V}\frac{1}{S_{\mathcal{G}_V}},$$

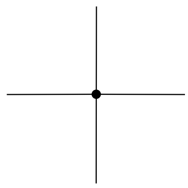
where $\mathcal{G}_V$ denotes all graphs with V vertices and $S_{\mathcal{G}_V}$ is the appropriate symmetry factor for each graph. We note there are twice as many propagators as vertices [20] . The factor S_V is

[20]Let us try to understand, in a more systematic way, how the interacting partition function may be calculated order by order in a diagrammatic way. Let us denote the number of vertices by V and the number of propagators as P. Note that each vertex has 4 derivatives of J and each propagator gives 2 powers of J. Thus a diagram with V vertices and P propagators has a net of $E = 2P - 4V$ remaining Js. If we wish to describe an n point-function, we therefore want $E = n$. For the partition function, where we set all of the J to zero, we only have contributions from $E = 0$ and so $P = 2V$ in all relevant diagrams. Using the diagram construction, we could write the partition function as

$$Z_\lambda(0) = \sum_{V\in\mathbb{Z}}(-\lambda)^V\left(\frac{1}{\alpha}\right)^{2V}\mathcal{S}_V \qquad \text{where} \qquad \mathcal{S}_V = \sum_{\mathcal{G}_V}\frac{1}{S_{\mathcal{G}_V}},$$

and $\mathcal{G}_V$ denotes all graphs with V vertices and $S_{\mathcal{G}_V}$ is the appropriate symmetry factor for each graph. An expression for the net contribution of the symmetry factor $\mathcal{S}_V$ can be derived from the partition function as is given in Appendix ?.

the symmetry factor for each diagram - the number of ways we can build the same diagram. Formally, it is the dimension of the automorphism group of the graph. The calculation of these is given below. The basic diagrammatic rules are

 $= \dfrac{1}{\alpha}$  $= -\lambda$ (3.7)

Calculating Symmetry Factors

[UNDER CONSTRUCTION]
Is there a systematic way to calculate symmetry factors? The safest thing to do is to go back to the path integral and make sure you have the factors right. However, in many cases it is possible to take a combinatoric shortcut.

Convergence and Asymptotics

Let's return to a simple model with action

$$S(\phi) = \phi^2 + \lambda\phi^4.$$

The partition function is a function of λ

$$Z_\lambda(0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\phi\, e^{-\phi^2 - \lambda\phi^4},$$

which exists for $\lambda \geq 0$ and it is trivial to show that $Z_0(0) = 1$. To simplify the exact formula below, we shall omit factors of 2 and 4!, although such combinatoric factors are helpful in simplifying expressions in perturbation theory. In fact the partition function can be computed exactly and is given by

$$Z_\lambda(0) = \frac{1}{\sqrt{\pi}} e^{\frac{1}{8\lambda}} \frac{K_{1/4}(1/8\lambda)}{2\sqrt{\lambda}},$$

where $K_n(x)$ is the modified Bessel function of the n'th kind. This is the full non-perturbative result and it is clearly not analytic at $\lambda = 0$. So what happens if we try to do perturbation theory about the free ($\lambda = 0$) theory? We can expect trouble as the $e^{\frac{1}{8\lambda}}$ contribution means that the partition function is not analytic[a] at $\lambda = 0$. We proceed as before and split the theory into a free and interacting part

$$Z_\lambda(0) = \frac{1}{\sqrt{\pi}} \int d\phi\, e^{-\phi^2} \sum_{k=0}^{\infty} \frac{(-\lambda\phi^4)^k}{k!}$$

and exchange the summation and integration. One can show that the series will now only converge asymptotically, rather than absolutely. This will not trouble us much in this course but it is worth keeping in mind. You can explore this further in the problem sheet.

[a] There is a nice physical argument, due to Freeman Dyson for why the failure of analyticity about $\lambda = 0$ should make us expect the perturbation series to not be absolutely convergent. You can find it here: https://journals.aps.org/pr/pdf/10.1103/PhysRev.85.631

3.3 Correlation Functions From Diagrams

The diagrams are a useful way to order the perturbative expansion of the partition function. Can something similar be done for correlation functions? The only new ingredient is to treat the J as sources. The diagrams relevant to an n-point correlation are those for which $E = 2P - 4V = n$. Returning to the $\langle\phi^2\rangle$ calculation, where we require $\{(V, P)\} = \{(0, 1), (1, 3), (2, 5), ...\}$. The $(V, P) = (0, 1)$ diagram is clearly the 'straight through' diagram of the free theory.

Normalization

It was useful to normalize the correlation functions of the free theory. Similarly, we shall define the n-point correlation function as for the interacting theory as

$$\langle \phi^n \rangle = \left[\frac{1}{Z_\lambda(0)} \frac{\partial^n}{\partial J^n} Z_\lambda(J) \right]_{J=0}$$

It is helpful then to have a perturbative expression for the reciprocal of the partition function

$$\frac{1}{Z_\lambda(0)} = \left(1 - \frac{\lambda}{8\alpha^2} + \frac{35}{384} \frac{\lambda^2}{\alpha^4} + ... \right)^{-1} = 1 + \frac{\lambda}{8\alpha^2} - \frac{29}{384} \frac{\lambda^2}{\alpha^4} + ... \tag{3.8}$$

2-point correlation function

To better understand the role of the partition function we shall derive an expression, to order λ^2, an expression for the two-point function (or propagator)

$$\langle \phi^2 \rangle = \frac{1}{Z_\lambda(0)} \frac{\partial^2}{\partial J^2} Z_\lambda \bigg|_{J=0}$$

The generating function is

$$Z_\lambda(J) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{-\lambda}{4!} \right)^r \left(\frac{\partial^4}{\partial J^4} \right)^r \exp\left(\frac{J^2}{2\alpha} \right)$$

so to order λ^2, we have

$$\begin{aligned} Z_\lambda(J) = & \left[1 - \frac{\lambda}{4!} \left(\frac{J^4}{\alpha^4} + \frac{6J^2}{\alpha^3} + \frac{3}{\alpha^2} \right) \right. \\ & \left. + \frac{1}{2} \left(\frac{-\lambda}{4!} \right)^2 \left(\frac{J^8}{\alpha^8} + \frac{28J^6}{\alpha^7} + \frac{210J^4}{\alpha^6} + \frac{420J^2}{\alpha^5} + \frac{105}{\alpha^4} \right) + ... \right] \exp\left(\frac{J^2}{2\alpha} \right) \end{aligned} \tag{3.9}$$

From the expressions (3.9) and (3.8) above we have

$$\begin{aligned} \frac{Z_\lambda(J)}{Z_\lambda(0)} = & \left(1 + \frac{\lambda}{8\alpha^2} + ... \right) \left[1 - \frac{\lambda}{4!} \left(\frac{J^4}{\alpha^4} + \frac{6J^2}{\alpha^3} + \frac{3}{\alpha^2} \right) \right. \\ & \left. + \frac{1}{2} \left(\frac{-\lambda}{4!} \right)^2 \left(\frac{J^8}{\alpha^8} + \frac{28J^6}{\alpha^7} + \frac{210J^4}{\alpha^6} + \frac{420J^2}{\alpha^5} + \frac{105}{\alpha^4} \right) + ... \right] Z_0(J) \end{aligned} \tag{3.10}$$

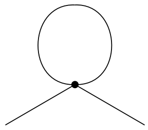
Collecting terms to order λ gives

$$\frac{Z_\lambda(J)}{Z_\lambda(0)} = Z_0(J) + \left[\frac{\lambda}{8\alpha^2} - \frac{\lambda}{4!} \left(\frac{J^4}{\alpha^4} + \frac{6J^2}{\alpha^3} + \frac{3}{\alpha^2} \right) \right] Z_0(J) + ...$$

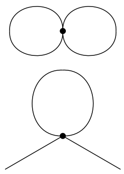
Differentiating twice (remembering to differentiate $Z_0(J)$ too) and setting $J = 0$ gives

$$\begin{aligned} \frac{1}{Z_\lambda(0)} \frac{\partial^2}{\partial J^2} Z_\lambda(J)\bigg|_{J=0} &= \frac{\partial^2}{\partial J^2} Z_0(J)\bigg|_{J=0} + \left[\frac{\lambda}{8\alpha^2} - \frac{\lambda}{4!}\left(\frac{J^6}{\alpha^6} + \frac{15J^4}{\alpha^5} + \frac{45J^2}{\alpha^4} + \frac{15}{\alpha^3}\right)\right] Z_0(J)\bigg|_{J=0} + \dots \\ &= \frac{1}{\alpha} + \frac{\lambda}{\alpha^3}\left(\frac{1}{8} - \frac{5}{8}\right) + \dots \\ &= \frac{1}{\alpha} - \frac{\lambda}{2\alpha^3} + \dots \end{aligned} \tag{3.11}$$

which is the contribution with the vacuum-vacuum amplitudes stripped out. The order λ terms comes form the single diagram

$$-\frac{\lambda}{2\alpha^3} =$$ 

where the factor of 2 is the symmetry factor of the diagram. Note that the diagram that includes a vacuum bubble does not contribute



Going to order λ^2, the only relevant contribution to $Z_\lambda(J)/Z_\lambda(0)$ is the term quadratic in J which, when differentiated, gives[21]

$$2 \times \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \frac{420}{\alpha^5} + 2 \times \frac{\lambda}{8\alpha^2}\left(-\frac{\lambda}{4!}\right)\frac{6}{\alpha^3} = \frac{2\lambda^2}{3\alpha^5}$$

This term can be reproduced by the diagrams in the figure below. And so the correlation

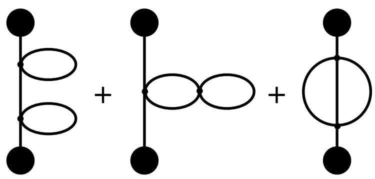


Figure 2. Order λ^2 2pt connected diagrams.

function to order λ^2 is

$$\langle \phi^2 \rangle = \frac{1}{\alpha} - \frac{\lambda}{2\alpha^3} + \frac{2\lambda^2}{3\alpha^5} + \dots$$

[21]This comes from collecting the order J^2 terms in (3.10) that are not part of $Z_0[J]$ and differentiating with respect to J. The order J^2 term in $Z_0[J]$ is ignored as it only contributes to the vacuum (disconnected vacuum bubbles) and so is cancelled in the normalization.

We note that this involves only the diagrams with the vacuum-vacuum contributions stripped out. We can see that only these diagrams contribute by writing the propagator as

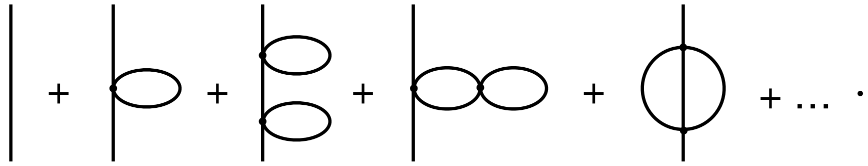


Figure 3. $\langle \phi^2 \rangle$ to two loops, with vacuum-to-vacuum contributions removed.

$$\langle \phi^2 \rangle = \frac{1}{\alpha} + \frac{(-\lambda)}{\alpha^3}\left(\frac{1}{2}\right) + \frac{(-\lambda)^2}{\alpha^5}\left(\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3!}\right) + ...$$

The Partition Function and Normalization

There are many other diagrams at this order that do not seem to contribute. Why only the diagrams listed above? For example, at order λ^2 we have the diagrams

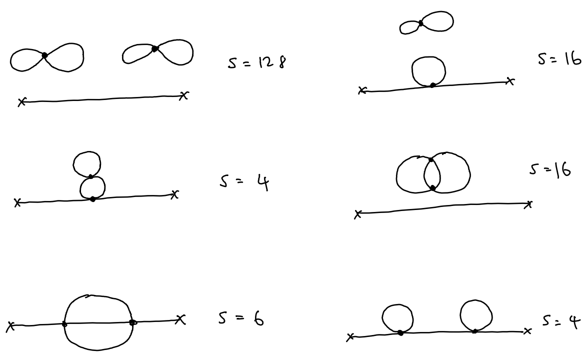


Figure 4. All 2-point diagrams at order λ^2. [Contains errors - E.G. middle diagram on RHS is 2.4!=48. (To Be Corrected)]

Only the ones that have no disconnected vacuum bubbles contribute to the propagator above. This hints at some hidden structure in the expressions. It is not hard to show that the full expression factorizes The normalization removes all of the vacuum $\rightarrow$ vacuum processes that arise solely due to the difference between the $\lambda = 0$ vacuum and the $\lambda \neq 0$ vacuum. And so dividing by $Z_\lambda[0]$, removes the purely vacuum-to-vacuum diagrams, leaving only the *connected* diagrams in figure 5.

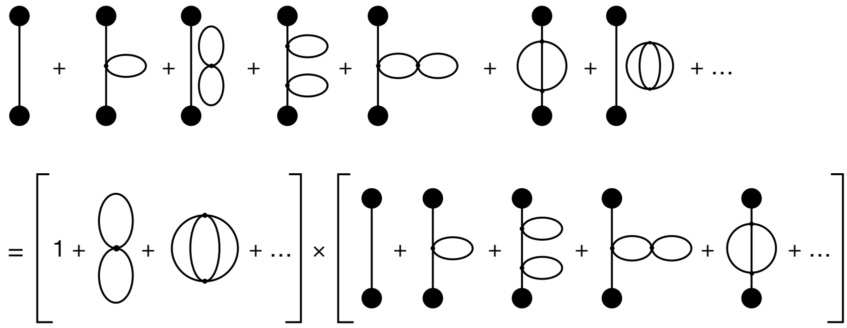


Figure 5. Diagrammatic representation of unnormalized $\langle\phi^2\rangle$ to one loop.

Feynman Rules

The results of these calculations can be concisely summarised in a set of **Feynman rules** for zero-dimensional ϕ^4 theory. To calculate $\langle\phi^n\rangle$, draw all diagrams with n external legs, except for those that contain a vacuum $\to$ vacuum sub-diagram. Assign factor of

- $-\lambda$ for each vertex.
- $1/\alpha$ for each line.
- Divide by the symmetry factor (dimension of the automorphism group of the graph). Remember external nodes are fixed and should not change under the automorphism. Different external points correspond to different physical objects - the automorphism is an over-counting, it should not relate different physical objects.

3.4 *Matrix Models and the large N limit

In this section we shall see how to introduce interactions in a slightly more systematic way. The key idea will be to use group theory to govern the structure of interactions. This will be provide an important set of examples in zero dimensions (matrix models) but our main goal will be to generalise this structure to higher dimensions and Yang-Mills theory.

We shall consider zero-dimensional models containing N^2 real degrees of freedom. Without loss of generality, we shall choose to arrange these degrees of freedom as components of an Hermitian matrix M_{IJ}. The only constraint we shall impose on the system is that we shall require that the theory be invariant under unitary transformations

$$M \to UMU^{\dagger}, \qquad U \in U(N).$$

The most general partition function for the theory is then

$$Z_N[M] = \int dM\, e^{-NTr(V(M))},$$

where dM is a measure on the space of Hermitian matrices[22] and $V(M)$ is general polynomial in M. Since the polynomial will transform as $V(UMU^\dagger) = UV(M)U^\dagger$, $Tr(V(M))$ is invariant under unitary transformations. This invariant is a prototype for the gauge symmetry we shall see later. All Hermitian matrices can be diagonalised by a unitary transformation[23]

$$M = UDU^\dagger$$

where $D =$ diag$\{\lambda_1, \lambda_2, ..., \lambda_N\}$ and $\lambda_i \in \mathbb{R}$. The N^2 degrees of freedom can therefore be written in terms of the N eigenvalues $\{\lambda_i\}$ and the $N^2 - N$ real degrees of freedom in the diagonalising matrix $U \in U(N)/U(1)^N$. Note that the elements of $U(N)$ that just multiply by diagonal phases play no role in the diagonalisation. We can write $U = e^{iH}$ where H_{IJ} is a traceless Hermitian matrix.

3.4.1 Eigenvalue repulsion

Since the theory is invariant under unitary transformations, there is a sense in which all of the physics is contained in the eigenvalues λ_i. We would like to write the partition function in terms of an integral over the λ_i. This change of variables will introduce a Jacobian. We shall see how to handle this sort of change of variables using the Faddeev-Popov construction in later sections but the calculation in zero dimensions is simple enough that we can proceed, at this stage, without inventing new techniques. The partition function is an integral over the space of M_{IJ} a line elemnt on this space is

$$ds^2 = Tr(dM\, dM^\dagger).$$

It is not hard to show that $dM = U(dD + i[dH, D])U^\dagger$, and so

$$ds^2 = \sum_I d\lambda_I^2 + \sum_{I \neq J} (\lambda_I - \lambda_J)^2 |dH_{IJ}|^2.$$

We then have

$$dM = \frac{1}{|S_N|} \prod_I d\lambda_I \prod_{I \neq J} |\lambda_I - \lambda_J||dH_{IJ}|,$$

where we have divided out by the obvious redundancy of permutation of the N eigenvalues by the permutation group S_N. The partition function becomes

$$\begin{aligned} Z_N[M] &= \frac{1}{|S_N|} \int \prod_I d\lambda_I \prod_{I \neq J} |\lambda_I - \lambda_J||dH_{IJ}|\, e^{-NTr(V(M))} \\ &= \frac{|U(N)|}{|U(1)^N \times S_N|} \int \prod_I d\lambda_I \prod_{I \neq J} |\lambda_I - \lambda_J|\, e^{-N\sum_{I=1}^N Tr(V(\lambda_I))}, \end{aligned} \tag{3.12}$$

[22] The precise form is not important but is easy to guess

$$dM = \prod_I dM_{II} \prod_{I<J} d(\Re M_{IJ}) d(\Im M_{IJ}).$$

[23] We shall assume the eigenvalues are distinct but the analysis does not require this.

where we have performed the integral over the H_{IJ} to give the volume of $U(N)/U(1)^N$. We can absorb this into a normalization and introduce an effective action for the eigenvalues so that the partition function becomes

$$Z_N[\lambda] = \int \prod_I d\lambda_I \, e^{-N^2 S_{\text{eff}}(\lambda)},$$

where

$$S_{\text{eff}}(\lambda) = \frac{1}{N}\sum_{I=1}^{N} Tr(V(\lambda_I)) - \frac{1}{N^2}\sum_{I\neq J} \ln|\lambda_I - \lambda_J|.$$

In higher dimensions, we would interpret the gradient of the potential as a force

$$\frac{dS_{\text{eff}}}{d\lambda_I} = \frac{1}{N}V'(\lambda_I) - \frac{1}{N^2}\sum_{J\neq I}\frac{1}{\lambda_I - \lambda_J}.$$

We might interpret the effect of the Jacobian term as a repulsive force, pushing the eigenvalues apart from one another. As such this is often referred to as eigenvalue repulsion.

We note that $\frac{1}{N}$ is playing the role of $\hbar$ here, so that $N \to \infty$ corresponds to the classical limit. The eigenvalue repulsion term is very clearly a quantum effect. We shall see in the following that perturbation theory simplifies dramatically in the large N limit.

3.4.2 Large N diagrammatics

Let's return to the description of the model in terms of Hermitian matrices M_{IJ} and consider the specific example

$$NV(M) = N\left(\frac{1}{2}\alpha M^2 + \frac{g}{4!}M^4\right).$$

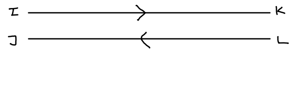
It is useful to absorb the factor of N to make contact with previous results and write this as

$$NV(M) = \frac{1}{2}\alpha\phi^2 + \frac{g/N}{4!}\phi^4, \qquad \phi_{IJ} = \sqrt{N}M_{IJ}$$

If we introduce a source J_{IJ}, we can proceed as before but just have to be careful with matrix indices. It is not hard to show that, for $g = 0$, the generating function is

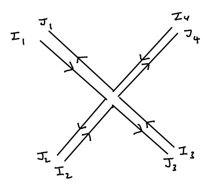
$$Z_N(J) = \exp\left(\frac{1}{2\alpha}tr(J^2)\right) = \exp\left(\frac{1}{2}J_{IJ}\frac{\delta_{IK}\delta_{JL}}{\alpha}J_{KL}\right),$$

so that the propagator is read off as



$$= \frac{\delta_{IK}\delta_{JL}}{\alpha} \tag{3.13}$$

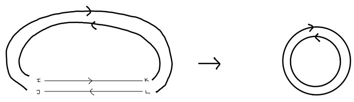
Setting $g > 0$ and proceeding as before, the quartic vertex is



$$= -\frac{g}{N}\delta_{I_1 J_2}\delta_{I_2 J_3}\delta_{I_3 J_4}\delta_{I_4 J_1}, \tag{3.14}$$

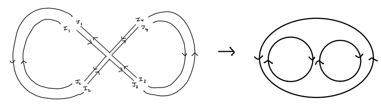
where we have introduced a double line notation that has a line for each index on the matrix Φ_{IJ} (and arrows to denote row and column indices).

Constructing vacuum diagrams is then straightforward as indicated below at one-loop by joining the ends of the propagator



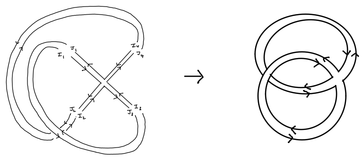
$$\sim \frac{N^2}{\alpha}. \tag{3.15}$$

The trace over all contracted indices produces gives a power of N^2. Note that, from two-loops on, we can contract the indices in qualitatively different ways, where the trace over all contracted indices produces gives a power of N^3, a power of N for each boundary. One possibility is



$$\sim N^3 \times \left(\frac{g}{N}\right) \times \left(\frac{1}{\alpha}\right)^2 = \frac{N^2 g}{\alpha^2}, \tag{3.16}$$

where there are three disconnected boundaries. Another is



$$\sim N \times \left(\frac{g}{N}\right) \times \left(\frac{1}{\alpha}\right)^2 = \frac{g}{\alpha^2}, \tag{3.17}$$

where there is only a single boundary. The general rules are

- Allocate $\frac{1}{\alpha}$ for each propagator.
- Allocate $\frac{g}{N}$ for each vertex.
- Include a factor of N for each closed boundary.
- divide by the symmetry factor

We observe that those diagrams that can be drawn on a plane ('planar diagrams') all scale as N^2. In particular, neglecting the symmetry factor (which is independent of N) the k-loop planar vacuum diagram is evaluated as

$$N^{k+1} \times \left(\frac{g}{N}\right)^{k-1} \times \left(\frac{1}{\alpha}\right)^{2k-2} = N^2 \left(\frac{g}{\alpha^2}\right)^{k-1},$$

for $k > 1$ and N^2/α for $k = 1$. The non planar diagrams are proportional to N^n where $n < 2$. As such, in the limit $N \to \infty$, it is only the planar diagrams that contribute and the theory simplifies dramatically.

One can go further. We have focused on vacuum diagrams but the analysis applies to correlation functions also, where we also find that the planar diagrams dominate in the large N limit. A general correlation function will include products of traces of matrices, such as $Tr(M^2)Tr(M^4)$. Such multi-trace objects are sub-leading in N limit and only the

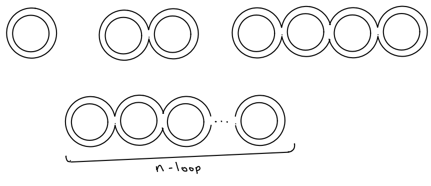


Figure 6. All planar vacuum diagrams are proportional to N^2.

single trace objects, such as $Tr(M^6)$ survive in the $N \to \infty$ limit. If we define the object $\mathcal{O}_n = Tr(M^n)$, then correlation functions between different $\mathcal{O}_n$ will vanish in the large N limit. There is a sense then that the theory becomes free in this limit as all correlation functions between the basic observables vanish in this limit. We observed that N^{-1} seemed to play the role of $\hbar$ in the eigenvalue analysis and so the large N limit appears to give a classical, free theory. These observations generalise to more realistic field theories, such as Yang-Mills theory, as we shall see in later chapters. They also have a profound interpretation in the context of the AdS/CFT correspondence.

4 Quantum Field Theory in Four Dimensions

We should be clear. What we mean by dimension is the as follows: If the action can be written in terms of a (local) Lagrangian density

$$S[\phi] = \int_{\mathcal{M}} d^D x\, \mathcal{L}(\phi),$$

the theory is said to be a D-dimensional quantum field theory defined on a manifold $\mathcal{M}$. Broadly speaking, the dimension is the number of parameters the objects ϕ are functions of. In thins sense, Quantum Mechanics is a $D = 1$ dimensional QFT, even though it may describe physics in three-dimensional space, plus time. Similarly, the worldsheet theory of string theory is a 2-dimensional QFT, even though the target space may be 10 or 26 dimensional[24].

From Functions to Functionals

The most obvious change is that the fields ϕ are now functions of a universal parameter - time, so $\phi \to \phi(\mathbf{x}, t)$. Our key objects, such as the action and generating function(al) are now functionals of $\phi(\mathbf{x}, t) = \phi(x)$ and $J(x)$ respectively. The objects appearing in the four-dimensional case have corresponding functionals

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial\phi, ...), \qquad Z[J] = \int \mathcal{D}\phi\, e^{\frac{i}{\hbar}\left(S[\phi] + \int d^4x\, J(x)\phi(x)\right)},$$

with time-ordered correlation functions given by

$$\langle T\{\phi(x_1)...\phi(x_n)\}\rangle := \frac{\langle\Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \left[\frac{(-i\hbar)^n}{Z[J]} \frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)}\right]_{J=0},$$

where $|\Omega\rangle$ is the vacuum state. The partition function is given by $Z[0] = \langle\Omega|\Omega\rangle$.

It is often useful to have in mind a reference vacuum, which we shall take to be $|0\rangle$, the vacuum of the corresponding free theory.

A Comment on The Vacuum

In previous discussions, we saw that the partition function contained information about vacuum-to-vacuum processes. In this sense, it seems sensible to define our vacuum $|\Omega\rangle$ such that

$$Z[0] = \langle\Omega|\Omega\rangle,$$

where an appropriate normalization is chosen. A first point to note is that the vacuum is different for different theories since, for an interacting theory, the partition function depends non-trivially on the coupling constants and other parameters. This also fits with the path

[24] A key point here is that, even though the worldsheet physics is described by a QFT, the target space physics in certainly not that of a QFT (modulo statements of holography).

integral description of a state. For all practical purposes in this course we shall take the $|\Omega\rangle$ as the lowest energy state of the Hamiltonian on the theory (although this might not always be possible in more general applications[25])

A given state may be written as

$$|\psi(t)\rangle = e^{-iHt}|\phi_i\rangle = e^{-iE_\Omega t}\langle\Omega|\phi_i\rangle|\Omega\rangle + \sum_{n\neq 0} e^{-iE_n t}\langle n|\phi_i\rangle|n\rangle,$$

where we have expanded in a basis of energy eigenstates. The minimum is taken to be the vacuum $|\Omega\rangle$ with energy E_Ω and we shall assume $E_\Omega < E_1 < E_2 ...$, i.e. there is a minimum positive energy. Taking the limit (using the usual $i\epsilon$ prescription used to define the propagator[26])

$$\lim_{t\to\infty(1+i\epsilon)} \frac{e^{-iHt}|\phi_i\rangle}{\langle\Omega|\phi_i\rangle} = e^{-iE_\Omega t}|\Omega\rangle. \tag{4.1}$$

We conclude that any initial condition in the infinite past or future is the vacuum and we take a normalized vacuum to vacuum correlation function to be

$$\frac{\langle\Omega|T\{\phi_1 ... \phi_n\}|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \lim_{t\to\infty(1+i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi_1 ... \phi_n\, e^{\frac{i}{\hbar}S[\phi]}}{\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}S[\phi]}},$$

where

$$S[\phi] = \int_{-t}^{t} dt' d^3\mathbf{x}\, \mathcal{L}(\phi(\mathbf{x}, t)).$$

The normalization given by dividing out by the partition function removes the factor of $\langle\Omega|\phi_i\rangle$ in (4.1).

4.1 Free, Massive Scalar Field Theory

Our starting point will be a free massive scalar field theory. The Lagrangian is

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

Our starting point for the generating functional is

$$Z[J] \propto \int \mathcal{D}\phi\, \exp\left[\frac{i}{\hbar}\int d^4x \Big(\mathcal{L}(\phi) + J(x)\phi(x)\Big)\right].$$

[25] This is not always the case and we may need so define our vacuum state in other ways. Another way to specify the vacuum could be the state that exhibits the all of the symmetries of the empty spacetime geometry (as we might do in de Sitter space). Happily, in flat Minkowski spacetime the lowest energy and maximally symmetric state coincide so there is an obvious choice for the vacuum.

[26] The Feynman $i\epsilon$ prescription specifies the contour in the propagator

$$\frac{i}{E^2 - \mathbf{p}^2 - m^2 + i\epsilon},$$

by placing the poles at $E = \pm\sqrt{\mathbf{p}^2 + m^2 - i\epsilon}$. This prescription is equivalent to defining the propagator to be

$$\frac{i}{(1+i\epsilon)E^2 - \mathbf{p}^2 - m^2}.$$

Which is equivalent to rotating the time axis $t \to (1+i\epsilon)t$. In other words, we can keep the poles on the real axis but integrate along a contour the 'tilts' off the real axis (thus unambiguously avoiding the poles).

From now on, we will set $\hbar = 1$ unless otherwise stated. We have seen how the basic structure of path integral perturbation theory works in zero dimensions and it may come as a relief to learn that all of that structure may be transplanted to higher dimensions with little fuss. Copying the zero-dimensional manipulations

$$\begin{aligned} Z[J] &\propto \int \mathcal{D}\phi \, \exp\left[i \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi(x) \right) \right] \\ &= \int \mathcal{D}\phi \, \exp\left[i \int d^4x \left(\frac{1}{2}\phi(-\partial^2 - m^2)\phi + J(x)\phi(x) \right) \right] \end{aligned} \tag{4.2}$$

where we have integrated by parts in the exponent to go from the first to the second line and there is a normalization that we shall fix later. In order to complete the square in the exponent, we need the Green's function for $\Box$. This may be computed by Fourier Transform and is given, in the Feynman $i\epsilon$ prescription as

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}.$$

so that

$$i(\Box_x + m^2 - i\epsilon)D_F(x-y) = \delta^4(x-y).$$

There are lots of propagators we could choose from, each of which satisfy this equation - why this one? The key point is that we are interested in *time-ordered* correlation functions. The Feynman propagator describes the time-ordered products of observables in a free quantum theory. We shall implicitly assume the $\epsilon \to 0$ limit. Completing the square in $\phi(x)$, we can write the exponent as

$$\int d^4x \Big(\mathcal{L}(\phi) + J(x)\phi(x) \Big) = -\frac{1}{2} \int d^4x \widetilde{\phi}(x) \left(\Box + m^2\right) \widetilde{\phi}(x) + \frac{i}{2} \int d^4x d^4y J(x) D_F(x-y) J(y)$$

where

$$\widetilde{\phi}(x) := \phi(x) - i \int d^4y D_F(x-y) J(y)$$

We can now integrate over $\widetilde{\phi}$ to get

$$Z_0[J] \propto Z_0[0] \exp\left(-\frac{1}{2} \int d^4x d^4y J(x) D_F(x-y) J(y) \right)$$

The obvious choice for the normalization is the one we made in the zero dimensional case (divide by $Z_0[0]$), we then have

Generating functional of the free scalar field

$$Z_0[J] = \exp\left(-\frac{1}{2} \int d^4x d^4y \, J(x) D_F(x-y) J(y) \right)$$

Consider the correlation function (assuming the normalisation $\langle 0|0\rangle = 1$)

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = \frac{1}{Z_\lambda[0]} \left(-i \frac{\delta^4}{\delta J^4(x_1)} \right) \left(-i \frac{\delta^4}{\delta J^4(x_2)} \right) Z_\lambda[J] \bigg|_{J=0}$$

Using the above expression for the generating functional, we find

$$\begin{aligned}\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle &= -\frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\exp\left(-\frac{1}{2}\int d^4x d^4y\, J(x)D_F(x-y)J(y)\right)\Big|_{J=0} \\ &= \frac{\delta}{\delta J(x_1)}\int d^4y\, D_F(x_2-y)J(y)\exp\left(-\frac{1}{2}\int d^4x d^4y\, J(x)D_F(x-y)J(y)\right)\Big|_{J=0} \\ &= D_F(x_1-x_2) \end{aligned} \tag{4.3}$$

so we see the two-point function is just the propagator, as we would expect. This also highlights the importance of using the Feynman propagator, which encodes the time-ordering of the correlation function[27]. It is not hard to show that the three-point function vanishes and the four-point function is

$$\begin{aligned}\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle &= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_4-x_2) \\ &\quad + D_F(x_1-x_4)D_F(x_1-x_3) \end{aligned} \tag{4.4}$$

which may be expressed diagramatically as in figure7.

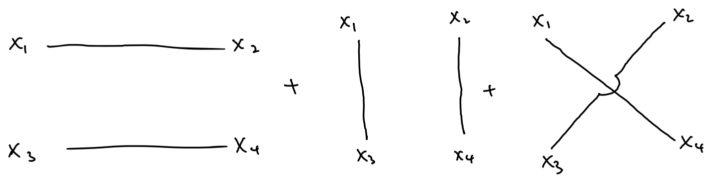


Figure 7. Four-point function in the free theory.

4.1.1 The Yukawa Potential

We can learn something from this theory already. Let us imagine that the scalar boson is the boson for some theory involving additional matter. The additional matter can be modeled by currents $J(x)$, which are sources for the scalar fields $\phi(x)$. This is clear from the classical equation of motion of the action with the source term

$$(\Box + m^2)\phi_J(x) = -J(x).$$

Let us further assume that the source is point-like and, in a particular frame, takes the form $J_1(x) = q_1\delta^3(\mathbf{x}-\mathbf{x}_1)$. The transition functions to and from a vacuum with source J, which we denote by $|0\rangle_J$, separated by a time T is given by

$$\langle e^{-iHT}\rangle_J = \frac{{}_J\langle 0|e^{-iHT}|0\rangle_J}{\langle 0|0\rangle} = \frac{1}{Z[0]}\int_{\phi_J(\mathbf{x}_i,0)}^{\phi_J(\mathbf{x}_f,T)}\mathcal{D}\phi\, e^{iS[\phi]+i\int J\phi} = \exp\left(-\frac{1}{2}\int d^4x d^4y\, J(x)D_F(x-y)J(y)\right).$$

We could take this as a definition of what we mean by the effective potential energy V_{12} between the two sources J_1 and J_2, i.e we define $\langle e^{-iHT}\rangle_J := e^{-iV_{12}T}$. Putting in the

[27]One expects that a more careful treatment of the path integral via time-slicing would directly lead to the appearance of the Feynman propagator.

delta-function sources gives us[28]

$$e^{-iV_{12}T} = \exp\left(-q_1 q_2 \int_0^T dt_1\, dt_2\, D_F(\mathbf{x}_1 - \mathbf{x}_2)\right)$$

Using the momentum space expression for the propagator, we find an expression for the energy

$$V_{12} = \frac{q_1 q_2}{iT} \int_0^T dt_1\, dt_2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x_1 - x_2)}$$

The t_2 (or t_1) integral sets $k_0 = 0$, leaving

$$V_{12} = -\frac{q_1 q_2}{T} \int_0^T dt_1 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}}{\mathbf{k}^2 + m^2 - i\epsilon}.$$

There is no time dependence in the integrand, so we can do the t_1 integral, which cancels with the factor of T in the denominator, giving

$$V_{12} = -q_1 q_2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}}{\mathbf{k}^2 + m^2}.$$

where we have dropped the ϵ as it plays no further role. Setting $r = |\mathbf{x}_1 - \mathbf{x}_2|$, this integral can be done by contour integration and gives

$$V_{12} = -\frac{q_1 q_2}{4\pi r} e^{-mr}.$$

The natural interpretation is that of a scalar potential between the two particles with currents $J_{1,2}$. We see the $m = 0$ case give a Coulomb-like potential, whilst $m > 0$ gives a screened potential. An early model of the strong interactions, considered the exchange of scalar pions between nucleons. Such a massive scalar interaction gives rise to this screened potential, often called the Yukawa potential.

Note that this is a classical calculation. $\hbar$ is absent. We have treated the mysterious source particles as classical sources but this does give a good, first order, feel for what is going on. We have assumed that the only interactions are between the scalar fields and the sources. We now move on to consider a theory in which the scalar field interacts with itself. We can extract this result from an interacting QFT by computing the 4-point amplitude. An example would be a Yukawa-type interaction of the form

$$\mathcal{L} = g\bar{\psi}\phi\psi.$$

The 2 fermion $\to$ 2 fermion amplitude has a current $J = \bar{\psi}\psi$ in the above discussion[29], giving a Yukawa-type potential for the interaction between two fermions.

28

29 The 4-point amplitude is of the form

$$\langle\bar{\psi}(x_1)\bar{\psi}(x_2)\psi(x_3)\psi(x_4)\rangle \sim g^2 \int d^4y d^4z S_F(x_1 - y) S_F(x_3 - y)\psi(x_3) D_F(y - z) S_F(x_2 - y) S_F(x_4 - y),$$

where $S_F(x_i - y)$ is a Feynman propagator for fermions.

A more general observation is that the potential emerged as the spatial Fourier transform of the momentum-space propagator. One might hope that quantum corrections to the potential could be calculated simply by including quantum corrections to the propagator. In general this will not be the full story as there will also be quantum corrections to the interaction vertex (affecting the coupling of the currents to the propagator). There are some situations where apparently magical cancellations occur due to symmetries of the theory and the lowest quantum correction to the potential may de determined from a quantum correction to the propagator.

4.2 Scalar ϕ^4 field theory in four dimensions

We now introduce a quartic interaction term, mirroring what we had in zero dimensions. The Lagrangian is

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4,$$

and the normalised generating functional of the full interacting ϕ^4 theory is

$$Z_\lambda[J] = \frac{1}{Z_0[0]}\int \mathcal{D}\phi \, \exp\left[i\int d^4x\left(\frac{1}{2}\phi(-\partial^2 - m^2)\phi - \frac{\lambda}{4!}\phi^4 + J(x)\phi(x)\right)\right]$$

Playing the same game as we did in zero dimensions, we expand the interaction term as a sum and exchange the summation and functional integration to get

$$\begin{aligned}
Z_\lambda[J] &= \frac{1}{Z_0[0]}\int \mathcal{D}\phi \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i\lambda}{4!}\int d^4y\,\phi^4(y)\right)^k \exp\left[i\int d^4x\left(\frac{1}{2}\phi(-\partial^2 - m^2)\phi + J(x)\phi(x)\right)\right] \\
&= \frac{1}{Z_0[0]}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-i\frac{\lambda}{4!}\right)^k\left((-i)^4\int d^4y\frac{\delta^4}{\delta J(y)^4}\right)^k \\
&\quad\times\int \mathcal{D}\phi \, \exp\left[i\int d^4x\left(\frac{1}{2}\phi(-\partial^2 - m^2)\phi + J(x)\phi(x)\right)\right] \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-i\frac{\lambda}{4!}\right)^k\left(\int d^4y\frac{\delta^4}{\delta J(y)^4}\right)^k Z_0[J] \qquad (4.5)
\end{aligned}$$

The interacting theory then has the generating functional given by

Generating functional for ϕ^4 theory

$$Z_\lambda[J] = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i\lambda}{4!}\right)^k\int d^4x_1...d^4x_k\left(\frac{\delta^4}{\delta J^4(x_1)}\right)...\left(\frac{\delta^4}{\delta J^4(x_k)}\right)Z_0[J]$$

We will denote the vacuum state of the interacting theory by $|\Omega\rangle$.The partition function for the interacting theory describes all vacuum-vacuum processes and so may be written as $Z_\lambda[0] = \langle\Omega|\Omega\rangle$. With this notation, a time-ordered correlation function of a string of scalar fields may be written as

Time-Ordered Correlation Functions

Time-ordered correlation functions are given by

$$\begin{aligned}\langle \phi(x_1)...\phi(x_n)\rangle &:= \frac{\langle\Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle}{\langle\Omega|\Omega\rangle} \\ &= \left[\frac{1}{Z_\lambda[J]}\left(-i\frac{\delta}{\delta J(x_1)}\right)...\left(-i\frac{\delta}{\delta J(x_n)}\right)Z_\lambda[J]\right]_{J=0}\end{aligned}$$

And, as discussed in the one-dimensional case, the time ordering is built in to the time-slicing definition of the path integral. We shall often adopt the compact notation

$$G_n(x_1,...,x_n) = \langle\phi(x_1)...\phi(x_n)\rangle,$$

where there is no danger of confusion (i.e. when we only have one type of operator insertion distinguished only by its position). Let us now look at some examples of correlation functions to get a feel for the origin of the Feynman rules

4.3 Calculation of Correlation Functions

To leading order, the generating functional is

$$Z_\lambda[J] = Z_0[J] - \frac{i\lambda}{4!}\int d^4y \frac{\delta^4}{\delta J(y)^4}Z_0[J] + ...$$

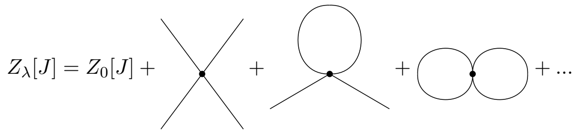
Let us focus on the order λ term

$$\begin{aligned}\int d^4y \frac{\delta^4 Z_0[J]}{\delta J(y)^4} &= \int d^4y \frac{\delta^3}{\delta J(y)^3}\left(-\int d^4z D_F(y-z)J(z)\right)Z_0[J] \\ &= \int d^4y \frac{\delta^2}{\delta J(y)^2}\left(-D_F(0) + \int d^4z_1 d^4z_2\, D_F(y-z_1)D_F(y-z_2)J(z_1)J_(z_2)\right)Z_0[J] \\ &= \int d^4y \frac{\delta}{\delta J(y)}\left(3D_F(0)\int d^4z\, D_F(y-z)J(z)\right. \\ &\quad \left. -\int d^4z_1 d^4z_2 d^4z_3\, D_F(y-z_1)D_F(y-z_2)D_F(y-z_3)J(z_1)J_(z_2)J(z_3)\right)Z_0[J] \\ &= \int d^4y\left\{3\Big(D_F(0)\Big)^2 + \int\prod_{i=1}^{4} d^4z_i D_F(y-z_i)J(z_i)\right. \\ &\qquad \left. -6D_F(0)\int\prod_{i=1}^{2} d^4z_i\, D_F(y-z_i)J(z_i)\right\}Z_0[J] \qquad (4.6)\end{aligned}$$

where

$$D_F(0) = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon},$$

is clearly divergent. We will spend a lot of time understanding such divergences later. For now, we ignore the fact that $D_F(0)$ appears not to exist. The three terms above may be understood diagrammatically below



We have seen similar diagrams already in zero dimensions. The vacuum diagram gives the leading order correction to the partition function

$$\begin{aligned} Z_\lambda[0] &= 1 - \frac{3i\lambda}{4!}\int d^4y \left(D_F(0)\right)^2 + ..., \\ &= 1 + \frac{1}{2^3}\left(-i\lambda\int d^4y\right)\left(D_F(0)\right)^2 + ..., \end{aligned} \tag{4.7}$$

which we have written in a suggestive way to help motivate the Feynman rules later. Briefly, the vertex is located at the point y is assigned a factor of $-i\lambda$. The location of the vertex, since it is arbitrary is integrated over and a propagator $D_F(y-y) = D_F(0)$ is given for each of the two lines that start and end on the vertex. This is doubly divergent. The divergence coming from the y integral is easy to understand, it is just an integral over the infinite volume of spacetime. This probably isn't meaningful in most physical applications so we can perhaps make sense of this. More problematic is the divergence $D_F(0)$ that arises from the momentum integral. Again this is not obviously a problem. Large momentum corresponds to short distance and, if we believe there is some short-distance (high-energy) scale above which our theory must be modified (perhaps by quantum gravity effects), then it is natural to regulate this divergence by some physical cut-off scale. We shall discuss this at great length in later chapters. For now we shall learn to live with such factors in our calculations.

As in zero dimensions, the only contributions to the connected 2-point correlation function comes from the middle diagram, which gives the contribution

$$\begin{aligned} &= (-i)^2\frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\left(-\frac{i\lambda}{4!}(-6)D_F(0)\int d^4y d^4z_1 d^4z_2\, D_F(y-z_1)D_F(y-z_2)J(z_1)J(z_2)\right) \\ &= -\frac{i\lambda}{2}D_F(0)\int d^4y\frac{\delta}{\delta J(x_1)}\left(\int d^4z_1\, D_F(y-z_1)D_F(y-x_2)J(z_1)\right) \\ &= -\frac{i\lambda}{2}D_F(0)\int d^4y\, D_F(y-x_1)D_F(y-x_2) \end{aligned} \tag{4.8}$$

We can relate this to the diagram as follows: the external lines end at the fixed points x_1 and x_2. The node is at the point y and carries a coupling contribution of $-i\lambda$. Two propagators stretch from the external points to the interaction vertex, giving a $D_F(x_i - y)$ each. The location of the vertex is not fixed so we integrate over all possible values. Finally we divide by the symmetry factor of the graph.

Example I: The 1PI amputated propagator and the mass shift

Let us look again at the integral (4.8) integral (ignoring for now that $D_F(0)$ doesn't exist). It is useful to denote

$$\Pi = \frac{\lambda}{2} D_F(0) = \frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon},$$

where the ellipsis denote terms of order λ^2 and higher. (4.8) may be evaluated

$$\begin{aligned} -i\Pi\int d^4y\, D_F(y-x_1)D_F(y-x_2) &= i\Pi_1\int d^4y \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{e^{-ip_1\cdot(x_1-y)}}{p_1^2-m^2+i\epsilon}\frac{e^{-ip_2\cdot(x_2-y)}}{p_2^2-m^2+i\epsilon} \\ &= \frac{i\Pi_1}{(2\pi)^4}\int d^4p_1 d^4p_2 \frac{e^{-i(p_1\cdot x_1-p_2\cdot p_2)}}{(p_1^2-m^2+i\epsilon)(p_2^2-m^2+i\epsilon)}\delta^4(p_1+p_2) \\ &= i\Pi_1\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{(p^2-m^2+i\epsilon)^2} \end{aligned} \tag{4.9}$$

where we have done the y integral to yield a delta-function. The two-point connected correlation function is then

$$\begin{aligned} \langle T\{\phi(x_1)\phi(x_2)\}\rangle_{\text{conn}} &= D_F(x_1-x_2) + i\Pi_1\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{(p^2-m^2+i\epsilon)^2}+\dots \\ &= \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x_1-x_2)} + i\Pi_1\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{(p^2-m^2+i\epsilon)^2}+\dots \\ &= \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x_1-x_2)}\left(1+\frac{\Pi_1}{p^2-m^2+i\epsilon}\right)+\dots \end{aligned} \tag{4.10}$$

where the ellipsis denote terms of higher order in λ. Although the motivation to do so may seem a bit of a stretch at the moment, we can write the term in parenthesis as

$$1+\frac{\Pi_1}{p^2-m^2+i\epsilon} = \left(1-\frac{\Pi_1}{p^2-m^2+i\epsilon}\right)^{-1}+\mathcal{O}(\lambda^2),$$

so that

$$\langle T\{\phi(x_1)\phi(x_2)\}\rangle_{\text{conn}} = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2-\Pi_1+i\epsilon}e^{-ip\cdot(x_1-x_2)}+\mathcal{O}(\lambda^2).$$

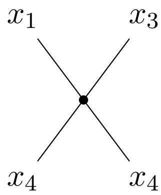
The key point is that the one-loop correction shifts the mass, so that the pole in the two-point function now lies at $p^2=m^2+\Pi(p^2)$. This means that, to first order[30] the qualitative form of the propagator is unchanged by quantum corrections. We shall see this phenomenon in zero dimensions when we calculate the quantum effective action $\Gamma(\Phi)$, where we will see a one-loop shift in the quadratic term. This highlights the fact that the quantities such as m and λ that appear in the classical action are not observable[31], only their quantum-corrected analogues that appear in correlation functions can be measured. As commented previously, this is very similar to the problem of knowing statistical parameters in classical probability distributions. Another troublesome feature is that, unless we make sense of $D_F(0)$, the shift in the mass is infinite!

[30] We shall see that it is true for all orders.

[31] Except, in principle, in the classical limit $\hbar\to 0$. However $\hbar\approx 1.05\times10^{-34}$Js so this is not available in practice.

Example II: The Four-point function

The contribution to the four-point function at order λ comes from the first term in figure (3.6), to give the Feynman diagram



where the central node is at the location y that we integrate over. The contribution is straightforwardly evaluated

$$
\begin{aligned}
&(-i)^4 \frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_3)} \int d^4y \left(-\frac{i\lambda}{4!}\int \prod_{i=1}^{4} d^4 z_i\, D_F(y-z_i)J(z_i)\right) \\
&= -i\lambda \int d^4y \prod_{i=1}^{4} D_F(y-x_i)
\end{aligned}
\tag{4.11}
$$

It is instructive to evaluate this

$$
-(i)^4 i\lambda \int d^4y \int \prod_{i=1}^{4} \frac{d^4p_i}{(2\pi)^4} \frac{1}{p_i^2 - m^2 + i\epsilon} \exp\left(i\sum_{i=1}^{4} p_i \cdot (y - x_i)\right)
$$

Performing the y integral gives a delta-function with support on $\sum_i p_i = 0$, i.e. momentum conservation

$$
-(2\pi)^4 i\lambda\delta^4\left(\sum_{i=1}^{4} p_i\right) \int \prod_{i=1}^{4} \frac{d^4p_i}{(2\pi)^4} \frac{i}{p_i^2 - m^2 + i\epsilon} \exp\left(-p_i \cdot x_i\right),
$$

which, by writing as,

$$
\int \prod_{i=1}^{4} \frac{d^4p_i}{(2\pi)^4} e^{-p_i\cdot x_i} \left(-(2\pi)^4 i\lambda\delta^4\left(\sum_{i=1}^{4} p_i\right) \widetilde{D}(p_1)\widetilde{D}(p_2)\widetilde{D}(p_3)\widetilde{D}(p_4)\right),
$$

can be seen as the Fourier transform of the momentum space amplitude

$$
-(2\pi)^4 i\lambda\delta^4\left(\sum_{i=1}^{4} p_i\right) \widetilde{D}(p_1)\widetilde{D}(p_2)\widetilde{D}(p_3)\widetilde{D}(p_4)
$$

where the momentum space Feynman propagator is

$$
\widetilde{D}(p) = \frac{i}{p^2 - m^2 + i\epsilon}.
$$

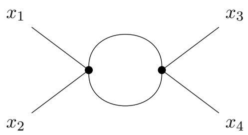
4.4 The Feynman rules from the path integral

We have now seen enough to state the spacetime Feynman rules

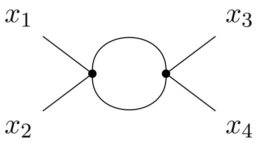
Spacetime Feynman Rules

- For each each propagator $D_F(x-y)$
- For each vertex $-i\lambda \int d^4y$
- Divide by the symmetry factor.

As an example, we can write down the contribution to the order λ^2 connected four-point function given in figure (**??**)



The blobs are vertices at the points y and z that are integrated over. Using the Feynman rules, the contribution is straightforwardly evaluated as



$$= \frac{(-i\lambda)^2}{2} \int d^4y d^4z \, D_F(y-z)^2 D_F(x_1-y) D_F(x_2-y) D_F(x_3-z) D_F(x_4-z).$$

where in the diagram on the left we have implicitly assumed integration over the points y and z. We now consider what this looks like in momentum space:

$$\begin{aligned}
&\frac{(-i\lambda)^2}{2} \int d^4y d^4z \, D_F(y-z)^2 D_F(x_1-y) D_F(x_2-y) D_F(x_3-z) D_F(x_4-z) \\
&= \frac{(-i\lambda)^2}{2} \int d^4y d^4z \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon} \frac{1}{q^2-m^2+i\epsilon} e^{-i(k+q)(y-z)} \\
&\quad \times \int \prod_{i=1,2} \frac{d^4p_i}{(2\pi)^4} \frac{e^{-ip_i\cdot(x_i-y)}}{p_i^2-m^2+i\epsilon} \times \int \prod_{j=3,4} \frac{d^4p_j}{(2\pi)^4} \frac{e^{-ip_j\cdot(x_j-z)}}{p_j^2-m^2+i\epsilon}
\end{aligned} \tag{4.12}$$

We can do the y and z integrals

$$\int d^4y \, e^{-iy\cdot(k+q-p_1-p_2)} = (2\pi)^4 \delta^4(k+q-p_1-p_2), \qquad \int d^4z \, e^{-iz\cdot(-k-q-p_3-p_3)} = (2\pi)^4 \delta^4(k+q+p_3+p_4),$$

gives

$$\begin{aligned}
&\frac{(-i\lambda)^2}{2} \int \prod_{i=1}^{4} \frac{d^4p_i}{(2\pi)^4} \frac{e^{-ip_i\cdot x_i}}{p_i^2-m^2+i\epsilon} \int d^4k \, d^4q \, \frac{\delta^4(k+q-p_1-p_2)}{k^2-m^2+i\epsilon} \frac{\delta^4(k+q+p_3+p_4)}{q^2-m^2+i\epsilon} \\
&= \frac{(-i\lambda)^2}{2} \delta^4\left(\sum_{i=1}^{4} p_i\right) \int d^4q \frac{1}{k^2-m^2+i\epsilon} \frac{1}{q^2-m^2+i\epsilon} \int \prod_{i=1}^{4} \frac{d^4p_i}{(2\pi)^4} \frac{e^{-ip_i\cdot x_i}}{p_i^2-m^2+i\epsilon},
\end{aligned}$$

where $k = p_1 + p_2 - q$ in the last line. We see that this is clearly the Fourier transform of the following momentum space expression

$$\frac{1}{2}(-i\lambda)^2(2\pi)^4\,\delta^4\left(\sum_{i=1}^{4} p_i\right)\prod_{i=1}^{4}\widetilde{D}_F(p_i)\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}\frac{i}{(p_1+p_2-q)^2-m^2+i\epsilon}.$$

We can amputate the external propagators by removing the factors of $\widetilde{D}_F(p_i)$ The (amputated) graph with all external legs removed is

$$\frac{1}{2}(-i\lambda)^2(2\pi)^4\,\delta^4\left(\sum_{i=1}^{4} p_i\right)\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}\frac{i}{(p_1+p_2-q)^2-m^2+i\epsilon}.$$

We have seen enough now to spot the Feynman rules for the theory in momentum space[32]

Momentum Space Feynman Rules

- For each line, include the Fourier transform of the associated propagator

$$\widetilde{D}_F(p) = \frac{i}{p^2-m^2+i\epsilon}$$

- At each vertex include a factor of

$$-i\lambda$$

  and impose momentum conservation at each vertex.

- For each loop, integrate over the unconstrained momentum

$$\int\frac{d^4q}{(2\pi)^4}$$

- Impose overall momentum conservation

$$(2\pi)^4\delta^4\left(\sum_i p_i\right),$$

  for all *external* momenta.

- Divide by the symmetry factor.

The third rule - integrating over all internal momentum lines - comes from the fact that the Fourier transform only acts on external states, so the integrals associated with these insertions are not removed when we do to momentum space.

[32]Alternatively, we could arrive at the momentum space Feynman rules by writing the classical action in terms of an integral over momentum space fields and derive the perturbation expansion of the path integral in momentum space directly.

As an example, consider the diagram in figure **??** above in momentum space

$$\frac{1}{2}(-i\lambda)^2 \int d^4k\, d^4q\, \delta^4(p_1 + p_2 - k - q)\delta^4(p_3 + p_4 + k + q)\widetilde{D}_F(k)\widetilde{D}_F(q) \prod_{i=1}^{4} \widetilde{D}_F(p_i).$$

The first delta function fixes $k + q = p_1 + p_2$ and so

$$\frac{1}{2}(-i\lambda)^2(2\pi)^4\, \delta^4\left(\sum_{i=1}^{4} p_i\right) \int \frac{d^4q}{(2\pi)^4}\, \widetilde{D}_F(p_1 + p_2 - q)\widetilde{D}_F(q) \prod_{i=1}^{4} \widetilde{D}_F(p_i).$$

which agrees with the expression found by the Fourier transform of the spacetime expression. Overall momentum conservation will always appear as a consequence of spacetime translation symmetry of the theory. We notice that the second delta function does not help us do the second integral, so we will always have momentum integral to do for each internal *loop*.

5 The Effective action and the Quantum Effective Action

In most cases the generating function $Z[J]$ is not the most useful object to work with. Here we shall introduce the effective action $W[J]$ and the quantum effective action $\Gamma(\Phi)$. The former generates connected diagrams only, thus removing a huge redundancy in the encoding of the theory. The latter further reduces redundancy by generating only the basic building blocks of the Feynman diagrams. Moreover $\Gamma[\Phi]$ gives insight into the effective physics of the quantum theory. In this section we shall return to our zero-dimensional toy model to introduce the main ideas before generalizing these constructions to four-dimensions.

5.1 Connected Correlation Functions and the Wilsonian effective action

We have seen that the normalization of the correlation function removes a large number of diagrams (the vacuum-vacuum processes) that do not contribute to interesting physical processes. The idea of distinguishing between the diagrams we consider important to a process and those that are in some sense superfluous, will be an important one. For two-point functions, it is only the vacuum bubbles that we would like to remove. All other diagrams seem to contribute in an interesting way; however, for higher point correlation functions matters are more complicated.

In higher dimensions, we shall think of the four-point function $\langle \phi^4 \rangle$ as a describing a process in which, for example, two particles scatter off each other (or perhaps one particle decays to three). In this case we are primarily interested in how the particles interact; however, there will be "disconnected diagrams" that describe, not the interaction but the corrections to the free particle propagator of each individual particle. The problem of contributions from The disconnected diagrams are a bit of a distraction as they don't really tell us how different particles interact with each other This is an important part of the process but it is one that is already described by the two-point function, i.e. it is in some sense not intrinsic to the four-point function.

There are a lot of disconnected diagrams floating around. Can we find away to systematically remove them?

The sheer number of disconnected graphs required for its evaluation makes $Z_\lambda(J)$ a rather cumbersome object top deal with. There is also the sense that the disconnected graphs aren't adding anything new, they are reproducing information that has already appeared at lower order. We would like to find a generating function for the connected graphs only. We have seen that normalizing by $Z_\lambda(0)$ removes the disconnected vacuum-vacuum diagrams leaving In this case, we see that only connected diagrams appear and we

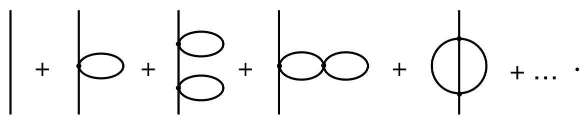


Figure 8. Connected 2-point diagrams.

have

$$\langle \phi^2 \rangle = \langle \phi^2 \rangle_{\text{conn}}.$$

More interesting is the 4-pt function which, when we normalize to remove the vacuum bubbles, leaves diagrams There are no longer vacuum bubbles, but there are still disconnected

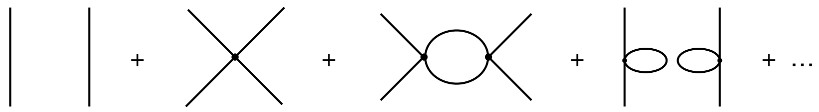


Figure 9. Connected 4-point diagrams.

diagrams. The expression has the form

$$\langle \phi^4 \rangle = \langle \phi^4 \rangle_{\text{conn}} + \Big(\langle \phi^2 \rangle \langle \phi^2 \rangle + 2 \text{ similar terms} \Big).$$

We know that

$$\langle \phi^n \rangle = \frac{1}{Z_\lambda(0)} \frac{\partial^n Z_\lambda(J)}{\partial J^n} \bigg|_{J=0}$$

To appreciate the fact that $Z(J)$ is a generating function for correlation functions, we can consider the Taylor expansion (assuming such an expansion exists) of $Z(J)$ about $J = 0$

$$\boxed{Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial^n Z(J)}{\partial J^n} \right]_{J=0} J^n = Z(0) \sum_{n=0}^{\infty} \frac{1}{n!} \langle \phi^n \rangle \, J^n.} \tag{5.1}$$

It is in this sense we think of the generating function as the generator of correlation functions. If the quantum field theory exists, there must be some sense in which a full set of correlation functions exists and so each term in the above series also exists (although the series may not converge). It is in this sense we say the generating function exists.

It would be helpful to find a generating function $W(J)$ for the connected diagrams, such that

$$W(J) \sim \sum_{n=0}^{\infty} \frac{1}{n!} \langle \phi^n \rangle_{\text{conn}} \, J^n. \tag{5.2}$$

We will make the following *claim*, which we shall prove later

Effective Action

The function $W_\lambda(J)$, defined by

$$Z_\lambda(J) = e^{-W_\lambda(J)}$$

generates connected diagrams, i.e.

$$\langle \phi^n \rangle_{\text{conn}} = - \frac{\partial^n W_\lambda(J)}{\partial J^n} \bigg|_{J=0}$$

Note that the issue of normalization is not ambiguous here as derivatives of $W(J)$ always give derivatives of $Z(J)$, divided out by $Z(J)$, for example (using primes to denote derivatives wrt J)

$$-W'''' = \frac{Z''''}{Z} - 4\left(\frac{Z'''}{Z}\right)\left(\frac{Z'}{Z}\right) + 12\left(\frac{Z''}{Z}\right)\left(\frac{Z'}{Z}\right)^2 - 6\left(\frac{Z'}{Z}\right)^4 - 3\left(\frac{Z''}{Z}\right)^2,$$

so dimensional analysis tells us that derivatives of W will always be homogenous in terms of Z. If we set $J = 0$ in this expression and assume all 1-pt and 3pt correlators vanish, we have

$$\langle \phi^4 \rangle_{\rm conn} = \langle \phi^4 \rangle - 3\langle \phi^2 \rangle \langle \phi^2 \rangle,$$

which is as we would expect for ϕ^4 theory.

The connected correlators have a more natural interpretation in terms of moments that measure changes about a mean. For example, consider the two-point connected correlator

$$\frac{\partial^2 W(J)}{\partial J^2} = -\frac{1}{Z(J)}\frac{\partial^2 Z(J)}{\partial J^2} + \left(\frac{1}{Z(J)}\frac{\partial Z(J)}{\partial J}\right)^2,$$

so

$$\langle \phi^2 \rangle_{\rm conn} = \langle \phi^2 \rangle - \langle \phi \rangle^2,$$

which, in a statistical theory, we would describe as the variance of the random variable ϕ. If we define $W_n = \langle \phi^n \rangle_{\rm conn}$, then

$$\boxed{W(J) = -\sum_{n=0}^{\infty} \frac{1}{n!} W_n J^n, \qquad W_n = -\left[\frac{\partial^n W(J)}{\partial J^n}\right]_{J=0} = \langle \phi^n \rangle_{\rm conn}}$$

Proof That $W(J)$ Generates Connected Diagrams

[Under Construction: See Prof. Osborne's lectures in the meantime.]

5.2 Quantum Effective Action

In this section we will briefly introduce a parameter g that will play a role very much like $\hbar$ but should not be confused with $\hbar$. To make the distinction clear, we will reintroduce $\hbar$ explicitly. And so, the generating function and Wilsonian effective actions are written as

$$Z(J) = \int d\phi \, e^{-\frac{1}{\hbar}(S(\phi)+\phi J)}, \qquad W(J) = -\hbar \ln(Z(J)),$$

so we see that, whilst $S(\phi) + \phi J$ does not depend on $\hbar$, $Z(J)$ and $W(J)$ depend on $\hbar$ in a complicated way. Indeed expanding $W(J)$ in powers of $\hbar$ is equivalent to a loop expansion of Feynman diagrams.

Legendre Transformations

As you may already know from thermodynamics, Legendre transforms are a great way of changing from a function of one dependent variable to another function of the conjugate variable. Suppose we have a function $W(J)$ which depends on the variable J. Clearly

$$dW(J) = \frac{\partial W(J)}{\partial J} dJ.$$

We then introduce a new variable Φ, defined by

$$\Phi := \frac{\partial W(J)}{\partial J},$$

so that $dW(J) = \Phi dJ$. This simply says that if we change J, we change W; i.e. W is a function of J. The interesting thing is is we then define a new object $E = \Phi J - W$. It follows that

$$dE = \Phi dJ + J d\Phi - dW = J d\Phi,$$

from the definition of Φ in terms of the derivative of W above. We see that a change in Φ gives rise to a change in E and so we conclude that E, so defined, is actually a function of Φ alone. The transformation has taken us from a function $W(J)$ to a function $E(\Phi)$. Given the conventions we have adopted for our quantum theory, it is actually more helpful to work in terms of the function $\Gamma(\Phi) = -E(\Phi)$, so we have

$$\Gamma(\Phi) + J\Phi = W(J), \qquad \Phi := \frac{\partial W(J)}{\partial J}.$$

One can then go on to prove many useful results including

$$\frac{\partial \Gamma}{\partial \Phi} + J + \frac{dJ}{d\Phi}\Phi = \frac{\partial W}{\partial \Phi} = \frac{dJ}{d\Phi}\frac{\partial W}{\partial J} = \Phi \frac{dJ}{d\Phi}, \tag{5.3}$$

and so

$$\frac{\partial \Gamma}{\partial \Phi} + J = 0$$

We introduce the mean field Φ, which is related to the Wilsonian effective action by

$$\Phi = \frac{\partial W}{\partial J} = \frac{1}{Z(J)} \int d\phi\, \phi\, e^{-(S(\phi)+J\phi)} = \langle \phi \rangle_J. \tag{5.4}$$

This relationship can be inverted so that it determines J in terms of Φ and, as such, allows us to think of the Wilsonian effective action as a function of Φ as $W(\Phi) = W(J(\Phi))$. This is reminiscent of the Legendre transform that takes us from the Lagrangian $L(x, \dot{x})$ to the Hamiltonian $H(p, x)$ in classical mechanics[33]. Note that, if we assume there are no one-point functions in our theory, then $\Phi = 0$ when $J = 0$. We define the Quantum effective action as

$$\Gamma(\Phi) = W(\Phi) - \Phi J \tag{5.5}$$

This is a Legendre transform which exchanges J as the dependent variable for Φ. We can define the vertex functions $\Gamma^{(n)}$ as

$$\Gamma(\Phi) = \sum_{n=1}^{\infty} \frac{1}{n!} \Gamma_n \Phi^n$$

so that

$$\Gamma_n = \frac{\partial^n}{\partial \Phi^n} \Gamma(\Phi)\bigg|_{\Phi=0}$$

This might seem a little strange at first but this is actually quite a natural change of variable. There is a sense in which we can think of J as a conjugate variable to ϕ. This is most noticeable when we use differentiation by J to introduce a ϕ into the path integral. There is a sense in which

$$\phi \sim -\frac{\partial}{\partial J}.$$

This is reminiscent of the relationship between momentum and position, with one realised as a derivative with respect to the other. What we are doing by introducing Φ is introducing a variable that is in some sense a quantum version of ϕ, that is a variable with which we can write an action in which all of the quantum effects are included.

The central question now is; what does $\Gamma(\Phi)$ compute?

5.2.1 1PI Diagrams

The answer is that the quantum effective action computes the (amputated) one-particle-irreducible diagrams. Since these diagrams are the basic building blocks of all Feynman diagrams, the tree-level vertices of the full quantum effective action compute the full quantum-corrected amplitudes. Let us see why this is the case.

[33] The idea there is to define the momentum p, conjugate to x as

$$p = \frac{\partial L}{\partial \dot{x}}.$$

We take this to then define $\dot{x}$ in terms of p, from which we construct the Hamiltonian

$$H(p, x) = p\dot{x}(p) - L(x, \dot{x}(p)).$$

In order to understand the role of $\Gamma(\Phi)$, let us use it as a generalised action and see what it computes. Treating $\Gamma(\Phi)$ as an action functional, we can introduce a corresponding Wilsonian effective action as

$$e^{-\frac{1}{g}W_\Gamma(J)} := \int d\phi \, \exp\left[-\frac{1}{g}\Big(\Gamma(\phi) + J\phi\Big)\right],$$

where $\Gamma(\phi)$ has the same functional form as (5.5) but we allow ϕ to be a free variable (i.e. it is distinct from Φ in (5.4)) we have introduced a fictitious parameter g, that plays a role much like $\hbar$. However, g should not be confused with $\hbar$. It is important to realise that $\Gamma(\phi)$, defined in (5.5) contains all loop effects. We also note that $W_\Gamma(J)$, is defined in the same way as the usual Wilsonian action and so the interpretation of $W_\Gamma(J)$ is that it is the generating function for connected diagrams with propagators and vertices given by $\Gamma(\phi)$. In particular (assuming one-point functions vanish), we can write

$$e^{-\frac{1}{g}W_\Gamma(J)} := \int d\phi \, \exp\left[-\frac{1}{g}\left(\frac{1}{2}\Gamma_2\phi^2 + \sum_{n>2}\frac{1}{n!}\Gamma_n\phi^n + J\phi\right)\right],$$

and $W_\Gamma(J)$ computes diagrams with propagators Γ_2^{-1} and vertices $\Gamma_{n>2}$. This seems a little odd, though. We have already said that $\Gamma(\phi)$ is the full quantum effective action with all order $\hbar$ effects included[34], and it looks like we are quantising the theory *again*, this time with g playing the role of $\hbar$. Clearly, the useful information is only contained in the 'classical' part, where we take g to zero. As a precursor to taking this limit, we note that we can write $W_\Gamma(J)$ as a loop expansion in g

$$W_\Gamma(J) = \sum_{\ell=0}^{\infty} g^\ell W_\Gamma^{(\ell)}(J)$$

To see the relationship between $W_\Gamma(J)$ and $W(J)$, we take the $g \to 0$ limit. We notice two things about this limit:

- $\lim_{g\to 0} W_\Gamma(J) = W_\Gamma^{(0)}(J)$, so only the tree level ($\ell = 0$) part of $W_\Gamma(J)$ survives the limit.
- In this limit the path integral is dominated by the extremum of the exponent $\Gamma(\phi)+J\phi$ (Hamilton's principle).

The first point means that we can identify the Wilsonian effective action with (the 'tree-level') $W_\Gamma^{(0)}(J)$.

The second point means that we evaluate the exponent on $\phi = \Phi$, where Φ is given by

$$\left[\frac{\partial \Gamma}{\partial \phi} + J\right]_{\phi=\Phi} = 0$$

i.e. we define the value of ϕ at the minimum to be[35] $\Phi(J)$. Taken with the first point, means we have

$$e^{-W_\Gamma^{(0)}(J)} = \exp\left[-\left(\Gamma(\Phi) + J\Phi\right)\right]$$

[34]This follows from treating $W(J)$ in (5.5) as an effective action.

[35]We note that at $J = 0$, $\Phi(0) = \phi_0$, the 'classical' value of the field wrt the action Γ.

and so $W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi$. However, $\Gamma(\Phi) + J\Phi$ is equal to the Wilsonian effective action $W(J)$ from the definition of Γ in (5.5) and so we see that

$$W_\Gamma^{(0)}(J) = W(J).$$

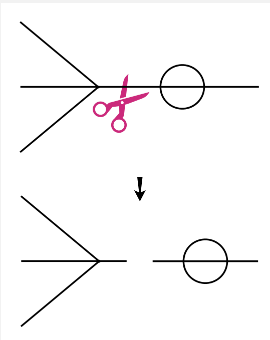
This is important. It says that the *tree-level* diagrams generated using the Feynman rules of $\Gamma(\phi)$ precisely reproduce the full connected quantum correlation functions, i.e. the Γ_n are the basic building blocks, or irreducible components, of the Feynman diagrams we have been looking for. We can make the idea of a basic irreducible component more precise by introducing notation:

Diagrammology

Connected Diagram A diagram made of just one piece. More specifically, one can move from one point to any other point on the diagram coninuously without leaving the diagram.

Amputated Diagram We say a diagram is amputated, if the external legs are removed.

1PI Diagram This is short for "one-particle irreducible". A One-Particle Irreducible diagram is a diagram which does not fall into two pieces if you cut one internal line. The diagram below gives an example of of a diagram that is not 1PI but may be divided into two 1PI parts.



Put another way, $\Gamma(\Phi)$ generates the full n-point amputated, 1PI diagrams.

5.2.2 Effective Vertices

As noted above, the relationship between the classical source J and mean field Φ is

$$\left[\frac{\partial \Gamma(\phi)}{\partial \phi} + J\right]_{\phi=\Phi} = 0.$$

If we set $J = 0$ in this expression, we find that the field $\Phi_0 := \Phi(0)$ describes an extremum of the quantum effective action. And it is our expectation[36] that this describes a minimum.

[36] In using the Legendre transformation, we have assumed Γ is convex.

We can then expand the quantum effective action around this minimum by introducing the quantum fluctuations $\Phi = \Phi_0 + \varphi$, where the fields φ describe the deviations from $J = 0$. The effective action may then be written as

$$\Gamma(\varphi) = \Gamma_0 + \frac{1}{2}\Gamma_2\varphi^2 + ... + \frac{1}{n!}\Gamma_n\varphi^n + ...$$

where

$$\Gamma_n = \left[\frac{\partial^n\Gamma(\Phi)}{\partial\Phi^n}\right]_{\Phi=\Phi_0},$$

and we note that the assumption that we are expanding around an extremum of Γ means that $\Gamma_1(\Phi_0) = 0$ by definition of Φ_0. The $\Gamma_n(\Phi_0)$ are called the vertices of the quantum effective action and these are derived by summing all of the n-point 1PI diagrams of the theory.

Comparing this with a conventional action, we expect $\Gamma_2(\Phi_0)$ to play the role of a kinetic term and we would expect its inverse $1/\Gamma_2(\Phi_0)$, to be related to the connected 2-point function. This is indeed what we find and we shall explore this action in greater detail in the context of ϕ^4 theory in four dimensions.

Effective Kinetic Terms: The relationship between W_2 and Γ_2

Assume we have solved for the mean field in terms of J, so that $\Phi = \Phi(J)$. Using the chain rule, we can write

$$\frac{\partial}{\partial J} = \frac{d\Phi}{dJ}\frac{\partial}{\partial\Phi}.$$

Substituting in the definition of Φ as the derivative of the effective action W, gives

$$\frac{\partial}{\partial J} = \frac{\partial^2 W(J)}{\partial J^2}\frac{\partial}{\partial\Phi}. \tag{5.6}$$

We can now apply this derivative to

$$\frac{\partial\Gamma(\Phi)}{\partial\Phi} = -J,$$

to find

$$\frac{\partial}{\partial J}\left(\frac{\partial\Gamma(\Phi)}{\partial\Phi}\right) = -\frac{\partial J}{\partial J} = -1. \tag{5.7}$$

Putting (5.6) (5.7) together yields

$$\frac{\partial^2 W(J)}{\partial J^2}\frac{\partial^2\Gamma(\Phi)}{\partial\Phi^2} = -1,$$

Recalling that

$$W_2 = \langle\phi^2\rangle_{\mathrm{conn}} = -\left[\frac{\partial^2 W(J)}{\partial J^2}\right]_{J=0},$$

we have

$$\boxed{W_2\Gamma_2 = 1.}$$

We see that Γ_2 is just the inverse propagator (connected 2-point function).

Effective Interaction Terms: The relationship between W_n and Γ_n for $n > 2$

We now study the relationship between connected correlation functions and effective interaction vertices. We shall do this explicitly for $n = 3$ and (hopefully) the general procedure for determining such relationships will be clear. Applying $\partial/\partial J$ to

$$\frac{\partial^2 W(J)}{\partial J^2}\frac{\partial^2\Gamma(\Phi)}{\partial\Phi\partial\Phi} = -1$$

gives

$$\frac{\partial}{\partial J}\left(\frac{\partial^2 W(J)}{\partial J^2}\frac{\partial^2\Gamma(\Phi)}{\partial\Phi^2}\right) = 0$$

The LHS can be expanded out using the chain rule, the RHS gives zero

$$\frac{\partial^3 W(J)}{\partial J^3}\frac{\delta^2\Gamma(\Phi)}{\partial\Phi^2} + \frac{\partial^2 W(J)}{\partial J^2}\frac{\partial}{\partial J}\left(\frac{\partial^2\Gamma(\Phi)}{\partial\Phi^2}\right) = 0.$$

Using

$$\frac{\partial}{\partial J} = \frac{\partial^2 W(J)}{\partial J^2}\frac{\partial}{\partial\Phi}$$

on the second term on the LHS gives

$$\frac{\partial^3 W(J)}{\partial J^3}\frac{\partial^2\Gamma(\Phi)}{\partial\Phi^2} + \frac{\partial^2 W(J)}{\partial J^2}\frac{\partial^2 W(J)}{\partial J^2}\frac{\partial^3\Gamma(\Phi)}{\partial\Phi^3} = 0.$$

We can write this more compactly as $W_3\Gamma_2 = -W_2^2\Gamma_3$. Multiplying through by W_2 gives

$$W_3W_2\Gamma_2 = -W_2^3\Gamma_3,$$

then using $\Gamma_2W_2 = 1$ on the LHS gives our final expression

$$\boxed{W_3 = -W_2^3\Gamma_3}$$

In other words

$$\langle\phi^3\rangle_{\text{conn}} = -\langle\phi^2\rangle_{\text{conn}}^3\langle\phi^3\rangle_{\text{1PI}}$$

This demonstrates that the full three-point function can be decomposed into a three-point 1PI diagram with connected two-point functions on its external legs. Note that this statement holds to all orders in perturbation theory.

Proceeding in the same way, we find a similar decomposition for the four-point function which we write schematically as

$$W_4 = W_2^4\Gamma_4 + W_2^5\Gamma_3^2 \tag{5.8}$$

This is much easier viewed as a diagram, as in figure 13 The interpretation is as for the

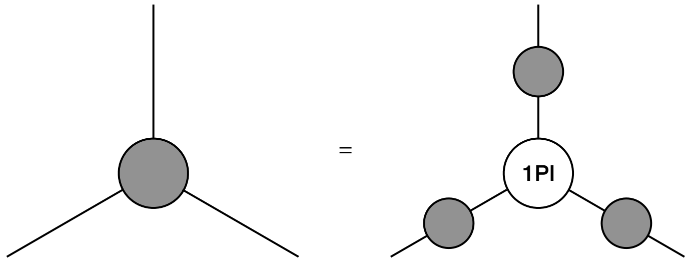


Figure 10. A pictorial version of the statement $W_3 = -W_2^3\Gamma_3$.

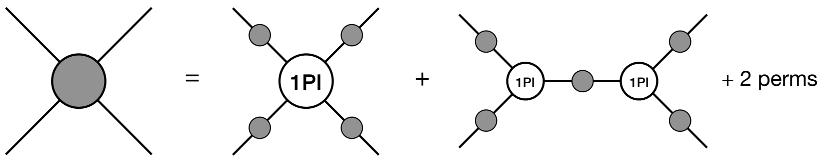


Figure 11. A pictorial version of the statement $W_4 = W_2^4\Gamma_4 + W_2^5\Gamma_3^2$.

three-point function and we conclude that the $\Gamma_n[\phi_{\rm cl}]$ are the 1PI diagrams.

Loops and $\hbar$

We have one power of $\hbar$ for each internal propagator and a power of $\hbar^{-1}$ for each vertex. External propagators add nothing. We shall see later that the number L of loops is given by

$$L = P - (V - 1)$$

So the overall $\hbar$ dependence in a diagram is

$$\hbar^{P-V+1} = \hbar^L.$$

Thus we see that the number of loops counts the powers of $\hbar$. We conclude that tree-level diagrams are describing classical physics and quantum effects are contained in the loop diagrams.

5.2.3 An Example: One-loop correction to Γ in ϕ^4 theory

Let's see how this works for our ϕ^4 theory up to fist order in λ

$$S(\phi) + J\phi = \frac{1}{2}\alpha\phi^2 + \frac{\lambda}{4!}\phi^4 + J\phi$$

so the generating function to order λ is

$$Z(J) = \left(1 - \hbar^3\frac{\lambda}{4!}\left(\frac{J^4}{\alpha^4\hbar^4} + \frac{6J^2}{\alpha^3\hbar^3} + \frac{3}{\alpha^2\hbar^2}\right) + \ldots\right)e^{\frac{J^2}{2\hbar\alpha}}$$

where we have been careful to put the $\hbar$ back in explicitly[37]. The Wilsonian effective action is then (using $\ln(1+x) = x + ...$)

$$W(J) = -\hbar \ln(Z(J)) = -\frac{J^2}{2\alpha} + \frac{\lambda}{4!}\frac{J^4}{\alpha^4} + \frac{\lambda\hbar}{4}\frac{J^2}{\alpha^3} + \frac{\hbar^2\lambda}{8\alpha^2} + ...$$

where we have truncated to first order in $\hbar$ and λ. The mean field is then given by

$$\Phi(J) := \frac{\partial W}{\partial J} = -\frac{J}{\alpha} + \frac{\lambda}{3!}\frac{J^3}{\alpha^4} + \frac{\lambda\hbar}{2}\frac{J}{\alpha^3} + ... = -\frac{J}{\alpha}\left(1 - \frac{\lambda\hbar}{2\alpha^2}\right) + \frac{\lambda}{3!}\frac{J^3}{\alpha^4} + ...$$

We can invert this (by plugging the equation back into itself) to get[38]

$$\begin{aligned}
J(\Phi) &= -\alpha\left(1 - \frac{\lambda\hbar}{2\alpha^2}\right)^{-1}\left(\Phi - \frac{J^3}{3!}\frac{\lambda}{\alpha^4}\right) + ... \\
&= -\alpha\left(1 + \frac{\lambda\hbar}{2\alpha^2}\right)\left(\Phi - \frac{J^3}{3!}\frac{\lambda}{\alpha^4}\right) + ... \\
&= -\alpha\Phi + \frac{\lambda}{3!}\frac{J^3}{\alpha^3} - \frac{\lambda\hbar}{2\alpha}\Phi + ... \\
&= -\alpha\Phi - \frac{\lambda}{3!}\Phi^3 - \frac{\hbar\lambda}{2\alpha}\Phi + ... \qquad (5.9)
\end{aligned}$$

and so, with a little work

$$W(\Phi) = \frac{\hbar^2\lambda}{8\alpha^2} - \frac{1}{2}\alpha\Phi^2 + \left(\frac{1}{4!} - \frac{1}{3!}\right)\lambda\Phi^4 - \frac{\lambda\hbar}{4\alpha}\Phi^2 + ...$$

The quantum effective action is then

$$\begin{aligned}
\Gamma(\Phi) &= W(\Phi) - J(\Phi)\Phi \\
&= \frac{\hbar^2\lambda}{8\alpha^2} - \frac{1}{2}\alpha\Phi^2 + \left(\frac{1}{4!} - \frac{1}{3!}\right)\lambda\Phi^4 - \frac{\lambda\hbar}{4\alpha}\Phi^2 - \left(-\alpha\Phi - \frac{\lambda}{3!}\Phi^3 - \frac{\hbar\lambda}{2\alpha}\Phi + ...\right)\Phi + ... \\
&= \frac{\lambda\hbar^2}{8\alpha^2} + \frac{1}{2}\left(\alpha + \frac{\hbar\lambda}{2\alpha}\right)\Phi^2 + \frac{\lambda}{4!}\Phi^4 + ... \qquad (5.10)
\end{aligned}$$

where the ellipsis denote terms of order λ^2. Setting $\hbar = 0$, the effective action reproduces the classical action but the terms linear in $\hbar$ give us the one-loop quantum correction. One of our goals with more realistic theories will be to calculate these quantum corrections.

A few brief comments on this result

- The one-loop quantum corrections do not break the $\Phi \to -\Phi$ symmetry.
- There is no correction to the Φ^4 coupling term at order λ. This is to be expected as the first connected diagram appears at one-loop and order λ^2.

[37]This is easy to do in practice as one can show that we get a factor of $\hbar$ for each loop in a diagram.
[38]Note that if we set $J = 0$ we get the classical equation of motion in the $\hbar \to 0$ limit, as we would expect.

- There is a one-loop correction to the propagator. We see above that

$$\Gamma_2 = \alpha + \frac{\hbar\lambda}{2\alpha}$$

and using the result $\Gamma_2 W_2 = 1$ tells us that

$$W_2 = \left(\alpha + \frac{\hbar\lambda}{2\alpha}\right)^{-1} = \frac{1}{\alpha} - \frac{\hbar\lambda}{2\alpha^3} + ...$$

to leading order in λ. This is exactly the connected 2-point function to one-loop.

- There is a vacuum contribution coming from the two-loop vacuum diagram with contribution $\frac{\lambda\hbar^2}{8\alpha^2}$.

- Note that we get one factor of $\hbar$ for each loop.

We make one final comment about the parameters of the theory. In the classical theory we might say that analogue of a mass-squared term in a scalar theory is α, so that if we asked what the mass of the particle is we would answer $\sqrt{\alpha}$. In practice, the theory is not classical and we can only infer the values of parameters from measurements. In this case, we would say that we can infer the values of our parameters α and λ by measuring the connected 2-point and 1PI 4-point correlation functions:

$$W_2 = \frac{1}{\alpha} + \frac{\hbar\lambda}{2\alpha^3} + ..., \qquad \langle\phi^4\rangle_{1PI} = \lambda + ...$$

We see that in practice this quantity, as measured by the two-point function receives quantum corrections. We learn an important lesson from this: there is a distinction between the abstract parameters we use to construct a theory and what we measure in practice. Only the latter is physically meaningful.

Summary

In summary:

Name	**Functional**	**The diagrams it generates**
Partition Function	$Z[0]$	Vacuum Bubbles
Generating Functional	$Z[J]$	Everything
Effective Action	$W[J] = -\frac{1}{\hbar}\ln(Z[J])$	Connected
Quantum Effective Action	$\Gamma[\Phi] = W[\Phi] - J\Phi$	Amputated 1PI

It is worth emphasising that:

- The classical action $S(\phi)$ gives the theoretical input that defines the theory (it is not observable).

- It is the effective actions $Z(J)$, $W(J)$ and $\Gamma(\Phi)$ that give the predictions of what we will actually observe.

5.3 The Quantum Effective Action and 1PI Diagrams in four dimensions

The generalization to higher dimensions is straightforward. Defining the mean field as

$$\Phi[J] := \frac{\delta W[J]}{\delta J(t)} = \frac{-i\hbar}{Z[J]}\frac{\delta Z[J]}{\delta J(t)} = \langle 0|\phi(t)|0\rangle_J .$$

In many cases, one can explicitly show that the mean field corresponds to the classical field in the limit $\hbar \to 0$. We note that Φ depends on the source $J(t)$. The quantum effective action is defined as the Legendre transform of $W[J]$

$$\Gamma[\Phi] = W[J] - \int d^4x\, J(x)\Phi(x)$$

The variation of this functional mirrors the classical Euler-Lagrange equations in the presence of a classical source

$$\frac{\delta S[\phi]}{\delta\phi(x)} = -J(x),$$

with ϕ replaced by Φ and $S[\phi]$ replaced by $\Gamma[\Phi]$. This can be seen as follows:

$$\begin{aligned}
\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} &= \frac{\delta W[J]}{\delta\Phi(x)} - \int d^4y\, \frac{\delta J(y)}{\delta\Phi(x)}\Phi(y) - J(x) \\
&= \int d^4y\, \frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\Phi(x)} - \int d^4y\, \frac{\delta J(y)}{\delta\Phi(x)}\frac{\delta W[J]}{\delta J(y)} - J(x) \\
&= -J(x).
\end{aligned} \tag{5.11}$$

And so we see that

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = -J(x),$$

as claimed. When $J(x) = 0$

$$\Phi(x) = \langle 0|\phi(x)|0\rangle := \phi_{\text{cl}}$$

is a constant. Thus:

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)}\bigg|_{\Phi=\phi_{\text{cl}}} = 0.$$

This is an extremum equation which we shall see is often easy to analyse to determine whether a symmetry is spontaneously broken.

5.3.1 Computing the Effective Vertices

In this section, we would like to relate the vertex functions Γ_n to the connected correlation functions. This gives us then a relationship between derivatives of $\Gamma[\Phi]$ and derivatives of $W[J]$. It is useful to write

$$\Gamma_n[\Phi] = \frac{\delta^n\Gamma[\Phi]}{\delta\Phi(x_1)...\delta\Phi(x_n)}, \qquad W_n[J] = \frac{\delta^n W[J]}{\delta J(x_1)...\delta J(x_n)}.$$

We shall find explicit relations between the two-point functions W_2 and Γ_2 and the three-point functions W_3 and Γ_3 to illustrate the relationship. Statements about higher point functions can be similarly derived. the relationship between the $W_n[J]$ and the connected correlation functions is

$$\langle T\{\phi(x_1)...\phi(x_n)\}\rangle_{\text{conn}} = (-i)^{n-1} W_n[0],$$

and we shall see that, for $n > 2$,

$$\langle T\{\phi(x_1)...\phi(x_n)\}\rangle_{1\text{PI}} = i\Gamma_n[\phi_{\text{cl}}].$$

5.4 Effective Kinetic Terms: The relationship between W_2 and Γ_2

Using the functional analogue of the chain rule, we can write

$$\frac{\delta}{\delta J(x)} = \int d^4y \, \frac{\delta \Phi(y)}{\delta J(x)} \frac{\delta}{\delta \Phi(y)}.$$

Substituting in the definition of $\Phi(x)$ as the derivative of the effective action $W[J]$, gives

$$\frac{\delta}{\delta J(x)} = \int d^4y \, \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \frac{\delta}{\delta \Phi(y)}. \tag{5.12}$$

We can now apply this derivative to

$$\frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x),$$

to find

$$\frac{\delta}{\delta J(x)} \left(\frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} \right) = -\frac{\delta J(x)}{\delta J(y)} = -\delta^4(x-y). \tag{5.13}$$

Putting (5.12) (5.13) together yields

$$\int d^4z \, \frac{\delta^2 W[J]}{\delta J(y)\delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z)\delta \Phi(x)} = -\delta^4(x-y),$$

And so

$$\boxed{\int d^4z \, W_2(y,z)\Gamma_2(z,x) = -\delta^4(x-y).}$$

which we can schematically write as $W_2\Gamma_2 = -1$. But

$$-i \left. \frac{\delta^2 W[J]}{\delta J(y)\delta J(z)} \right|_{J=0} = G_2(y-z), \qquad \left. \frac{\Gamma[\Phi]}{\delta \Phi(z)\delta \Phi(x)} \right|_{\Phi=\phi_{\text{cl}}} = \Gamma_2(z,x)$$

where we recall that setting $J = 0$ in $W_n[J]$ is equivalent to setting Φ equal to the constant ϕ_{cl} in $\Gamma_n[\Phi]$. We have written $G_2(x,y)$ for the connected two-point function. The relationship between Γ_2 and G_2 becomes more transparent in momentum space. Writing $G_2(x,y)$ as

$$G_2(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\epsilon} e^{-ip\cdot(x-y)},$$

where $M^2 = m^2 + \Pi_1$, we have

$$i\int d^4z\, G_2(y,z)\Gamma_2(z,x) = -\int d^4z \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{\widetilde{\Gamma}(q)}{p^2 - M^2 + i\epsilon}e^{-ip\cdot(y-z)}e^{-iq\cdot(z-x)} = -\delta^4(x-y)$$

Performing the z integration gives a $\delta^4(p-q)$, which we then use to do the q integration, giving

$$\int \frac{d^4p}{(2\pi)^4}\frac{\widetilde{\Gamma}(p)}{p^2 - M^2 + i\epsilon}e^{-ip\cdot(y-x)} = \delta^4(x-y),$$

and so we see that

$$\widetilde{\Gamma}(p) = p^2 - M^2 + i\epsilon.$$

5.4.1 Effective Interaction Terms: The relationship between W_n and Γ_n

We now study the relationship between connected correlation functions and effective interaction vertices. We shall do this explicitly for $n = 3$ and (hopefully) the general procedure for determining such relationships will be clear.

Applying $\delta/\delta J(w)$ to

$$-\int d^4z\, \frac{\delta^2 W[J]}{\delta J(y)\delta J(z)}\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(x)} = \delta^4(x-y)$$

gives

$$\frac{\delta}{\delta J(w)}\int d^2z\, \frac{\delta^2 W[J]}{\delta J(y)\delta J(z)}\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(x)} = 0$$

The LHS can be expanded out using the chain rule, the RHS gives zero

$$\int d^4z\, \frac{\delta^3 W[J]}{\delta J(w)\delta J(y)\delta J(z)}\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(x)} + \int d^4z\, \frac{\delta^2 W[J]}{\delta J(y)\delta J(z)}\frac{\delta}{\delta J(w)}\left(\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(x)}\right) = 0.$$

Using

$$\frac{\delta}{\delta J(w)} = \int d^4\sigma\, \frac{\delta^2 W[J]}{\delta J(w)\delta J(\sigma)}\frac{\delta}{\delta\Phi(\sigma)}.$$

The LHS gives

$$\int d^4z\, \frac{\delta^3 W[J]}{\delta J(w)\delta J(y)\delta J(z)}\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(x)} + \int d^4z d^4\sigma\, \frac{\delta^2 W[J]}{\delta J(y)\delta J(z)}\frac{\delta^2 W[J]}{\delta J(w)\delta J(\sigma)}\frac{\delta^3\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(x)\delta\Phi(\sigma)} = 0.$$

We can write this more compactly as

$$\int d^4z\, W_3(w,y,z)\Gamma_2(z,x) = -\int d^4\sigma d^4z\, W_2(y,z)W_2(\sigma,w)\Gamma_3(z,x,\sigma).$$

Multiplying through by

$$\int d^4x\, W_2(x,\rho)$$

gives

$$\int d^4z\, W_3(w,y,z)', \int d^4x\, W_2(x,\rho)\Gamma_2(z,x) = -\int d^4x d^4\sigma d^4z\, W_2(x,\rho)W_2(y,z)W_2(\sigma,w)\Gamma_3(z,x,\sigma),$$

then using

$$\int d^4x\, \Gamma_2(z,x)W_2(x,\rho) = -\delta^4(z-\rho),$$

on the LHS gives our final expression

$$\boxed{W_3(x,y,z) = \int d^4w_1 d^4w_2 d^4w_3\, W_2(x,w_1)W_2(y,w_2)W_2(z,w_3)\Gamma_3(w_1,w_2,w_3).}$$

If we interpret Γ_3 as giving rise to a 1PI three-point function, we have the diagrammatic description of this statement in figure 12. This demonstrates that the full three-point function can be decomposed into a three-point 1PI diagram with connected two-point functions on its external legs. Note that this statement holds to all orders in perturbation theory.

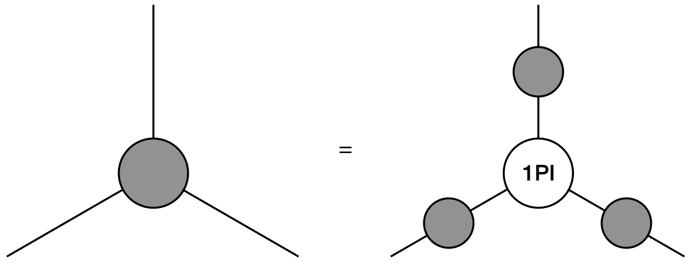


Figure 12.

Proceeding in the same way, we find a similar decomposition for the four-point function which we write schematically as

$$\begin{aligned} W_4(x,y,z,w) = &\int d^4u_1...d^4u_4\, W_2(x,u_1)W_2(y,u_2)W_2(z,u_3)W_2(w,u_4)\Big[\Gamma_4(u_1,u_2,u_3,u_4) \\ &+\int d^4v_1 d^4v_2\Big(\Gamma_3(u_1,u_2,\rho_1)W_2(\rho_1,\rho_2)\Gamma_3(u_3,u_4,\rho_2) \\ &\qquad +\Gamma_3(u_1,u_3,\rho_1)W_2(\rho_1,\rho_2)\Gamma_3(u_4,u_1,\rho_2) \\ &\qquad +\Gamma_3(u_1,u_4,\rho_1)W_2(\rho_1,\rho_2)\Gamma_3(u_2,u_3,\rho_2)\Big)\Big] \end{aligned} \tag{5.14}$$

This is much easier viewed as a diagram, as in figure 13 The interpretation is as for the three-point function and we conclude that the $\Gamma_n[\phi_{\rm cl}]$ are the 1PI diagrams.

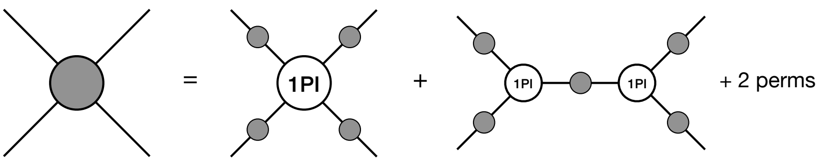


Figure 13.

We conclude by noting that the Quantum Effective Action is most simply written in momentum space. We present it below in all its glory

The Quantum Effective Action

The Quantum Effective Action in d-dimensional Euclidean space isc

$$\Gamma[\phi] = -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\widetilde{\Phi}(-p)\Big(p^2+m^2+\Pi(p^2)\Big)\widetilde{\Phi}(p)$$
$$+\sum_{n=3}^{\infty}\frac{1}{n!}\int\frac{d^dp_1}{(2\pi)^d}...\int\frac{d^dp_n}{(2\pi)^d}(2\pi)^d\delta\left(\sum_{i=1}^n p_i\right)\;\Gamma_n(p_1,...,p_n)\widetilde{\Phi}(p_1)...\widetilde{\Phi}(p_n)$$

where

$$\widetilde{\Phi}(p) = \int d^dx\, e^{-ik\cdot x}\Phi(x).$$

The *tree-level* diagrams generated by $\Gamma[\Phi]$ generate the *full* quantum scattering amplitudes of the original theory.

Let us compare this with a very general class of classical actions for a scalar field in momentum space.

$$S[\phi] = -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\tilde{\phi}(-p)\Big(p^2+m^2\Big)\tilde{\phi}(p)+\sum_{n>2}\frac{\lambda_n}{n!}\int\frac{d^dp_1}{(2\pi)^d}...\int\frac{d^dp_n}{(2\pi)^d}(2\pi)^d\delta\left(\sum_{i=1}^n p_i\right)\;\widetilde{\phi}(p_1)...\widetilde{\phi}(p_n)$$

A few comments are in order

- The functions $\Gamma_n(p_1,...,p_n)$ play the role of the coupling constants λ_n in the Lagrangian.
- The $\Gamma_n(p_1,...,p_n)$ depend on momentum. When we Fourier transform to configuration space the momentum p_i will be transformed to a derivative which will act on the Fourier transform of the field $\widetilde{\Phi}(p_i)$. As such, the quantum effective action will generally involve higher derivative terms.
- The effective coupling constants, as we would normally think of them, therefore arise as $\lambda_n \sim \Gamma_n(0,...,0)$, i.e. the vertex functions evaluated at zero external momentum.
- The effective mass is thus $m^2+\Pi(0)$. As we shall see, the remaining parts of $\Pi(p^2)$ are more naturally absorbed into a wave-function rescaling.

6 Calculating the QEA using momentum cut-off regularization

In this section we start to address the fact that, at loop level, we start to see divergences appear in the Feynman diagram calculations. This is not something we saw in the zero-dimensional theory and appears due to integration over unconstrained parameters (momenta) in loop diagrams. These divergences are real and need to be understood. Before we can make physical sense of such divergences, we need to get a better understanding of what is causing the divergence and exactly *how* a diagram[39] is diverging. Ideally, we would like to isolate a parameter for which the diagram is finite at generic generic values of the parameter but we see the divergence appear as the parameter takes the physically relevant value. This is possible and the process is called regularization.

Our goal will be to compute, to one-loop order, the propagator and quartic vertex ($\Gamma_2$and Γ_4 respectively) of the quantum effective action for ϕ^4 theory in four dimensions. Recall the QEA is given by

$$\Gamma[\phi] = \sum_{n=2}^{\infty} \frac{1}{n!} \int \frac{d^d p_1}{(2\pi)^d} ... \int \frac{d^d p_n}{(2\pi)^d} (2\pi)^d \delta\left(\sum_{i=1}^{n} p_i\right) \Gamma_n(p_1, ..., p_n) \widetilde{\Phi}(p_1) ... \widetilde{\Phi}(p_n)$$

where

$$\widetilde{\Phi}(p) = \int d^d x \, e^{-ik\cdot x} \Phi(x).$$

6.1 The UV Regulator

As we shall see in later sections, a natural regulator is the largest momentum we allow a particle in the internal loop to have. In Minkowski spacetime, there is no reason to constraint the momentum running through the loop; however, we can impose a cut-off momentum Λ and only integrate loop momenta up to Λ. The physical situation is recovered at $\Lambda \to \infty$, whereupon the diagram diverges. What is interesting to us in this section is to find the how the two-point and four-point functions depend on the cut-off.

6.1.1 The $\Gamma^{(2)}$ Divergence

We start with the 2-pt correlation function. We saw in zero dimensions that the quantum effective action received a quantum correction to Γ_2 at order λ

$$\Gamma_2 = \alpha + \frac{\hbar\lambda}{2\alpha} + ...$$

In zero dimensions there is no kinetic term and we can think of α as playing the role of the mass-squared of a scalar theory. As such, this correction can be viewed as a shift in the mass we observe

$$\alpha \to \alpha + \frac{\hbar\lambda}{2\alpha} + ...$$

This highlights an important fact about quantum theories; the parameters we actually measure generally do not take the same values as those in the classical action. A more

[39] We shall rather confusingly refer to both the Feynman diagram itself and the contribution to the correlation function it represents as the 'diagram'. Since they are in one-to-one correspondence, hopefully no meaningful ambiguity will arise.

straightforward (although less direct) way to see this quantum correction is through the connected correlation function

$$W_2 = \Gamma_2^{-1} = \frac{1}{\alpha} - \frac{\hbar\lambda}{2\alpha^3} + ...$$

We will study the analogous one-loop effect in the four-dimensional theory. By consideration of the Feynman diagrams, we can easily see that at order λ^2 there is a one-loop correction to Γ_4, giving the shift

$$\lambda \to \lambda - \frac{3\lambda^2\hbar}{2\alpha^2} + ...,$$

and so the coupling constant we actually measure, if we compute $\langle\phi^4\rangle_{\text{conn}}$ is not the λ we enter into the classical action but some quantum corrected value.

We computed the connected two-point correlation function to first order in λ and found

$$\langle\phi(x_1)\phi(x_2)\rangle_{\text{conn}} = D_F(x_1 - x_2) - \frac{i\lambda}{2} D_F(0) \int d^4y\, D_F(y - x_1) D_F(y - x_2) + ...$$

where we recall that the contribution

$$D_F(0) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}$$

diverges.

Using the Feynman rules to go beyond one-loop, the full propagator $G_2(x_1, x_2) = \langle\phi(x_1)\phi(x_2)\rangle_{\text{conn}}$ may be expressed as in figure 14 . This diagrammatic series can be help-

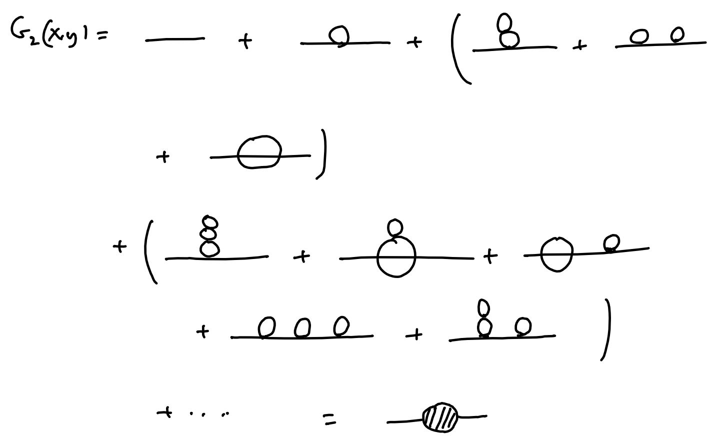


Figure 14. The connected propagator.

fully re-ordered in terms of the free propagator the 1PI amputated two-point function. We will introduce the notation

$$\Pi = i\Big(\text{Amputated 1PI}\Big) = \frac{\lambda}{2} D_F(0) + \mathcal{O}(\lambda^2).$$

This is illustrated diagramatically in 15.

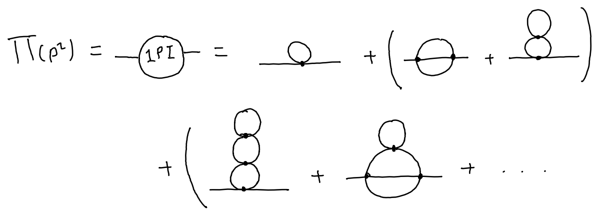


Figure 15. Π The sum of 1PI diagrams contributing to the propagator.

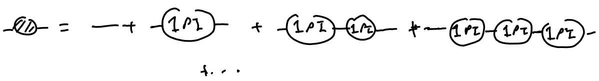


Figure 16. The full propagator in terms of 1PI components.

You can hopefully convince yourself that all of the diagrams are in figure 14 may be written in terms of Π, joined by free propagators as shown in figure 16. Once again, we see that the diagrammatic expression highlights structure that we might otherwise have missed.

It is simpler to work in momentum space, where we will also denote the amputated 1PI contribution by $\Pi(p^2)$ (hopefully without causing confusion), allowing for the fact that at higher orders the sum of amputated 1PI diagrams will be momentum-dependent. In momentum space the sum of the diagrams in figure 16 may be written as

$$\begin{aligned} -iG_2(p^2) &= \frac{1}{p^2 - m^2 + i\epsilon} + \frac{1}{p^2 - m^2 + i\epsilon}\,\Pi(p^2)\,\frac{1}{p^2 - m^2 + i\epsilon} \\ &\quad + \frac{1}{p^2 - m^2 + i\epsilon}\,\Pi(p^2)\,\frac{1}{p^2 - m^2 + i\epsilon}\,\Pi(p^2)\,\frac{1}{p^2 - m^2 + i\epsilon} + \ldots \\ &= \sum_{r=0}^{\infty} \frac{1}{p^2 - m^2 + i\epsilon}\left(\Pi(p^2)\,\frac{1}{p^2 - m^2 + i\epsilon}\right)^r, \end{aligned} \tag{6.1}$$

which is a geometric progression that we can sum to give

$$G_2(p^2) = \frac{i}{p^2 - m^2 - \Pi(p^2)}.$$

The effective vertex $\Gamma^{(2)}$ is the inverse of the correlation function, so

$$\Gamma_2(p^2) = p^2 - m^2 - \Pi(p^2).$$

One way to define the mass that we measure in an experiment is as the pole in the two-point function. As such, the physical mass is not given by the parameter m we put in by hand in the classical action but by

$$M^2(p^2) = m^2 + \Pi(p^2).$$

We also see the somewhat uprising result that the mass we measure appears to be momentum-dependent!

6.2 Wick Rotation and Euclidean QFT

Correlation functions $G_n(\phi(\mathbf{x}_1, t_1), \phi(\mathbf{x}_2, t_2).., \phi(\mathbf{x}_n, t_n))$ are functions of the boundary data, which usually includes the locations $\{\mathbf{x}, t\}$ of the incoming and outgoing fields. Alternatively, we can work in momentum space and think of the corelation functions as functions of external four-momenta. In general these corelation functions will have zeros and poles at physically interesting values of the physical parameters and it is interesting to ask if we can analytically continue these functions to the complex plane, where we allow Minkowski space (or momentum space) to be complexified[40]. Though formulating QFT on complex spacetime is of interest in its own right, we will take a less radical approach and consider the Wick rotation

$$t \to \tau = it,$$

which takes us from Minkowski space with line element

$$ds^2 = dt^2 - d\mathbf{x}^2$$

to

$$ds^2 = -d\tau^2 - d\mathbf{x}^2.$$

We shall see that this is a useful thing to do in evaluating loop integrals. We can then rotate back to Minkowski space once we hae the answer. This is a funny thing to do and one has to be careful of the subtleties. If we want to be more careful we need to think in detail about the analyticity of the correlation functions. There is a theorem of Osterwalder and Schrader[41] that tells us when we can expect Wick rotation to give us a well-defined isomorphism between a QFTs on Minkowski and Euclidean spaces. The argument goes something like this:

1. The Osterwalder-Schrader theorem tells us the conditions under which correlation functions on spacetime are equivalent to correlation functions in Euclidean space.

2. The Wightman reconstruction theorem tells us that knowing all vacuum-vacuum correlation functions is equivalent to knowing the quantum fields themselves.

3. Thus, if the Osterwalder-Schrader conditions are met, the Minkowski and Euclidean theories are isomorphic.

In this course we will always assume that Wick rotation will cause us no problems. To evaluate this Π at leading order in momentum space, we go to Euclidean signature[42].

[40]This idea led to an alternative approach to QFT, known as the Analytic S-Matrix, in which analyticity and physical principles (such as unitarity) were used to attempt to 'bootstrap' a theory. This had only limited success at the time and was quickly superceded by what we would consider to be conventional non-abelian gauge theory but it has recently found to be a useful idea. This seems to be another example of the observation that, in theoretical physics, no good idea goes to waste.

[41]Konrad Osterwalder, Robert Schrader, Axioms for Euclidean Green's functions, Comm. Math. Phys. Volume 31, Number 2 (1973), 83-112

[42]The conventions section before the Introduction provides a useful summary.

In Euclidean space ($\mathbb{R}^4$) we take

$$x_E = it, \qquad \int_{\mathbb{R}^{1,3}} d^4x = i\int_{\mathbb{R}^4} d^4x_E,$$

then, if we also define $ip^0 = p_4$, we see

$$p\cdot x = p_0x^0 - \mathbf{p}\cdot\mathbf{x} = -ip_0x_4 - \mathbf{p}\cdot\mathbf{x} = -p_4x_4 - \mathbf{p}\cdot\mathbf{x} := -p_E\cdot x_E$$

and so phases change sign. The Euclidean propagator is

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+m^2}e^{ip\cdot(x-y)} \tag{6.2}$$

Note the absence of the $i\epsilon$. We have rotated the contour of integration onto the imaginary axis so we easily avoid poles in our prescription.

The momentum space Feynman rules are similar to those we found in zero dimensions (where we adopted Euclidean conventions):

Momentum Space Feynman Rules in Euclidean Signature

- For each line, include the Fourier transform of the associated propagator

$$\widetilde{D}_F(p) = \frac{1}{p^2+m^2}$$

- At each vertex include a factor of

$$-\lambda$$

  and impose momentum conservation at each vertex.

- For each loop, integrate over the unconstrained momentum

$$\int\frac{d^4q}{(2\pi)^4}$$

- Impose overall momentum conservation

$$(2\pi)^4\delta^4\left(\sum_i p_i\right),$$

  for all *external* momenta.

- Divide by the symmetry factor.

Changing between signatures is, for the most part, just a matter of keeping track of conventions.

6.2.1 Γ_2 at one-loop

At one-loop, in Euclidean signature, the vacuum polarization diagram is

$$\widetilde{\Pi}_1 = -\frac{\lambda}{2}\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2},$$

As we shall see below, this integral does not exist. In particular the integration over $\mathbb{R}^4$ diverges. We can regulate this divergence by imposing a maximum upper value for the momentum in the theory of $0 \geq k^2 < \Lambda^2$. Imposing this cut-off amounts to integrating over a solid ball $\mathcal{B}_\Lambda \subset \mathbb{R}^4$ of radius Λ. We note that the integral scales like $k^4/k^2 = k^2$ and so is expected diverge as Λ^2 as we remove the cut-off, i.e. as $\Lambda \to \infty$. We shall see this is the case in detail below. To regulate this integral we shall introduce a momentum cut-off Λ, so that $k^2 \leq \Lambda^2$. We can recover the usual expression for Π_1 by taking the $\Lambda \to \infty$ limit, however the integral diverges in this limit (as we shall see) so it will not be in our interests to take such limits until we have an agreed upon way to handle such divergences. We can write the integral over $\mathbb{R}^4$ in four-dimensional spherical polar coordinates

$$d^4k = k^3\, dk\, d\Omega_4$$

where $d\Omega_4$ is an *area* element of the four-dimensional sphere, whose total area we shall denote by S^4. To do the integral

$$\int_{\mathcal{B}_\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}$$

we clearly need to learn something about computing areas of spheres in arbitrary dimensions.

Areas of (co-dimension one) Spheres in d Dimensions

S_d is defined as the area of the unit sphere in d dimensions, so that the integral of a spherically symmetric function $f(r)$ in d dimensions may be written as

$$\int_{\mathcal{B}_\Lambda} f(r)d^dx = \int_{\mathcal{B}_\Lambda} f(r)r^{d-1}dr\, d\Omega_d = S_d\int_0^\Lambda f(r)r^{d-1}dr,$$

where $d\Omega_d$ is the angular integration measure[43]. We use the standard Gaussian integral $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$ to write

$$\begin{aligned}\sqrt{\pi}^d &= \int_{(\mathbb{R}^4)^d}\prod_{i=1}^d\left(dx_i\, e^{-x_i^2}\right)\\ &= S_d\int_0^\infty dr\, r^{d-1}e^{-r^2}\\ &= \frac{S_d}{2}\int_0^\infty du\, u^{\frac{d}{2}-1}e^{-u}\\ &= \frac{1}{2}S_d\,\Gamma(d/2), \end{aligned} \tag{6.3}$$

[43]For example, in $d=3$, $d\Omega_3 = \sin(\theta)d\theta d\phi$.

where we have written the integral in terms of d-dimensional spherical polar coordinates in the second equality, made a change of variable $u = r^2$ in the penultimate line and used the definition of the Gamma function

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}.$$

in the last. Thus, we have

Areas of (co-dimension one) Spheres in d Dimensions

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

Some useful properties of Gamma functions are

$$\Gamma(n) = (n-1)! \qquad n \in \mathbb{Z}_+.$$

A useful result for us is

$$\Gamma(z) = \frac{1}{z} - \gamma + ...$$

where $\gamma := 0.57721...$ is the Euler-Mascheroni constant and ellipsis denote terms regular in z.

The result we need is then

$$\int d\Omega_4 = S_4 = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2,$$

since $\Gamma(n) = (n-1)!$ if $n \in \mathbb{Z}$. Consider then the integral

$$\int_0^\Lambda dk \frac{k^3}{k^2 + m^2},$$

and let $u = k^2/m^2$. We then have that

$$\begin{aligned}
\int_0^\Lambda dk \frac{k^3}{k^2+m^2} &= \frac{m^2}{2} \int_0^{\Lambda^2/m^2} \frac{udu}{1+u} \\
&= \frac{m^2}{2} \int_1^{1+\Lambda^2/m^2} dw \frac{w-1}{w}, \qquad \text{where we introduce } w = 1+u \\
&= \frac{m^2}{2} \int_1^{1+\Lambda^2/m^2} dw - \frac{m^2}{2} \int_1^{1+\Lambda^2/m^2} \frac{dw}{w} \\
&= \frac{m^2}{2} \left[\frac{\Lambda^2}{m^2} - \ln\left(1 + \frac{\Lambda^2}{m^2}\right) \right].
\end{aligned} \tag{6.4}$$

And so we have the 1-loop result

$$\widetilde{\Pi}_1(p^2) = -\frac{\lambda}{2} \int_{\mathcal{B}_\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} = -\frac{\lambda}{32\pi^2} \left(\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right) \right)$$

where we have denoted the solid ball of radius Λ in $\mathbb{R}^4$ as $\mathcal{B}_\Lambda$. Thus, to 1-loop, the effective vertex is

$$\boxed{\Gamma_2(p^2) = p^2 + m^2 + \frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right] + ...,}$$

where the ellipsis denote terms of order λ^2 and higher. $\widetilde{\Gamma}_2(p^2)$ clearly diverges as we remove the cut-off ($\Lambda \to \infty$) but the value of this result is that we see precisely *how* it diverges in this limit.

6.3 Γ_4 at one-loop

The 4pt function may be written in terms of (amputated) 1PI diagrams as given in figure 17. We only care about the 1-loop contribution so we are interested in the diagrams shown

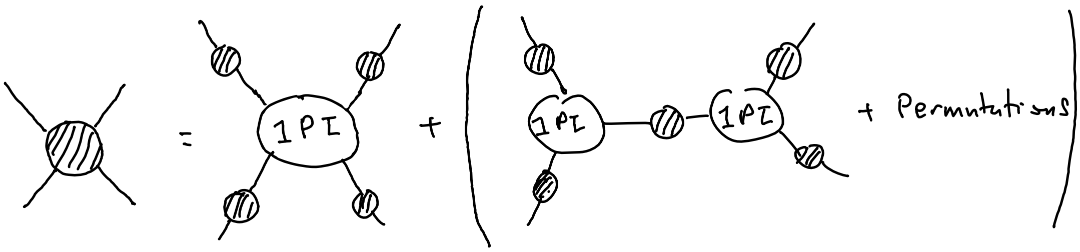
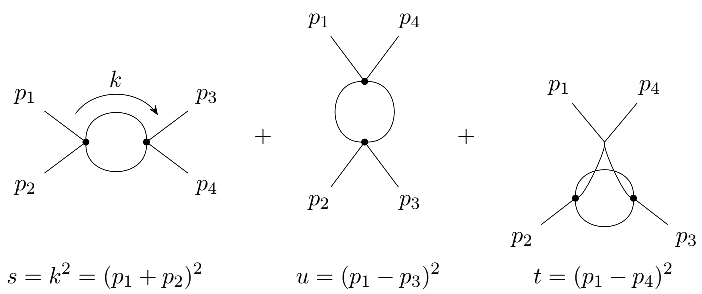
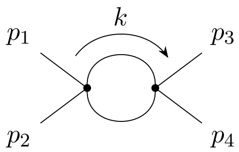


Figure 17.

in figure (**??**).



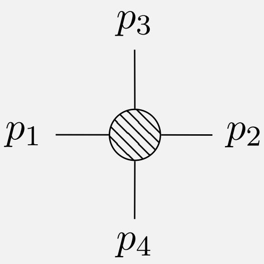
We shall often abbreviate these three diagrams as:



+permutations,

where it is the external momenta that are undergoing the permutation. A comment on notation:

Mandelstam Variables



In the diagram above we imagine Euclidean 'time' going from left to right. The invariant momenta are usefully labelled as

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_4)^2, \qquad u = (p_1 - p_3)^2.$$

It is helpful to remember s as the total ingoing (or outgoing) momentum.

The divergence we are studying comes from the loop integral and is independent of the external momenta. Thus we set the $p_i = 0$, at which point all of the diagrams give the same contribution, and compute

$$\begin{aligned} \Gamma_4(0,0,0,0) &= \lambda - 3 \times \frac{(-\lambda)^2}{2} \int_{\mathcal{B}_\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m^2)^2} \\ &= \lambda - \frac{3\lambda^2}{2} \frac{2\pi^2}{(2\pi)^4} \int_0^\Lambda dk \frac{k^3}{(k^2+m^2)^2}. \end{aligned} \tag{6.5}$$

Let $u := k^2/m^2$ and then $w := 1 + u$ as before

$$\begin{aligned} \Gamma_4(0,0,0,0) &= \lambda - \frac{3\lambda^2\pi^2}{2(2\pi)^4} \int_0^{\Lambda^2/m^2} du \frac{u}{(1+u)^2} \\ &= \lambda - \frac{3\lambda^2}{32\pi^2} \int_1^{1+\Lambda^2/m^2} dw \frac{w-1}{w^2} \\ &= \lambda - \frac{3\lambda^2}{32\pi^2} \int_1^{1+\Lambda^2/m^2} dw \left(\frac{1}{w} - \frac{1}{w^2} \right) \\ &= \lambda - \frac{3\lambda^2}{32\pi^2} \left[\ln\left(1 + \frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2} \right]. \end{aligned} \tag{6.6}$$

In summary

$$\boxed{\Gamma_4(0,0,0,0) = \lambda - \frac{3\lambda^2}{32\pi^2} \left[\ln\left(1 + \frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2} \right]}$$

Again, this is divergent when the cut-off is removed.

6.4 Γ_6 at one-loop

There is no six-point coupling in the classical action so Γ_6 measures a purely quantum effect. In other words, the leading 1PI contribution to Γ_6 occurs at one-loop. The diagram is

[DIAGRAM]

and the integral is

$$\Gamma_6(0,0,0,0,0,0) \sim -i\frac{(-\lambda)^3}{(2\pi)^4}2\pi^2 \int_0^\Lambda dk \frac{k^3}{k^2+m^2{}^3},$$

where we have set all external momenta to zero and have rotated to Euclidean space as above. We have not bothered making the combinatoric factors explicit. Using the same substitutions as for the Γ_4 calculation, we find

$$\Gamma_6(0,0,0,0,0,0) \sim -i\frac{(-\lambda)^3}{8m^2\pi^2}\left[\frac{1}{2}+\frac{1}{2}\left(1+\frac{\Lambda}{m^2}\right)^{-2} - \left(1+\frac{\Lambda}{m^2}\right)^{-1}\right].$$

What is notable about this is that we find a finite answer in the $\Lambda \to \infty$ limit. In hindsight, this is to be expected as the integral is of the form

$$\int \frac{d^4k}{k^6} \sim \Lambda^{-2},$$

and so is not divergent. In fact it is clear that all one-loop diagrams with $n > 2$ of this kind will be convergent. What is also notable is that the answer is not zero - there is a non-trivial quantum effect!

In d dimensions, the diagrams of the given form with $2n$ external legs will behave like

$$\int \frac{d^dk}{(k^2)^{2n}} \sim \Lambda^{d-4n},$$

and so we see that the UV properties of the theory depend on the dimension of spacetime. We expect the leading divergence to be logarithmic when $d = 4n$.

Discussion

In summary, we have seen that

$$\Gamma_2(p^2) = p^2 + m^2 + \frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(1+\frac{\Lambda^2}{m^2}\right)\right] + ...,$$

$$\Gamma_4(0,0,0,0) = \lambda - \frac{3\lambda^2}{32\pi^2}\left[\ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right] + ...$$

It is therefore possible to tame the divergences by introducing a high-energy cut-off into the theory. The question we now face is whether to take $\Lambda \to \infty$. In this limit the above expressions diverge. Can we make sense of the situation in which Λ is large, but finite?

- The presence of a cut-off is not particularly troublesome. We have seen that the path integral is defined as the continuum limit of a lattice construction (time and space splicing), so the large-momentum cut-off reflects the minimum lattice spacing. Just as Newtonain physics is not a good model to all energies, we also do not expect our QFT (on flat spacetime) to be good descriptions of nature to all energies. At some point gravitational effects will become important and we must abandon quantum field theory for something else. The presence of the cut-off Λ simply makes this apparent. The reason we did not have to worry about this before QFT is that, unlike classical physics, processes in QFT include contributions from all energies ia the loop integrals.

- What is a cause for concern is the apparent explicit dependence on a particular cut-off Λ in the correlation functions. This seems to suggest that, in order to compute correlation functions in QFT, we need to know details about the physics at the highest of energies. This is contrary to experience in science; we expect to be able to accurately capture low energy physics using effective theories that are ignorant of what happens at high energies. Put differently, we expect there to be a decoupling of scales in the physics.

This is clearly a deep issue that we need a better understanding of. In the next sections, we shall study the dependence on the effective low energy theory on the high energy cut-off. We shall see that renormalizable theories do indeed exhibit a decoupling of scales and will understand why it is such theories that play such an important role at low energies.

7 Wilsonian renormalization and effective field theory

In the path integral approach to QFT, the basic objects of study, such as the generating functionals $Z[J]$, $W[J]$ and the quantum effective action $\Gamma[\Phi]$ can all be written as functional integrals over the fields that appear in the action functional. Th structure makes it particularly simple to study the cumulative effect of one field on another. For example, if we have a theory with two fields ϕ and χ and action functional $S[\phi, \chi]$, we might only wish to study the correlation functions of the ϕ field and so consider the generting functional

$$Z[J] = \frac{1}{Z[0]} \int \mathcal{D}\phi \mathcal{D}\chi \, e^{\frac{i}{\hbar} S[\phi,\chi] + \frac{i}{\hbar} \int J\phi}.$$

Performing the χ integral then gives the generating functional in terms of the Wilsonian effective action for ϕ,

$$Z[J] = \frac{1}{Z[0]} \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} W[\phi] + \frac{i}{\hbar} \int J\phi},$$

where

$$e^{\frac{i}{\hbar} W[\phi]} = \int \mathcal{D}\chi \, e^{\frac{i}{\hbar} S[\phi,\chi]}.$$

The Feynman diagrams derived from this effective theory will only have external ϕ propagators but the χ fields will still give rise to internal lines in the diagrams and so the interactions between the ϕ and χ fields will still affect the correlation functions, even though χ no longer appears in the effective action. This notion of integrating out degrees of freedom to give effective actions that describe the same physics will be at the heart of our attempts to better understand renormalization in this chapter.

7.1 A toy Model in Zero Dimensions

The Wilsonian effective action $W(J)$ naturally appears when we start to integrate out degrees of freedom. Suppose that we have a theory involving two interacting fields ϕ and χ, with action

$$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4!}\phi^4 + \frac{g}{4}\phi^2\chi^2 + ...$$

The generating function is

$$Z(J, K) = \int d\phi d\chi \, e^{-S(\phi,\chi) + J\phi + K\chi},$$

where J and K are sources for ϕ and χ respectively. Suppose we are in some regime (e.g. $m << M$) where the effects of χ are small and we want to integrate out the χ to get an effective model for ϕ. We set $K = 0$ and define the effective action $W(\phi)$ for ϕ as

$$e^{-W_{\text{eff}}(\phi)} = \int d\chi \, e^{-S(\phi,\chi)},$$

so that the generating function for ϕ is

$$Z(J) = \int d\phi \, e^{-W_{\text{eff}}(\phi) + \phi J},$$

so we see the that the function

$$W_{\text{eff}} = -\ln\left(\int d\chi\, e^{-S(\phi,\chi)}\right),$$

is just the Wilsonian effective action for χ but with ϕ^2 as the source.

In the previous section, we considered integrating out heavy fields to construct an effective action of the light degrees of freedom. In this section, we shall do something similar in QFT. Instead of integrating out different species of field, we will integrate out high momentum states of a given field to acquire a low energy effective action of the theory. Given that we strongly suspect that quantum field theory cannot be a fundamental theory, especially in the context of gravitational physics, this is of more than academic interest.

7.2 Cut-off independence and the Callan-Symanzik equation for $Z[0]$.

If we consider the theory defined with cut-off Λ_0. This cut-off might be physically motivated in some way. For example, it could be a scale at which new physics enters; the threshold for creating a new particle of rest mass of order Λ_0 or the point at which QFT no longer accurately describes the physics (say at the string or Planck scale). We then introduce a second cut-off $\Lambda << \Lambda_0$. The effective action given by integrating out all modes with momentum between some new cut-off Λ and Λ_0 is given by

$$e^{-S_\Lambda} = \int_\Lambda^{\Lambda_0} \mathcal{D}\phi\, e^{-S_{\Lambda_0}[\phi]}.$$

If we then integrate out the remaining fields, we get the partition function of the theory. Similarly, the effective action at some lower cut-off Λ' is given by integrating out fields of momentum greater than Λ'. The partition function that we would get if we integrate out all fields in one go is the same in either case

$$Z_\Lambda[0] = \int_0^\Lambda \mathcal{D}\phi\, e^{-S_\Lambda[\phi]} = \int_0^{\Lambda'} \mathcal{D}\phi\, e^{-S_{\Lambda'}[\phi]} = Z_{\Lambda'}[0].$$

These are both equal to $Z_{\Lambda_0}[0]$. Since the RHS does not depend on Λ and the LHS does not depend on Λ', we have the general result that

$$\Lambda \frac{d}{d\Lambda} Z_\Lambda[0] = 0$$

In general the partition function will be a function of the couplings $g_i = (m^2, \lambda, ...)$, which in turn may depend on the cut-off choice, so we have

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \Lambda \frac{dg_i}{d\Lambda}\frac{\partial}{\partial g_i}\right) Z_\Lambda[0] = 0,$$

and so we anticipate that for general theories, the couplings will change with the scale at which they are measured.

7.3 The 1-Loop Effective action for ϕ^4 theory

The generating functional in Euclidean signature in d-dimensions, is

$$Z_\Lambda[J] = \mathcal{N} \int \mathcal{D}\phi_\Lambda \, \exp\left[- \int d^d x \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda_0}{4!}\phi^4 + J\phi \right) \right].$$

where Λ denotes a momentum cut-off, so that

$$\mathcal{D}\phi_\Lambda = \prod_{|k|\leq\Lambda} d\phi(k)$$

i.e. we integrate up to the momentum Λ and $\mathcal{N}$ is a choice of normalization. We treat m_0 and λ_0 as bare parameters (independent of Λ). We shall take the first steps in finding a low energy effective theory by integrating over a 'shell' of fields $\hat{\phi}$ defined as

$$\hat{\phi}(k) = \begin{cases} \phi(k) & \text{for } b\Lambda \leq |k| < \Lambda, \\ 0 & \text{otherwise} \end{cases}$$

where b is a number between 0 and 1 so that; if $b = 1$ no fields are integrated and if $b = 0$ all fields are integrated. The idea is to treat each momentum k as defining a different field $\phi(k)$ and to integrated out those fields in a particular range. We then write[44]

$$\phi(x) \to \phi(x) + \hat{\phi}(x),$$

where $\phi(x)$ now denotes a field whose Fourier Transform has momentum $|k| < b\Lambda$. Substituting this into the generating functional

$$\begin{aligned} Z_\Lambda[J] = \frac{1}{Z_\Lambda[0]} \int \mathcal{D}\phi_{b\Lambda}\mathcal{D}\hat{\phi} \, \exp\Bigg[-S[\phi] - \int d^d x \Bigg(& \partial\hat{\phi}\cdot\partial\phi + m_0^2\phi\hat{\phi} + \frac{1}{2}(\partial\hat{\phi})^2 + \frac{1}{2}m_0^2\hat{\phi}^2 \\ & + \lambda_0 \left[\frac{1}{4!}\hat{\phi}^4 + \frac{1}{3!}\phi^3\hat{\phi} + \frac{1}{4}\hat{\phi}^2\phi^2 + \frac{1}{3!}\hat{\phi}^3\phi \right] \Bigg) \Bigg]. \end{aligned} \tag{7.2}$$

where

$$S[\phi] = \int d^d x \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{\lambda_0}{4!}\phi^4 \right).$$

Note that Fourier components with different momenta are orthogonal and so

$$\int d^d x \, \phi(x)\hat{\phi}(x) = 0, \qquad \int d^d x \, \partial\hat{\phi}(x)\cdot\partial\phi(x) = 0,$$

leading to a simplification of the action. As before, we can write this as an effective action for the lower energy ϕ fields by integrating out the $\hat{\phi}$ fields

$$Z_{b\Lambda}[J] = \frac{1}{Z_{b\Lambda}[0]} \int \mathcal{D}\phi_{b\Lambda} \, e^{-W[\phi] - \int J\phi},$$

[44] This split is justified as follows

$$\begin{aligned} \phi(x) \to \int_0^\Lambda dk \, \phi(k) e^{ik\cdot x} &= \int_0^{b\Lambda} dk \, \phi(k) e^{ik\cdot x} + \int_{b\Lambda}^{\Lambda} dk \, \phi(k) e^{ik\cdot x} \\ &= \int_0^\Lambda dk \left(\phi(k) + \hat{\phi}(k) \right) e^{ik\cdot x} := \phi(x) + \hat{\phi}(x), \end{aligned} \tag{7.1}$$

where what we mean by $\phi(x)$ should be clear from context.

where

$$e^{-W[\phi]} = \int \mathcal{D}\hat{\phi} \, \exp\left[-S[\phi] - S_0[\hat{\phi}] - S_{\text{int}}[\phi, \hat{\phi}]\right].$$

where

$$S_0[\hat{\phi}] = \frac{1}{2} \int d^d x (\partial\hat{\phi})^2$$

and

$$S_{\text{int}}[\phi, \hat{\phi}] = \int d^d x \left(\frac{1}{2} m_0^2 \hat{\phi}^2 + \frac{\lambda_0}{4!} \left[\hat{\phi}^4 + 4\phi^3\hat{\phi} + 6\hat{\phi}^2\phi^2 + 4\hat{\phi}^3\phi \right] \right).$$

For the purposes of integrating out the $\hat{\phi}$ fields, we treat the ϕ fields as fixed, background, fields and the $\hat{\phi}$ and the dynamical quantum fluctuation. We shall treat $S_{\text{int}}[\phi, \hat{\phi}]$ (including the mass term, for which we assume $m_0^2 << \Lambda^2$) as an interaction term and treat the evaluation of the path integral perturbatively.

We take m_0 and λ_0 to be perturbatively small and we write

$$\begin{aligned} S_0[\hat{\phi}] \to S_0[\hat{\phi}] + \int d^d x \hat{J}(x)\hat{\phi}(x) &= \frac{1}{2} \int d^d k (\partial\hat{\phi}(x))^2 + \int d^d x \hat{J}(x)\hat{\phi}(x) \\ &\to \frac{1}{2} \int d^d x d^d y \, \hat{J}(x)\hat{D}_F(x-y)\hat{J}(y) \end{aligned}$$

where the last term arises, in the usual way, by appropriately normalizing the generating functional and the restricted propagator is

$$\hat{D}_F(x-y) = \int_{b\Lambda}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} e^{ik\cdot(x-y)},$$

where the limits of the integral note that the momentum associated with $\hat{\phi}$ lies between $b\Lambda$ and Λ. Notice that we are treating the mass term for the high momentum modes as a perturbation. In the interaction term, we make the usual replacement

$$\hat{\phi}(x) \to -\frac{\delta}{\delta\hat{J}(x)}.$$

The effective action is then

$$e^{-W[\phi]} = e^{-S[\phi]} \exp\left(-S_{\text{int}}[\phi, \delta/\delta\hat{J}]\right) \exp\left(-\frac{1}{2} \int d^d x d^d y \, \hat{J}(x)\hat{D}_F(x-y)\hat{J}(y) \right)\Bigg|_{\hat{J}=0} \tag{7.3}$$

and

$$S_{\text{int}}[\phi, \delta/\delta\hat{J}] = \int d^d x \left(\frac{1}{2} m_0^2 \frac{\delta^2}{\delta\hat{J}(x)^2} + \frac{\lambda_0}{4!} \left[\frac{\delta^4}{\delta\hat{J}^4(x)} + 4\phi^3(x)\frac{\delta}{\delta\hat{J}(x)} + 6\phi^2(x)\frac{\delta^2}{\delta\hat{J}(x)^2} + 4\phi(x)\frac{\delta^3}{\delta\hat{J}(x)^3} \right] \right).$$

7.3.1 The Mass Shift

If we ignore all terms in S_{int} except

$$\frac{\lambda_0}{4}\phi^2\hat{\phi}^2.$$

Focusing on this term, the relevant contribution to the generating functional is

$$\frac{\lambda_0}{4} \int d^d x \, \phi^2(x) \frac{\delta^2}{\delta\hat{J}(x)\delta\hat{J}(x)} \exp\left(-\frac{1}{2} \int d^d x d^d y \, \hat{J}(x)\hat{D}_F(x-y)\hat{J}(y) \right)\Bigg|_{\hat{J}=0} = \frac{\lambda_0}{4} \int d^d x \, \phi^2(x)\hat{D}_F(0).$$

We see that this term gives rise to a change in the mass in the effective theory at one-loop:

$$m_0^2 \to m_0^2 + \frac{\lambda_0}{2}\hat{D}_F(0).$$

We can evaluate the propagator

$$\begin{aligned}\hat{D}_F(0) &= \int_{b\Lambda}^{\Lambda} \frac{d^dk}{(2\pi)^d}\frac{1}{k^2} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{b\Lambda}^{\Lambda}\frac{d^dk}{(2\pi)^d}k^{d-3} \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{\Lambda^{d-2}(1-b^{d-2})}{(2\pi)^d(d-2)}.\end{aligned} \tag{7.4}$$

We may then write

$$\frac{\lambda_0}{4}\int d^dx\,\phi(x)^2\frac{\delta^2}{\delta\hat{J}(x)\delta\hat{J}(x)}\exp\left(-\frac{1}{2}\int d^dxd^dy\,\hat{J}(x)\hat{D}_F(x-y)\hat{J}(y)\right)\bigg|_{\hat{J}=0} = \frac{1}{2}\int d^dx\,\mu\phi(x)^2,$$

where

$$\mu = \frac{\lambda_0\Lambda^{d-2}}{(4\pi)^{d/2}}\frac{(1-b^{d-2})}{(d-2)}\frac{1}{\Gamma(d/2)}$$

We see this gives rise to a shift in the mass

$$m_0^2 \to m^2 = m_0^2 + \mu.$$

The correction may be expressed diagramatically as

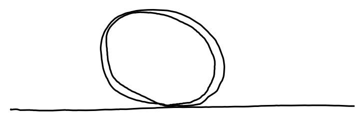


Figure 18.

where the double line notation refers to a high momentum mode that we integrate out.

7.3.2 The Coupling Constant Shift and Higher Dimensional Operators

Similar arguments can be applied to the effect of this term at order λ^2. The contribution we are interested in is

$$S_{\text{int}} = \int d^dx\left(\frac{\lambda_0}{4!}6\phi^2(x)\frac{\delta^2}{\delta\hat{J}(x)^2}\right).$$

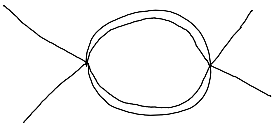
Thus the λ_0 power series expansion of $e^{-S_{\text{int}}}$ contains the term quadratic in λ_0

$$e^{-S_{\text{int}}}Z_0[\hat{J}]\Big|_{\hat{J}=0} = \left(-6\frac{\lambda_0}{4!}\right)^2\int d^dxd^dy\,\phi^2(x)\phi^2(y)\frac{\delta^4}{\delta\hat{J}(x)^2\delta\hat{J}(y)^2}Z_0[\hat{J}]\Big|_{\hat{J}=0} + \ldots$$

where

$$Z_0[\hat{J}] = \exp\left(-\frac{1}{2}\int d^dz_1d^dz_2\hat{J}(z_1)\hat{D}_F(z_1-z_2)\hat{J}(z_2)\right)$$

This may be written as the diagram



$$= \left(-\frac{\lambda_0}{4}\right)^2 \int d^d x d^d y\, \phi(x)^2 \phi(y)^2 \hat{D}_F(x-y)^2$$

$$= \frac{\lambda_0^2}{16} \int d^d x d^d y\, \phi(x)^2 \phi(y)^2\, \frac{d^d p}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot(x-y)}}{k^2} \frac{e^{ip\cdot(x-y)}}{p^2}. \quad (7.5)$$

To do the x or y integrals we can expand one of these fields as a power series. Let

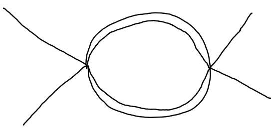
$$\phi(y) = \phi(x) + (x^\mu - y^\mu)\partial_\mu \phi(x) + ...$$

and so

$$\int d^d y\, \phi(x)^2 \phi(y)^2 e^{iy\cdot(p-k)} = \phi(x)^4 \int d^d y\, e^{iy\cdot(p-k)} + \text{terms involving } \partial\phi(x)$$

$$= (2\pi)^d \delta^4(p-k)\phi(x)^4 + \text{terms involving } \partial\phi(x) \quad (7.6)$$

and so we have



$$= \frac{\lambda_0^2}{16} \int d^d x d^d y \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \frac{1}{k^2} \frac{1}{p^2} e^{ik\cdot(x-y)} e^{ip\cdot(y-x)} + ...$$

$$= \frac{\lambda_0^2}{16} \int d^d x\, \phi(x)^4 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} + ... = -\frac{\zeta}{4!} \int d^d x\, \phi(x)^4 + ... \quad (7.7)$$

where we have defined

$$\zeta := -4! \frac{\lambda_0^2}{16} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4}$$

We see this amounts to a shift in the coupling of the form

$$\lambda_0 \to \lambda_0 = \lambda_0 + \zeta.$$

Let us focus on the integral

$$\zeta = -4! \frac{\lambda_0^2}{16} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} = -4! \frac{\lambda_0^2}{16} \frac{2\pi^{d/2}}{(2\pi)^4 \Gamma(d/2)} \int_{b\Lambda}^{\Lambda} dk\, k^{d-5} = -4! \frac{\lambda_0^2}{16} \frac{2\pi^{d/2}}{(2\pi)^4 \Gamma(d/2)} \frac{\Lambda^{d-4}(1-b^{d-4})}{d-4}.$$

In $d=4$ this is

$$\zeta = \frac{3\lambda_0^2}{16\pi^2} \ln(b).$$

In addition to this contribution, we have higher derivative terms appearing in the Lagrangian. We therefore conclude that

- Integrating out higher momentum modes leads to shifts of the coupling constants - the 'data' that define the theory - appearing in the effective action.
- In addition to terms of the kind we expect, we also see higher derivative terms appearing. How do we make sense of these?

7.3.3 The β-function and Landau poles

A quantity of interest is the β-function - a measure of the rate of change in a coupling with changes in (the logarithm) energy scale

$$\beta_\lambda = \Lambda \frac{d\lambda}{d\Lambda},$$

If we define $\Lambda = b\Lambda_0$, we have found that the effective coupling changes as

$$\lambda(\Lambda) = \lambda_0 + \frac{3\lambda_0^2}{16\pi^2} \ln\left(\frac{\Lambda}{\Lambda_0}\right) + ...$$

which can be inverted to give

$$\lambda_0 = \lambda - \frac{3\lambda^2}{16\pi^2} \ln\left(\frac{\Lambda}{\Lambda_0}\right) + ...$$

The coupling λ_0 is arbitrary and does not depend on the cut-off Λ we choose to calculate at. This fact gives us a way of determining the Λ-dependence of the coupling $\lambda(\Lambda)$. We have that

$$0 = \Lambda \frac{d\lambda_0}{d\Lambda} = \beta_\lambda - \frac{3\lambda}{8\pi^2}\beta_\lambda \ln\left(\frac{\Lambda}{\Lambda_0}\right) - \frac{3\lambda^2}{16\pi^2} + ...,$$

where we have used the definition of β_λ above. Rearranging gives

$$\begin{aligned} \beta_\lambda &= \left(1 - \frac{3\lambda}{8\pi^2} \ln\left(\frac{\Lambda}{\Lambda_0}\right)\right)^{-1} \frac{3\lambda^2}{16\pi^2} + ... \\ &= \left(1 + \frac{3\lambda}{8\pi^2} \ln\left(\frac{\Lambda}{\Lambda_0}\right) + ...\right) \frac{3\lambda^2}{16\pi^2} + ... \end{aligned} \tag{7.8}$$

and so, to leading order, the beta-function is

$$\beta_\lambda = \frac{3\lambda^2}{16\pi^2} + ...,$$

where the ellipsis denote higher order contributions. This can be integrated to give

$$\lambda'(\Lambda') = \frac{\lambda(\Lambda)}{1 - \frac{3}{16\pi^2}\lambda(\Lambda)\ln(\Lambda'/\Lambda)}.$$

This is a key result. It tells us that the coupling we measure depends on the energy at which we measure it. This is sometimes called the running of the coupling. A few comments:

- The point where the coupling seems to diverge at finite energy is often called the Landau Pole. It is difficult to say much about this as we really need to understand such poles non-perturbatively.

- The existence of the pole is due to the sign of the Beta-function. QED and ϕ^4 theory isn four dimensions have positive beta-function and so have Landau poles. The one-loop beta-function for QCD is negative and so there is no Landau pole in that theory.

7.4 The Energy-Dependence of Measurement

When you study probability theory for the first time you spend a lot of time learning the key probability distributions (Bernoulli, Binomial, Poisson, Normal,...) and you learn how to compute moments in terms of the probability density function. The probability density function will be determined by a finite set of parameters. For example, the normal distribution with density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

is determined by two parameters; the mean μ and standard deviation σ. This is all fine. The problems start when you take your first course on statistics and you do not, apriori, know what μ and σ are. You need to take a sample and perhaps you use some understanding of estimators to try to find a well-motivated value for these parameters.

We are in a similar situation in Quantum Field Theory; we have a Lagrangian with a number of parameters in it, such as

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4,$$

where the parameters are m, λ and maybe an overall scale for the wavefunction. How do you know what values to use for these parameters? What does it mean to measure a physical parameter in a Quantum Field Theory? In some sense the story is the same as for the probability distributions above; you cannot directly measure the values of these quantities you have to infer their value from correlation functions. What makes this a thorny, and ultimately very rewarding, problem is that the values of the parameters you infer depend on the energy scale at which you measure them! The body of knowledge that makes sense of all this is called *Renormalization Theory*.

8 The Renormalization Group

We started with a theory with cut-off Λ and, after integrating out high energy modes, we ended up with a theory with cut-off $b\Lambda < \Lambda$, where $0 < b \leq 1$. To compare the original and final theories we can rescale

$$x \to x' = bx, \qquad k \to k' = k/b,$$

so that $\Lambda' = \Lambda/b$. In order for the action to be independent of b after the rescaling, we need to also rescale the fields and couplings by their length dimension. The dimension of the field $\phi(x)$ is given by the canonically normalised kinetic term

$$S_0[\phi] = \frac{1}{2}\int d^d x\, \partial_\mu\phi\partial^\mu\phi$$

The measure has dimension d and so for S_0 to be dimensionless,

$$\phi'(x') = b^{1-\frac{d}{2}}\phi(x), \qquad d^d x = d^d x' b^{-d}.$$

Our starting point is a general theory with cut-off Λ

$$S_\Lambda[\phi] = \int d^d x \left(\frac{1}{2}\lambda_{2,2}\, \partial_\mu\phi\partial^\mu\phi + \sum_{M,N} \lambda_{M,N}\, \partial^M \phi^N \right)$$

where the interaction term is given schematically and includes M derivatives and N powers of the field (in any combinations). We see then that (taking $[x]_{\text{length}} = +1$)

$$[\lambda_{M,N}]_{\text{length}} = M + N\left(\frac{d}{2} - 1\right) - d,$$

so that under the rescaling, the couplings scale as

$$\lambda'_{M,N} = b^{M+N(\frac{d}{2}-1)-d}\lambda_{M,N}.$$

Whether this coupling grows or shrinks under the rescaling (remember $0 < b \leq 1$) depends on the details of the interaction, encoded in the numbers M and N and also the dimension of the spacetime. It is interesting to note that, in $d = 2$ the N-dependence drops out and a special role is played by by couplings with two derivatives. This is a hint that non-linear sigma models

$$S[\phi] = \frac{1}{2}\int d^2 x\, g_{\mu\nu}(\phi)\partial^\mu\phi\partial^\nu\phi,$$

such as play a central role in string theory, are of unique significance.

8.1 Iterating the Renormalization Group Flow

We can use the above starting point as the basis of an iterative process that describes how we flow form a theory at high energies to a low energy theory. Along the way the couplings will change. Let us discuss the iterative process in some detail:

- Start with theory with cut-off Λ and[45] $\lambda_{2,2} = 1$

$$S_\Lambda[\phi] = \int d^dx \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_{M,N} \lambda_{M,N}\, \partial^M \phi(x)^N \right)$$

- Integrate out momentum shell $b\Lambda < k < \Lambda$, resulting in shifts of couplings and effective theory with cut-off $b\Lambda$

$$S_{b\Lambda}[\phi] = \int d^dx \left(\frac{1}{2}(1+\Delta Z)\, \partial_\mu\phi(x)\partial^\mu\phi(x) + \sum_{M,N} \Big(\lambda_{M,N} + \Delta\lambda_{M,N}\Big)\, \partial^M \phi(x)^N \right).$$

  We have denoted $\Delta\lambda_{2,2}$ by ΔZ to signify the special role it plays as a wavefunction renormalization.

- Rescale: $x = x'/b$ and raise the cut-off from $b\Lambda$ Λ

$$S_\Lambda[\phi] = \int d^dx'\, b^{-d} \left(b^2 \frac{1}{2}(1+\Delta Z)\, \partial'_\mu\phi(x')\partial'^\mu\phi(x') + \sum_{M,N} b^M \Big(\lambda_{M,N} + \Delta\lambda_{M,N}\Big)\, \partial'^M \phi(x')^N \right).$$

- Rescale fields and couplings according to length dimension

$$\phi'(x') = b^{1-\frac{d}{2}}(1+\Delta Z)^{\frac{1}{2}}\phi(x'), \qquad \lambda'_{M,N} = b^{M+N(\frac{d}{2}-1)-d}\lambda_{M,N}$$

  and so the action becomes

$$S_\Lambda[\phi'] = \int d^dx' \left(\frac{1}{2}\partial'_\mu\phi'(x')\partial'^\mu\phi'(x') + \sum_{M,N} (1+\Delta Z)^{-\frac{N}{2}} \Big(\lambda'_{M,N} + \Delta\lambda'_{M,N}\Big)\, \partial'^M \phi'(x')^N \right).$$

The new low energy theory is written in terms of the effective couplings $\widetilde{\lambda}_{M,N}$ so that

$$S_\Lambda[\phi'] = \int d^dx' \left(\frac{1}{2}\partial'_\mu\phi'(x')\partial'^\mu\phi'(x') + \sum_{M,N} \widetilde{\lambda}_{M,N}\, \partial'^M \phi'(x')^N \right).$$

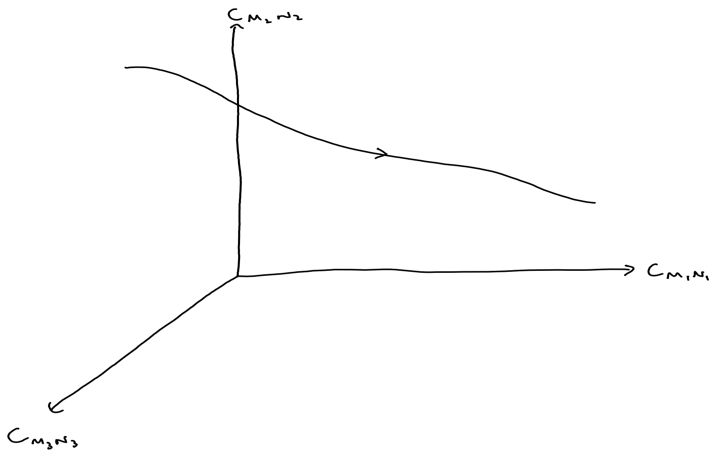
where

$$\boxed{\widetilde{\lambda}_{M,N} = (1+\Delta Z)^{-\frac{N}{2}} b^{M+N(\frac{d}{2}-1)-d}\Big(\lambda_{M,N} + \Delta\lambda_{M,N}\Big)}$$

This action is of the same form as the one we started with. We can now imagine repeating this process over and over again. This gives us a way of seeing the effect of taking the low energy limit of a theory, whilst keeping the cut-off the same. Taking b infinitesimally close

[45] We can take this to define the initial normalization of the fields.

to 1, this can be made into a continuous process, or flow in the space of real scalar theories in four dimensions.



This continuously generated flow in theory space is known as the *renormalization group flow.* NB: This process of integrating out high energy modes is clearly a one-way process. Thus the action of the renormalization group is one-way. As such, the renormalization group is not a group at all, as there is not unique inverse for each element. One can picture this in the same way that one studies the evolution of a plane autonomous system in phase space.

Fixed Points

There can be fixed points of the flow in the space of theories. For example, if

$$S[\phi] = \frac{1}{2}\int d^dx\, \partial_\mu\phi\partial^\mu\phi,$$

we have seen that that this will flow to the same theory

$$S[\phi] = \frac{1}{2}\int d^dx'\, \partial'_\mu\phi'\partial'^\mu\phi'.$$

This example of a fixed point is called the Gaussian fixed point. Near the Gaussian fixed point, the couplings are small and the classical scaling dimension reliably describes the behaviour of the couplings. Far from the fixed point, we will have to accommodate the quantum effects described in the beta-function.

8.2 Evolution of Couplings Under Renormalization Group Flow

Of particular interest is understanding fate of qualitatively new interaction terms that appear under the flow, i.e for terms which have $C_{M,N} = 0$ in the initial theory. Let us start with ϕ^4 theory

$$\lambda_{2,2} = 1, \qquad \lambda_{0,4} = \frac{\lambda}{4!}, \qquad \lambda_{0,2} = \frac{m^2}{2}.$$

and all other $\lambda_{M,N} = 0$. We assume (as is true at one-loop) that $\Delta Z = 0$ to leading order. Consider the effect of the iteration

$$\lambda_{M,N} \to (1+\Delta Z)^{-N/2} b^{M+N-4}\Big(\lambda_{M,N} + \Delta\lambda_{M,N}\Big) \approx b^{M+N-4}\Big(\lambda_{M,N} + \Delta\lambda_{M,N}\Big),$$

where we have fixed $d = 4$.

Relevant, Irrelevant and Marginal Deformations

- The $\lambda\phi^4$ term has $M = 0$ and $N = 4$. We see that $[\lambda] = 0$ and so

$$\lambda \to \lambda + \Delta\lambda.$$

  Thus, under iterations, the evolution of this constant depends on the details of $\Delta\lambda$. Such an operator, that scales as b^0 is called **marginal** and we need the full details of the quantum corrections to determine how it scales.

- The $m^2\phi^2$ term has $M = 0$ and $N = 2$. We see that $[m^2] = -2$ and so

$$m^2 \to b^{-2}(m^2 + \Delta m^2)$$

  Since $0 < b < 1$, $b^{-2} > 1$ and so this contribution grows with each iteration of the flow. We call such operator insertions in the theory, which increase under the flow **relevant**.

- Consider the term $\widetilde{\lambda}\phi^2(\partial\phi)^2$, which does not appear in the original theory but is generated when we integrate out high momentum modes. This term has $M = 2$ and $N = 4$. We see that $[\widetilde{\lambda}] = +2$ and so

$$\widetilde{\lambda} \to b^2\widetilde{\lambda}$$

  Since $0 < b < 1$, $b^2 < 1$ and so this contribution reduces with each iteration of the flow. We call such operator insertions in the theory, which reduce under the flow **irrelevant**.

More generally, we see that for a term to not be irrelevant in $d = 4$, we require

$$N + M - 4 \leq 0.$$

If we also require $M \geq 0$ (local interactions) and $N > 0$ (polynomial interactions), then the possibilities are $(M, N) \in \{(3,1),(2,2),(2,1),(1,3),(1,2),(1,1),(0,4),(0,3),(0,2),(0,1)\}$ which correspond to terms of the form

$$\partial^3\phi, \qquad (\partial\phi)^2 \qquad \partial^2\phi, \qquad \phi^2\partial\phi, \qquad \phi\partial\phi, \qquad \partial\phi, \qquad \phi^4, \qquad \phi^3, \qquad \phi^2, \qquad \phi.$$

Of these, some clearly add nothing new[46] and some violate the $\phi \to -\phi$ symmetry of the

[46]Total derivative terms drop out of the action, $\phi\partial\phi$ and ϕ terms can be removed by completing the square.

original theory. Of those that remain we only have

$$\phi^2, \qquad (\partial\phi)^2, \qquad \phi^4,$$

which all appeared in the original theory. We conclude that the additional complicated terms that arose when we integrated out the high-energy modes are suppressed in the low-energy theory and so may be neglected. In other words, the theory has the same qualitative form at low energies. Some comments:

- QFT may be qualitatively different at different energy scales.
- The self-similarity of renormalizable theories means that higher energy scales decouple from low energy physics. This fact is what makes experimental science possible!

8.3 *The Exact Renormalization Group Equation

Splitting the action with cut-off Λ_0 into kinetic and interaction terms we can write in momentum space

$$S[\phi,\Lambda_0] = \int \frac{d^4p}{(2\pi)^4}\left(-\frac{1}{2}\widetilde{\phi}(p)(p^2+m^2)K^{-1}(p^2/\Lambda_0)\widetilde{\phi}(-p) + J(p)\widetilde{\phi}(-p)\right) + S_{\text{int}}[\phi,\Lambda_0].$$

We have introduced a cut-off in momentum space by modifying the propagator

$$\frac{1}{p^2+m^2} \to \widetilde{D}_\Lambda(p) = \frac{K(p^2/\Lambda_0)}{p^2+m^2},$$

where $K(p^2/\Lambda_0) = 1$ for $p^2 < \Lambda_0^2$ and decays very quickly to zero for $p^2 > \Lambda_0^2$. This smooth cut-off is different form the hard cut-off we considered previously but its role, in suppressing high energy modes, is the same. With this regularization, only modes below the cut-off will propagate. The reason we do not want a restrict ourselves to a hard cut-off is that we will want to differentiate this function.

We have been studying the effect of changing the cut-off on the interaction term. Defining the generating functional

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\, e^{-S[\phi,\Lambda_0]},$$

We can integrate out modes with energies $\Lambda < p^2 < \Lambda_0$, thus lowering the the cut-off from Λ_0 to Λ gives rise to

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\, e^{-S[\phi,\Lambda]},$$

The clear requirement that $Z[J]$ is independent of Λ is

$$\begin{aligned} 0 &= \Lambda\frac{d}{d\Lambda}Z[J] \\ &= \int \mathcal{D}\phi\left[\int \frac{d^4p}{(2\pi)^4}\left(-\frac{1}{2}\widetilde{\phi}(p)\widetilde{\phi}(-p)(p^2+m^2)\Lambda\frac{\partial}{\partial\Lambda}K^{-1}(p^2/\Lambda_0) + J(p)\widetilde{\phi}(-p)\right)\right. \\ &\quad \left. +\Lambda\frac{d}{d\Lambda}S_{\text{int}}[\phi,\Lambda]\right] e^{-S[\phi,\Lambda]}. \end{aligned} \tag{8.1}$$

This equation is satisfied if the the interaction part of the action changes with Λ as goverened by

The Polchinski Equation

$$\Lambda \frac{d}{d\Lambda} S_{\text{int}} = -\int d^4x d^4y \left(\frac{\delta S_{\text{int}}}{\delta \phi(x)} D_\Lambda(x,y) \frac{\delta S_{\text{int}}}{\delta \phi(y)} - D_\Lambda(x,y) \frac{\delta^2 S_{\text{int}}}{\delta \phi(x) \delta \phi(y)} \right).$$

This is called the Polchinski equation and describes how the interaction terms in the action of the renormalization group flow. To get an intuition for this, we introduce the parameter $t = \ln(\Lambda)$ and we find that

$$\frac{\partial}{\partial t} e^{-S_{\text{int}}} = -\int d^4x d^4y D_\Lambda(x,y) \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} e^{-S_{\text{int}}}.$$

This is nothing more than a functional heat equation for the "time" variable $t = \ln(\Lambda)$ and the functional Laplacian

$$\Delta = \int d^4x d^4y D_\Lambda(x,y) \frac{\delta^2}{\delta \phi(x) \delta \phi(y)}.$$

The analogue of the diffusion coefficient is played by the function that imposes the high momentum cut-off in the propagator.

If the interaction term is written as

$$S_{\text{int}} = \sum_{m,n} \frac{g_{mn}}{n!} \int d^4x \partial^m \Phi^n(x),$$

we can think of $\{\partial^m \Phi^n(x)\}$ as a basis in theory space and the c_{mn} as components of the vector that describes the particular interaction Lagrangian. What is of interest is how these coefficients evolve over renormalization time' $t = \ln(\Lambda)$. Solution to the heat equation usually decay as e^{-t} but the space in question here is the space of couplings which does not have a Euclidean metric. As such it is possible to have exponentially increasing and decreasing behaviour close to a fixed point, describing relevant and irrelevant deformations of the theory.

9 Renormalization Using Counter-terms

The Wilsonian perspective on renormalization is profound. It tells us how to make sense of the divergences in Quantum Field Theory from the perspective of low-energy effective field theory. Moreover, it explains how we are able to do physics at all and why the Standard Model is written in terms of renormalizable theories. The drawback to this perspective is that it can be difficult to do actual predictive calculations. In this section we introduce a prescription that allows us to calculate efficiently without having to resort to the momentum-splitting arguments of Wilsonian Renormalization.

Our treatment is pedagogical rather than historical and runs opposite to that of most textbooks. The counter-term approach predates Wilsonian renormalization; however, given that renormalizations is a potentially confusing topic, it is helpful to know where you are going to appreciate the route taken.

9.1 Counter-terms: The basic idea

Let us collect together some things we have learned about ϕ^4 theory in four dimensions

1. Divergences appear in loop diagrams. These divergences come from allowing the momenta in the loops to be arbitrarily large. This is a UV divergence.

2. We can tame these divergences by regularization. An intuitive example is to introduce a large-momentum cut-off.

3. We can ϕ^4 theory in four-dimensions belongs to a special class of theories in which the only effect of integrating out the higher momentum modes on the low energy theory is to shift the couplings and rescale the fields. If the theory with cut-off Λ_0 has Lagrangian
$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4,$$
then the theory with cut-off $\Lambda < \Lambda_0$ is described by
$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \\ &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4,\end{aligned} \tag{9.1}$$
where ϕ is a rescaling of ϕ_0
$$\phi_0(x) = Z^{\frac{1}{2}}\phi(x),$$
and the shifting in the couplings is calculated in perturbation theory. The perspective we had in the previous section was that we knew the high energy theory and wanted to find an effective theory of the low energy physics. This is not the situation we usually find ourselves in. We usually only have access to the low energy physics. In the following sections we will develop a method that does not require us to know what the coupling λ_0 and Lagrangian $\mathcal{L}_0$ is. Instead we will introduce additional terms in the Lagrangian that play the role of $\Delta\lambda$ and fix these additional terms by comparison with experiment.

Working at the level of the Lagrangian

Remarkably, we can impose these subtractions directly at the level of the Lagrangian - they can be understood as corrections to the local theory. This is because the infinite shifts amount to changes in the values of the coupling constants of the theory. Being able to apply the subtractions at the level of the Lagrangian means that we can treat this in perturbation theory. This is very helpful.

Our starting point is what we shall refer to as the 'bare' (i.e. unobservable) Lagrangian for ϕ^4 theory:

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4.$$

The quantities appearing here such as m_0 are not those we measure as they receive quantum corrections. The mass is shifted and infinite amount by Π. Thus, we should distinguish between the parameters that appear in the bare Lagrangian and those that are actually measured. We introduce a function Z_ϕ, which describes the renormalization of the wave-function and relates ϕ_0 to the physically 'observable' field ϕ as

$$\phi_0(x) = Z^{\frac{1}{2}}\phi(x).$$

The power of Z_ϕ used in the definition is chosen to allow Z_ϕ to be fixed by the normalization of the single particle wavefunction.

$$\langle 0|\phi|1\rangle = 1.$$

The Lagrangian then becomes

$$\mathcal{L}_0 = \frac{1}{2}Z_\phi(\partial\phi)^2 + \frac{1}{2}Z_\phi m_0^2\phi^2 + \frac{\lambda_0}{4!}Z_\phi^2\phi^4.$$

The claim is that this Lagrangian may be written in terms of physical parameters m and λ in terms of counter-terms

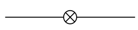
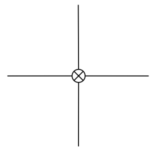
$$\begin{aligned} \mathcal{L}_0 = & \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 && \Big\} \text{ Seed theory} \\ & +\hbar\left(\frac{1}{2}\delta_{Z_\phi}(\partial\phi)^2 + \frac{1}{2}\delta_{m^2}\phi^2 + \frac{\delta_\lambda}{4!}\phi^4\right) && \Big\} \text{ Counterterms} \end{aligned} \tag{9.2}$$

where

$$\delta_{Z_\phi} = Z_\phi - 1, \qquad \delta_{m^2} = Z_\phi^2 m_0^2 - m^2, \qquad \delta_\lambda = Z_\phi^2\lambda_0 - \lambda.$$

The $\hbar$ in front of the counter-term Lagrangian highlights the fact that the divergences, which the counter-terms cancel, first appear at one-loop (i.e. order $\hbar$). We will return to setting $\hbar = 1$ from now on

The question is, can we find $\delta_{Z_\phi}, \delta_{m^2}, \delta_\lambda$ such that this is all true? The short answer is; Yes. The long answer is yes, but the terms formally involve the cancellation of infinities, e.g. both δ_{m^2} and $Z_\phi^2 m_0^2$ are infinite as $\Lambda \to \infty$ but are delicately calibrated such that m^2 is finite and corresponds to what we measure in experiment. Thus we fix this finite number of counter-terms by comparing with a finite number of measurable quantities.

 $= \quad -p^2\delta_Z - \delta_{m^2},$  $= \quad -\delta_\lambda$

We can incorporate the effect of subtracting off the divergences by introducing these counter-terms as additional Feynman diagrams in perturbation theory.

The divergences only begin to appear when we start integrating over closed momentum loops in diagrams. Since $\hbar$ counts loop order, we conclude that the counter-terms must be of order $\hbar$.

What has this got to do with Wilsonian renormalization? Suppose we want to compute a correlation function of external states of momentum p. We regulate the calculation by introducing a cut-off Λ. We can now do this calculation in two equivalent ways

- Calculate directly using the original action with counter-terms.
- Integrate out the momentum modes between Λ and p to get an effective action and then calculate the correlation functions using this effective action.

Both approaches are equivalent. In the first approach, we include all effects of the momentum integration in one go and the loop integrations contribute to shifts in the coupling constants. In the second approach, the shifts in the coupling constants are already incorporated into the modified couplings of the effective action. We see the Wlisonian approach gives us a new perspective on renormalization.

The procedue is as follows:

1. Write down a Lagrangian and choose a **regularization prescription** (e.g. momentum cut-off, dimensional regularization)
2. Add **counter-terms** to the Lagrangian
$$\mathcal{L}_{\text{c.t.}} = \hbar\left(\frac{1}{2}\delta_{Z_\phi}(\partial\phi)^2 + \frac{1}{2}\delta_{m^2}\phi^2 + \frac{\delta_\lambda}{4!}\phi^4\right).$$
3. Choose a **renormalization scheme**, that is conditions that will be used to fix the counter-terms. For example one could require that
   - At energy p, the pole in the connected two-point function is at the the physical mas we measure.
   - The residue of this pole is 1.
   - 1PI 4-point diagram takes the value λ_{eff} in the limit all external momenta vanish.
4. Once the counter-terms are fixed at a given orfder in perturbation theory, one can make predictions at that order in pertutbation theory.

9.2 Renormalization of Γ_2

We found the divergence in the propagator

$$\widetilde{\Gamma}_2(p^2) = p^2 + m^2 + \frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right] - \delta_{m^2} - p^2\delta_Z ...,$$

where Λ is a cut-off. The expression diverges when we remove the cut-off and send $\Lambda \to \infty$, unless the counter-terms are chosen such as to cancel this divergence.

What criteria should we use to fix the counter-terms. Pretty much any sensible choice is fine. In the next section we will make choices that make the calculations simple. Here we will make a choice that have a clear physical intuition. We define the renormalized Π_1 at one-loop as $\widetilde{\Gamma}_2(p^2) = p^2 + m^2 - \widetilde{\Pi}_{\text{ren}}$ where

$$\widetilde{\Pi}_{\text{ren}} = \widetilde{\Pi} + \delta_{m^2} + p^2\delta_Z,$$

and

$$\widetilde{\Pi} := -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right] + ...$$

On-Shell Renormalization Condition

Natural renormalization conditions are

$$\Pi_{\text{ren}}\big|_{p^2 = m^2_{\text{phys}}} = 0, \qquad \frac{d}{dp^2}\Pi_{\text{ren}}\bigg|_{p^2 = m^2_{\text{phys}}} = 0,$$

where m_{phys} is the physical mass that we measure. This is often called the 'on-shell' prescription.

The beauty of this prescription is that the we fix the counter-terms such that, in the Minkowski case, the pole in the propagator is when $p^2 = m^2$. If we define the pole to be at the physical mass, this prescription sets

$$m = m_{\text{phys}},$$

as we would have in classical physics.

The second renormalization condition tells us that, at one loop

$$\delta_Z = 0.$$

This is a quirk of the theory and there is a non-trivial wavefunction renormalization at two loops. The first renormalization condition tells us that

$$\delta_{m^2} = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right].$$

9.3 Renormalization of Γ_4

The location of the pole in the propagator is a natural definition for the physical mass-squared. Similarly, we could use the residue of the this pole to fix a choice of wavefunction renormalization. How should we fix the counterterm for the coupling λ? A natural choice of coupling (as good as any) would be to take the coupling constant to be that value that is measured at very large distances. This corresponds to low momentum and so we could take our physical value of the coupling to be given by the value of $\widetilde{\Gamma}_4$ at zero momentum. Thus or third and final renormalization condition is (in Euclidean signature)

$$\widetilde{\Gamma}_4(0,0,0,0) = \lambda,$$

in the limit $\Lambda \to \infty$. The actual value we measure will vary with momentum in a way that we will describe at the end of this chapter when we discuss the beta-function.

And so our next task is to choose the remaining counter-terms such that the divergence in $\widetilde{\Gamma}_4(0,0,0,0)$ is also cancelled. The counter-term, at one-loop is $-\delta\lambda$. Recall that

$$\widetilde{\Gamma}_4^{\text{1-loop}}(0,0,0,0) = -\frac{3\lambda^2}{32\pi^2}\left[\ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right].$$

Using the fact that

$$\begin{aligned}\ln\left(1+\frac{\Lambda^2}{m^2}\right) &= \ln\left(\frac{\Lambda^2}{m^2}\right) + \ln\left(1+\frac{m^2}{\Lambda^2}\right)\\ &= \ln\left(\frac{\Lambda^2}{m^2}\right) + \frac{m^2}{\Lambda^2} - \frac{1}{2}\left(\frac{m^2}{\Lambda^2}\right)^2 + ..., \end{aligned} \tag{9.3}$$

we see that, as $\Lambda \to \infty$, the divergence is coming from the first term and so, in the limit of large Λ

$$\lim_{\Lambda\to\infty} \widetilde{\Gamma}_4^{\text{1-loop}}(0,0,0,0) = -\frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - 1\right],$$

and so a natural choice to make for δ_λ would be

$$\delta_\lambda = -\frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - 1\right] + \mathcal{O}(\lambda^2)$$

which removes the divergent part of $\Gamma^{(4)}(0,0,0,0)$ as $\Lambda \to \infty$. At tree-level $\lambda = \Gamma^{(4)}(0,0,0,0)$. We could then introduce the 1-loop effective coupling λ_{eff} as

$$\begin{aligned}\lambda_{\text{eff}} &:= \Gamma^{(4)}(0,0,0,0) = \lambda + \Gamma_1^{(4)}(0,0,0,0) + \delta\lambda + ...\\ &= \lambda - \frac{3\lambda^2}{32\pi^2}\left[\ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right] + \frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - 1\right] + ...\\ &= \lambda - \frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) + \ln\left(1+\frac{m^2}{\Lambda^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right] + \frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - 1\right] + ...\\ &= \lambda - \frac{3\lambda^2}{32\pi^2}\left[\ln\left(1+\frac{m^2}{\Lambda^2}\right) + \frac{m^2}{\Lambda^2+m^2}\right] + ... \end{aligned} \tag{9.4}$$

In summary

$$\boxed{\lambda_{\text{eff}} = \lambda - \frac{3\lambda^2}{32\pi^2}\left[\ln\left(1 + \frac{m^2}{\Lambda^2}\right) + \frac{m^2}{\Lambda^2 + m^2}\right]}$$

Note that,

$$\lim_{\Lambda\to\infty} \lambda_{\text{eff}} = \lambda,$$

so that, as the cut-off is removed, the effective coupling tends to the renormalized coupling. As such, this renormalization procedure leads to one-loop calculations, where we can safely remove the cut-off.

The keys points:

- We can fix the counter-terms at each order in λ by imposing a finite number of measurable conditions (thus fixing a finite number of physical parameters).
- For a renormalizable theory, the divergences can be absorbed into a redefinition of the parameters. Qualitatively new interaction terms are are not introduced.

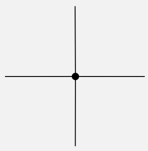
One might like to think of this as akin to using the Feynman diagrams to remove the vacuum bubble divergences by a careful choice of normalization of the generating functional. What we are doing here is more subtle but there are some qualitative similarities.

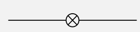
10 Dimensional Regularization

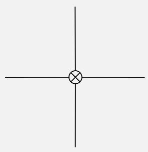
Whilst the momentum regularization is intuitive, it can be rather cumbersome in practice. Moreover, the addition of the cut-off energy scale may break symmetries in the expression that may otherwise be useful to exploit. A more convenient, but somewhat esoteric alternative is dimensional regularization. We will return to Minkoswski space for this section, although the ideas apply equally well to Euclidean theories. To evaluate diagrams it will be helpful to have the Feynman rules to hand, now augmented with counter-terms.

Momentum Feynman Rules in Minkowski Spacetime for ϕ^4 Theory

 $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

 $= -i\lambda$

 $= i(p^2\delta_Z - \delta_{m^2})$

 $= -i\delta_\lambda$

Another (and more widespread) method of regularization is dimensional regularization. In this case we pretend, instead of working in 4 dimensions, we are working in $d = 4 - \epsilon$ dimensions where we take ϵ to be small (and eventually zero). Unlike moentum regularization, dimensional regularization is not very intuitive. Its strength lies in the fact that many important features of a theory, that are broken by a regulator, are preserved in dimensional regularization. Obviously, those features of a theory that are sensitive to the dimension of spacetime (such as the form of gamma matrices) need to be treated carefully.

A comment on mass dimension

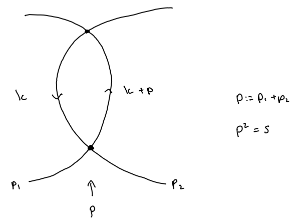
In four dimensions, the coupling λ in ϕ^4 theory is dimensionless. This is no longer true in $d = 4 - \epsilon$ dimensions, where λ has mass dimension $[\lambda] = \epsilon$. We can introduce a dimensionless coupling g and a dimensionful parameter μ such that

$$\lambda = \mu^\epsilon g.$$

The presence of μ thus gives a measure of how far from four dimensions we are. All physical results should not depend on μ.

10.1 One-loop Γ_4 in dimensional regularization

To compute $\delta\lambda$ in dimensional regularization, we first consider the 1-loop contribution



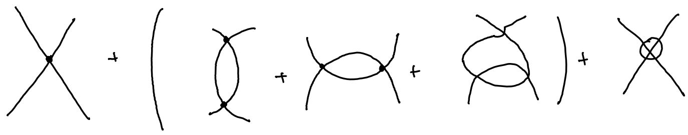
$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} := (-i\lambda)^2 iV(p^2), \tag{10.1}$$

defines $V(p^2)$, where

$$p = p_1 + p_2 = -p_3 - p_4.$$

The full four-point amplitude is

$$i\mathcal{M}(p_1, p_2 \to p_3, p_4) =$$



where the last term is the counter-term contribution at one-loop (remember that we have no need of counterterms at tree level, so the they first enter at one-loop).

Thus, we have, to one-loop

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2 \left[iV(s) + iV(t) + iV(u) \right] - i\delta_\lambda + ...$$

where

$$V(p^2) = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} \frac{1}{(k+p)^2 - m^2}.$$

To begin, we need to find the generalization of our divergent expressions from 4 to d dimensions. In order to evaluate integrals of this kind, it is useful to use:

Feynman Parameterization

This is a very useful trick for doing integrals of the kind that repeatedly occur in dimensional regularization

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}$$

so that

$$\int dk \frac{1}{A(k)B(k)} = \int_0^1 dx \left(\int dk \frac{1}{[xA(k) + (1-x)B(k)]^2} \right),$$

and, for suitably simple $A(k)$, $B(k)$, these integrals can be performed.

The integral becomes

$$
\begin{aligned}
V(p^2) &= \frac{i}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}\frac{1}{(k+p)^2-m^2} \\
&= \frac{i}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{[x[(k+p)^2-m^2]+(1-x)(k^2-m^2)]^2}
\end{aligned}
\tag{10.2}
$$

But

$$
x[(k+p)^2-m^2]+(1-x)(k^2-m^2) = k^2+2xk\cdot p+xp^2-m^2,
$$

so

$$
V(p^2) = \frac{i}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+2xk\cdot p+xp^2-m^2)^2}
$$

Let us introduce the variable $\ell_\mu := k_\mu + xp_\mu$, so that

$$
k^2+2xk\cdot p+xp^2-m^2 = \ell^2+x(1-x)p^2-m^2.
$$

We now go to Euclidean space to actually do the integral and define

$$
\ell_E^0 = -i\ell^0, \qquad d^4\ell = id^4\ell_E.
$$

Note that there is no Wick rotation of the external momenta (p_μ) as we still want to think of this process as happening in Minkowski space. The Wick rotation used here is just a trick to enable us to evaluate the integral. The expression for $V(p^2)$ becomes

$$
V(p^2) = -\frac{1}{2}\int_0^1 dx \int_{\mathbb{R}^4}\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2-x(1-x)p^2+m^2)^2}.
$$

We evaluate this in d-dimensional spherical polar coordinates and use the fact that the area of a d-sphere is

$$
S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},
$$

so that we just have the radial integral to do

$$
V(p^2) = -\frac{1}{2}\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^1 dx \int_0^\infty d\ell_E \frac{\ell_E^{d-1}}{(\ell_E^2-x(1-x)p^2+m^2)^2}.
$$

In order to do this integral, we define $\Delta := m^2-x(1-x)p^2$, so the ℓ_E integral becomes

$$
\int_0^\infty d\ell_E \frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^2} = \frac{1}{2}\int_0^\infty d(\ell_E^2)\frac{(\ell_E^2)^{d/2-1}}{(\ell_E^2+\Delta)^2}
\tag{10.3}
$$

since this in an integral in terms of ℓ_E^2, we can find a better parameterisation. We choose

$$
y(\ell_E) := \frac{\Delta}{\ell_E^2+\Delta}, \qquad \text{then} \qquad dy(\ell_E) = -\frac{\Delta}{(\ell_E^2+\Delta)^2}d(\ell_E^2) = -\frac{y^2}{\Delta}d(\ell_E^2)
$$

We also note that $y(0)=1$ and $y(\infty)=0$. The integral becomes

$$
\int_0^\infty d\ell_E \frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^2} = \frac{1}{2}\Delta^{\frac{d}{2}-2}\int_0^1 dy\, y^{1-\frac{d}{2}}(1-y)^{\frac{d}{2}-1}.
$$

Comparing this with

The Euler Beta function

$$B(a,b) = \int_0^1 dy\, y^{a-1}(1-y)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

and taking $a = 2 - d/2$ and $b = d/2$, we see that

$$\int_0^\infty d\ell_E \frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^2} = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)}.$$

Since $\Gamma(2) = 1$, we can put this back into our previous expressions to give

$$\begin{aligned}\int_{\mathbb{R}^4} \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} &= \frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)} \times \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \Gamma(2-d/2)\Gamma(d/2) \\ &= \frac{1}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\Gamma(2-d/2), \end{aligned} \tag{10.4}$$

and so

$$V(p^2) = -\frac{1}{2}\int_0^1 dx \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{(m^2 - x(1-x)p^2)^{2-d/2}}$$

Now assume $d = 4-\epsilon$, then $2 - d/2 = \epsilon/2$ and

$$\Gamma(2-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon),$$

where γ is the Euler-Mascheroni constant, so that

$$\begin{aligned} V(p^2) &= -\frac{1}{2}\int_0^1 dx \frac{1}{(4\pi)^{2-\epsilon/2}}\frac{\frac{2}{\epsilon}-\gamma}{(m^2-x(1-x)p^2)^{\epsilon/2}} + \ldots \\ &= -\frac{1}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma\right)\int_0^1 dx\left(\frac{4\pi}{\Delta(p^2)}\right)^{\frac{\epsilon}{2}} + \ldots \end{aligned} \tag{10.5}$$

We can expand in small powers of ϵ to give[47] In four dimensions $V(p^2)$ would be dimensionless but in $4-\epsilon$ dimensions it has mass-dimension $[V(p^2)] = \epsilon$. The combination $\lambda V(p^2)$ is dimensionless and so we consider

$$\mu^\epsilon V(p^2) = -\frac{1}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma\right)\int_0^1 dx\left(\frac{4\pi\mu^2}{\Delta(p^2)}\right)^{\frac{\epsilon}{2}} + \ldots$$

Taking the limit as $\epsilon \to 0$, we have

$$\mu^\epsilon V(p^2) = -\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{\Delta(p^2)}\right) + \mathcal{O}(\epsilon)\right)$$

[47] A useful result is

$$a^{-\epsilon/2} = e^{-\frac{\epsilon}{2}\ln(a)} = 1 - \frac{\epsilon}{2}\ln(a) + \ldots$$

We are interested in

$$\Gamma_4(s,t,u) = \lambda + \lambda^2\Big(V(s)+V(t)+V(u)\Big) + \delta_\lambda + \ldots$$

In $d = 4-\epsilon$ dimensions, the associated dimensionless object is

$$\boxed{\begin{aligned}\mu^{-\epsilon}\Gamma_4(s,t,u) &= g + \frac{\mu^\epsilon g^2}{2}\Big(V(s)+V(t)+V(u)\Big) + \mu^\epsilon\delta_g + \ldots \\ &= g - \frac{g^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + \ln\left(\frac{(4\pi\mu^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right)\right) + \delta_g + \ldots\end{aligned}}$$

where

$$\Delta(p^2) = m^2 - x(1-x)p^2,$$

and where we note in the top line that the counter-term appears at order g^2 and has mass-dimension $-\epsilon$.

10.2 One-loop calculation of Γ_2 using dimensional regularization

The connected two-point function is

$$W_2(p^2) = \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\Big(-i\Pi(p^2)\Big)\frac{i}{p^2-m^2} + \ldots$$

where we define the renormalized self-energy as[48]

$$-i\Pi(p^2) := -\frac{i\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} + i\Big(p^2\delta_{Z_\phi} - \delta_{m^2}\Big) + \mathcal{O}(\lambda^2),$$

This can be summed to all orders[49] to give

$$W_2(p^2) = \frac{i}{p^2-m^2}\sum_{r=0}^{\infty}\left(\Pi(p^2)\frac{i}{p^2-m^2}\right)^r = \frac{i}{p^2-m^2}\left(1-\frac{\Pi(p^2)}{p^2-m^2}\right)^{-1} = \frac{i}{p^2-m^2-\Pi(p^2)}.$$

The quadratic part of the quantum effective action is given by $W_2(p^2)\Gamma_2(p^2) = i$, and so

$$\Gamma_2(p) = p^2 - m^2 - \Pi(p^2).$$

We will now evaluate $\Pi(p^2)$ to one-loop order in dimensional regularization. Consider the integral first, going to Euclidean coordinates[50]

$$\begin{aligned}\int d^dk\frac{1}{k^2-m^2} &= -i\int d^dk_E\frac{1}{k_E^2+m^2} \\ &= -i\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty d^dk_E\frac{k_E^{d-1}}{k_E^2+m^2} \\ &= -i\frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty d^d(k_E^2)\frac{(k_E^2)^{d/2-1}}{k_E^2+m^2}.\end{aligned} \tag{10.6}$$

[48] Note that Π is defined with the same factor of $-i$ as the mass counter-term.

[49] We saw this earlier and it seemed a little dodgy to treat this as a geometric progression since the one-loop term diverges. We see the resolution here - the counter-term will be chosen such that the renormalized Π is small compared to p^2-m^2.

[50] For convenience: $k_E^0 = -ik^0$, $d^dk = id^dk_E$, $k^2-m^2 = -(k_E^2+m^2)$.

This is of a similar form to the integral (10.3) but with $\Delta = m^2$, so the same techniques for evaluating the integral will apply. Let:

$$y = \frac{m^2}{k_E^2 + m^2},$$

so that

$$\int_0^\infty dk_E \frac{k_E^{d-1}}{k_E^2 + m^2} = \frac{1}{2} m^{d-2} \int_0^1 dy\, y^{-\frac{d}{2}} (1-y)^{\frac{d}{2}-1} = \frac{1}{2} \frac{1}{(m^2)^{1-\frac{d}{2}}} \frac{\Gamma(1-d/2)\Gamma(d/2)}{\Gamma(1)}$$

and so (using the fact that $\Gamma(1) = 1$)

$$-i\Pi(p^2) = -i\frac{\lambda}{2} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-d/2)}{(m^2)^{1-\frac{d}{2}}} + i\Big(p^2 \delta_{Z_\phi} - \delta_{m^2}\Big).$$

If we take $d = 4 - \epsilon$, and write $\lambda = \mu^\epsilon g$, then

$$-i\Pi(p^2) = -\frac{igm^2}{32\pi^2} \left(\frac{\mu^2}{4\pi m^2}\right)^{\frac{\epsilon}{2}} \Gamma(\epsilon/2 - 1) + i\Big(p^2 \delta Z_\phi - \delta m^2\Big).$$

Expanding out for small ϵ using

$$\left(\frac{\mu^2}{4\pi m^2}\right)^{\frac{\epsilon}{2}} = 1 + \frac{\epsilon}{2} \ln\left(\frac{\mu^2}{4\pi m^2}\right) + ..., \qquad \Gamma(\epsilon/2 - 1) = -\frac{2}{\epsilon} + \gamma - 1 + ...,$$

we have

$$\Pi(p^2) = \frac{gm^2}{32\pi^2} \left(-\frac{2}{\epsilon} + \gamma - 1 - \ln\left(\frac{\mu^2}{4\pi m^2}\right) + ...\right) - \Big(p^2 \delta_{Z_\phi} - \delta_{m^2}\Big).$$

Putting it all back together we find

$$\boxed{\Gamma_2(p^2) = p^2 - m^2 - \frac{gm^2}{32\pi^2} \left(-\frac{2}{\epsilon} + \gamma - 1 - \ln\left(\frac{\mu^2}{4\pi m^2}\right) + ...\right) + \Big(p^2 \delta_{Z_\phi} - \delta_{m^2}\Big) + ...}$$

10.3 Summary

It is helpful to collect the results in one place. The one-loop quantum effective action for ϕ^4 theory in $d = 4 - \epsilon$ dimensions is

$$\begin{aligned}\Gamma[\Phi] = &-\frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \widetilde{\Phi}(-p) \Gamma_2(p^2) \widetilde{\Phi}(p) \\ &+ \sum_{n=3}^{\infty} \frac{1}{n!} \int \frac{d^dp_1}{(2\pi)^d} ... \int \frac{d^dp_n}{(2\pi)^d} (2\pi)^d \delta\left(\sum_{i=1}^{n} p_i\right) \Gamma_n(p_1, ..., p_n) \widetilde{\Phi}(p_1) ... \widetilde{\Phi}(p_n)\end{aligned}$$

where

$$\widetilde{\Phi}(p) = \int d^dx\, e^{-ik\cdot x} \Phi(x).$$

At one-loop in dimensional regularization, the first two vertices are

$$\Gamma_2(p^2) = p^2 - m^2 - \frac{gm^2}{32\pi^2}\left(-\frac{2}{\epsilon} + \gamma - 1 - \ln\left(\frac{\mu^2}{4\pi m^2}\right) + ...\right) + \left(p^2\delta_{Z_\phi} - \delta_{m^2}\right) + ...$$

and

$$\mu^{-\epsilon}\Gamma_4(s,t,u) = g - \frac{g^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + \ln\left(\frac{(4\pi\mu^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right)\right) + \delta_g + ...$$

where (s,t,u) are related to (p_1, p_2, p_3, p_4) in the usual way.

11 Renormalization Schemes

We are in the business of subtracting off divergent parts of Feynman diagrams to give finite results. Of course, when we remove the divergent parts we can also subtract off various finite contributions to the diagrams. The creates an ambiguity in how renormalization occurs in practice. Different choices are known as different renormalization schemes. We will consider three popular examples, given below

11.1 The Space of theories

Recall

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 && \Big\} \text{ Seed theory} \\ &+\frac{1}{2}\delta_{Z_\phi}(\partial\phi)^2 + \frac{1}{2}\delta_{m^2}\phi^2 + \frac{\delta_\lambda}{4!}\phi^4 && \Big\} \text{ Counterterms}\end{aligned} \tag{11.1}$$

The theory we have been discussing is described by the 'bare' theory $\mathcal{L}_0$. This statement hides a lot of ambiguity since, until we specify how the parameters are split between (m, λ, Z) and the counter-terms we cannot sensibly do anything with the theory. As we have seen, we will always require the counter-terms to cancel divergences at each order in perturbation theory but, other than this one requirement, there is nothing else to fix precisely how we split the bare terms into (m, λ, Z) and $(\delta_{m^2}, \delta_\lambda, \delta_Z)$. We are free to choose. Since we cannot do anything with the quantum field theory until we have i) introduced a regulator, and ii) fixed renormalization conditions, there are two ways in which the splitting of $\mathcal{L}_0$ is not unique:

- The choice of renormalization scale μ affects (m, λ, Z) and $(\delta_{m^2}, \delta_\lambda, \delta_Z)$ but does not affect quantities appearing in $\mathcal{L}_0$.
- We can think of the renormalization flow as the flow along a line with parameter μ (or Λ), embedding in the space of theories. The bare quantities are constants of the 'motion' or flow

$$\lambda_0(\lambda, m, Z; \mu) = \lambda_0(\lambda', m', Z'; \mu'), \qquad m_0(\lambda, m, Z; \mu) = m_0(\lambda', m', Z'; \mu').$$

- The renormalization scheme - the rules for how we divide up the finite parts between (m, λ, Z) and $(\delta_{m^2}, \delta_\lambda, \delta_Z)$ - needs to be chosen.

In some sense, we can think of this ambiguity as describing a whole family of theories that live in a three-dimensional parameter space labeled by the parameters (m, λ, Z), with one point in the space given by a particular choice. This gives us a new way of understanding the running of the coupling. λ_0 is fixed but the division of λ_0 into λ and δ_λ changes with μ. This is what the running of the coupling λ is telling us; it is telling us how we move around the space of theories described by different splittings of the bare quantities into (m, λ, Z) and $(\delta_{m^2}, \delta_\lambda, \delta_Z)$ as we change the energy μ. The running of this coupling is measured by a (scheme-dependent) quantity called the beta function, defined for λ as

$$\beta_\lambda = \mu\frac{\partial\lambda}{\partial\mu}.$$

In the following sections we shall see how to calculate the beta function in perturbation theory.

11.2 The mass-shell scheme

This is probably the easiest scheme to motivate on physical grounds. As a reminder the expressions we want to renormalize are

$$\Gamma_2(p^2) = p^2 - m^2 - \frac{gm^2}{32\pi^2}\left(-\frac{2}{\epsilon} + \gamma - 1 - \ln\left(\frac{\mu^2}{4\pi m^2}\right) + ...\right) + \left(p^2\delta_{Z_\phi} - \delta_{m^2}\right) + ...$$

and

$$\mu^{-\epsilon}\Gamma_4(s,t,u) = g - \frac{g^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + \ln\left(\frac{(4\pi\mu^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right)\right) + \delta_g + ...$$

11.2.1 Fixing δ_Z and δ_{m^2} in the mass-shell scheme.

What renormalization conditions should we impose to fix the counter-terms δ_{Z_ϕ} and δ_{m^2}? Two obvious choices are to define the pole of the propagator $\widetilde{G}_2(p)$ to be at the value $p^2 = m^2$. Since the value of m is then the physical mass, this is sometimes called the *on-shell renormalization scheme.* A second is to fix the residue of that pole to be i. In this way the renormalized propagator will have the same essential features as the free propagator. The condition $\Pi(p^2) = 0$ when $p^2 = m^2$ will guarantee that the pole of the propagator is at $p^2 = m^2$. A meromorphic function $f(z)$ with pole at $z = z_0$ has residue

$$\lim_{z\to z_0}(z - z_0)f(z),$$

and so the specification of the residue is

$$\lim_{p^2\to m^2}(p^2 - m^2)\widetilde{G}_2(p) = \lim_{p^2\to m^2}\frac{i(p^2 - m^2)}{p^2 - m^2 - \Pi(p^2)}$$

Since numerator and denominator vanish in the limit, we use l'Hopital's rule

$$\lim_{p^2\to m^2}(p^2 - m^2)\widetilde{G}_2(p) = \lim_{p^2\to m^2}\frac{i}{1 - \frac{d\Pi(p^2)}{dp^2}}.$$

This will give the same residue as the free propagator if $\frac{d\Pi(p^2)}{dp^2} = 0$ when $p^2 = m^2$. As such, the renormalization conditions we choose are

Pole :

$$\Pi(p^2) = 0, \qquad \text{when} \qquad p^2 = m^2.$$

Residue :

$$\frac{d}{dp^2}\Pi(p^2) = 0, \qquad \text{when} \qquad p^2 = m^2$$

These choices ensure that, at the energy scale chosen (the scale fixed by the physical mass of the field) the two-point function has the same pole and residue as a free propagator (with renormalized wavefunction and mass). We now impose these conditions on

$$\Pi(p^2) = \frac{gm^2}{32\pi^2}\left[-\frac{2}{\epsilon} + \gamma - 1 - \ln\left(\frac{\mu^2}{4\pi m^2}\right)\right] - \left(p^2\delta_{Z_\phi} - \delta_{m^2}\right)$$

We see immediately that the diverging quantity has no p^2 dependence (a quirk of ϕ^4 theory at one-loop) and so we set

$$\delta_{Z_\phi} = 0.$$

The requirement that we identify m^2 with m^2_{phys}, the physical mass at the pole gives

$$\delta_{m^2} = -\frac{gm^2}{32\pi^2}\left[-\frac{2}{\epsilon} + \gamma - 1 - \ln\left(\frac{\mu^2}{4\pi m^2}\right)\right]$$

We then have

$$\Gamma_2(p^2) = p^2 - m^2 + ...$$

where we identify m in the renormalized Lagrangian as the physical mass m_{phys}. There will of course be momentum-dependent contributions at higher loops.

Fixing δ_λ in the mass-shell scheme

To fix δ_g a natural choice is to require that Γ_4 takes a fixed value $g(M^2)$ when all ingoing and outgoing momenta take the same invariant value $p^2 \sim M^2$ (but not necessarily co-linear) so that $s = t = u = M^2$ and also that we identify this coupling with the parameter in our renormalized Lagrangian g. We also assume $M >> m$. This then gives

$$\begin{aligned}\mu^{-\epsilon}\Gamma_4(M^2, M^2, M^2) = g(M^2) &= g - \frac{g^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + \ln\left(\frac{(4\pi\mu^2)^3}{\Delta(M^2)^3}\right) + ...\right) + \delta_g + ... \\ &= g - \frac{g^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + 3\ln\left(\frac{4\pi\mu^2}{\Delta(M^2)}\right) + ...\right) + \delta_g + ...\end{aligned}$$

So we make the choice

$$\delta_g = \frac{g^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + 3\ln\left(\frac{4\pi\mu^2}{\Delta(M^2)}\right) + ...\right).$$

The renormalized $\Gamma_4(s, t, u)$ is then

$$\mu^{-\epsilon}\Gamma_4(s, t, u) = g - \frac{g^2}{32\pi^2}\int_0^1 dx\ \ln\left(\frac{\Delta(M^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right) + ...$$

We can now set ϵ to zero

$$\Gamma_4(s, t, u) = \lambda - \frac{\lambda^2}{32\pi^2}\int_0^1 dx\ \ln\left(\frac{\Delta(M^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right) + ...$$

and we identify λ as the coupling measured at energy scale M.

The Beta-function

We briefly comment on the Beta-function. The quantum effective action does not care at which energy scale we define the couplings in the renormalized Lagrangian, it only cares about the about the bare couplings given by the sum of renormalized and counter-term Lagrangians. So we should have that

$$M\frac{d}{dM}\Gamma[\Phi]=0.$$

Since there is no wavefunction renormalization at the loop order we are working at, this means that we expect

$$M\frac{d}{dM}\Gamma_n=0+...$$

where the ellipsis are terms at two-loop and beyond. As such we require this be be true for the $n=4$ calculation above (it is trivially true for Γ_2). What does it mean to impose this condition? Consider

$$\begin{aligned}M\frac{d}{dM}\Gamma_4 &= \beta_\lambda-\beta_\lambda\frac{\lambda}{16\pi^2}\int_0^1 dx\left(\ln\left(\frac{\Delta(M^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right)+...\right)\\ &\quad -\frac{\lambda^2}{32\pi^2}M\frac{d}{dM}\int_0^1 dx\left(\ln\left(\frac{\Delta(M^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right)+...\right)=0\end{aligned}$$

where

$$\beta_\lambda=M\frac{d\lambda}{dM}.$$

Neglecting terms of order m/M, it is straightforward to show that

$$\beta_\lambda=\frac{3\lambda^2}{16\pi^2}+...$$

as found before in the Wilsonian approach. It does not matter at what energy scale we fix the coupling λ in the renormalized Lagrangian, the coupling will depend on energy in such a way that the physics, embodied by $\Gamma[\Phi]$, is invariant.

11.3 Minimal and Modified Minimal subtraction schemes

These are less physically intuitive than the mass-shell scheme but what they lose in transparency they gain in calculational simplicity.

MS Scheme

This is shorthand for *minimal subtraction.* In this case we only subtract the divergent part of the diagram. Recall that

$$\Gamma_2(p^2)=p^2-m^2-\frac{gm^2}{32\pi^2}\left(-\frac{2}{\epsilon}+\gamma-1-\ln\left(\frac{\mu^2}{4\pi m^2}\right)+...\right)+\left(p^2\delta_{Z_\phi}-\delta_{m^2}\right)+...$$

and

$$\mu^{-\epsilon}\Gamma_4(s,t,u)=g-\frac{g^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon}-3\gamma+\ln\left(\frac{(4\pi\mu^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right)\right)+\delta_g+...$$

Only requiring the counter-terms to remove the divergent part of the diagram gives

$$\delta_{Z_\phi} = 0, \qquad \delta_{m^2} = \frac{gm^2}{16\pi^2\epsilon}, \qquad \delta_g = \frac{3g^2}{16\pi^2\epsilon}.$$

We then have the renormalized quantities

$$\Gamma_2(p^2) = p^2 - m^2 - \frac{gm^2}{32\pi^2}\left(\gamma - 1 - \ln\left(\frac{\mu^2}{4\pi m^2}\right) + ...\right)$$

and

$$\Gamma_4(s,t,u) = g - \frac{g^2}{32\pi^2}\int_0^1 dx\left(-3\gamma + \ln\left(\frac{(4\pi\mu^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right)\right) + ...$$

We note that

- Once we have subtracted off the divergences, we can take the $\epsilon \to 0$ limit without trouble.
- As with the mass-shell scheme, the renormalized quantties still depend on an arbitrary scale parameter; μ in the case of the MS scheme and M in the case of the mass-shell scheme.

$\overline{\text{MS}}$ Scheme

This is shorthand for modified minimal subtraction, where we include the various additional finite terms that naturally appear in the evaluation of the diverging diagram such as γ and $\ln(4\pi)$.

$$\delta_{m^2} = \frac{gm^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + 1 + \ln(4\pi)\right], \qquad \delta_Z = 0.$$

$$\delta_g = \frac{g^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + 3\ln(4\pi)\right)$$

at one-loop.

11.3.1 The Beta-function revisited

Since there is no wavefunction renormalization at one-loop, the beta function comes directly from the statement that

$$\mu\frac{d}{d\mu}\Gamma_4(s,t,u) = 0.$$

In the MS scheme

$$\Gamma_4 = g - \frac{g^2}{32\pi^2}\int_0^1 dx\left(-3\gamma + \ln\left[\frac{(4\pi\mu^2)^3}{\Delta(s)\Delta(t)\Delta(u)}\right]\right) + ...$$

We are only interested in the μ-dependence, so

$$\Gamma_4 = g - \frac{3g^2}{16\pi^2}\ln(\mu) + ...$$

And so

$$0 = \mu\frac{d}{d\mu}\Gamma_4(s,t,u) = \beta_g - \beta_g\frac{3}{8\pi^2}\ln(\mu) - \frac{3g^2}{16\pi^2} + ...$$

which gives, to leading order

$$\beta_g = \frac{3g^2}{16\pi^2},$$

as found previously[51]. Again the beta-function equation can be solved to relate the coupling g at different values of μ

$$g(\mu') = \frac{g(\mu)}{1 - \frac{3}{16\pi^2} g(\mu) \ln(\mu'/\mu)}.$$

Note that, to leading order the couplings for different values of μ are related by

$$g(\mu') = g(\mu) + \frac{3g^2(\mu)}{16\pi^2} \ln\left(\frac{\mu'}{\mu}\right) + ...$$

The difference in the coupling as we change μ is, to leading order

$$\Delta g = -\frac{3g^2}{16\pi^2} \ln\left(\frac{\mu'}{\mu}\right).$$

Recalling the counter-term in the MS scheme, the difference in the counter-term determined at different values of μ is

$$\Delta\delta_g = \frac{g^2}{32\pi^2}\Big(\ln(\mu'^6) - \ln(\mu^6)\Big) = \frac{3g^2}{16\pi^2} \ln\left(\frac{\mu'}{\mu}\right).$$

What does this tell us? It illustrates the fact that a change in the renormalization scheme (different choices of δ_g, related by a finite constant) can be absorbed into a change in μ. We know that the choice of μ is not physically significant and so this shows in a concrete example how the choice of renormalization scheme does not matter. We notice that the constant of the flow $g_0 = g + \delta_g$ is preserved by the combined effect of the change in g and δ_g.

11.4 Regularization Independence and the Callan-Symanzik Equation

In both examples of regularization and renormalization scheme we have used (mass-shell and MS), we were forced to introduce two un-physical quantities

- A quantity for which the divergence re-appears in some limit of that quantity: $\epsilon \to 0$ in dimensional regularization and $\Lambda \to \infty$ in momentum cut-off regularization. These limits are taken in the physical calculations so these quantities do not appear.
- An arbitrary energy (or mass) scale. In dimensional regularization we needed to introduce μ so that the expression we wrote down still made dimensional sense in $4 - \epsilon$ dimensions. In the mass-shell scheme we saw all μ-dependence canceled out but at the cost of introducing the energy scale - M at which the counter-terms are defined.

[51] It should not come as a surprise that the beta function for λ and g are the same in the limit $\epsilon \to 0$. Indeed, it is simple to show that

$$\mu \frac{d\lambda}{d\mu} = \epsilon \mu^\epsilon g + \mu^\epsilon \mu \frac{dg}{d\mu},$$

and so

$$\beta_\lambda = \epsilon\lambda + \mu^\epsilon \beta_g.$$

Thus the beta functions for λ and g coincide.

Physics should not depend on the values of such arbitrary quantities. In this section and the next we shall see that it does not.

We have argued in previous sections that the partition function is independent of regularization. This is manifest by the fact that we now define the partition function in terms of the Lagrangian $\mathcal{L}_0$, which is defined independently of the cut-off scale and renormalization scheme we use. The separate seed and counter-term Lagrangians do depend on the regularization choice, but their sum does not. It is then clear that

$$Z[0] = \int \mathcal{D}\phi_0 \, e^{i \int d^dx \, \mathcal{L}_0},$$

is independent of the regularization. It follows then that the generating functional

$$Z[J_0] = \int \mathcal{D}\phi_0 \, e^{i \int d^dx \, (\mathcal{L}_0 + J_0 \phi_0)},$$

is also invariant. Under renormalization, the wavefunction is changed as

$$\phi_0(x) = Z_\phi^{\frac{1}{2}} \phi(x) := (1 + \delta_{Z_\phi})^{\frac{1}{2}} \phi(x).$$

The source term for the $\phi(x)$ in the generating functional then scales under renormalization as

$$J_0(x) = Z_\phi^{-\frac{1}{2}} J(x),$$

so that

$$\int d^dx \, \phi_0(x) J_0(x) = \int d^dx \, \phi(x) J(x),$$

is invariant. Z_ϕ is not regularization independent. As such correlation functions of the $\phi(x)$ will transform under renormalization group flow as we will now show.

The bare correlation functions

$$G_n^{(0)}(x_1, ..., x_n) = \langle \Omega | T\{\phi_0(x_1)...\phi_0(x_n)\} | \Omega \rangle_{\text{conn}}$$

cannot depend on the renormalization scale μ as it does not depend on the renormalized couplings. By contrast, the renormalized correlation functions

$$G_n(x_1, ..., x_n) = \langle \Omega | T\{\phi(x_1)...\phi(x_n)\} | \Omega \rangle_{\text{conn}}$$

do depend on μ. Since the wavefunction renormalization relates these fields $\phi_0(x) = Z_\phi^{\frac{1}{2}} \phi(x)$, we have that

$$G_n(x_1, ..., x_n) = Z_\phi^{-\frac{n}{2}} G_n^{(0)}(x_1, ..., x_n). \tag{11.2}$$

What happens if we change the renormalization scale μ? We know that $G^{(0)}(x_1, ..., x_n)$ is independent of the choice of μ, and so

$$\mu \frac{d}{d\mu} G_n^{(0)}(x_1, ..., x_n) = \mu \frac{d}{d\mu} \left(Z_\phi^{\frac{n}{2}} G_n(x_1, ..., x_n) \right) = 0$$

i.e

$$\mu \frac{d}{d\mu} G_n(x_1, ..., x_n) + n\gamma G_n(x_1, ..., x_n) = 0,$$

where we have defined the quantum anomalous dimension

$$\gamma := \frac{1}{2}\frac{\mu}{Z_\phi}\frac{\partial Z_\phi}{\partial \mu}.$$

The correlation function will depend on couplings g_i, which will depend on μ and so we have the Callan-Symanzik equation

The Callan-Symanzik equation

For a correlation function $G_n(x_1, ..., x_n)$ with anomalous dimension γ_a for the field a, couplings g_i and renormalization scale μ,

$$\left(\mu\frac{\partial}{\partial \mu} + \sum_i \beta_{g_i}\frac{\partial}{\partial g_i} + \sum_a \gamma_a\right) G_n(x_1, ..., x_n) = 0,$$

where $\beta_{g_i} = \mu\frac{dg_i}{d\mu_i}$ is the beta-function for the coupling g_i.

This is an example of the Callan-Symanzik equation. Applied to massless ϕ^4 theory, this equation is

$$\left(\mu\frac{\partial}{\partial \mu} + \beta_\lambda\frac{\partial}{\partial \lambda} + n\gamma\right) G_n(x_1, ..., x_n) = 0.$$

A few comments are in order:

- β and g are functions of g and so may be determined order by order in perturbation theory.
- β and γ do not depend on μ or n, so they are universal properties of the theory that do not depend on our choice of renormalization scale or the particular correlation function under consideration.

At one-loop there is no wavefunction renormalization of ϕ^4 theory but we can calculate the leading order beta-function.

11.5 Renormalization Scheme-independence

We can think of the flow of the renormalization group as the embedding of a line $\mathscr{L}$ into the theory space $\mathcal{M}$. If the theory space has local coordinates $\mathcal{X}^I = m, g, Z$ and we parameterise the line by μ, the flow can be thought of as an embedding

$$\mathcal{X} : \mathscr{L} \to \mathcal{M},$$

or $\mathcal{X}^I = \mathcal{X}^I(\mu)$.

In this section, we will argue that a change in renormalization scheme is nothing more than a reparameterization of the embedding of the flow line into theory space and so does not correspond to a physical choice.

We start with a theory with Lagrangian

$$\mathcal{L}_0 = \mathcal{L}_b + \mathcal{L}_{c.t.},$$

where $\mathcal{L}_b$ is the Lagrangian we use to write down our naive Feynman rules, e.g. $\mathcal{L}_b = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$, and $\mathcal{L}_{c.t.}$ is a counter-term Lagrangian. As always, the counter-terms remove divergences arising from the naive perturbation theory. A choice of renormalization scheme has been used to define the precise splitting of the bare mass and coupling into the observed value and the counter-term.

Now imagine we make a different choice of renormalization scheme and so the counter-terms differ by a finite constant amount (the divergent part must be the same),

$$\mathcal{L}_{c.t.} = \mathcal{L}'_{c.t.} + \Delta\mathcal{L}_{c.t.}$$

The Lagrangian in this scheme is

$$\mathcal{L}_0 = \mathcal{L}_b + \Delta\mathcal{L}_{c.t.} + \mathcal{L}'_{c.t.} := \mathcal{L}'_b + \mathcal{L}'_{c.t.},$$

where $\mathcal{L}'_b := \mathcal{L}_b + \Delta\mathcal{L}_{c.t.}$ is the new seed theory Lagrangian. We can write $\Delta\mathcal{L}_{c.t.}$ in the form[52]

$$\Delta\mathcal{L}_{c.t.} = \frac{1}{2}\Delta_Z(\partial\phi)^2 - \frac{1}{2}\Delta_{m^2}\phi^2 - \frac{1}{4!}\Delta_\lambda\phi^4,$$

and so we can write

$$\mathcal{L}'_b = \frac{1}{2}(\partial\phi')^2 - \frac{1}{2}m'^2\phi'^2 - \frac{\lambda'}{4!}\phi'^4,$$

where

$$m'^2 = (1+\Delta_z)(m^2 + \Delta_{m^2}), \qquad \lambda' = (1+\Delta_z)^2(\lambda + \Delta_\lambda)$$

where we have rescaled

$$\phi'(x) = (1+\Delta_Z)^{\frac{1}{2}}\phi(x).$$

This should feel reminiscent of the discussions of Wilsonian renormalization, which followed from a change in the energy cut-off and lead to renormalization group flow. We see that the change in the renormalization scheme may be interpreted as (alternatively, compensated by) a change in the way in which we parameterize the embedding $\mathcal{X} : \mathscr{L} \to \mathcal{M}$ with both $\mathcal{X}^I(\mu)$ and $\mathcal{X}'^I(\mu')$ representing the same point. Since the physical observables do not depend on μ, the change in renormalization scheme does not change the physics.

[52] Perhaps a more indicative notation would be $\Delta\delta_\lambda$ for the change in δ_λ, instead Δ_λ etc. Indicative is not always best.

12 Quantum Electrodynamics

QED is a theory of fermions and photons. We will need to understand how to describe both kinds of object in the path integral formalism. We start with fermions.

12.1 Fermions and Grassmann Variables

We shall introduce fermions in this simplest of settings and return to them in more realistic models later. This section summarises some of the more useful results involving Grassmann variables that are used to describe fermions, including the Faddeev-Popov ghosts involved gauge-fixing of the path inetegral which will be discussed late in the course.

We would like to treat fermionic fields in in a theory in a similar way to bosonic ones, i.e. correlation functions of fermions should be given by an expression of the form

$$\langle\psi(x_1)...\psi(x_n)\rangle \sim \left[\frac{1}{Z[\eta]}\frac{\delta^n Z[\eta]}{\delta\eta(x_1),,,\delta\eta(x_n)}\right]_{\eta=0}, \qquad Z[\eta] \sim \int \mathcal{D}\psi\, e^{\frac{i}{\hbar}(S[\psi]+\int\psi\cdot\eta)}$$

for some classical sources $\eta(x_i)$. The starting point is to recall that fermionic insertions into correlation functions anti-commute and so the sources we differentiate with respect to must also anti-commute and so the fields themselves. Thus, we need to work in terms of objects that naturally anti-commute (rather than commute). The objects in the path integral are not operators, so we need to introduce a new type of number - the Grassmann numbers.

The key point is that Grassmann variables anticommute

$$\theta_1\theta_2 = -\theta_2\theta_1.$$

This means that $\theta^2 = 0$ and that a function of a standard coordinate x and a single Grassmann 'coordinate' θ can at most be linear in θ

$$F(x,\theta) = f(x) + g(x)\theta,$$

where

$$f(x) := F(x,0), \qquad g(x) := \left.\frac{\partial F(x,\theta)}{\partial\theta}\right|_{\theta=0}.$$

We often think of the coordinates (x^μ, θ^a) as being local coordinates on a superspace $\mathbb{R}^{d|N}$, where $\mu = 1,2,..,d$ and $a = 1,2,...,N$.

Integration

For a single Grassmann variable, the most general function we can write down is $F(x,\theta) = f(x) + g(x)\theta$ so to integrate, we only need to learn how to integrate linear functions of θ. We will ask that our measure $d\theta$ is translation invariant in the fermionic directions, so that $d(\theta+\xi) = d\theta$ for a constant Grassmann number ξ. This means that

$$\int \theta d\theta = \int (\theta+\xi)d(\theta+\xi) = \int \theta d\theta + \xi\int d\theta$$

where the first equality holds by redefining the dummy integration variable and the second by linearity. We therefore conclude that

$$\int d\theta = 0.$$

By a conventional choice of scaling we can choose the normalisation of the measure and we define Grassmann integration according to

$$\boxed{\int d\theta = 0, \qquad \int d\theta \; \theta = 1.}$$

Then we have integrals such as

$$\int_{\mathbb{R}^{1|1}} dx d\theta \, F(x,\theta) = \int_{\mathbb{R}^{1|1}} dx d\theta \, (f(x) + \theta g(x)) = \int_{\mathbb{R}} dx \, g(x).$$

Differentiation

Differentiation proceeds in the obvious way

$$\boxed{\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu, \qquad \frac{\partial x^\mu}{\partial \theta^a} = 0, \qquad \frac{\partial \theta^a}{\partial x^\nu} = 0, \qquad \frac{\partial \theta^a}{\partial \theta^b} = \delta!_b.}$$

Notice that there is then a correspondence between integration and differentiation

$$\int \mathrm{d}\theta \leftrightarrow \frac{\partial}{\partial \theta},$$

and so we have statements such as

$$\int \mathrm{d}\theta \; F(x,\theta) = \left.\frac{\partial F(x,\theta)}{\partial \theta}\right|_{\theta=0} = g(x).$$

The theory of the integration of Grassmann variables is called *Berezin Integration.*

Gaussians, Determinants and Pfaffians

It is not too hard to prove the following results:

$$\int \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \; e^{-\theta_1 M \theta_2} = \int \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \; (1 - \theta_1 M \theta_2) = M,$$

where we note that the ordering of the terms in the measure now matters[53] since $\mathrm{d}\theta_1 \, \mathrm{d}\theta_2 = -\, \mathrm{d}\theta_2 \, \mathrm{d}\theta_1$. In particular, two results that are of particular interest to us and may be thought of as a Grassmann analogue of the result

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \; \exp\Big(-\frac{1}{2} x^T M x \Big) = \frac{(2\pi)^{n/2}}{\sqrt{\det(M)}},$$

[53]There is a subtlety here. We would usually assume for bosonic (x, y) that $\mathrm{d}^2 x \, \mathrm{d}x \, \mathrm{d}y = \mathrm{d}y \, \mathrm{d}x$; however, if we think of the measure as the volume form on $\mathbb{R}^2$, then $\mathrm{d}x \wedge \mathrm{d}y = -\,\mathrm{d}y \wedge \mathrm{d}x = \mathrm{d}^2 x * 1$. There is a natural generalistion to Grassmann variables involving commuting $\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 = \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_1$.

where M is a symmetric matrix and x is a vector in $\mathbb{R}^n$, are

$$\int_{\mathbb{R}^{0|n}\times\mathbb{R}^{0|n}} \mathrm{d}^n\theta\,\mathrm{d}^n\eta\;\exp\Big(-\theta^T M\eta\Big) = \det(M),$$

where η and θ are both Grassmann-valued vectors and we adopt the convention $\mathrm{d}^n\theta = \mathrm{d}\theta_1\,\mathrm{d}\theta_2 ...\,\mathrm{d}\theta_n$. Another useful result is

$$\int_{\mathbb{R}^{0|n}} \mathrm{d}^n\theta\;\exp\Big(-\frac{1}{2}\theta^T A\theta\Big) = \sqrt{\det(A)} = \mathrm{Pf}(A),$$

where A is an antisymmetric matrix and n is taken to be even. The integral vanishes if n is odd. The object $\mathrm{Pf}(A)$ is known as the Pfaffian of A and is nonvanishing only on antisymmetric matrices, where it takes the form $\sqrt{\det(A)}$.

12.2 QED in the Path Intergral Formalism

In this section we apply our understanding gained from the quantization of ϕ^4 theory to QED. The classical action for QED is

$$S[\psi,\bar{\psi},A] = \int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi\right)$$

where $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$ and the covariant derivative is

$$\not{D} = \not{\partial} + ie\not{A},$$

we adopt the usual slash notation $\not{A}(x) := \gamma^\mu A_\mu(x)$ and the electromagnetic field strength is given by

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The action enjoys a classical Abelian gauge symmetry, under which the fields transform as

$$\begin{aligned}\psi(x) &\to e^{i\alpha(x)}\psi(x)\\ \bar{\psi}(x) &\to \bar{\psi}(x)e^{-i\alpha(x)}\\ A_\mu(x) &\to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)\end{aligned} \tag{12.1}$$

where $\alpha(x)$ is an arbitrary real function and e is the coupling constant of the theory (usually the electron charge). We will not need to choose an explicit representation for the gamma-matrices but a useful one to have in mind is the Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix},$$

where σ^i are the Pauli matrices.

12.3 Path Integrals for Fermions

The action for the fermions is

$$S[\psi, \bar{\psi}] = \int d^4x\, \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

The generating functional is then

$$Z_0[\eta, \bar{\eta}] = \frac{1}{Z[0]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{iS[\psi,\bar{\psi}] + i\int \bar{\psi}\cdot\eta + i\int \psi\cdot\bar{\eta}},$$

where the subscript denotes the fact that this is a free theory. The key point is that ψ, $\bar{\psi}$, η and $\bar{\eta}$ are all taken to be anti-commuting *Grassmann* fields. The propagator, defined by

$$(i\partial\!\!\!/ - m)S_F(x-y) = i\delta^4(x-y),$$

can be found in the usual way, by going to momentum space, where it is found to be

$$\widetilde{S}_F(p) = \frac{i}{p\!\!\!/ - m} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2}.$$

Fourier transforming then gives[54]

$$\boxed{S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}.}$$

As with the scalar field theory studied previously, the generating functional for the free theory may be written entirely in terms of the sources. Proceeding as before, we find

$$\boxed{Z_0[\eta, \bar{\eta}] = \exp\left(-\int d^4x d^4y\, \bar{\eta}(x) S_F(x-y)\eta(y)\right).}$$

and we have

$$S_F(x-y) = \langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle = \frac{1}{Z_0[0]} \frac{\delta^2 Z_0[\eta,\bar{\eta}]}{\delta\bar{\eta}(x)\delta\eta(y)}\bigg|_{\eta=\bar{\eta}=0}.$$

12.3.1 Path Integrals for Vector Fields

Our starting point is the functional integral

$$\int \mathcal{D}A\, e^{iS[A] + i\int A_\mu J^\mu}, \qquad S[A] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}.$$

Integrating by parts, the action may be written as

$$S[A] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x\, A_\mu(x)\Big(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\Big)A_\nu(x).$$

[54]Our convention is that

$$S_F(x-y) = \langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle.$$

It is illuminating to write this in momentum space

$$S[A] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\widetilde{A}_\mu(k)\Big(-g^{\mu\nu}k^2 + k^\mu k^\nu\Big)\widetilde{A}_\nu(k).$$

The operator

$$\mathcal{P}^{\mu\nu} := -g^{\mu\nu}k^2 + k^\mu k^\nu,$$

projects out states of the form $\widetilde{A}_\mu(k) = k_\mu\alpha(k)$ and so such states are not propagating degrees of freedom of the theory. In position space, we see that fields of the form $A_\mu(x) = \partial_\mu\alpha(x)$ do not propagate. This is problematic, as this indicates that the operator $\mathcal{P}^{\mu\nu}$ has a non-trivial kernel and so, we cannot invert it to find the Feynman propagator for the photon. i.e. the Green's function equation

$$\Big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\Big)D_F^{\nu\rho}(x-y) = i\delta^\rho_\mu\delta^4(x-y).$$

This is a classical problem, not a quantum one but it means that the functional integral is badly divergent for configurations of the form $A_\mu(x) = \partial_\mu\alpha(x)$ as $e^{iS[A]} = 1$

$$\int \mathcal{D}A\,e^{iS[A]}\bigg|_{A=\partial\alpha} = \int \mathcal{D}\alpha.$$

The problem is related to the gauge symmetry of the theory. The action[55] is invariant under the transformations

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x).$$

The kernel of $\mathcal{P}^{\mu\nu}$ is spanned by those configurations that are gauge equivalent to $A_\mu(x)\sim 0$. Our strategy to deal with this problem will be to take a slice through the configuration space of $A_\mu(x)$ so that only one representative of the equivalence class $A_\mu(x)\sim A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)$ is integrated over.

The Fadeev-Popov Determinant

We shall insert the number 1, written in a complicated way, into the path integral[56]

$$1 = \int \mathcal{D}\alpha\,\delta\Big(G(A^\alpha)\Big)\,\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right),$$

[55] We shall assume that the functional integral measure is also invariant. It is not hard to show that the classical equations of motion $\partial_\mu\partial_\nu F^{\mu\nu} = \partial_\mu J^\mu = 0$ by antisymmetry of $F_{\mu\nu}$. Using the current conservation $\partial_\mu J^\mu = 0$ means that the presence of the source term does not break gauge invariance.

[56] This is the functional analogue of the one-dimensional identity

$$1 = \int da\,\delta\Big(g(a)\Big)\,\frac{dg(a)}{da}.$$

For n variables we have

$$1 = \int\left(\prod_{i=1}^n da_i\right)\delta^n\Big(g_i(a_j)\Big)\,\det\left(\frac{\partial g_i}{\partial a_j}\right). \tag{12.2}$$

The above expression is the functional (continuum) limit of this expression.

where[57]

$$A^\alpha_\mu(x) := A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x).$$

The delta functional $\delta(G(A^\alpha))$ enforces a gauge choice $G(A^\alpha) = 0$ which selects out one representative of each gauge equivalence class. Inserting (12.2) into the functional integral, and ignoring the source term for now,

$$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]}\,\delta\Big(G(A^\alpha)\Big),$$

where we shall choose a $G(A)$ such that[58]

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det\left(\frac{\partial^2}{e}\right),$$

is independent of the field $A_\mu(x)$ and the function $\alpha(x)$ and so can be pulled out of both integrals[59]. Next, we perform a gauge transformation in the integral from $A_\mu(x)$ to the specific gauge $A^\alpha_\mu(x)$. Classical gauge invariance ensures that $S[A] = S[A^\alpha]$ and we shall assume that $\mathcal{D}A = \mathcal{D}A^\alpha$

$$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha \int \mathcal{D}A^\alpha\, e^{iS[A^\alpha]}\,\delta\Big(G(A^\alpha)\Big),$$

Now all vector fields in the functional integral are called A^α. We rename the dummy integration variable $A^\alpha(x) \to A(x)$

$$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]}\,\delta\Big(G(A)\Big).$$

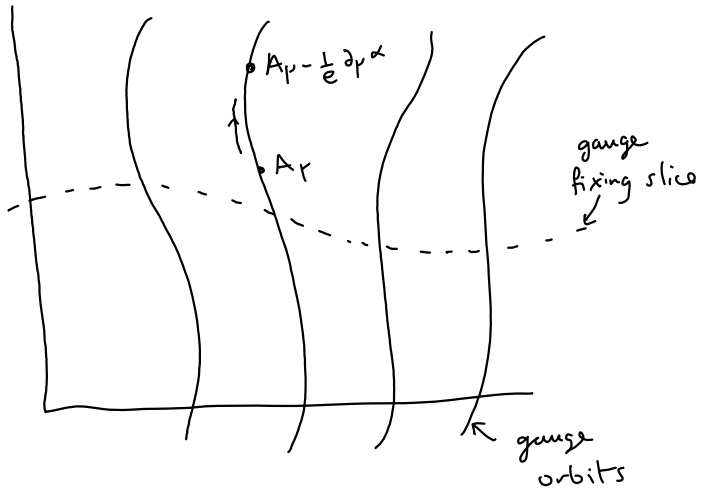
The functional integration over A as been split into an integral over gauge-inequivalent configurations multiplied by an integral over the space of gauge parameters $\alpha(x)$, with a

[57]What do we mean by the determinant of an operator? Since this operator could be taken to act on a vector space, we define the determinant as the product of the eigenvalues of the operator. We exclude the zero modes (those directions in the kernel do not contribute to the image and so do not contribute to the path integral).

[58]Lorentz gauge $\partial \cdot A = 0$, would be one such choice but we shall find it useful to fix to a one-parameter family of gauges that includes Lorentz gauge.

[59]We will see things are not so simple for Yang-Mills theories.

Jacobian to keep the change of variables in the measure honest.



Without loss of generality, we specify a gauge fixing function

$$G(A) = \partial^\mu A_\mu(x) - \omega(x),$$

where $\omega(x)$ is any scalar function and we have chosen to work in a generalization of Lorentz gauge $\partial^\mu A_\mu(x) = \omega(x)$. Without loss of generality, we can average over the functions $\omega(x)$ with a Gaussian weighting

$$\int \mathcal{D}A\, e^{iS[A]} = \left(N(\xi)\int \mathcal{D}\omega\, e^{-i\int \frac{\omega^2}{2\xi}}\right)\int \mathcal{D}A\, e^{iS[A]}$$

The functional determinant becomes $\det(\partial^2/e)$ and so

$$\begin{aligned}\int \mathcal{D}A\, e^{iS[A]} &= N(\xi)\int \mathcal{D}\omega \exp\left[-i\int d^4x\, \frac{\omega^2}{2\xi}\right]\det\left(\frac{\partial^2}{e}\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A\, e^{iS[A]}\,\delta\Big(\partial^\mu A_\mu(x) - \omega(x)\Big)\\ &= N(\xi)\det\left(\frac{\partial^2}{e}\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A\, e^{iS[A]}\exp\left[-\frac{i}{2\xi}\int d^4x\left(\partial^\mu A_\mu(x)\right)^2\right] \qquad (12.3)\end{aligned}$$

where we have used the gauge-fixing delta functional to do the $\omega(x)$ integral. There are a number of unimportant factors in the above expression. In particular the factors

$$N(\xi)\det\left(\frac{\partial^2}{e}\right)\left(\int \mathcal{D}\alpha\right)$$

will all cancel out in the normalization of any correlation functions, leaving our generating functional

$$Z[J] = \int \mathcal{D}A\, e^{iS[A]+iS_{\text{gf}}[A]+i\int J\cdot A}, \qquad (12.4)$$

where

$$S_{\text{gf}}[A] = -\frac{1}{2\xi}\int d^4x\left(\partial^\mu A_\mu(x)\right)^2.$$

Notice that if we take the limit $\xi \to 0$, we have

$$\lim_{\xi \to 0} N(\xi) \int \mathcal{D}\omega \, e^{-i\int \frac{\omega^2}{2\xi}} = \int \mathcal{D}\omega \, \delta(\omega) = 1,$$

which fixes $\omega(x) = 0$ and the gauge choice is Lorentz gauge. In general, different choices of ξ correspond to different gauges

- Landau (or Lorentz) Gauge $\xi = 0$
- Feynman Gauge $\xi = 1$

but other choices may be useful. We will tend to work in Feynman gauge and set $\xi = 1$. The expression for general ξ is often referred to as R_ξ gauge[60].

The gauge-fixing term is quadratic in $A_\mu(x)$ and so inclusion of the gauge fixing term $S_{\text{gf}}[A]$ modifies the momentum space Green's function equation to

$$\left(-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) \frac{k_\mu k_\nu}{k^2}\right) \widetilde{D}_F^{\nu\rho}(k) = i\delta_\mu^\rho,$$

which now has the solution

$$\widetilde{D}_F^{\nu\rho}(k) = \frac{-i}{k^2 + i\epsilon} \left(g^{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right).$$

In Feynman gauge this is

$$\widetilde{D}_F^{\nu\rho}(k) = \frac{-i g^{\mu\nu}}{k^2 + i\epsilon}$$

In position space, the general R_ξ gauge propagator is

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} \left(g^{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) e^{-ik\cdot(x-y)}$$

The generating functional for the (free) photon theory is then

$$\boxed{Z_0[J] = \exp\left(-\frac{1}{2}\int d^4x d^4y \, J_\mu(x) D_F^{\mu\nu}(x-y) J_\nu(y)\right)}$$

Putting the results of the last two sections together, we can construct the path integral for QED. The Lagrangian may be written as

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi,$$

where the interaction terms between the two sectors is

$$\mathcal{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu A_\mu \psi.$$

The path integral description of this theory is straightforward. The generating functional for the interacting theory may be written as

[60] R is for Renormalization.

$$\boxed{Z[J,\eta,\bar\eta] = \frac{1}{Z_0[0]}\exp\left[-ie\int d^4x\left(\frac{\delta}{\delta J^\mu(x)}\right)\left(\frac{\delta}{\delta\eta_\alpha(x)}\right)\gamma^\mu_{\alpha\beta}\left(\frac{\delta}{\delta\bar\eta_\beta(x)}\right)\right] Z_0[J]\,Z_0[\eta,\bar\eta].}$$

Using this generating functional, we can derive the Feynman rules of QED.

12.4 Regularization and Renormalization of QED

We return to the QED Lagrangian, but with m and e replaced by the unobservable bare mass and charge m_0 and e_0 respectively

$$\mathcal{L}_0 = -\frac{1}{4}F_0^{\mu\nu}F_{0\mu\nu} + \bar\psi_0(i\partial\!\!\!/ - m_0)\psi_0 - e_0\bar\psi_0 A\!\!\!/_0\psi_0,$$

the bare fields are related to the observed fields by the wavefunction renormalization

$$\psi_0(x) = Z_2^{1/2}\psi(x) \qquad A_0(x) = Z_3^{1/2}A,$$

so that the Lagrangian may be written in terms of renormalized fields

$$\mathcal{L}_0 = -\frac{Z_3}{4}F^{\mu\nu}F_{\mu\nu} + Z_2\bar\psi(i\partial\!\!\!/ - m_0)\psi - e_0 Z_2\sqrt{Z_3}\bar\psi A\!\!\!/\psi,$$

The charge coupling renormalization is related to the observed charge[61]

$$e_0 Z_2 Z_3^{1/2} := eZ_1$$

so that

$$\mathcal{L}_0 = -\frac{Z_3}{4}F^{\mu\nu}F_{\mu\nu} + Z_2\bar\psi(i\partial\!\!\!/ - m_0)\psi - eZ_1\bar\psi A\!\!\!/\psi.$$

We now split this Lagrangian into an observed Lagrangian and a counter-term part

$$\begin{aligned}\mathcal{L}_0 = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar\psi(i\partial\!\!\!/ - m)\psi - e\bar\psi\gamma^\mu A_\mu\psi && \Big\}\text{Classical Theory}\\ &-\frac{\delta_3}{4}F^{\mu\nu}F_{\mu\nu} + \bar\psi(\delta_2 i\partial\!\!\!/ - \delta_m)\psi - e\delta_1\bar\psi\gamma^\mu A_\mu\psi && \Big\}\text{Counter-terms}\end{aligned} \tag{12.5}$$

Comparing with the previous expression, we have the definitions

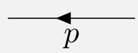
$$\delta_2 = Z_2 - 1, \qquad \delta_3 = Z_3 - 1,$$

$$\delta_m = Z_2 m_0 - m, \qquad \delta_1 = Z_1 - 1 = \frac{e_0}{e}Z_2 Z_3^{1/2} - 1.$$

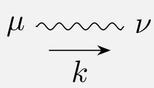
Perturbation theory with this Lagrangian then reduced to a diagrammatic calculus with the following Feynman diagrams:

[61] Observed at large distances (low momentum).

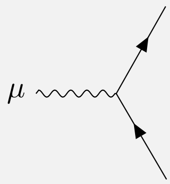
Momentum Space Feynman Rules in Feynman Gauge



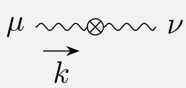
$$= \frac{i}{\not{p} - m + i\varepsilon}$$



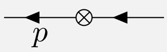
$$= -\frac{i\eta_{\mu\nu}}{k^2 + i\varepsilon}$$



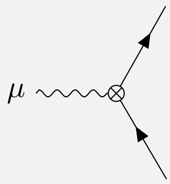
$$= -ie\gamma^\mu$$



$$= -i(\eta^{\mu\nu}k^2 - k^\mu k^\nu)\delta_3$$



$$= i(\not{p}\delta_2 - \delta_m)$$

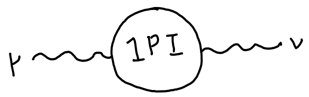


$$= -ie\gamma^\mu\delta_1$$

- Impose momentum conservation at each vertex.
- Integrate $\int \frac{d^4k}{(2\pi)^4}$ over all unconstrained (loop) momenta
- Overall momentum conservation $(2\pi)^4\delta^4(\sum_i p_i)$
- Divide by the symmetry factor of the graph
- Include a factor of -1 for each fermion loop
- Introduce a relative $-$ sign between diagrams that have external fermions exchanged

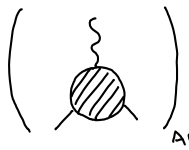
The last two rules are features that enter when dealing with Grassmann fields.

We fix the counter-terms using the two-point and three point (1PI amputated) graphs



$$= i\Pi^{\mu\nu}(p)$$

$$= -i\Sigma(\not p)$$



Amputated

$$= -ie\Gamma^{\mu}(p',p)$$

12.5 Dimensional regularization of one-loop divergences and renormalization in the MS scheme

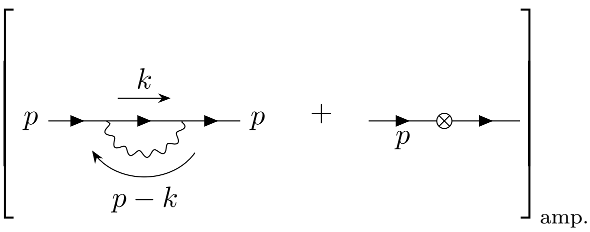
At one-loop there are three divergent diagrams that we need to regularize and evaluate. We shall choose to work in dimensional regularization but similar results would follow using a hard momentum cut-off. We introduce an arbitrary dimension-full constant μ with mass dimension $[\mu] = +1$, we can then recover a dimensionless coupling g as

$$e = \mu^{\frac{\epsilon}{2}} g,$$

where we are working in $d = 4 - \epsilon$ dimensions.

12.5.1 Electron Self-Energy

The two-point function has the same form as the ϕ^4 theory propagator with 1PI component which we shall call $-i\Sigma$. In particular, we would like to evaluate the contribution to order e^2. We want to evaluate the amputated diagram



$$= -i\Sigma_2(p) + i(\not p\delta_2 - \delta_m),$$

where $-i\Sigma_2(p)$ denotes the order e^2 contribution to $-i\Sigma(p)$ and is given by

$$-i\Sigma_2(p) = (-ig)^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\gamma^{\mu}\frac{i(\not k + m)}{k^2 - m^2 + i\varepsilon}\gamma_{\mu}\frac{-i}{(p-k)^2 + i\varepsilon}$$

and we define the renormalized self-energy as

$$-i\Sigma_2^{\text{ren}}(p) = -i\Sigma_2(p) + i(\not{p}\delta_2 - \delta_m).$$

The details of the calculation are similar to those we have already seen and are given in the Appendix. After a lot of algebra, we find

$$\begin{aligned} -i\Sigma_2^{\text{ren}}(p) \;=\; & -i\frac{g^2\mu^{d-4}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\, \frac{(4-\epsilon)m-(2-\epsilon)x\not{p}}{[(1-x)m^2-x(1-x)p^2]^{2-d/2}} \\ & + i(\not{p}\delta_2 - \delta_m) \end{aligned}$$

It is useful to have the expression in the limit of small ϵ. Using

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + ..., \qquad \left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} = \exp\left[\frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{\Delta}\right)\right] = 1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{\Delta}\right) + ...,$$

we find

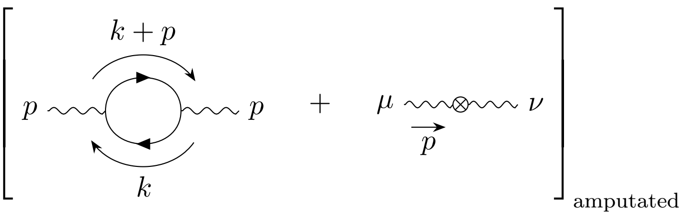
$$\begin{aligned} -i\Sigma_2^{\text{ren}}(p) \;=\; & i\frac{g^2}{8\pi^2}\not{p}\int_0^1 dx\, x\left(\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{\Delta}\right) + 1\right) \\ & -i\frac{g^2}{4\pi^2}m\int_0^1 dx\left(\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{\Delta}\right) - \frac{1}{2}\right) + i(\not{p}\delta_2 - \delta_m) \end{aligned} \qquad (12.6)$$

where $\Delta = (1-x)m^2 - x(1-x)p^2$. The MS scheme then gives

$$\delta_2 = -\frac{g^2}{8\pi^2\epsilon}, \qquad \delta_m = -\frac{mg^2}{2\pi^2\epsilon}$$

12.5.2 Photon Self-Energy (Vacuum Polarization)

The amputated, renormalized, diagram is



$$= \quad i\Pi_{\mu\nu}(p) - i(\eta^{\mu\nu}p^2 - p^\mu p^\nu)\delta_3$$

We define the renormalized self-energy as

$$i\Pi_{\mu\nu}^{\text{ren}}(p) = i\Pi_{\mu\nu}(p) - i(\eta^{\mu\nu}p^2 - p^\mu p^\nu)\delta_3,$$

where

$$i\Pi_{\mu\nu}(p) = (-ie)^2(-1)\int\frac{d^dk}{(2\pi)^d}\text{Tr}\left[\gamma_\mu\frac{i(\not{k}+m)}{k^2-m^2}\gamma_\nu\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\right]$$

The -1 factor comes from the fermion loop. This expression may be written as

$$i\Pi_{\text{ren}}^{\mu\nu}(p) = \left(p^2g^{\mu\nu} - p^\mu p^\nu\right)i\Pi_{\text{ren}}(p^2),$$

where

$$\Pi_{\text{ren}}(p^2) = -\frac{8g^2}{(4\pi)^{d/2}}\mu^{4-d}\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\, x(1-x)\left(\frac{1}{m^2-x(1-x)p^2}\right)^{2-\frac{d}{2}} - \delta_3$$

and $\Delta := m^2 - x(1-x)p^2$. We now need to isolate the diverging part of this expression. Let $d - 4 - \epsilon$ and we use the fact that

$$\Gamma(2-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + ...$$

where the ellipsis denote terms which vanish as $\epsilon \to 0$. We can then write

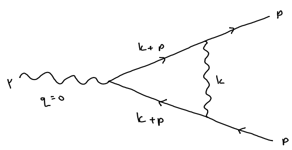
$$\begin{aligned}\Pi_{\text{ren}}(p^2) &= -\frac{g^2}{2\pi^2}\left(\frac{2}{\epsilon}-\gamma+...\right)\int_0^2 dx\, x(1-x)\left(1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{m^2-x(1-x)p^2}\right)+...\right) - \delta_3 \\ &= -\frac{g^2}{2\pi^2}\int_0^2 dx\, x(1-x)\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2-x(1-x)p^2}\right)+...\right] - \delta_3 \qquad (12.7)\end{aligned}$$

Working in the MS scheme we then choose

$$\delta_3 = -\frac{g^2}{6\pi^2\epsilon}$$

Three-Point Vertex

The amputated diagram is



$$= -ie\Gamma^\mu(p)$$

The renormalized vertex is

$$-ie\Gamma^\mu_{\text{ren}}(p) = -ie\Gamma^\mu(p) - ie\gamma^\mu\delta_1,$$

where

$$-i\Gamma^\mu(p) = (-ie)^3\int\frac{d^dk}{(2\pi)^d}\gamma^\lambda\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\gamma^\mu\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\gamma^\rho\left(-\frac{ig_{\lambda\rho}}{k^2}\right).$$

There is a quite a lot of work to be done to evaluate this graph but the general principles are the same as with other Feynman integrals and the details may be found in the Appendix. The result is

$$\begin{aligned}-i\Gamma^\mu(p) &= -\frac{ig^2\mu^{d-4}\,\Gamma(2-d/2)}{(4\pi)^{d/2}}\int_0^1 dx\frac{1}{\Delta^{2-d/2}}\Big[(d-2)(1-x)\gamma^\mu \\ &\quad +x(1-x)(4-d)\Big[(2-d)(1-x)\not{p}+dm\Big]p^\mu\Delta\Big] \qquad (12.8)\end{aligned}$$

where $\Delta = xm^2 - x(1-x)p^2$ here. The $4-d$ factor in the second term renders that term finite in the $\epsilon \to 0$ limit. We shall denote this contribution by

$$-iF^\mu(p) = -\frac{ie^2}{4\pi^2}\left((1-x)\not{p}-2m\right)p^\mu\Delta.$$

Expanding about small ϵ as above, we find

$$-i\Gamma^{\mu}_{\text{ren}}(p) = -\frac{ig^2\gamma^{\mu}}{8\pi^2}\int_0^1 dx(1-x)\left(\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi\mu^2}{\Delta}\right)-1\right) - iF^{\mu}(p) - i\gamma^{\mu}\delta_1 + \ldots$$

The MS scheme gives

$$\delta_1 = -\frac{g^2}{8\pi^2\epsilon}.$$

12.6 The Beta Function in the Minimal Subtraction (MS) Scheme

Our motivation is the calculation of the beta function for $e = \mu^{\epsilon}g(\epsilon)$, given by

$$\beta_e = \mu\frac{\partial}{\partial\mu}e.$$

We see that

$$\mu\frac{\partial}{\partial\mu}$$

is an operator (the Euler operator) that measures how a quantity scales with μ. Classically we would expect a quantity $\mathcal{O}$ with mass dimension D to satisfy

$$\mu\frac{\partial}{\partial\mu}\mathcal{O} = D\mathcal{O},$$

In the case here $D = \epsilon/2$. In a quantum theory, we expect some corrections to this

$$\mu\frac{\partial}{\partial\mu}\mathcal{O} = D\mathcal{O} + \beta_{\mathcal{O}},$$

where $\beta_{\mathcal{O}}$ measures the quantum effects (i.e. vanishes as $\hbar \to 0$). By working in terms of the dimensionless quantity $g = \mu^{-D}\mathcal{O}$, we strip off the classical scaling contribution and work with the quantum corrections.

The scale μ is unconnected to any physical scale and is an artifact of dimensional regularization (in the same way the momentum cut-off was an artifact of momentum regularization). As such, any physical quantity should not depend on μ.

Our goal will be to compute the beta function for e using minimal subtraction. The strategy we adopt will be to find a relationship between e_0 and e and use the fact that

$$\mu\frac{d}{d\mu}e_0 = 0,$$

to determine $e(\mu)$ through its beta-function. If we recall the quantum, classical and counter-term Lagrangians

$$\begin{aligned}
\mathcal{L}_0 &= -\frac{1}{4}F_0^{\mu\nu}F_{0\mu\nu} + \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 - e_0\bar{\psi}_0 A\!\!\!/_0\psi_0 && \Big\}\text{Quantum Theory} \\
&= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^{\mu}A_{\mu}\psi && \Big\}\text{Classical Theory} \\
&\quad -\frac{\delta_3}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\delta_2 i\partial\!\!\!/ - \delta_m)\psi - e\delta_1\bar{\psi}\gamma^{\mu}A_{\mu}\psi && \Big\}\text{Counter-terms}
\end{aligned} \tag{12.9}$$

where

$$\psi_0(x) = Z_2^{1/2}\psi(x), \qquad A_0(x) = Z_3^{1/2}A, \qquad e_0 Z_2 Z_3^{1/2} := eZ_1,$$

so that

$$\delta_2 = Z_2 - 1, \qquad \delta_3 = Z_3 - 1, \qquad \delta_1 = Z_1 - 1 = \frac{e_0}{e} Z_2 Z_3^{1/2} - 1, \qquad m_0 = \frac{m + \delta_m}{Z_2}$$

We see that

$$e_0 = \frac{Z_1}{Z_2\sqrt{Z_3}} e \approx (1+\delta_1)(1-\delta_2)(1-\delta_3/2)e \approx \left(1 + \delta_1 - \delta_2 - \frac{1}{2}\delta_3\right) e, \tag{12.10}$$

to leading order in the counter-terms. We collected here the counter-terms in the MS scheme as found previously

$$\boxed{\delta_1 = -\frac{g^2}{8\pi^2\epsilon}, \qquad \delta_2 = -\frac{g^2}{8\pi^2\epsilon}, \qquad \delta_3 = -\frac{g^2}{6\pi^2\epsilon}, \qquad \delta_m = -\frac{mg^2}{2\pi^2\epsilon},}$$

We observe that $\delta_1 = \delta_2$. This is not a coincidence and can be shown to be a consequence of a Ward identity.

We saw above that $\delta_1 = \delta_2$ and so we have, terms of the dimensionless coupling the simplified expression

$$e_0 = \mu^{\frac{\epsilon}{2}}\left(1 - \frac{1}{2}\delta_3\right) g = \mu^{\frac{\epsilon}{2}}\left(g + \frac{g^3}{12\pi^2\epsilon}\right)$$

We now require that

$$\mu\frac{d}{d\mu}e_0 = \frac{\epsilon}{2}\mu^{\epsilon/2}\left(g + \frac{g^3}{12\pi^2\epsilon}\right) + \mu^{\epsilon/2}\beta_g\left(1 + \frac{g^2}{4\pi^2\epsilon}\right) = 0,$$

rearranging gives

$$\begin{aligned} \beta_g &= -\frac{\epsilon}{2}\left(1 + \frac{g^2}{4\pi^2\epsilon}\right)^{-1}\left(g + \frac{g^3}{12\pi^2\epsilon}\right) \\ &\approx -\frac{\epsilon}{2}\left(1 - \frac{g^2}{4\pi^2\epsilon}\right)\left(g + \frac{g^3}{12\pi^2\epsilon}\right) \\ &\approx \frac{g^3}{12\pi^2} \end{aligned} \tag{12.11}$$

where we have only kept leading terms in the coupling and have taken ϵ to zero at the end. Putting the factor of $\hbar$ back in we find

$$\boxed{\beta_g = \frac{\hbar g^3}{12\pi^2} + ...}$$

where the ellipsis denote terms of higher order in $\hbar$. It is worth pointing out that the exact calculation of the beta-function in perturbation theory depends on the renormalization scheme being adopted. This makes sense as the scheme determines what parts of the couplings in $\mathcal{L}_0$ are contained in the classical and counter-term Lagrangians. This in term determines the coupling we are working at and hence the form of each term in our perturbation series. In otyer words the choice of scheme effectively reorganises the perturbation series.

Again, this equation can be integrated to give the running of the electron charge

$$e^2(\mu) = \frac{e^2(M)}{1 + \frac{e^2(M)}{6\pi^2}\ln(M/\mu)}$$

We can also define a scale Λ_{QED} at which the coupling seems to diverge[62]

$$1 + \frac{e^2(M)}{6\pi^2}\ln(M/\Lambda_{QED}) = 0$$

One can then take this as the fundamental reference scale and write the coupling at energy μ as

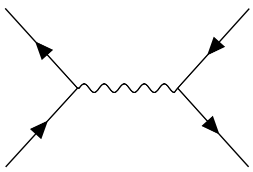
$$e^2(\mu) = \frac{6\pi^2}{\ln(\Lambda_{QED}/\mu)}$$

We see then that, as with ϕ^4 theory, the coupling in QED reduces at large distances and increases as short distances. This is in accordance with our intuition of the running of the coupling as coming from vacuum polarization. Such a divergence seems to occur at $\Lambda_{QED} \approx 10^{286}$ GeV, well above any experiments we can come close to.

12.7 Renormalization: Some Physical Intuition

We have seen that the inclusion of counter-terms has a chance of absorbing the infinities into the definition of the QFT.

To get a feel for why this is so consider the problem of inferring a charge from scattering at tree level

$i\mathcal{M} =$ 

We adopt the interaction picture and take the Hamiltonian to be $\widehat{H} = \widehat{H}_0 + \widehat{V}$, where we assume $\widehat{H}_0$ is the free Hamiltonian. The potential is something that we are used to defining in non-relativistic quantum mechanics, where the transition from the initial to final state is given by

$$\langle \psi_f | U_I(t_f - t_i) | \psi_i \rangle$$

[62]Historically, this is known as a Landau pole. The converse of this concern for the divergence of the theory in the UV, is that the theory seems to be trivial in the IR - as the coupling goes to zero at the energy goes to zero, leaving a non-interacting theory.

where $U_I(t_f - t_i)$ is the interaction picture operator

$$U_I(t_f - t_i) = \exp\left(-i\int_{t_i}^{t_f} \widehat{V}(s)ds\right)$$

where we assume the interaction term in the Hamiltonian is just the potential and the time evolution of the initial and final states is given by the free Hamiltonian $|\psi(t)\rangle = e^{-i\widehat{H}_0 t}|\psi(0)\rangle$. The S-Matrix is just given by the limit of taking the initial and final states to be in the infinite past and future respectively. To leading order, the S-Matrix is

$$\begin{aligned} S_{if} &= \langle\psi_f|U_I(t_f - t_i)|\psi_i\rangle \\ &= \delta_{if} - i\int_{-\infty}^{\infty} ds\langle\psi_f|\widehat{V}(s)|\psi_i\rangle + ... \end{aligned} \tag{12.12}$$

If we focus on 2 particle $\to$ 2 particle scattering of the form[63]

$$|\mathbf{p}_1, \mathbf{p}_2\rangle \to |\mathbf{p}'_1, \mathbf{p}'_2\rangle,$$

with the initial and final states describing plane waves with energy E_i, then we have

$$S_{if} = \delta_{if} - i2\pi\delta(E_i - E_f)\langle\mathbf{p}'_1, \mathbf{p}'_2|\widehat{V}|\mathbf{p}_1, \mathbf{p}_2\rangle + ...$$

and it is not hard to show that, for the plane wave asymptotic states

$$S_{if} = \delta_{if} - i(2\pi)^4\delta(E_i - E_f)\delta^3\Big(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2\Big)\widetilde{V}(\mathbf{q}) + ...$$

where $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}'_1 = -\mathbf{p}_2 + \mathbf{p}'_2$ is the momentum transfer. We see that the Fourier transform of the potential is related to the 1PI momentum space amplitude. Our strategy will be to find a non-relativistic limit of the 1PI process and then take the Fourier transform to get the potential. Here are some examples:

ϕ^4 Theory

At tree-level, the 1PI four-point amplitude is

$$i\mathcal{M} = -i\lambda$$

We then identify $\widetilde{V}(\mathbf{q}) = \lambda$ and see that the position space potential is what is sometimes called a contact interaction

$$V(\mathbf{x}) \sim \lambda\delta^3(\mathbf{x}).$$

ϕ^3 Theory

Consider next the scalar theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\phi^2 - \frac{g}{3!}\phi^3.$$

[63] A more detailed derivation of this same result, using the Lipmann-Schwinger solution is presented in the Appendix.

Since the interaction vertices are cubic, at tree-level the amplitude requires the propagator and so we can see that, for the four-point 1PI diagrams

$$\widetilde{V}(\mathbf{q}) \sim \frac{g^2}{q^2 - m^2}.$$

In the non-relativistic limit the four-momenta are approximately $p_\mu \approx (m, \mathbf{p})$ and so $q_\mu = (E_1 - E_1', \mathbf{q}) \approx (0, \mathbf{q})$ and so

$$\widetilde{V}(\mathbf{q}) \sim \frac{g^2}{\mathbf{q}^2 + m^2}.$$

The potential then has the behaviour

$$V(\mathbf{x}) \sim g^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 + m^2} = g^2 \frac{e^{-mr}}{4\pi r},$$

where $|\mathbf{x}| = r$.

Comments

We can see something of a pattern from the above examples, where the propagator is playing the key role. There are two interesting limits we can consider. If we take the mass of the propagator to infinity[64]

$$V(\mathbf{x}) \sim g^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{m^2} = \frac{g^2}{m^2}\delta^3(\mathbf{x}),$$

and we recover the contact interaction of the ϕ^4 theory.

If instead we take the mass of the propagating particle to zero, then we find a Coulomb-like potential

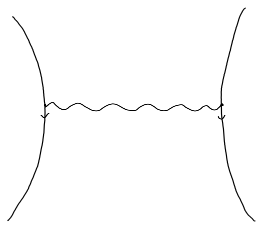
$$V(\mathbf{x}) \sim g^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2} = \frac{g^2}{4\pi r},$$

consistent with what we would expect for the propagation of a massless photon.

A final comment is that the propagator will receive contributions at higher loop order. There will also be other diagrams that modify the vertex. As such, the potential will be modified. In particular the mass and couplings will be shifted as we take these quantum effects into account.

12.7.1 The Uehling Potential

How do we measure the charge of the electron? We scatter other electrons/positrons off it. At tree-level, the scattering amplitude is

$$i\mathcal{M} =$$ 

[64] The calculation of the Fourier transform is greatly facilitated by spotting that in spherical polar coordinates $\mathbf{q}\cdot\mathbf{x} = qr\cos(\theta)$, where θ is the angle between the vectors.

and we can (using the Born rule) take the Fourier transform of this amplitude to get the Coulomb potential. We know that other processes occur, governed by more complicated diagrams which correct the amplitude and so give corrections to the Coulomb potential. The effects of these processes, whose strength is momentum-dependent, can be interpreted as a screening of the original Coulomb potential.

The object of study is therefore the 4-point 1PI process. At one loop there are three types of diagram, one with a photon loop on each external fermion line to include a factor of $\Sigma_2(p^2)$ on each leg, one with a modification to each vertex by $-ie\Gamma(p)$ and one with a fermion loop inserted into the propagator. One can show that the first two types of diagram cancel each other (this can again be understood as a consequence of a Ward-Takahashi identity). This leaves only the last process - vacuum polarization - to take into account. The propagator is then (in Feynman gauge)

$$\begin{aligned} G^{\mu\nu}(p) &= -\frac{i\eta^{\mu\nu}}{p^2} + \left(-\frac{i\eta^{\mu\lambda}}{p^2}\right) i\Pi^{\text{ren}}_{\lambda\rho}(p) \left(-\frac{i\eta^{\rho\nu}}{p^2}\right) + ... \\ &= -\frac{i\eta^{\mu\nu}}{p^2}\Big(1 + \Pi_{\text{ren}}(p)\Big) + \frac{p^\mu p^\nu}{p^4} i\Pi_{\text{ren}}(p) + ... \end{aligned} \tag{12.13}$$

We are interested in the amputated four-point amplitude. The four-point amplitude is given by contracting the propagator with fermion currents $j^\mu = -ig\bar{\psi}\gamma^\mu\psi$. Current (fermion number) conservation in momentum space tells us $p_\mu j^\mu = 0$ and so the last terms involving the $\frac{p^\mu p^\nu}{p^4}$ terms will vanish. The amputated four point amplitude is then

$$i\mathcal{M} = (-ig\gamma_\mu)iG^{\mu\nu}(p)(-ig\gamma_\nu) = -\frac{g^2}{p^2}\Big(1 + \Pi_{\text{ren}}(p)\Big) + g^2\frac{\cancel{p}^2}{p^4}\Pi_{\text{ren}}(p) + ...$$

The $p^\mu p^\nu$ terms are sub-leading and we shall ignore them. The momentum-space potential is then

$$\widetilde{V}(p) = \frac{g^2}{p^2}\Big(1 + \Pi_{\text{ren}}(p)\Big).$$

We have evaluated the vacuum self-energy diagram

$$\Pi_{\text{ren}}(p^2) = -\frac{g^2}{2\pi^2}\int_0^2 dx\, x(1-x)\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2 - x(1-x)p^2}\right) + ...\right] - \delta_3$$

In this case we shall make the physically motivated choice $\delta_3 = \Pi(0)$; i.e. the counter-term subtracts off the vacuum polarization measured in the IR limit. This gives the renormalized expression

$$\begin{aligned} \Pi_{\text{ren}}(p^2) &= -\frac{g^2}{2\pi^2}\int_0^2 dx\, x(1-x)\ln\left(\frac{m^2}{m^2 - x(1-x)p^2}\right) \\ &= \frac{g^2}{2\pi^2}\int_0^2 dx\, x(1-x)\ln\left(1 - \frac{p^2}{m^x}x(1-x)\right) \end{aligned} \tag{12.14}$$

12.7.2 The non-relativistic limits and the Lamb shift

We can make the assumption $p^2 \ll m^2$ in the $\widetilde{V}(p)$ and approximate

$$\int_0^1 dx\, x(1-x)\ln\left(1 - \frac{p^2}{m^2}x(1-x)\right) \approx \int_0^1 dx\, x(1-x)\left(-\frac{p^2}{m^2}x(1-x)\right) = -\frac{p^2}{30m^2}$$

The momentum space potential then becomes

$$\widetilde{V}(p) \approx \frac{e^2}{p^2} = \frac{e^4}{60\pi^2 m^2}.$$

This last term is a constant and the Fourier transform is a delta-function. The potential in this limit is then

$$V(r) = -\frac{e^2}{4\pi r} - \frac{e^4}{60\pi^2 m^2}\delta(r).$$

This is in the non-relativistic limit and so we can use this modified Coulomb potential in quantum mechanical perturbation theory. The shift in the ground state energy will be

$$\Delta E = \langle\psi|\Delta V|\psi\rangle \approx -\frac{e^4}{60\pi^2 m^2}|\psi(0)|^2.$$

This relativistic correction to the energy level of the hydrogen atom is known as the Lamb shift.

12.7.3 Charge screening and Running of the Coupling

Working in the (static) limit $p_0 \approx 0$, we can take the Fourier transform and integrate out the angular directions to get

$$V(r) = -\frac{e^2}{4\pi r}\left(1 + \frac{e^2}{6\pi^2}\int_1^\infty du\, e^{-2mur}\left(1 + \frac{1}{2u^2}\right)\frac{\sqrt{u^2-1}}{u^2}\right)$$

We consider this in two regimes[65]

$r \ll \frac{1}{m}$

$$V(r) \approx \frac{e^2}{4\pi r}\left(1 + \frac{e^2}{12\pi^2}\left[-2\ln(mr) - 2\gamma - \frac{5}{3} + ...\right]\right)$$

In this limit the effective coupling grows at short distance.

$r \gg \frac{1}{m}$

$$V(r) \approx \frac{e^2}{4\pi r}\left(1 + \frac{e^2}{16(\pi mr)^{\frac{3}{2}}}e^{-2mr} + ...\right)$$

We see in this latter limit that the long-distance Coulomb potential receives a Yukawa-like term due to the electron loop which is damped at large distance.

For QED , the coupling runs as shown in figure 19. An obvious question arises: what are we talking about when we say that the fine structure constant is

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}?$$

[65] Where the integral formula for the Euler-Mascheorni constant

$$\gamma = -\int_0^\infty dx\, \ln(x)e^{-x},$$

has been used.

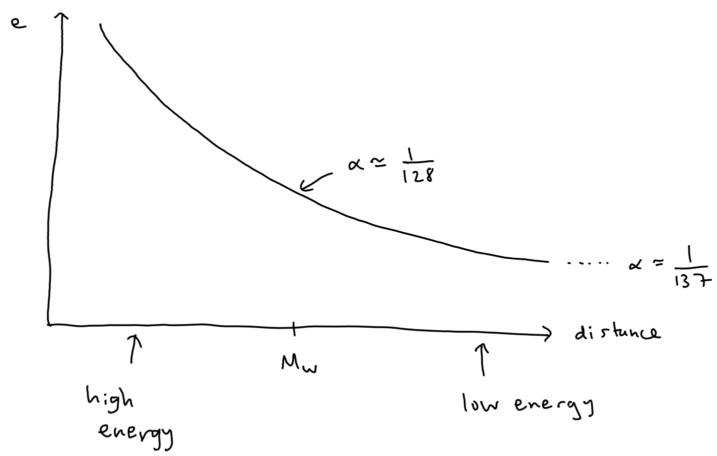


Figure 19. The running of the coupling in QED.

To make this precise, we need to specify an energy scale. We usually define the fine structure constant as the square of the completely screened electron charge; i.e. the value measured by an observer at infinity (corresponding to zero momentum transfer). In other words we measure the low energy effective charge.

We can understand the change in the coupling with energy in terms of a screening charge. What follows is a semi-classical caricature, but can be a useful conceptual tool in our attempts to understand QFT. Near an electron, virtual electron/positron pairs jump are continuously created and destroyed. The positrons are attracted to the electron and the virtual electrons repelled, so that an effective sea of virtual charge surrounds the original electron we want to measure and screens the 'bare' charge e_0, which is fundamentally unmeasurable.

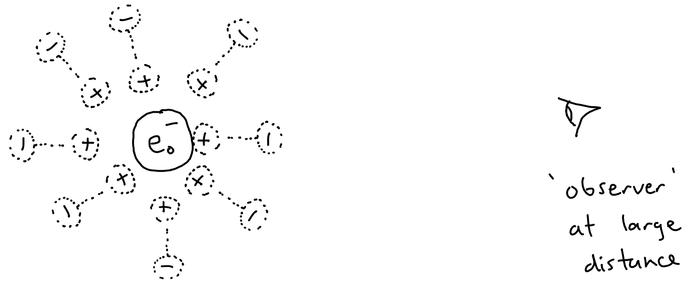


Figure 20. A sea of virtual electron/positron pairs screen the charge.

At higher energies (shorter distances), scattering processes probe the cloud of virtual particle pairs surrounding the electron (the screen) and we measure a slightly higher value for α. For example, at the energy of the W-boson (about 81 GeV or a distance of 2×10^{-17}m),

$$\alpha(M_W) \approx \frac{1}{128}.$$

13 Yang-Mills Theory

In this section we we begin our study of Yang-Mills theory[66], a cousin of electromagnetism in which the $U(1)$ gauge symmetry is generalised to a non-abelian gauge group (such as $SU(N)$). In QED, we had the gauge symmetry

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi,$$

where the covariant derivative is

$$D_\mu = \partial_\mu + ieA_\mu,$$

and the fermions and gauge boson transform as

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha(x)}, \qquad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x).$$

This is $U(1)$ because $e^{i\alpha(x)} \in U(1)$. How do we see the $U(1)$ action on $A_\mu(x)$? If we define $U = e^{i\alpha(x)}$, $U^\dagger = e^{-i\alpha(x)}$ (note that $U^\dagger U = 1$), then

$$\psi(x) \to U\psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}(x)U^\dagger, \qquad D_\mu(x) \to UD_\mu U^\dagger.$$

To see the last statement, consider

$$\begin{aligned} D_\mu &\to UD_\mu U^\dagger \\ &= e^{i\alpha(x)}\Big(\partial_\mu + ieA_\mu(x)\Big)e^{-i\alpha(x)} \\ &= \partial_\mu + ieA'_\mu(x) \qquad \text{if} \qquad A'_\mu(x) := A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x). \end{aligned} \tag{13.1}$$

13.1 Non-Abelian Gauge Theory

We have seen how $U \in U(1)$ appears in QED. What happens if we allow U to take values in some more general group G? We shall be mainly interested in $G = SU(N)$, which we can think of the group of $N \times N$ unitary matrices ($U^\dagger U = I$) with $\det(U) = 1$. There are $N^2 - 1$ generators, T_a for such a group. We shall require the gauge fields to transform such that there is a covariant derivative that transforms as

$$D_\mu(x) \to UD_\mu U^\dagger, \qquad D_\mu = \partial_\mu - igA_\mu$$

this will require our gauge fields to be in the adjoint representation of the group. This is the representation that the generators live in. As such we can write

$$(A_\mu)^i{}_j(x) = A^a_\mu(x)(T_a)^i{}_j,$$

where T_a (for $a = 1, 2, ...N^2 - 1$) is a basis of generators, which satisfy the Lie algebra

$$[T_a, T_b] = if_{ab}{}^c T_c.$$

[66]Sometimes referred to as Yang-Mills-Shaw theory, recognising the simultaneous, but unpublished discovery by Ron Shaw (see https://www.maths.ed.ac.uk/ v1ranick/shaw.pdf).

Associativity of the algebra

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0$$

follows from the Jacobi identity

$$f_{ab}{}^{d} f_{cd}{}^{e} + f_{bc}{}^{d} f_{ad}{}^{e} + f_{ca}{}^{d} f_{bd}{}^{e} = 0. \tag{13.2}$$

The Lie group is a manifold and the Lie algebra encodes the local structure of the group. The $f_{ab}{}^{c}$ are the structure constants.

Tha most straightforward generalisation of QED leads to fermions transforming in the fundamental representation[67]

$$\psi_i(x) \to U_i{}^j \psi_j(x),$$

where $i = 1, 2, .., N$. More generally, tensors of the Lie group transform covariantly according to their index structure

$$\psi_i(x) \to U_i{}^j \psi_j(x), \qquad \bar{\psi}^i(x) \to \bar{\psi}^j(x)(U^\dagger)_j{}^i, \qquad (D_\mu)_i{}^j \to U_i{}^k (D_\mu)_k{}^l (U^\dagger)_l{}^j.$$

The gauge field transforms as a connection, rather than a tensor,

$$(A_\mu)_i{}^j \to U_i{}^k (A_\mu)_k{}^l (U^\dagger)_l{}^j + \frac{i}{g} U_i{}^k \partial_\mu (U^\dagger)_k{}^j.$$

Infinitesimal transformations

We can write

$$U_i{}^j = \exp\Big(i\alpha^a(x) T_a\Big)_i{}^j,$$

For $SU(N)$, the T_a comprise a (traceless, Hermitian) basis of generators. The covariant derivative $D_\mu = \partial_\mu - igA_\mu$ transforms, to first order in $\alpha^a(x)$ as

$$\begin{aligned} D_\mu &\to U D_\mu U^\dagger \\ &\approx (1 + i\alpha^a(x)T_a)\Big(\partial_\mu - igA_\mu^b T_b\Big)(1 - i\alpha^c(x)T_c) \\ &\approx \partial_\mu - igA_\mu^a T_a + g\alpha^a(x) A_\mu^b(x)[T_a, T_b] - i\partial_\mu \alpha^a T_a \\ &= \partial_\mu - ig(A')_\mu^a T_a, \end{aligned} \tag{13.3}$$

where

$$\boxed{(A')_\mu^a = A_\mu^a + \frac{1}{g} D_\mu \alpha^a, \qquad D_\mu \alpha^a = \partial_\mu \alpha^a + g f_{bc}{}^{a} A_\mu^b \alpha^c.}$$

We can also construct a field strength tensor $F_{\mu\nu}$ as

$$[D_\mu, D_\nu] = -igF_{\mu\nu}.$$

If you have studied General Relativity, this definition may seem familiar. We can treat the gauge boson as a connection in a bundle over spacetime. The field strength is nothing more

[67] This is the smallest representation of the group.

than the curvature tensor for this connection. Alternatively, it is just the minimal covariant extension of the Faraday tensor. Taking this definition, we can find an expression of $F_{\mu\nu}$ in terms of A_μ:

$$\begin{aligned}[D_\mu, D_\nu] &= [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \\ &= -ig\Big(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\Big), \end{aligned} \tag{13.4}$$

so that

$$\boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{bc}{}^a A^b_\mu A^c_\nu}$$

where $F_{\mu\nu} = F^a_{\mu\nu} T_a$ and so the field strength also transforms in the adjoint representation

$$F_{\mu\nu} \to U F_{\mu\nu} U^\dagger.$$

The infinitesimal transformations of the field strength and fundamental representation fermions is

$$\boxed{\delta F^a_{\mu\nu} = f_{bc}{}^a F^b_{\mu\nu}\alpha^c, \qquad \delta\psi_i = i\alpha^a (T_a)_i{}^j \psi_j.}$$

Similarly, matter in the adjoint representation $\phi_i{}^j(x) = \phi^a(x)(T_a)_i{}^j$ transforms as

$$\phi \to U\phi U^\dagger, \qquad \delta\phi^a = f_{bc}{}^a \phi^b \alpha^c.$$

To summarise

Conventions

The field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].$$

In terms of the commutation relations $[T_a, T_b] = i f_{ab}{}^c T_c$, this is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{bc}{}^a A^b_\mu, A^c_\nu.$$

Our conventions for the Yang-Mills gauge field and field strength gauge transformations are

$$A'_\mu = U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger, \qquad F'_\mu = U F_{\mu\nu} U^\dagger.$$

13.2 Thinking Geometrically*

There is an elegant way to understand these structures in term of the geometry of fibre bundles. With this perspective, we start with a finbre bundle - a manifold that looks locally like the product of two manifolds $E = M \times G$, endowed with a projection $\pi : E \to M$ to the 'base' manifold M. We take $M = \mathbb{R}^{3,1}$ to be spacetime and G some vector space on which the group acts. The field strength is the curvature of this connection. This has striking parallels with gravity in the vierbein formalism. There we introduce a vierbein $e_\mu{}^a(x)$ that relates a general curved metric $g_{\mu\nu}(x)$ to the Minkowski metric in the tangent space to at a point.

$$g_{\mu\nu}(x) = e_\mu{}^a(x) e^b{}_\nu(x)\eta_{ab}.$$

In the absence of torsion, we can define a spin connection $\omega_a{}^b = \omega_{a\mu}{}^b dx^\mu$ entirely in terms of the vielbein

$$de^a + \omega^a{}_b \wedge e^b = 0,$$

where $e^a(x) = e_\mu{}^a(x)dx^\mu$. From this, we define the curvature two-form

$$R_a{}^b = \frac{1}{2}R_{\mu\nu a}{}^b dx^\mu \wedge dx^\nu = d\omega_a{}^b + \omega_a{}^c \wedge \omega_c{}^b.$$

In this sense we can think of the curvature tensor as the field strength for an $SO(3,1)$ gauge symmetry - the ability to independently perform a Lorentz transformation at each point in spacetime. The more familiar Riemann curvature tensor is given by $R_{\mu\nu\lambda}{}^\rho = R_{\mu\nu a}{}^b e^a{}_\lambda e_b{}^\rho$.

13.3 The Lagrangian

By analogy with the QED action and with a minor modification, we can write down a Lagrangian for the Yang-Mills theory, coupled to fermions in the fundamental representation

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\not{D} - m)\psi,$$

where the trace is taken over the matrix indices. We define

$$\mathrm{Tr}(T_aT_b) = (T_a)_i{}^j(T_b)_j{}^i = C(\mathrm{Adj})\delta_{ab}, \qquad \text{where} \qquad C(\mathrm{Adj}) = N,$$

We absorb this factor of N into the definition of the field strength. Another useful invariant is

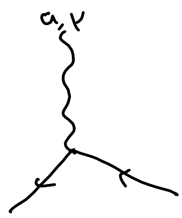
$$(T^2)_i{}^j := \sum_a (T_aT_b)_i{}^j = C(r)\delta_i{}^j.$$

The Lie algebra indices can be raised and lowered by δ^{ab} and δ_{ab} respectively. We denote $f_{abc} := \delta_{ad}f_{bc}{}^d$ which is antisymmetric in all three indices.

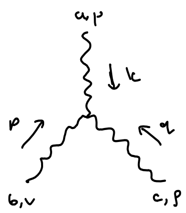
We can expand the Lagrangian out to give

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \\ &= \frac{1}{2}A^a_\mu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A_\nu - gf_{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} - \frac{1}{4}g^2 f_{eab}f_{cd}{}^e A^a_\mu A^b_\nu A^{c\mu}A^{d\nu} \\ &\quad + \bar{\psi}(i\not{\partial} - m)\psi + gA^a_\mu\bar{\psi}^i\gamma^\mu(T_a)_i{}^j\psi^j \end{aligned} \tag{13.5}$$

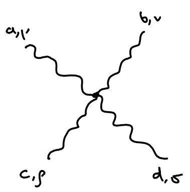
We see that the non-abelian structure leads to self-interaction cubic and quartic terms for the gauge boson. We shall come to the gauge boson propagator in short order. The interaction vertex Feynman rules in momentum space can be straightforwardly deduced from the momentum space action and are given by



$$= ig\gamma^\mu T^a$$



$$= gf^{abc}\Big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\mu\rho}(q-k)^\nu\Big]$$



$$= -ig^2\Big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})$$

$$+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\Big]$$

Finally, a comment about terminology. The gauge boson of QED is called the photon. Generalising the terminology used in QCD, we call the gauge boson of the Yang-Mills theory, the *gluon*.

13.4 Fadeev-Popov and BRST Quantization

The free part of the pure Yang-Mills Lagrangian is

$$\mathcal{L} = \frac{1}{2} A^a_\mu \Big(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu \Big) A_\nu + ...$$

This is identical in structure to the Maxwell Lagrangian, the only difference being the presence of the gauge indices. This means that this is really (for $G = SU(N)$) $N^2 - 1$ copies of the Maxwell Lagrangian. And so, the problems that arose with finding a propagator for the photon also arise for the gluon. In particular, a gluon of the form $A^a_\mu(x) = \partial_\mu f^a(x)$, for some Lie-algebra valued function $f^a(x)$ is in the kernel of the differential operator

$$\mathcal{P}^{\mu\nu} := g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu,$$

and so we cannot invert this operator on a space that includes such gluon configurations. The solution is as with QED; we gauge-fix using the Fadeev-Popov determinant technique. As with QED, the path integral over the space of gluons may be written as

$$\int \mathcal{D}A\, e^{iS[A]} = N(\xi) \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]+iS_{\text{gf}}[A]} \det\left(\frac{\delta G^a(A^\alpha)}{\delta\alpha^b} \right)$$

where the gauge-fixing part of the action is

$$S_{\text{gf}} = \frac{1}{2\xi} \int d^4x \Big(\partial^\mu A^a_\mu(x) \Big)^2$$

We choose the gauge condition

$$G^a(A) = \partial^\mu A^a_\mu(x) - \omega^a(x) = 0$$

for an arbitrary Lie-algebra valued function $\omega^a(x)$. We can absorb

$$N(\xi) \int \mathcal{D}\alpha$$

into the normalisation of the functional. In the QED case, we also absorbed the determinant into the normalization; however, we cannot do that here as the determinant explicitly depends on the gluon. In the gauge-fixing term we have

$$(A^\alpha)^a_\mu(x) = A^a_\mu(x) + \frac{1}{g}\partial_\mu \alpha^a(x) + f_{bc}{}^a A^b_\mu(x)\alpha^c(x) = A^a_\mu(x) + \frac{1}{g} D_\mu \alpha^a(x),$$

as such

$$\frac{\delta G^a(A^\alpha)}{\delta\alpha^b} = \frac{1}{g}\partial^\mu (D_\mu)^a{}_b$$

where $(D_\mu)^a{}_b = \delta^a_b \partial_\mu + g f_{cb}{}^a A^c_\mu$. The path integral becomes

$$\int \mathcal{D}A\, e^{iS[A]} \approx \int \mathcal{D}A\, e^{iS[A]+iS_{\text{gf}}[A]} \det\left(\frac{\delta G^a(A^\alpha)}{\delta\alpha^b} \right)$$

How, in practice, should we deal with this determinant term? In particular, how should the presence of such a term be incorporated into our perturbative calculations? We can take some inspiration from the finite dimensional standard integrals

$$\int_{\mathbb{R}^n} d^n x\, e^{-\frac{1}{2}x^i M_{ij} x^j} = \frac{(2\pi)^{n/2}}{\sqrt{\det M}}$$

and the Grassmann integral

$$\int d^n\theta\, e^{-\frac{1}{2}\theta^i A_{ij}\theta^j} = \sqrt{\det(A)},$$

where the measure is defined as $d^n\theta = \prod_{i=1}^n d\theta^i$. The key point to note here is that the right hand side of these equations is inverted if we exchange bosonic integration variables with fermionic ones. Using the related result

$$\int d^n\bar{\theta} d^n\theta\, e^{-\bar{\theta}^i A_{ij}\theta^j} = \det(A), \qquad d^n\bar{\theta} d^n\theta := \prod_{i=1}^n d\bar{\theta}^i d\theta^i,$$

gives us a way to write the determinant as a functional integral. We introduce Lie-algebra valued Grasmann scalar fields $c^a(x)$ and $\bar{c}^a(x)$ in the adjoint representation and write the determinant as

$$\det\left(\frac{1}{g}\partial^\mu D_\mu\right) = \int \mathcal{D}\bar{c}\mathcal{D}c\, \exp\left(i\int d^4x\, \bar{c}^a(-\partial^\mu D_\mu)_{ab}c^b\right),$$

where the factor of g^{-1} is absorbed into the definition of the fields c^a and $\bar{c}^a$. Two comments are in order regarding these new Grassmann fields

- c^a and $\bar{c}^a$ are scalar fields (they have no spin or spacetime indices)
- They are fermionic (anticommuting variables)

Taken together, these two observations mean that these fields violate the spin-statistics theorem. Thus, they are not physical degrees of freedom. We call the *ghosts* and they can be interpreted as constraints, rather than degrees of freedom.

The path integral can then be written as

$$\int \mathcal{D}\bar{c}\,\mathcal{D}c\,\mathcal{D}A\, e^{iS[A]+iS_{\text{gf}}[A]+iS_{\text{gh}}[c,\bar{c},A]}$$

where the ghost action is given by

$$S_{\text{gh}}[c,\bar{c},A] = \int d^4x\,\Big(-\bar{c}^a\partial^2 c_a - g f_{ab}{}^c\bar{c}^a\partial^\mu(A^c_\mu c_b)\Big)$$

which can be thought of as a free part; $-\bar{c}^a\partial^2 c_a$, plus an interaction term $-gf_{ab}{}^c\bar{c}^a\partial^\mu(A^c_\mu c_b)$. The Lagrangian for the ghosts has the same kinetic term as a massless Klein-Gordon field and so the ghost propagator is

$$\delta^{ab}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2}e^{-ik\cdot(x-y)}.$$

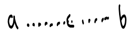
The full Lagrangian is

$$\boxed{\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{\psi}(i\not{D} - m)\psi + \bar{c}^a(-\partial^\mu D_\mu)_{ab}c^b}$$

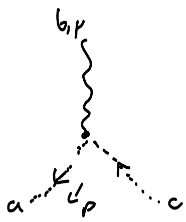
The discussion of the gluon propagator mirrors that of the photon in QED, so in Feynman gauge, we have the additional momentum space Feynman rules

$$= \frac{i\delta^a_b}{k^2} g^{\mu\nu}$$

and the ghosts contribute



$$= \frac{i\delta^a_b}{p^2}$$



$$= -g f^{abc} p^\mu$$

13.5 BRST Symmetry

Two questions arise from the Fadeev-popov prescription

- What does the gauge-fixed theory know about gauge symmetry?
- How do we see the ghosts as constraints?

The answer to both questions can be found by considering BRST symmetry[68]. We can introduce an auxilliary scalar field $B^a(x)$, sometimes called the Nakanishi-Lautrup field to give the Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D} - m)\psi + \bar{c}^a(-\partial^\mu D_\mu)_{ab}c^b - \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu$$

The $B^a(x)$ equation of motion is

$$\xi B^a = \partial^\mu A^a_\mu.$$

B^a is not dynamical and can be integrated out to give the gauge-fixed Lagrangian previously. This Lagrangian has the following rigid symmetry:

$$\begin{aligned}
\delta_Q A^a_\mu &= \theta D^{ab}_\mu c_b \\
\delta_Q \psi &= ig\theta c^a T_a \psi \\
\delta_Q c^a &= -\frac{1}{2} g\theta f_{bc}{}^a c^b c^c \\
\delta_Q \bar{c}^a &= \theta B^a \\
\delta_Q B^a &= 0
\end{aligned} \tag{13.6}$$

[68] Becchi, Rouet, Stora and Tyutin.

where θ is an anti-commuting constant. The Lagrangian of the theory is invariant under this transformation, called the BRST symmetry. This is a rigid symmetry, rather than a gauge symmetry as θ is constant. The first two transformations above are just the gauge transformations of the gluon and fermion with gauge parameter $\alpha^a(x) = \theta c^a(x)$ and so the action $S[A]$ is trivially invariant. The invariant of the other terms may be seen as follows:

We define the BRST charge Q by the transformations above

$$\delta_Q := \theta Q$$

where Q is necessarily anti-commuting.

The BRST charge

An explicit functional expression for the BRST charge is

$$Q = \int d^4x \left(D_\mu^{ab} c_a \frac{\delta}{\delta A_\mu^b} + igc^a (T_a)^i{}_j \psi^j \frac{\delta}{\delta \psi^i} + B^a \frac{\delta}{\delta \bar{c}^a} - \frac{g}{2} f_{abc} c^a c^b \frac{\delta}{\delta c_c} \right). \quad (13.7)$$

We can clearly identify the generator of infinitesimal gauge transformations of the gauge field and fermions as

$$\int d^4x \mathcal{T}_a(x) \alpha^a(x)$$

where

$$\mathcal{T}_a(x) = (D_a^b)_\mu \frac{\delta}{\delta A_\mu^b} + ig(T_a)^i{}_j \psi^j \frac{\delta}{\delta \psi^i}$$

One can then show that these generators satisfy the algebra

$$[\mathcal{T}_a(x), \mathcal{T}_b(y)] = if_{ab}{}^c \mathcal{T}_c(y) \delta^4(x-y).$$

The BRST operator takes the form

$$Q = \int d^4x \left(\mathcal{T}_a c^a - \frac{1}{2} g f_{abc} c^a c^b \frac{\delta}{\delta c_c} + B^a \frac{\delta}{\delta \bar{c}^a} \right).$$

The result $Q^2 = 0$ may then be seen to follow from the Jacobi identity (13.2).

In terms of the BRST charge, we have that

$$\begin{aligned}
Q(A_\mu^a) &= D_\mu^{ab} c_b \\
Q(\psi) &= igc^a T_a \psi \\
Q(c^a) &= -\frac{1}{2} g f_{bc}{}^a c^b c^c \\
Q(\bar{c}^a) &= B^a \\
Q(B^a) &= 0
\end{aligned} \quad (13.8)$$

Since Q is anti-commuting, we expect that $Q^2(\Phi) := Q(Q(\Phi)) = 0$ for any field Φ. This can be verified explicitly by direct calculation. For example,

$$Q^2(c^a) = Q\left(-\frac{1}{2}gf_{bc}{}^a c^b c^c\right) = -gf_{b[c}{}^a f_{de]}{}^b c^c c^d c^e = 0$$

by virtue of the Jacobi identity[69]

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] \equiv 0 \qquad \Longrightarrow \; f_{[ab}{}^d f_{c]d}{}^e = 0.$$

We note that, if we define (what we shall call the gauge-fixing fermion)

$$\Psi(x) := \bar{c}_a\left(\partial^\mu A^a_\mu(x) - \frac{\xi}{2}B^a(x)\right)$$

then

$$Q(\Psi) = -\bar{c}^a\partial^\mu D_\mu c_a - \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu$$

and so the total action may be written as

$$\begin{aligned} \mathcal{S}[A, c, \bar{c}, \psi] &:= S[A] + S_{\text{gf}}[A] + S_{\text{gh}}[c, \bar{c}, A] \\ &= S[A] + \int d^4x\, Q(\Psi) \end{aligned} \tag{13.9}$$

Acting with Q then gives

$$\delta_Q\mathcal{S}[A, c, \bar{c}, \psi] = \delta_Q S[A] + \epsilon\int d^4x\, Q^2(\Psi) = 0$$

The first term vanishes by gauge symmetry, as discussed above and the second term vanishes by virtue of the fact that $Q^2 = 0$ on all fields in the theory. As such the gauge-fixed action is BRST-invariant.

Note that this analysis is at the classical level. It is possible that, in the quantum theory, we might find $Q^2 \neq 0$. This occurs in string theory away from the critical dimension and, as in that case, this signifies that the symmetry is broken at the quantum level. This is an anomaly.

13.6 BRST Cohomology and Physical States

In general the gauge-fixing fermion can be written as

$$\boxed{\Psi(x) = \bar{c}_a\left(G^a(x) - \frac{\xi}{2}B^a(x)\right).}$$

Under a BRST transformation, this has the expected form of ghost and gauge-fixing term

$$\{Q, \Psi(x)\} = -\bar{c}_a\left(\frac{\delta G^a}{\delta A^b_\mu}\right)D_\mu c^b + B_a G^a - \frac{\xi}{2}B^2.$$

[69] This can be taken as a statement of associativity.

How can we see gauge-invariance of the theory? A change in gauge, which we will write as ΔG^a corresponds to a change in the gauge-fixing fermion

$$\Delta\Psi = \bar{c}_a \Delta G^a,$$

which gives rise to a change in the full action

$$\Delta S[A, c, \bar{c}, B] = \int d^4x \{Q, \Delta\Psi(x)\} = \left\{Q, \int d^4x \Delta\Psi(x)\right\}$$

and so, in a path integral, computing the transition between some initial and final physial states

$$\langle\Phi_f|\Phi_i\rangle = \int_{\Phi(t_i)=\Phi_i}^{\Phi(t_f)=\Phi_f} \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}B\, e^{iS[A,c,\bar{c},B]} := \int_i^f \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}B\, e^{iS[A,c,\bar{c},B]},$$

a change in the gauge $\Delta G^a(x)$ leads to a change in the correlation function

$$\begin{aligned}
\langle\Phi_f|\Phi_i\rangle + \Delta\langle\Phi_f|\Phi_i\rangle &= \int_i^f \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}B\, e^{iS[A,c,\bar{c},B]+i\Delta S[A,c,\bar{c},B]} \\
&= \int_i^f \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}B\, e^{iS[A,c,\bar{c},B]}\Big(1 + i\Delta S[A, c, \bar{c}, B] + ...\Big)
\end{aligned}$$

The change in the amplitude is then

$$\Delta\langle\Phi_f|\Phi_i\rangle = i\int d^4x \langle\Phi_f|\{Q, \Delta\Psi(x)\}|\Phi_i\rangle.$$

For this to vanish for arbitrary changes of gauge, i.e. for all $\Delta\Psi$ requires

$$\langle\Phi_f|Q = 0 = Q|\Phi_i\rangle.$$

For Hermitian Q, i.e. $Q = Q^\dagger$, this is simply summarised by the statement that physical states must satisfy

$$Q|\Phi\rangle = 0,$$

i.e. that all physical, that is gauge invariant, states are in the kernel of Q.

Since $Q^2 = 0$, one simply was to ensure a state is in the kernel of Q is if it is exact in Q, i.e. $|\Phi'\rangle = Q|\lambda\rangle$ for some state $|\lambda\rangle$. It is easy to see that all such states will be orthogonal to any physical state

$$\langle\Phi|\Phi'\rangle = \langle\Phi|Q|\lambda\rangle = 0,$$

by virtue of $|\Phi\rangle$ being a physical state. As such correlation functions of fields including states of the form $Q|\lambda\rangle$ will vanish identically. Clearly, we are not interested in such states. It is easy to see what such states are not really physical. If we focus on the matter fields only, the BRST transformation is simply a gauge transformation with parameter $\lambda(x) = \epsilon c^a(x)$ (the ghosts are treated as arbitrary functions). As such the BRST transformations on the matter fields encode the gauge transformations and we see the change

$$|\Phi\rangle \to |\Phi\rangle + Q|\lambda\rangle$$

is simply a gauge transformation. Thus, Q-exact states are pure gauge.

As such, the states $|\Phi\rangle$ are physical (gauge-invariant, modulo gauge transformations) if

$$\boxed{|\Phi\rangle \in \frac{\text{Ker(Q)}}{\text{Im(Q)}}}$$

This set is called the *Cohomology* of Q. Thus, the BRST symmetry gives us a precisse definition of what it is to be a physical state in the theory.

13.7 A concrete example

Let us consider this construction for QED (or the $g \to 0$ limit of Yang-Mills). $B(x)$ appears algebraically in the action so we can integrate it out and replace it with

$$B(x) = \frac{1}{\xi}\partial^\mu A_\mu,$$

and the BRST transformations are

$$Q(A_\mu) = \partial_\mu c, \qquad Q(c) = 0, \qquad Q(\bar{c}) = \frac{1}{\xi}\partial^\mu A_\mu.$$

What does it mean for a state to live in the cohomology of Q? Consider the mode expansion of the gauge field

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\Big(a^\dagger_\mu(k)e^{ik\cdot x} + a_\mu(k)e^{-ik\cdot x}\Big).$$

A general momentum space state can be built as

$$|\varepsilon\rangle = \varepsilon^\mu(k)a^\dagger_\mu(k)|\Omega\rangle.$$

The Fourier transform gives the position space state. The requirement that this state is in the kernel of Q is

$$Q|\varepsilon\rangle = k_\mu\varepsilon^\mu(k)c(k)|\Omega\rangle = 0 \qquad \Longrightarrow \qquad k_\mu\varepsilon^\mu(k) = 0.$$

i.e.

$$|\varepsilon\rangle \in \text{Ker}(Q) \implies k_\mu\varepsilon^\mu(k) = 0.$$

The fact that $Q^2 = 0$, implies the identification

$$|\varepsilon\rangle \sim |\varepsilon\rangle + |\alpha\rangle, \qquad |\alpha\rangle \in \text{Im}(Q).$$

We note that

$$Q(\bar{c}(k)) = \frac{1}{\xi}k^\mu a^\dagger_\mu(k),$$

and so the state

$$|\alpha\rangle = \tilde{\alpha}(k)k^\mu a^\dagger_\mu(k)|\Omega\rangle \in \text{Im}(Q),$$

thus we have the identification

$$\varepsilon^\mu(k)a^\dagger_\mu(k)|\Omega\rangle \sim \varepsilon^\mu(k)a^\dagger_\mu(k)|\Omega\rangle + \tilde{\alpha}(k)k^\mu a^\dagger_\mu(k)|\Omega\rangle,$$

or

$$\varepsilon^\mu(k) \sim \varepsilon^\mu(k) + k^\mu\tilde{\alpha}(k).$$

This is simply gauge equivalence in momentum space for a free gauge field.

Thus

$$|\varepsilon\rangle = \varepsilon^\mu(k)a^\dagger_\mu(k)|\Omega\rangle \in \mathrm{Cohom}(Q),$$

if

- $k^\mu\varepsilon_\mu = 0$ (No longitudinal polarization)
- $\varepsilon^\mu \sim \varepsilon^\mu + k^\mu\tilde{\alpha}$ (Gauge-invariance)

We can further understand these results by choosing a basis for ε_μ. Let us introduce a new light-cone basis, related to the standard basis by a vielbein ${\varepsilon_\mu}^I$, where

$${\varepsilon_\mu}^I{\varepsilon_\nu}^J\eta_{IJ} = \eta_{\mu\nu}$$

i.e.

$$2\varepsilon^+_\mu\varepsilon^-_\nu - \varepsilon^i_\mu\varepsilon^j_\mu\delta_{ij} = \eta_{\mu\nu}.$$

We can define vectors along the light-cone

$$\varepsilon^+_\mu = (k_0, \mathbf{k}) = k_\mu, \qquad \varepsilon^-_\mu = (k_0, -\mathbf{k}),$$

we can think of ε^+ and ε^- as 'forward' and 'backward' polarizations along the light cone. The basis is completed by introducing transverse polarizations ε^i. Thus $\varepsilon_\mu = C_I\varepsilon^I_\mu$

$$\varepsilon_\mu := C_+\varepsilon^+_\mu + C_-\varepsilon^-_\mu + C_i\varepsilon^i_\mu = C_+k_\mu + C_-\varepsilon^-_\mu + C_i\varepsilon^i_\mu.$$

Given that $k^2 = 0$ on shell, the condition $\varepsilon_\mu k^\mu = 0$, requires $C_- = 0$.

The equivalence

$$\varepsilon_\mu \sim \varepsilon_\mu + k_\mu\tilde{\alpha},$$

means that $C_+ \sim C_+ + \tilde{\alpha}$ and so C_+ is pure gauge and is not physical (this polarization is in $\mathrm{Im}(Q)$). As such the only physical polarizations are the transverse ones

$$|\varepsilon\rangle = \varepsilon^i a^\dagger_i(k)|\Omega\rangle,$$

as expected since the forward and backward polarizations are identified with un-physical ghost states.

13.8 Large N

The basic degrees of freedom in Yang-Mills theory is a gluon vector field $A^a_\mu(T_a)^i{}_j$. For gauge group $U(N)$, this describes, at each point in spacetime, an $N\times N$ Hermitian matrix of vectors. We can imitate the large N analysis we saw in the matrix models in Yang-Mills.

To begin, it will be helpful to absorb a factor of g into the definition of the vector fields so that $A^a_\mu \to \frac{1}{g}A^a_\mu$. As such the field strength is redefined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \to \frac{1}{g}F^a_{\mu\nu} = \frac{1}{g}\Big(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\Big).$$

The action then becomes

$$S[A] = -\frac{1}{2g^2} Tr(F_{\mu\nu}F^{\mu\nu}).$$

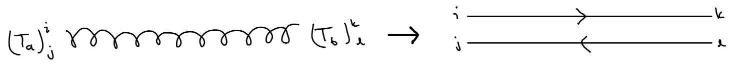
This choice of variables has the virtue of having all of the coupling dependence explicitly as a pre-factor to the action. The field strength no longer carries a dependence on the coupling g and the gauge transformations do not involve g. We next introduce the 'tHooft coupling

$$\lambda := g^2 N.$$

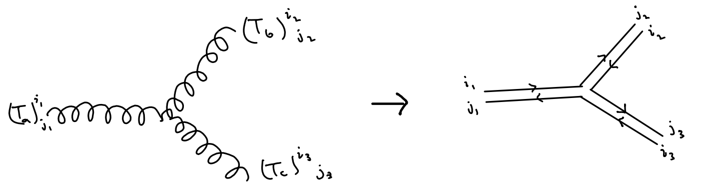
giving the action

$$S[A] = -\frac{N}{2\lambda} Tr(F_{\mu\nu}F^{\mu\nu}).$$

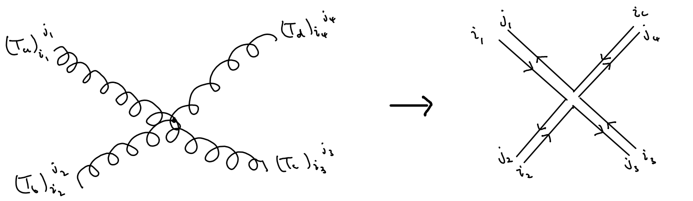
This is now starting to look like the Matrix model action and we see that there is a resemblance between the role played by $\hbar$ and that played by N. The propagator picks up a factor of λ/N and the cubic and quartic vertices both acquire weights of N/λ and the Feynman rules may be written in double line notation



$$\sim \frac{\lambda}{N} \tag{13.10}$$

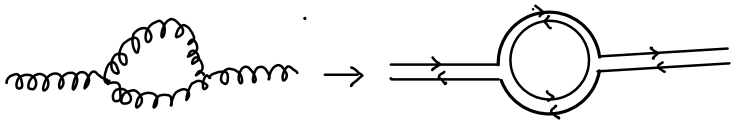


$$\sim \frac{N}{\lambda} \tag{13.11}$$



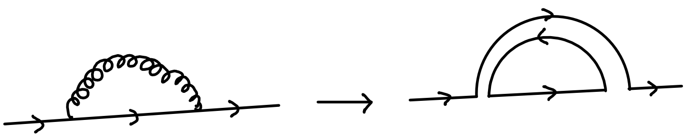
$$\sim \frac{N}{\lambda} \tag{13.12}$$

For example, the correction to the gluon propagator at one loop is given by the diagram



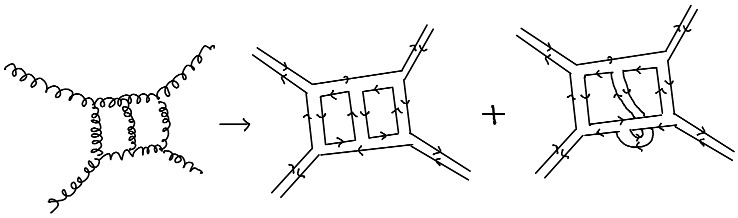
$$\sim \frac{\lambda^2}{N}. \tag{13.13}$$

The fermions, if they are in the fundamental representation will be represented by a single line (as they carry only one index) and so the one-loop correction to the fermion propagator is



$$. \tag{13.14}$$

At two loops, we start to get planar and non-planar contributions, corresponding to the differing ways we can trace over indices in the diagram. An example of a two-loop contribution to the gluon four-point function is



. (13.15)

The planar diagram contributes λ^5/N^3, whilst the non-planar diagram has λ^5/N^5. This the non-planar diagram is suppressed relative to the planar one in the large N limit.

14 Asypmtotic Freedom

We now consider the renormalization, to one-loop of $SU(N)$ Yang-Mills theories with fermionic matter. For $N = 3$, this describes Quantum Chromodynamics, the theory of the string force coupled to matter. We will keep N general and so are really discussing general Yang-Mills matter theories but we shall refer to all such theories, regardless of the value of N as Quantum Chromodynamics, or QCD for short. The gauge-fixed Lagrangian is

$$\boxed{\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{\psi}(i\not{D} - m)\psi + \bar{c}^a(-\partial^\mu D_\mu)_{ab}c^b}$$

where $a = 1, 2, ...N$. The Feynman rules for this theory are described in the previous chapter. In addition we need to add counter-terms and a prescription to regularize the theory. Our choice will be dimensional regularization and the counter-terms we explore below.

The full quantum Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(F^a_{0\mu\nu})^2 + \frac{1}{2\xi_0}(\partial^\mu A^a_{0\mu})^2 + \bar{\psi}_0(i\not{\partial} - m)\psi_0 + ig_0\bar{\psi}\not{A}_0\psi_0 + \bar{c}^a_0(-\partial^\mu D_\mu)_{ab}c^b_0 \\ &\sim \frac{1}{2}A^{a\mu}_0\Big(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\Big)A^{a\nu}_0 - g_0 f^{abc}A^{a\mu}_0 A^{b\nu}_0\partial_\mu A^c_{0\nu} - \frac{1}{4}g_0^2 f^{abe}f^{cd}{}_e A^{a\mu}_0 A^{b\nu}_0 A^c_{0\mu}A^d_{0\nu} \\ &\quad -\partial^\mu\bar{c}^a_0\partial_\mu c^a_0 + g_0 f^{abc}A^c_{0\mu}\partial^\mu\bar{c}^a_0\, c^b_0 + \frac{1}{2\xi_0}A^{a\mu}_0\partial_\mu\partial_\nu A^{a\nu}_0 \\ &\quad + i\bar{\psi}_0\not{\partial}\psi_0 - m\bar{\psi}_0\psi_0 + g_0 A^a_\mu\bar{\psi}_0\gamma^\mu T_a\psi_0. \end{aligned} \tag{14.1}$$

Introducing the wavefunction renormalization

$$A^a_{0\mu}(x) = \sqrt{Z_3}A^a_\mu(x), \qquad \psi_0(x) = \sqrt{Z_2}\psi(x), \qquad c^a_0(x) = \sqrt{\tilde{Z}_2}c^a(x), \qquad \xi_0 = Z_3\xi$$

we can write this Lagrangian in terms of a bare term and counter-terms

$$\begin{aligned}\mathcal{L}_b &= \frac{1}{2}Z_3\, A^{a\mu}\Big(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\Big)A^{a\nu} - Z_{3,g}\, g f^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \frac{1}{4}Z_{g,4}\, g^2 f^{abe}f^{cd}{}_e A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu \\ &\quad -\tilde{Z}_2\,\partial^\mu\bar{c}^a\partial_\mu c^a + \tilde{Z}_1\, g f^{abc}A^c_\mu\partial^\mu\bar{c}^a\, c^b + \frac{1}{2\xi}A^{a\mu}\partial_\mu\partial_\nu A^{a\nu} \\ &\quad + iZ_2\,\bar{\psi}\not{\partial}\psi - Z_m\, m\bar{\psi}\psi + Z_1\, gA^a_\mu\bar{\psi}\gamma^\mu T_a\psi + \mathcal{L}_{c.t.}, \end{aligned} \tag{14.2}$$

where $\mathcal{L}_{c.t.}$ contains the counter-terms. g_0 appears in four places in the Lagrangian and so we have potentially a bare coupling that gets renormalized in four different ways

$$g_0 = \frac{Z_{3,g}}{Z_3^{\frac{3}{2}}}g\,\mu^{\epsilon/2}, \qquad g_0^2 = \frac{Z_{4,g}}{Z_3^2}g^2\mu^\epsilon, \qquad g_0 = \frac{\tilde{Z}_1}{Z_3^{\frac{1}{2}}\tilde{Z}_2}g\,\mu^{\epsilon/2}, \qquad g_0 = \frac{Z_1}{Z_3^{\frac{1}{2}}Z_2}g\,\mu^{\epsilon/2}$$

Are all of these g's the same? One can show that in fact they are. To do so requires proving a set of non-abelian versions of the Ward-Takahashi identities, called Slavnov-Taylor identities[70]. Unfortunately to work through this proof will take us beyond what we

[70] These follow straightforwardly from the above identities and may be written as

$$\frac{Z_{3,g}}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_2}, \qquad Z_{4,g} = \frac{Z_{3,g}^2}{Z_3}.$$

have time to consider in this course. Essentially it is a consequence of gauge symmetry (or BRST symmetry in the gauge-fixed form). Instead, we shall assume all the expressions for g above are equivalent. This greatly reduces the possible range of counter-terms we need to consider. Let us assume we calculate (say to one-loop order) the counter-terms Z_1, Z_2, Z_3 and $\tilde{Z}_2$. We can then use the relationships above to determine the remaining counter-terms. As such we can write the counter-term Lagrangian in the simple form

$$\mathcal{L}_{\text{c.t.}} = -\frac{\delta_3}{4}(F^a_{\mu\nu})^2 + i\delta_2\bar{\psi}\partial\!\!\!/\psi - \delta_m\bar{\psi}\psi - \tilde{\delta}_2\bar{c}^a(\partial^\mu D_\mu)_{ab}c^b,$$

where $Z_i = 1 + \delta_i$. The relationships between counter-terms also gives a variety of ways to compute the beta-function for the theory. If we are interested in pure Yang-Mills, then any of the first three expressions can be used to compute the beta function. We shall use the last expression, which includes fermion wavefunction renormalization, to compute the beta function of QCD.

The gauge-fixing term, proportional to ξ^{-1} does not get renormalized as the scaling of ξ exactly matches that of A^a_μ.

14.1 One-Loop Divergences in QCD

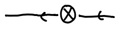
In this section we take the first steps towards calculating the one-loop beta function for non-abelian gauge theories, coupled to fermions. As with QED, we introduce counter-terms to balance the divergences at a particular energy scale and allow us to define, in terms of correlation functions, what our coupling constants and wavefunctions normalizations are as a function of energy scale. We introduce three[71] counterterms:

- δ_1 is the three-point vertex counter-term
- δ_2 is the fermion wavefunction counter-term
- δ_3 is the gluon wavefunction counter-term

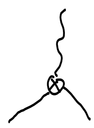
We shall work in a limit in which the masses of our fermions can be taken to be zero and so we will not worry about mass renormalization. The counter-terms may be defined in terms of the additional momentum space Feynman rules

[71] One might wonder whether we need counter-terms for the ghosts. Since we will only be working at one-loop, we will be able to avoid this additional complication, although in principle it is there.

$$= -i(k^2 g^{\mu\nu} - k^\mu k^\nu)\delta_{ab}\delta_3$$



$$= i\not{p}\delta_2$$

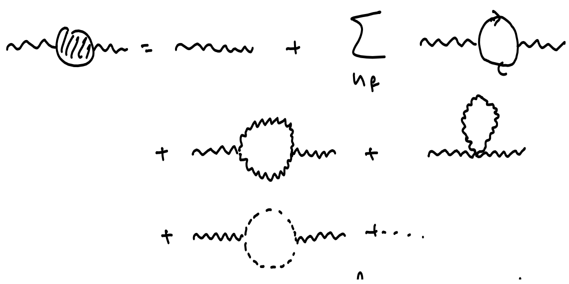


$$= ig\gamma^\mu T_a \delta_1$$

The first step is to compute the self-energy of the gluon

14.2 Gluon Self-Energy

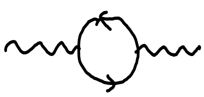
In terms of Feynman diagrams, the gluon two-point function may be written as



where the sum is over the fermion species (n_f) that couple to the Yang-Mills field. We consider these terms one-by-one.

14.2.1 Fermion Loop

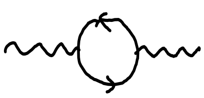
This is a similar calculation to the analogous diagram in QED, but with the insertion of generators in the fundamental representation at each vertex. For a single fermion species propagating in the loop



$$:= i\Pi_2^{\mu\nu}(q)$$

$$= -(ig)^2 \mathrm{Tr}(T_a T_b) \int \frac{d^d k}{(2\pi)^d} \mathrm{tr}\left[\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma^\nu \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\right] \qquad (14.3)$$

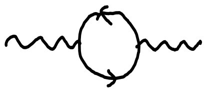
where the factor of -1 at the beginning comes from the fact there is a fermion loop, the Tr is a trace over group indices and tr is a trace of gamma matrix (spinor) indices. We note that $\mathrm{Tr}(T_a T_b) = C(r)\delta_{ab}$, where $C(r)$ denotes the index of the representation r. In the calculations that follow, we shall neglect fermion masses as such terms do not affect the UV divergences we are studying and so we need to evaluate



$$\approx -g^2 C(r)\delta_{ab} \int \frac{d^4 k}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^\mu(\not{k})\gamma^\nu(\not{k}+\not{q})\right]}{k^2(k+q)^2}$$

where we choose to work in $d = 4 - \epsilon$ dimensions.

Assuming that there are n_f species of fermion, which (except for masses, which we neglect) are all the same,

$$\sum_{\text{fermions}}$$


$$= \Big(g^{\mu\nu}q^2 - q^\mu q^\nu\Big)\delta_{ab}C(r)i\Pi(q^2),$$

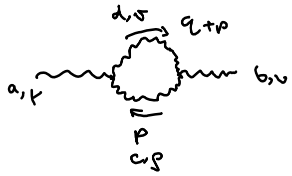
where the sum is taken over the fermions in the loop and

$$\Pi(q^2) = -\frac{8g^2}{(4\pi)^{d/2}}n_f\Gamma(2-d/2)\int_0^1 dx\,\frac{2x(1-x)}{[x(x-1)q^2]^{2-d/2}}$$

It is encouraging to see the factor of $g^{\mu\nu}q^2 - q^\mu q^\nu$ in the expression for the gluon propagator. Our understanding of this factor is that it arises from the gauge invariance of the theory and we would not expect this to be broken by perturbative effects. In QED, a contribution f this kind would be all that we could expect at one loop. In Yang-Mills, the gluon self-interactions allow for other contributions.

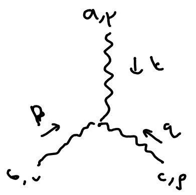
14.2.2 Cubic Interaction Gluon Loop

The one-loop contribution coming from the gluon cubic interactions is given by the single diagram



$$= \frac{1}{2}g^2 f^{acd}{f_{cd}}^b \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2}\frac{-i}{(p+q)^2}N^{\mu\nu}$$

where we have used propagators for gluons in the Feynman gauge and the vertex

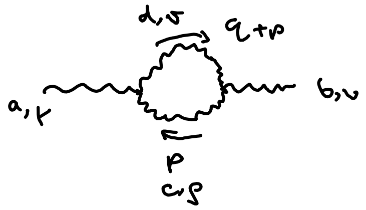


$$= gf^{abc}\Big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\Big].$$

The numerator is

$$N^{\mu\nu} = \Big[g^{\mu\rho}(q-p)^\sigma + g^{\rho\sigma}(2p+q)^\mu + g^{\sigma\mu}(-q-2q)^\rho\Big] \times \Big[\delta^\nu_\rho(p-q)_\sigma + g_{\rho\sigma}(-2p-q)^\nu + \delta^\nu_\sigma(p+2q)_\rho\Big]$$

We go to Euclidean space to evaluate the integral and using standard integrals, we find the somewhat horrific expression



$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(\mathrm{Adj})\delta_{ab} \int_0^1 dx \, \frac{F^{\mu\nu}(q,x)}{[x(x-1)q^2]^{2-d/2}}$$

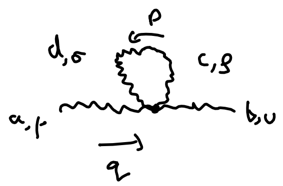
where

$$\begin{aligned} F^{\mu\nu}(q,x) = &\ \Gamma(1-d/2)g^{\mu\nu}q^2 \left[\frac{3}{2}(d-1)x(1-x)\right] \\ &+\Gamma(2-d/2)g^{\mu\nu}q^2 \left[\frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2\right] \\ &-\Gamma(2-d/2)q^\mu q^\nu \left[\left(1-\frac{d}{2}\right)(1-2x)^2 + (1-x)(2-x)\right] \end{aligned} \tag{14.4}$$

This looks awful and there is not the expected prefactor present. We shall see that, combined with the remaining two diagrams, the final expression simplifies dramatically.

14.2.3 Quartic Interaction Gluon Loop

The one-loop contribution coming from the gluon quadratic interactions is given by the single diagram



$$= \frac{1}{2}(-ig^2)H^{ab\mu\nu\rho\sigma} \int \frac{d^4p}{(2\pi)^4} \frac{-ig_{\rho\sigma}}{p^2}$$

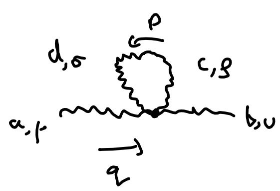
The symmetry factor of 1/2 is included and

$$\begin{aligned} H^{ab\mu\nu\rho\sigma} &= \delta^{cd}\Big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\Big] \\ &= \delta^{cd}\Big[f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\Big] \\ &= 2C_2(\mathrm{Adj})\delta^{ab}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \end{aligned} \tag{14.5}$$

where in the second line we have used the fact that $\delta^{cd}f^{cde} = 0$ in the first term in the brackets and $f^{acd}f^{bcd} = C_2(\mathrm{Adj})\delta^{ab}$ has been used in the last line. Contracting with $g_{\rho\sigma}$ gives

$$g_{\rho\sigma}H^{ab\mu\nu\rho\sigma} = 2C_2(\mathrm{Adj})\delta^{ab}g_{\rho\sigma}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) = 2C_2(\mathrm{Adj})\delta^{ab}g^{\mu\nu}(d-1),$$

so we have

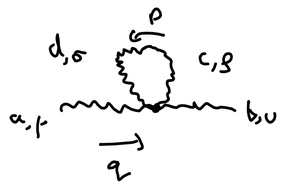


$$= -g^2 C_2(\mathrm{Adj})\delta^{ab}g^{\mu\nu}(d-1) \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2}$$

In $d = 4$ this integral gives zero but we will not discard it. Instead, we shall use it to write zero in a very helpful way. Let us insert

$$1 = \frac{(p+q)^2}{(p+q)^2},$$

and use Feynman parameterization to write the integral in terms of the variable $\ell_\mu = p_\mu + xq_\mu$ as in the other expressions. After some straightforward algebra, we find



$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(\mathrm{Adj})\delta^{ab} \int_0^1 dx \, \frac{J^{\mu\nu}(x,q)}{[x(x-1)q^2]^{2-d/2}},$$

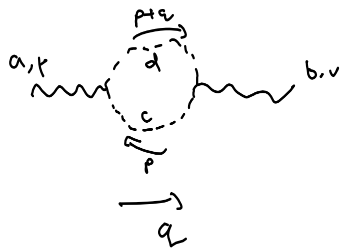
where

$$\begin{aligned} J^{\mu\nu}(x,q) \; = \; & -\Gamma(1-d/2)g^{\mu\nu}q^2 \left[\frac{1}{2}d(d-1)x(1-x)\right] \\ & -\Gamma(2-d/2)g^{\mu\nu}q^2 \left[(d-1)(1-x)^2\right] \end{aligned} \tag{14.6}$$

This looks unwieldy, but we can now see the possibility of cancellations between this and the term constructed from cubic vertices.

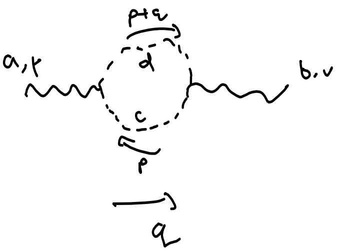
14.2.4 Ghost Loop Contribution

There is one final contribution. This comes from the ghost fields running in the loop.



$$= (-1)g^2 f^{dac} f^{cbd} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2} \frac{i}{(p+q)^2} (p+q)^\mu p^\nu$$

where we note the factor of -1 due to the fact that the ghosts are fermionic so we have a fermion loop. Proceeding as before, the integral can be evaluated to give



$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(\mathrm{Adj})\delta^{ab} \int_0^1 dx \, \frac{i}{p^2} \frac{K^{\mu\nu}(x,q)}{[x(x-1)q^2]^{2-d/2}},$$

where

$$K^{\mu\nu}(x,q) = \Gamma(1-d/2)G^{\mu\nu}q^2\left[\frac{1}{2}x(1-x)\right] + \Gamma(2-d/2)q^\mu q^\nu\left[x(1-x)\right] \tag{14.7}$$

which has a similar structure to the previous two terms.

14.2.5 Putting It All Together

Let us consider the sum of the three diagrams

$$= \frac{ig^2}{(4\pi)^{d/2}}C_2(\mathrm{Adj})\delta^{ab}\int_0^1 dx\,\frac{\mathcal{X}^{\mu\nu}(x,q)}{\Delta^{2-d/2}},$$

where

$$\mathcal{X}^{\mu\nu}(x,q) = F^{\mu\nu}(x,q) + J^{\mu\nu}(x,q) + K^{\mu\nu}(x,q)$$

is given by the previous expressions for each amplitude added together. The explicit expression for $\mathcal{X}^{\mu\nu}(x,q)$ looks intractable; however, if we collect coefficients of the Gamma functions, we start to see some simplifications emerge.

There are several tricks we can use to simplify $\mathcal{X}^{\mu\nu}(x,q)$

- The Gamma function identity $\Gamma(1-d/2) = \frac{\Gamma(2-d/2)}{1-d/2}$ can be used to write each term in the expression in terms of $\Gamma(2-d/2)$.
- The integral over x is invariant under the exchange $x \leftrightarrow 1-x$. This means that, under the integral, we can make simplifications such as

$$2-x \sim 1+x$$

or

$$x = \frac{1}{2}x + \frac{1}{2}x \sim \frac{1}{2}x + \frac{1}{2}(1-x) = \frac{1}{2}.$$

and so terms such as

$$\frac{1}{2}(d-2)(-2x+1) \sim 0.$$

After much tedious algebraic manipulation, one finds that

$$\mathcal{X}^{\mu\nu}(x,q) = \left(q^2g^{\mu\nu} - q^\mu q^\nu\right)\Gamma(2-d/2)\left[\left(1-\frac{d}{2}\right)(1-2x)^2 + 2\right],$$

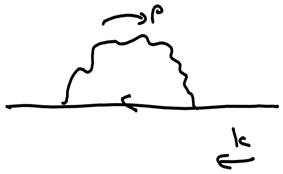
and we see the expected projective pre-factor $\mathcal{P}^{\mu\nu} := q^2g^{\mu\nu} - q^\mu q^\nu$ emerges, giving the final result

$$\boxed{= \frac{ig^2}{(4\pi)^{d/2}} C_2(\mathrm{Adj})\delta^{ab}\Big(q^2 g^{\mu\nu} - q^\mu q^\nu\Big) \int_0^1 dx \, \frac{\Gamma(2-d/2)}{[x(x-1)q^2]^{2-d/2}} \left[\left(1-\frac{d}{2}\right)(1-2x)^2 + 2\right]}$$

It is worth emphasising that the ghost term was crucial in giving the correct projective term to the sum of diagrams.

14.3 Fermion Self-Energy

The one-loop correction to the fermion propagator is much simpler to calculate as there in only one diagram to evaluate



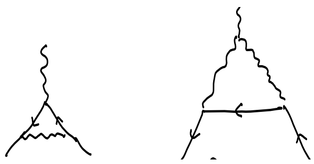
$$= \int \frac{d^4p}{(2\pi)^4}(ig)^2\gamma^\mu T_a \frac{i(\not p + \not k)}{(p+k)^2}\gamma_\mu T_a \frac{-i}{p^2}$$

By now the way to evaluate such diagrams should be straightforward. The final result is

$$\frac{ig^2}{(4\pi)^{d/2}} C_2(r) \not k \, \Gamma(2-d/2) \int_0^1 dx \, \frac{(1-x)(d-2)}{[x(x-1)k^2]^{2-d/2}}.$$

14.4 One-Loop Vertex Corrections

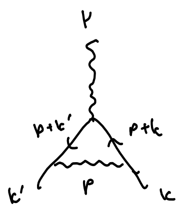
At one loop, there are two diagrams that contribute to the 2 fermion, 1 gluon vertex.



We shall treat these diagrams in turn. Since the evaluation of the diagrams does not involve any fundamentally new ideas, we shall be very brief.

Fermion-Fermion-Gluon Vertex Loop

The first diagram we shall discuss has a divergence coming from a loop involving two fermion lines and a gluon. The diagram is



$$= ig^3 T_b T_a T_b \int \frac{d^4p}{(2\pi)^4} \frac{\gamma^\nu(\not p + \not k')\gamma^\mu(\not p + \not k)\gamma_\nu}{(p+k')^2(p+k)^2p^2}$$

Note that we can write

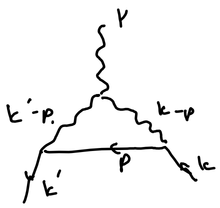
$$\begin{aligned} T_bT_aT_b &= T_bT_bT_a + T_b[T_a, T_b] = C_2(r)T_a + if_{ab}{}^c T_b T_c \\ &= C_2(r)T_a - \frac{1}{2} f_{ab}{}^c f_{bc}{}^d T_d = \Big(C_2(r) - \frac{1}{2}C_2(\mathrm{Adj})\Big)T_a. \end{aligned} \tag{14.8}$$

so the diagram gives

$$ig^3 T_a \left(C(r) - \frac{1}{2} C_2(\mathrm{Adj}) \right) \int \frac{d^4 p}{(2\pi)^4} \frac{\gamma^\nu (\not{p} + \not{k}') \gamma^\mu (\not{p} + \not{k}) \gamma_\nu}{(p+k')^2 (p+k)^2 p^2}$$

14.4.1 Fermion-Gluon-Gluon Vertex Loop

We also have the diagram



$$= \int \frac{d^4 p}{(2\pi)^4} (ig\gamma_\nu T_b) \frac{i\not{p}}{p^2} (ig\gamma_\rho T_c) \frac{-i}{(k'-p)^2} \frac{-i}{(k-p)^2} (igf^{abc}) N^{\mu\nu\rho}$$

where

$$N^{\mu\nu\rho} = g^{\mu\nu}(2k' - k - p)^\rho + g^{\nu\rho}(-k' - k + 2p)^\mu + g^{\rho\mu}(2k - k' - p)^\nu$$

It will be useful to write

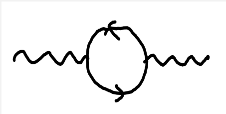
$$f^{abc} T_b T_c = \frac{i}{2} f^{abc} f_{bc}{}^{d} T_d = \frac{i}{2} C_2(\mathrm{Adj}) T^a,$$

so that we have the integral

$$\frac{ig^3}{2} C_2(\mathrm{Adj}) T^a N^{\mu\nu\rho} \int \frac{d^d p}{(2\pi)^d} \frac{\gamma_\nu \not{p} \gamma_\rho}{p^2 (k'-p)^2 (k-p)^2}.$$

Summary of divergences

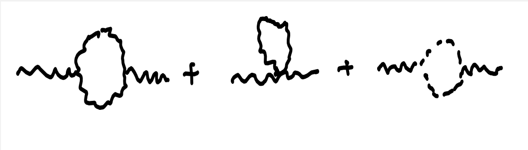
Gluon Polarization

$$\sum_{\text{fermions}}$$

$$= \Big(g^{\mu\nu}q^2 - q^\mu q^\nu\Big)\delta_{ab}C(r)i\frac{-g^2}{6\pi^2\epsilon} + ...,$$

where $C(r)$ is given by

$$tr(T_aT_b) = (T_a)_i{}^j(T_b)_j{}^i = C(r)\delta_{ab}.$$

In this, the fundamental representation, $C(r) = \frac{1}{2}$. The other terms in the gluon polarization are


$$= \frac{ig^2}{8\pi^2}C_2(\text{Adj})\delta^{ab}\Big(q^2g^{\mu\nu} - q^\mu q^\nu\Big)\frac{5}{3\epsilon} + ...$$

where $C_2(r)$, the quadratic Casimir, is given by

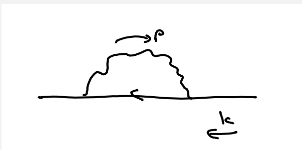
$$(T_a)_i{}^k(T_a)_k{}^j = C_2(r)\delta_i{}^j.$$

For the Adjoint representation, where $(T_a)_b{}^c = f_{ab}{}^c$, of $SU(N)$, we have $C_2(\text{Adj}) = N$.

Putting them together gives

$$i\Big(q^2g^{\mu\nu} - q^\mu q^\nu\Big)\delta^{ab}\frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(\text{Adj}) - \frac{4}{3}n_fC(r)\right) + ...$$

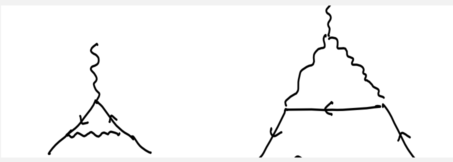
Fermion self-energy



$$= \frac{ig^2}{8\pi^2\epsilon}C_2(r)\not{k} + ...$$

Vertex

The two one-loop diagrams may be written as


$$= \frac{ig^3}{8\pi^2\epsilon}\Big(C_2(r) + C_2(\text{Adj})\Big)T_a\gamma^\mu + ...$$

14.5 Counter-terms in the MS Scheme

We can use the asymptotic expansion of the Gamma-functions to extract the divergent parts of the diagrams. We require

MS scheme counter-terms

Including the counter-terms, the renormalized one-loop contributions to Z_1, Z_2 and Z_3 are:

Gluon polarization

$$i\Big(q^2 g^{\mu\nu} - q^\mu q^\nu\Big)\delta^{ab}\frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(\text{Adj}) - \frac{4}{3}n_f C(r)\right) - i\Big(q^2 g^{\mu\nu} - q^\mu q^\nu\Big)\delta^{ab}\delta_3 + \ldots = \text{Finite},$$

which, in the MS scheme, gives

$$\delta_3 = \frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(\text{Adj}) - \frac{4}{3}n_f C(r)\right).$$

Fermion self-energy

$$\frac{ig^2}{8\pi^2\epsilon}C_2(r)\not{k} + i\not{k}\delta_2 \ldots = \text{Finite}$$

which, in the MS scheme, gives

$$\delta_2 = -\frac{g^2 C_2(r)}{8\pi^2\epsilon}.$$

Fermion-gluon vertex

$$\frac{ig^3}{8\pi^2\epsilon}\Big(C_2(r) + C_2(\text{Adj})\Big)T_a\gamma^\mu + ig\gamma^\mu T_a\delta_1 \ldots = \text{Finite}$$

which, in the MS scheme, gives

$$\delta_1 = -\frac{g^2}{8\pi^2\epsilon}\Big(C_2(r) + C_2(\text{Adj})\Big).$$

14.6 The Beta-Function and Asymptotic Freedom

We now bring all of these results together to derive the beta-function for non-abelian gauge theories. Collecting the MS scheme counter-terms in one place for convenience

$$\boxed{\begin{aligned}\delta_1 &= -\frac{g^2}{8\pi^2\epsilon}\Big(C_2(r)+C_2(\text{Adj})\Big)\\ \delta_2 &= -\frac{g^2 C_2(r)}{8\pi^2\epsilon}\\ \delta_3 &= \frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(\text{Adj})-\frac{4}{3}n_f C(r)\right)\end{aligned}} \tag{14.9}$$

We may then use the relationship between the bare and renormalized couplings. This has the same form as we saw in QED

$$g_0 = \frac{Z_1}{Z_2\sqrt{Z_3}}\mu^{\frac{\epsilon}{2}}g \approx \left(1+\delta_1-\delta_2-\frac{1}{2}\delta_3\right)\mu^{\frac{\epsilon}{2}}g.$$

The above expressions give

$$\delta_1-\delta_2-\frac{1}{2}\delta_3 = -\frac{g^2}{16\pi^2\epsilon}\left(\frac{11}{3}C_2(\text{Adj})-\frac{4}{3}n_f C(r)\right). \tag{14.10}$$

And so, the condition that

$$g_0 = \mu^{\frac{\epsilon}{2}}g\left(1-\frac{g^2}{16\pi^2\epsilon}\left(\frac{11}{3}C_2(\text{Adj})-\frac{4}{3}n_f C(r)\right)+\ldots\right)$$

is independent of μ, i.e. that

$$\mu\frac{d}{d\mu}g_0 = 0,$$

gives

$$\boxed{\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(\text{Adj})-\frac{4}{3}n_f C(r)\right)}$$

What is remarkable in this is that, depending on the gauge group, the types and representations of matter, this beta function can be positive or negative. Unlike ϕ^4 theory or QED, which had positive beta-functions, a theory with negative beta-function would be such that the coupling decreased as we went to higher energies/shorter distances.

For the case of $SU(N)$, the quadratic casimirs and indices for the fundamental and adjoint representations are

$$C_2(\text{Adj}) = N, \qquad C(\text{Adj}) = N,$$

$$C_2(\text{Fund}) = \frac{N^2-1}{2N}, \qquad C(\text{Fund}) = \frac{1}{2}.$$

and so for $SU(N)$ with n_f identical massless fermions in the fundamental representation

$$\beta_{QCD}(g) = -\frac{g^2}{(4\pi)^2}\left(\frac{11}{3}N-\frac{2}{3}n_f\right)$$

so if $n_f < \frac{11}{2}N$, the beta function is negative and the coupling will decrease at short distances. This phenomenon is known as *asymptotic freedom*. Note that for QCD, $N = 3$ and so if

$$n_f < 16\frac{1}{2}$$

we have asymptotic freedom. To the best of our knowledge, we have six types of quark $n_f = 6$ and so at short distances we predict the theory to be asymptotically free. This is what we see in experiment in deep inelastic scattering, where we can probe the quark structure of matter and trust perturbation theory. At larger distances/lower energies the coupling grows and we can no longer trust perturbation theory. In particular, our one-loop calculation of the beta-function is no longer a reliable guide to the behaviour of the beta-function when $g \sim \mathcal{O}(1)$. Nonetheless, it is suggestive of the confining nature of QCD at low energies that we observe in nature. To show that QCD is actually predicting confinement, we would need other methods.

Once way to view what is going on is in terms of the competing effects of the terms in the beta-function. These effects could be described as screening and anti-screening:

- Virtual quark vacuum polarization screens the gluon charge
- Virtual gluons act in the opposite way to enhance the apparent charge.

which effect dominates depends on details of the theory, such as the number of fermion species available.

There are also examples of special theories where the matter content and representations are such that the beta-function is zero. If this holds to all orders of perturbation theory, then the theory is said to be conformal. Such theories are very special and examples include the worldsheet QFT of string theory and $\mathcal{N} = 4$ Supersymmetric Yang-Mills in four dimensions, which plays a starring role in the AdS/CFT correspondence.

A Zero-dimensional Symmetry Factor from the Partition Function

In this Appendix, we derive an expression for the symmetry factor $S_{\mathcal{G}_V}$ for the zero-dimensional theory from the partition function

$$Z_\lambda(0) = \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{+\infty} d\phi \, e^{-\frac{\alpha}{2}\phi^2 - \frac{\lambda}{4!}\phi^4}.$$

Since the integrand is symmetric under the exchange $\phi \to -\phi$, we take ϕ to have values between 0 and ∞ and multiply the integral by two. To make contact with perturbation theory we can expand the quartic term in a power series

$$\begin{aligned} Z_\lambda(0) &= 2\sqrt{\frac{\alpha}{2\pi}} \int_0^\infty d\phi \, e^{-\frac{\alpha}{2}\phi^2} \sum_{V\in\mathbb{Z}} \frac{1}{V!} \left(\frac{-\lambda\phi^4}{4!}\right)^V \\ &= \sqrt{\frac{2\alpha}{\pi}} \sum_{V\in\mathbb{Z}} \frac{1}{V!} \left(\frac{-\lambda}{4!}\right)^V \int_0^\infty d\phi \, \phi^{4V} \, e^{-\frac{\alpha}{2}\phi^2} \end{aligned} \tag{A.1}$$

where the last step is very suspect. Changing variables

$$x := \frac{\alpha}{2}\phi^2,$$

which gives

$$\begin{aligned} Z_\lambda(0) &= \sqrt{\frac{2\alpha}{\pi}} \sum_{V\in\mathbb{Z}} \frac{1}{V!} \left(\frac{-\lambda}{4!}\right)^V \frac{1}{\alpha} \left(\frac{2}{\alpha}\right)^{2V-\frac{1}{2}} \int_0^\infty dx \, x^{2V-\frac{1}{2}} \, e^{-x} \\ &= \frac{1}{\sqrt{\pi}} \sum_{V\in\mathbb{Z}} \left(\frac{2}{\alpha}\right)^{2V} \frac{1}{V!} \left(\frac{-\lambda}{4!}\right)^V \Gamma\left(2V + \frac{1}{2}\right) \end{aligned} \tag{A.2}$$

as follows from the definition of the Γ-function:

$$\Gamma\left(2V + \frac{1}{2}\right) = \int_0^\infty dx \, x^{(2V+\frac{1}{2})-1} e^{-x}.$$

Since V is an integer, we have that

$$\Gamma\left(2V + \frac{1}{2}\right) = \frac{(4V)!\sqrt{\pi}}{4^{2V}(2V)!},$$

so we have

$$Z_\lambda(0) = \sum_{V\in\mathbb{Z}} \left(\frac{-\lambda}{\alpha^2}\right)^V \frac{1}{(4!)^V V!} \frac{(4V)!}{2^{2V}(2V)!}$$

We can understand this diagrammatically; how? We write this as

$$Z_\lambda(0) = \sum_{V\in\mathbb{Z}} \frac{1}{S_V} (\lambda)^V \left(\frac{1}{\alpha}\right)^P,$$

where

$$S_V^{-1} = \frac{1}{(4!)^V V!} \times \frac{(4V)!}{2^P P!}$$

is the symmetry factor and $P = 2V$ is the number of propagators (recalling that the non-zero contribution only comes from $E = 2P - 4V = 0 \implies P = 2V$). The symmetry factor is a count of the number of ways we can build the diagram. We can think of S as the number of operations we can do that leave the diagram invariant.

B One-loop Calculations in QED

The one-loop divergent diagrams are computed in dimensional regularization. We will need some technical results.

B.1 Gamma Matrix Identities in d Dimensions

In $d = 4 - \epsilon$ dimensions, the trace identities are slightly modified[72]

$$\begin{aligned}
\gamma^\mu \gamma_\mu &= d \\
\gamma^\mu \gamma^\nu \gamma_\mu &= (2-d))\gamma^\nu \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4g^{\nu\rho} + (d-2)\gamma^\nu \gamma^\rho \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= 2\gamma^\rho \gamma^\sigma \gamma^\nu - 2\gamma^\nu \gamma^\sigma \gamma^\rho - (d-2)\gamma^\nu \gamma^\rho \gamma^\sigma
\end{aligned} \tag{B.1}$$

B.2 Useful Integrals

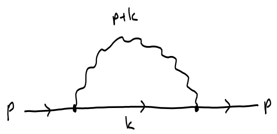
We will also make use of the integrals

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{B.2}$$

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{\ell_E^2}{(\ell^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-d/2-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{B.3}$$

B.3 The Electron Self Energy

We want to evaluate the diagram



$$= \frac{i(\not{p}+m)}{p^2-m^2}\Big(-i\Sigma_2(p)\Big)\frac{i(\not{p}+m)}{p^2-m^2}$$

where $-i\Sigma_2(p)$ denotes the order e^2 contribution to $-i\Sigma(p)$ and is given by

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \gamma^\mu \frac{i(\not{k}+m)}{k^2 - m^2 + i\varepsilon} \gamma_\mu \frac{-i}{(p-k)^2 + i\varepsilon}$$

We use Feynman parameterization

$$\frac{1}{k^2-m^2}\frac{1}{(p-k)^2} = \int_0^1 dx \frac{1}{[(1-x)(k^2-m^2) + x(p-k)^2]^2}$$

[72] The proof of these identities is straightforward and follows from $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $g^\mu{}_\mu = \delta^\mu{}_\mu = d$, which implies that $\gamma^\mu \gamma_\mu = d$. For example:

$$\gamma^\mu \gamma^\nu \gamma_\mu = (2g^{\mu\nu} - \gamma^\nu \gamma^\mu)\gamma_\mu = 2\gamma^\nu - d\gamma^\nu = -(2-\epsilon)\gamma^\nu .$$

We can simplify the denominator by completing the square in k:

$$(1-x)(k^2-m^2)+x(p-k)^2 = (k-xp)^2+xp^2-x^2p^2-(1-x)m^2 \tag{B.4}$$

Let $\ell_\mu := k_\mu - xp_\mu$, then the denominator becomes

$$(1-x)(k^2-m^2)+x(p-k)^2 = \ell^2-\Delta \qquad \text{where} \qquad \Delta := [x(x-1)p^2+(1-x)m^2].$$

We therefore have

$$-i\Sigma_2(p) = -e^2\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\frac{\gamma^\mu(\not\ell + x\not p + m)\gamma_\mu}{(\ell^2-\Delta)^2}$$

Let us deal with the spinor terms in the numerator

$$\gamma^\mu \not\ell \gamma_\mu = \gamma^\mu\gamma^\nu\ell_\nu\gamma_\mu = -\ell_\nu(2-\epsilon)\gamma^\nu = -(2-\epsilon)\not\ell,$$

and so

$$-i\Sigma_2(p) = -e^2\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\frac{[-(2-\epsilon)(\not\ell + x\not p)+(4-\epsilon)m]}{(\ell^2-\Delta)^2}$$

The term linear in $\not\ell$ vanishes once the integration is performed, leaving

$$-i\Sigma_2(p) = -e^2\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\frac{[-(2-\epsilon)x\not p+(4-\epsilon)m]}{(\ell^2-\Delta)^2}$$

We perform the ℓ integral in Euclidean space by defining (as usual) $\ell_E^0 := -i\ell^0$, so that

$$\begin{aligned}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2} &= i\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} \\ &= i\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty \frac{d\ell_E}{(2\pi)^d}\frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^2}.\end{aligned} \tag{B.5}$$

We now use the integral we have seen many times before (and proven in section ?)

$$\int_0^\infty \frac{d\ell_E}{(2\pi)^d}\frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^2} = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-d/2}\frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)},$$

so that we have

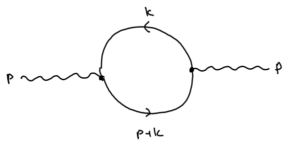
$$-i\Sigma_2(p) = -i\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx\, \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}[-(2-\epsilon)x\not p+(4-\epsilon)m]$$

where

$$\Delta := [x(x-1)p^2+(1-x)m^2].$$

B.4 Vacuum Polarization (Photon self-Energy)

This diagram one-loop divergent diagram is



$$= i\Pi_{\mu\nu}(p)$$

where

$$i\Pi_{\mu\nu}(p) = (-ie)^2(-1)\int \frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[\gamma_\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma_\nu \frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\right]$$

The factor of -1 is due to the fermion loop. We study the trace

$$\begin{aligned}\mathrm{tr}[\gamma_\mu(\not{k}+m)\gamma_\nu(\not{k}+\not{q}+m)] &= \mathrm{tr}[\gamma_\mu\not{k}\gamma_\nu(\not{k}+\not{q})] + m\mathrm{tr}[\gamma_\mu\gamma_\nu(\not{k}+\not{q})] + m\mathrm{tr}[\gamma_\mu\not{k}\gamma_\nu] + m^2\mathrm{tr}[\gamma_\mu\gamma_\nu] \\ &= (4-\epsilon)[k^\mu(k^\nu+q^\nu) - g^{\mu\nu}[k\cdot(k+q)-m^2] + k^\nu(k^\mu+q^\mu)].\end{aligned}$$

so that

$$i\Pi_{\mu\nu}(p) = -4e^2\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu(k^\nu+q^\nu)+k^\nu(k^\mu+q^\mu)-g^{\mu\nu}[k\cdot(k+q)-m^2]}{(k^2-m^2)[(k+q)^2-m^2]}$$

The next step is to deal with the denominator by using a Feynman parameterization

$$\begin{aligned}\frac{1}{(k^2-m^2)[(k+q)^2-m^2]} &= \int_0^1 dx\frac{1}{(k^2+2xk\cdot q+xq^2-m^2)^2} \\ &= \int_0^1 dx\frac{1}{(\ell^2+x(1-x)q^2-m^2)^2},\end{aligned} \tag{B.6}$$

where we have defined $\ell_\mu := k_\mu + xq_\mu$. In terms of ℓ, the numerator becomes

$$\begin{aligned}k^\mu(k^\nu+q^\nu)+k^\nu(k^\mu+q^\mu)-g^{\mu\nu}[k\cdot(k+q)-m^2] &= 2\ell^\mu\ell^\nu - 2x(1-x)q^\mu q^\nu \\ &\quad -g^{\mu\nu}[\ell^2-x(1-x)q^2-m^2] \\ &\quad + \text{terms linear in } \ell.\end{aligned} \tag{B.7}$$

The terms linear in ℓ do not contribute to the integral and so we have

$$i\Pi_{\mu\nu}(p) = -4e^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{2\ell^\mu\ell^\nu-2x(1-x)q^\mu q^\nu-g^{\mu\nu}[\ell^2-x(1-x)q^2-m^2]}{(\ell^2-\Delta)^2}$$

where

$$\Delta := m^2 - x(1-x)q^2.$$

Since $g^{\mu\nu}$ is the only rank two isotropic tensor, we may substitute

$$\int d^d\ell\,\ell^\mu\ell^\nu f(\ell^2) = \frac{1}{d}\int d^d\ell\,g^{\mu\nu}\ell^2 f(\ell^2)$$

for some function f. We then have

$$i\Pi_{\mu\nu}(p) = -4e^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\frac{2}{d}g^{\mu\nu}\ell^2-2x(1-x)q^\mu q^\nu-g^{\mu\nu}[\ell^2-x(1-x)q^2-m^2]}{(\ell^2-\Delta)^2}$$

We now go to Euclidean space to do the integral (note $\ell^2 = -\ell_E^2$) to give

$$i\Pi_{\mu\nu}(p) = -4ie^2 \int_0^1 dx \int \frac{d^d\ell_E}{(2\pi)^d} \left[-\left(\frac{2}{d} - 1\right) \frac{\ell_E^2}{(\ell_E^2 + \Delta)^2} + \frac{G^{\mu\nu}}{(\ell_E^2 + \Delta)^2} \right]$$

where we have defined

$$G^{\mu\nu} := -2x(1-x)q^\mu q^\nu + g^{\mu\nu}[x(1-x)q^2 + m^2]$$

Using the standard integrals

$$i\Pi_{\mu\nu}(p) = -4ie^2 \int_0^1 dx \left[\left(1 - \frac{d}{2}\right) \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(1-d/2)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} + \frac{G^{\mu\nu}}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \right]$$

Using the property of Gamma functions, we can write

$$\Gamma(1-d/2) = \frac{\Gamma(2-d/2)}{1-\frac{d}{2}}$$

and so

$$i\Pi_{\mu\nu}(p) = -\frac{4ie^2}{(4\pi)^d/2} \Gamma(2-d/2) \int_0^1 dx \frac{-\Delta g^{\mu\nu} + G^{\mu\nu}}{\Delta^{2-d/2}}$$

All that remains is to simplify the numerator:

$$-\Delta g^{\mu\nu} + G^{\mu\nu} = 2x(1-x)\Big(g^{\mu\nu}q^2 - q^\mu q^\nu\Big), \tag{B.8}$$

so that

$$i\Pi_{\mu\nu}(p) = -\frac{4ie^2}{(4\pi)^d/2} \Gamma(2-d/2) \Big(g^{\mu\nu}q^2 - q^\mu q^\nu\Big) \int_0^1 dx \frac{2x(1-x)}{\Delta^{2-d/2}}.$$

This may be conveniently written as

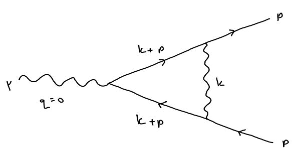
$$i\Pi_{\mu\nu}(p) = \Big(g^{\mu\nu}q^2 - q^\mu q^\nu\Big) i\Pi(q^2),$$

where

$$i\Pi(q^2) = -\frac{8ie^2}{(4\pi)^d/2} \Gamma(2-d/2) \int_0^1 dx \frac{x(1-x)}{[m^2 - x(1-x)q^2]^{2-d/2}}$$

B.5 Electron Vertex Function

The divergent one-loop diagram is



$$= -i\Gamma^\mu(p)$$

where

$$-i\Gamma^\mu(p) = (-ie)^3 \int \frac{d^dk}{(2\pi)^d} \gamma^\lambda \frac{i(\not{k} + \not{p} + m)}{(k+p)^2 - m^2} \gamma^\mu \frac{i(\not{k} + \not{p} + m)}{(k+p)^2 - m^2} \gamma^\rho \left(-\frac{ig_{\lambda\rho}}{k^2}\right)$$

To evaluate this digram, we need the trace identities

$$
\begin{aligned}
\gamma^\lambda \not{A}\gamma^\mu \not{A}\gamma_\lambda &= (2-d)(2A^\mu \not{A} - \gamma^\mu A^2) \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= dg^{\nu\rho} \\
\gamma^\lambda\gamma^\mu\gamma_\lambda &= (2-d)\gamma^\mu
\end{aligned}
\tag{B.9}
$$

We find

$$
\begin{aligned}
\gamma^\lambda(\not{k}+\not{p}+m)\gamma^\mu(\not{k}+\not{p}+m)\gamma_\lambda &= (2-d)\Big[2(k^\mu+p^\mu)(\not{k}+\not{p}) - \gamma^\mu(k+p)^2\Big] \\
&\quad +(2-d)m^2\gamma^\mu + 2md(k^\mu+p^\mu) \\
&= -(2-d)\gamma^\mu[(k+p)^2-m^2] + 2\Big[(2-d)(\not{k}+\not{p})+dm\Big](k^\mu+p^\mu)
\end{aligned}
\tag{B.10}
$$

and so

$$
-i\Gamma^\mu(p) = -e^3\int\frac{d^dk}{(2\pi)^d}\left[\frac{-(2-d)\gamma^\mu}{k^2[(k+p)^2-m^2]} + \frac{2\Big[(2-d)(\not{k}+\not{p})+dm\Big](k^\mu+p^\mu)}{k^2[(k+p)^2-m^2]^2}\right]
$$

To deal with the denominator, we need to generalize out Feynman prescription

Feynman Parametrization II

$$
\begin{aligned}
\frac{1}{AB^2} &= -\frac{\partial}{\partial B}\frac{1}{AB} = -\frac{\partial}{\partial B}\int_0^1 dx\frac{1}{[(1-x)A+xB]^2} \\
&= \int_0^1 dx\frac{2x}{[(1-x)A+xB]^3}
\end{aligned}
\tag{B.11}
$$

Using this and taking $A = k^2$ and $B = (k+p)^2 - m^2$;

$$
\frac{1}{k^2[(k+p)^2-m^2]^2} = \int_0^1 dx\frac{2x}{[(1-x)k^2 + x[(k+p)^2-m^2]]^3}
$$

but

$$
(1-x)k^2 + x[(k+p)^2-m^2] = (k+xp)^2 - \Delta,
$$

where

$$
\Delta := xm^2 - x(1-x)p^2, \qquad \ell_\mu := k_\mu + xp_\mu
$$

We then have

$$
-i\Gamma^\mu(p) = -e^3\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\left[-\frac{(2-d)\gamma^\mu}{(\ell^2-\Delta)^2} + \frac{4x\Big[(2-d)(\not{\ell}+(1-x)\not{p})+dm\Big](\ell^\mu+(1-x)p^\mu)}{(\ell^2-\Delta)^3}\right]
$$

The numerator in the second term may be written as

$$
\begin{aligned}
\Big[(2-d)(\not{\ell}+(1-x)\not{p})+dm\Big](\ell^\mu+(1-x)p^\mu) &= (2-d)\not{\ell}\ell^\mu + (2-d)[(1-x)\not{p}+dm](1-x)p^\mu \\
&\quad + \text{terms linear in } \ell,
\end{aligned}
\tag{B.12}
$$

which, under the integral, is equivalent to

$$(2-d)\gamma^\mu \frac{\ell^2}{d} + (1-x)(2-d)\Big[(1-x)\not{p} + dm\Big](1-x)p^\mu.$$

Putting this back in and using the standard integrals to do the momentum space integrals in ℓ_E, we have

$$\begin{aligned} -i\Gamma^\mu(p) = & -\frac{ie^3\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 dx \frac{1}{\Delta^{2-d/2}} \Big[(d-2)(1-x)\gamma^\mu \\ & +x(1-x)(4-d)\Big[(2-d)(1-x)\not{p} + dm\Big]p^\mu \Delta\Big] \end{aligned} \tag{B.13}$$

The second term, which includes the factor $4-d=\epsilon$, is finite as $\epsilon \to 0$, so we focus on the first term which contains the divergence. The divergent part is thus

$$\boxed{-i\Gamma^\mu(p) = -\frac{ie^3\gamma^\mu\Gamma(2-d/2)}{8\pi^2} \int_0^1 dx \, \frac{(1-x)}{[xm^2 - x(1-x)p^2]^{2-d/2}} + \text{ finite}}$$

C *The LSZ Reduction Formula: From Correlation Functions to Scattering Amplitudes

What do we actually measure in particle physics experiments? This is an important question as, of all the things we could learn about a QFT, the answer to this question provides some focus on what we should care about. In a collider we will start with some particles which we allow to interact with each other and then we measure which particles emerge from the interaction and with what properties. The interaction is local and the cluster decomposition principle[73] tells us that, if we can assume the initial and final states are far enough apart, we can take the initial and final states as being effectively free. Describing the initial and final states and momentum eigenstates, the natural object to study in an experiment is the transition amplitudes betweeen one set of momentum eigenstates initially and a final set of momentum eigenstates. This is the S-Matrix

$$\mathcal{S}_{if} = \langle f|i\rangle = \langle k_1, k_2, .., k_m|\mathcal{S}|k_{m+1}, k_{m+2}, ...k_n\rangle.$$

Thus S describes the evolution of the system through the interaction and $\mathcal{S}_{if}$ are the matrix elements of the operator $\mathcal{S}$ written in a Fock space basis. One could think of this as being a measure of how much the effects of an interaction over an infinite time rotate the Fock space basis.

This is almost what we want to study. In the case of a free theory, we expect $|i\rangle = |f\rangle$ and so $\mathcal{S} = 1$. This is not interesting. Instead we shall study the *transfer matrix* $\mathcal{T}$, given by

$$\mathcal{S} = 1 + i\mathcal{T}.$$

[73] This says that experiments carried out far from each other cannot influence each other. In terms of correlation functions, we could express this sentiment in terms of the factorization of correlation functions

$$\lim_{|x|\to\infty} \langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \langle \mathcal{O}(x)\rangle\langle \mathcal{O}(0)\rangle.$$

In what follows we shall assume that $|i\rangle \neq |f\rangle$ and so can refer to $\mathcal{S}$ and $\mathcal{T}$ almost ointerchangably[74]. We shall see that the scattering amplitudes may be written as

$$\langle k_1, k_2, .., k_m | i\mathcal{T} | k_{m+1}, k_{m+2}, ...k_n\rangle = i(2\pi)^4 \mathcal{M}_{if} \delta^4 \left(\sum k\right),$$

where the delta function imposes momentum conservation in the process and the matrix elements $\mathcal{M}_{if}$ can be determined from time-ordered correlation functions. The LSZ reduction formula is a remarkable equation relating time-ordered correlation functions $\langle T\{\phi(x_1)...\phi(x)_n\}\rangle$ to scattering amplitudes.

It is worth noting that the probability of this transition, given by $|\langle f|i\rangle|^2$, formally diverges since

$$|\langle f|i\rangle|^2 \sim |\mathcal{M}_{if}|^2 \left(\delta^4\left(\sum k\right)\right)^2 \sim |\mathcal{M}_{if}|^2 \delta^4(0)\delta^4\left(\sum k\right),$$

and $\delta^4(0)$ is not defined. The root of this problem is that, to get to this momentum space expression for the amplitude, we have integrated over the infinite volume of spacetime and so

$$\delta^4(0) \sim \int \frac{d^4x}{(2\pi)^4}.$$

In reality, the volume would be large but finite, giving a sensible result. We shall often find that we have to regulate the fact that we are working in infinite volume in order to achieve finite results below.

C.1 Asymptotic States

We shall stick with massive scalar fields as our illustrative example and find an expression for the asymptotic states. Asyptotically we assume the field is free and so has a mode expansion

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E}\Big(a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\Big)$$

where

$$E^2 = \mathbf{k}^2 + m^2, \qquad k\cdot x = Et - \mathbf{k}\cdot\mathbf{x}.$$

One can show that

$$\int d^3\mathbf{k}\, e^{ik\cdot x}\phi(x) = \frac{1}{2E}\Big(a(\mathbf{k}) + e^{2iET}a^\dagger(-\mathbf{k})\Big)$$

and

$$\int d^3\mathbf{k}\, e^{ik\cdot x}\partial_0\phi(x) = -\frac{i}{2E}\Big(a(\mathbf{k}) - e^{2iET}a^\dagger(-\mathbf{k})\Big),$$

so that

$$\begin{aligned} a(\mathbf{k}) &= \int d^3\mathbf{k}\, e^{ik\cdot x}\Big(i\partial_0\phi(x) + E\phi(x)\Big) \\ a^\dagger(\mathbf{k}) &= \int d^3\mathbf{k}\, e^{-ik\cdot x}\Big(-i\partial_0\phi(x) + E\phi(x)\Big). \end{aligned} \tag{C.1}$$

[74]The arguments we will use have subtleties if the momenta are allowed to be equal, so we shall assume this is not the case. One can show the final result holds in the general case.

In the free theory, a one-particle state is

$$|k\rangle = a^\dagger(\mathbf{k})|\Omega\rangle,$$

where $|\Omega\rangle$ is the vacuum[75] where $\langle\Omega|\Omega\rangle = 1$ and $a(\mathbf{k})|\Omega\rangle = 0$, $\forall\mathbf{k}$.

We will run into more infinite volume problems if we don't rgularize the wavefunctions in some way. As such we modify the free fields by introducing Gaussian wavepackets for the i'th particle: $a^\dagger(\mathbf{k}) \to a_i^\dagger$, where

$$a_i^\dagger := \int d^3\mathbf{k}\, f_i(\mathbf{k}) a^\dagger(\mathbf{k}),$$

where $f_i(\mathbf{k})$ is a Gaussian envlope

$$f_i(\mathbf{k}) = \frac{1}{\sqrt{4\pi\sigma^2}} \exp\left(-\frac{(\mathbf{k}-\mathbf{k}_i)^2}{4\sigma^2}\right).$$

The normalization is such that $\lim_{\sigma\to 0} f_i(\mathbf{k}) = \delta^3(\mathbf{k}-\mathbf{k}_i)$, a limit we shall take to remove the effect of the Gaussian spreading. The operator $a_i^\dagger$ is in the Heisenberg (or ineraction) picture and we evolve it forward in time using the interaction Hamiltonian to get $a_i^\dagger(t)$ as some specified time t. The wavepacket spreads out as time increases but the overlap may be taken to be zero (the particles well-seperated) in the infinite past.

We shall focus on the 2 particle $\to$ 2 particle scattering and define the initial state as

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t) a_2^\dagger(t)|\Omega\rangle,$$

and the final state as

$$|f\rangle = \lim_{t\to+\infty} a_3^\dagger(t) a_4^\dagger(t)|\Omega\rangle,$$

where all of the k_i are distinct. Again, these expressions are assumed true asymptotically (in the infinite past and fiuture). In perturbation theory, we can evolve from the infinite past to the infinite future using the interaction Hamiltonian.

To find an expression for the S-matrix we make use of the fundamental theorem of calculus

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{+\infty} dt\, \partial_0 a_1^\dagger(t),$$

But, using the explicit expressions (C.1), we can write this as

$$\begin{aligned} a_1^\dagger(+\infty) - a_1^\dagger(-\infty) &= -i\int d^3\mathbf{k}\, f_1(\mathbf{k}) \int d^4x\, e^{-ik\cdot x}\left(\partial_0^2 + E^2\right)\phi(x) \\ &= -i\int d^3\mathbf{k} f_1(\mathbf{k}) \int d^4x\, e^{-ik\cdot x}\left(\partial_0^2 - \mathbf{k}^2 + m^2\right)\phi(x) \end{aligned} \tag{C.2}$$

[75] In the free theory we usually denote $\Omega\rangle$ by $|0\rangle$. The LSZ formula is non-perturbative and so the results will hold in the full interactiong theory which has vacuum $|\Omega\rangle$. To pre-emt this, we shall use $|\Omega\rangle$ to denote the vacuum in this section.

Integrating by parts[76] we find

$$
\begin{aligned}
a_1^\dagger(+\infty) - a_1^\dagger(-\infty) &= -i\int d^3\mathbf{k}\, f_1(\mathbf{k}) \int d^4x\, e^{-ik\cdot x}(\partial_0^2 - \nabla^2 + m^2)\phi(x) \\
&= -i\int d^3\mathbf{k}\, f_1(\mathbf{k}) \int d^4x\, e^{-ik\cdot x}(-\Box + m^2)\phi(x)
\end{aligned} \tag{C.3}
$$

$\phi(x)$ appering in the integrand is defined at all points in spacetime and so is not necessarily free. In a fully free theory $(\Box - m^2)\phi(x) = 0$, we would find

$$a_1^\dagger(+\infty) = a_1^\dagger(-\infty),$$

and that would be that. However, for an interacting theory this is not the case. For example, for ϕ^4 theory, we have

$$(\Box - m^2)\phi(x) = \frac{\lambda}{3!}\phi^3(x),$$

and, in general, we have

$$a_1^\dagger(+\infty) = a_1^\dagger(-\infty) - i\int d^3\mathbf{k}\, f_1(\mathbf{k}) \int d^4x\, e^{-ik\cdot x}(\Box - m^2)\phi(x).$$

Similarly

$$a_1(+\infty) = a_1(-\infty) - i\int d^3\mathbf{k}\, f_1(\mathbf{k}) \int d^4x\, e^{ik\cdot x}(\Box - m^2)\phi(x).$$

C.2 The S-Matrix

The S-matrix element we are interested in is then

$$\langle f|i\rangle = \langle\Omega|a_3(+\infty)a_4(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)|\Omega\rangle.$$

This is already time-ordered and so we can trivially write this as

$$\langle f|i\rangle = \langle\Omega|T\{a_3(+\infty)a_4(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)\}|\Omega\rangle.$$

Putting the above expressions for the asymptotic states into the correlation function, we see that the time ordering moves all of the $a_i^\dagger(+\infty)$ to the left, where they annihilate on $\langle\Omega|$ and all of the $a_i(-\infty)$ to the right, where they annihilate on $|\Omega\rangle$. Thus, if we write

$$a_3(+\infty) =_3 (-\infty) + K_3, \qquad a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + K_1$$

and so on, we find

$$\langle f|i\rangle = \langle\Omega|T\{K_1K_2K_3K_4\}|\Omega\rangle,$$

[76]This is where the value of the Gaussian envelope comes in. The integration by parts produces boundary term which includes

$$\lim_{|x|\infty}\int d^3\mathbf{k}\, e^{ik\cdot x} f_1(\mathbf{k})\phi(x) = 0,$$

due to the Gaussian $f_1(\mathbf{k})$. As such, the fact that we are working in infinite volume causes no problems.

and so

$$\langle f|i\rangle = (-i)^4 \int d^3\mathbf{k}_1...d^3\mathbf{k}_4 \int d^4x_1...d^4x_4\, e^{-i(k_1\cdot x_1+k_2\cdot x_2-k_3\cdot x_3-k_4\cdot x_4)}$$
$$\times \left(\prod_{j=1}^{4} f_i(\mathbf{k}_i)(\Box_j - m^2)\right) \langle\Omega|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle. \tag{C.4}$$

If we now remove the Gaussian envelope by taking $\sigma \to 0$ so that the f_i tend to delta functions which kill the $\mathbf{k}_i$ integrals to give

The LSZ Formula

$$\langle k_3, k_4|i\mathcal{T}|k_1, k_2\rangle = (-i)^4 \int d^4x_1...d^4x_4\, e^{-i(k_1\cdot x_1+k_2\cdot x_2-k_3\cdot x_3-k_4\cdot x_4)}$$
$$\times \left(\prod_{j=1}^{4}(\Box_j - m^2)\right) \langle\Omega|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle. \tag{C.5}$$

One of the remarkable things about this formula is that, since all of the momenta are on-shell, it has what appears to be a product of zeros. This formula only give s a non-zero result if the Green's function has a simple pole when any of the particles goes on-shell. We can indeed see that each of these Green's functions does have a simple pole if we write them in what is called the *Lehman representation.*

This expression is the Fourier transform of the momentum space Correlation function $\widetilde{G}_4(k_1, k_2, k_3, k_4)$ so we can write this as

The Momentum Space LSZ Formula

$$\langle k_3, k_4|i\mathcal{T}|k_1, k_2\rangle$$
$$= (-i)^4 \prod_{i=1,2}^{4} \sqrt{\frac{1}{2E_i}(2\pi)^3(k_i - m^2)} \prod_{j=3,4}^{4} \sqrt{\frac{1}{2E_j}(2\pi)^3(k_j - m^2)\widetilde{G}_4(k_1, k_2, k_3, k_4)}.$$

C.3 Scattering Examples

We now see how this works in our favorite example of ϕ^4 theory. In particular, we shall see that

$$\langle k_3, k_4|i\mathcal{T}|k_1, k_2\rangle = i(2\pi)^4\mathcal{M}_{if}\delta^4(k_1 + k_2 - k_3 - k_4) \tag{C.6}$$

for $\mathcal{M}_{if}$ that we shall determine in terms of correlation functions.

Order λ^0

This is the free case. The Time-ordered correlation function is

$$G_4^{(0)}(x_1, x_2, x_3, x_4) = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4)$$
$$+D_F(x_1 - x_4)D_F(x_2 - x_3) \tag{C.7}$$

where the Feynman propagator $D_F(x-y)$ satisfies

$$(\Box_x + m^2)D_F(x-y) = -i\delta^4(x-y).$$

Substituting this into (C.5) gives zero, as there are onlt two propagators in each term, which is not enough to soak up the four zeros from the $\Box + m^2 \sim 0$. This is as expected, as the free theory has $\mathcal{T} = 0$.

Order λ^1

The first interaction effect comes in at leading order in λ. The correlation function is

$$G_4^{(1)}(x_1, x_2, x_3, x_4) = (-i\lambda)\int d^4y D_F(x_1-y)D_F(x_2-y)D_F(x_3-y)D_F(x_4-y)$$

corresponding to the diagram

[DIAGRAM]

We then have

$$\begin{aligned}
\langle k_3, k_4 | i\mathcal{T} | k_1, k_2\rangle &= i^4(-i\lambda)\int d^4y \int d^4x_1 ... d^4x_4\, e^{-i(k_1\cdot x_1 + k_2\cdot x_2 - k_3\cdot x_3 - k_4\cdot x_4)} \\
&\quad \times \left(\prod_{j=1}^{4}(\Box_j - m^2)D_F(x_j - y)\right) \\
&= -i\lambda \int d^4y \int d^4x_1 ... d^4x_4\, e^{-i(k_1\cdot x_1 + k_2\cdot x_2 - k_3\cdot x_3 - k_4\cdot x_4)} \prod_{i=1}^{4}\delta^4(x_i - y) \\
&= -i\lambda \int d^4y\, e^{-iy\cdot(k_1+k_2-k_3-k_4)} \\
&= -i\lambda(2\pi)^4\delta^4(k_1+k_2-k_3-k_4), \qquad \text{(C.8)}
\end{aligned}$$

which has the expected form with $\mathcal{M} = -i\lambda$.

Order λ^2

To next order we have three connected contributions coming from the diagrams

Let us focus on the contribution coming from the first diagram. The correlation function associated with this diagram is

$$\begin{aligned}
G_4^{(2)}(x_1, x_2, x_3, x_4) &= \frac{(-i\lambda)^2}{2}\int d^4y d^4z\, DF(x_1-y)D_F(x_2-y)D_F(y-z)D_F(y-z) \\
&\quad \times D_F(x_3-z)D_F(x_4-z), \qquad \text{(C.9)}
\end{aligned}$$

so that the LSZ formula (C.5) gives

$$\begin{aligned}
&(-i)^4\frac{(-i\lambda)^2}{2}\int d^4yd^4zd^4x_1...d^4x_4\, e^{-i(k_1\cdot x_1 + k_2\cdot x_2 - k_3\cdot x_3 - k_4\cdot x_4)}\left(\prod_{j=1}^{4}(\Box_j - m^2)\right) \\
&\times DF(x_1-y)D_F(x_2-y)D_F(y-z)D_F(y-z)D_F(x_3-z)D_F(x_4-z) \\
&= (-i)^4\frac{(-i\lambda)^2}{2}\int d^4yd^4zd^4x_1...d^4x_4\, e^{-i(k_1\cdot x_1 + k_2\cdot x_2 - k_3\cdot x_3 - k_4\cdot x_4)}\Big(D_F(z-y)\Big)^2 \\
&\times \prod_{i=1,2}\delta^4(x_i - y)\prod_{j=3,4}\delta^4(x_j - z) \qquad \text{(C.10)}
\end{aligned}$$

Using

$$D_F(y-z) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip(y-z)} \frac{i}{p^2 - m^2 + i\epsilon},$$

we have

$$\frac{(-i\lambda)^2}{2} \int d^4y d^4z\, e^{-i(k_1+k_2)\cdot y} e^{i(k_3+k_4)\cdot z} \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \prod_{j=1,2} e^{ip_j(y-z)} \frac{i}{p_j^2 - m^2 + i\epsilon}$$

Performing the y and z integrals gives

$$\frac{(-i\lambda)^2}{2} \int d^4p_1 d^4p_2 \delta^4(k_1 + k_2 + p_1 + p_2)\delta^4(k_3 + k_4 + p_1 + p_2) \prod_{j=1,2} \frac{i}{p_j^2 - m^2 + i\epsilon}$$

Integrating over p_2 and renaming $p_1 := p$, we have

$$\frac{(-i\lambda)^2}{2}(2\pi)^4\delta^4(k_1 + k_2 - k_3 - k_4) \int d^4p \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{(p + k_3 + k_4)^2 - m^2 + i\epsilon}.$$

we see then that the contribution to $\mathcal{M}_{if}$ is

$$\frac{(-i\lambda)^2}{2} \int d^4p \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{(p + k_3 + k_4)^2 - m^2 + i\epsilon}.$$

From this we can read off the momentum space Feynman rules for the scattering amplitudes

D The Beta Function from Callan-Symanzik

In the lectures we discussed how to calculate the Beta function of QED using the MS scheme. In this Appendix we show how to arrive at this result using renormalization conditions and the Callan-Symanzik equation. The renormalization conditions we choose are fixed at mass scale M, so that $p^2 = M^2$ (rather than on-shell renormalization)

$$\begin{aligned}
\Sigma(\not{p}) &= 0 && \text{Fixes the electron mass} \\
\frac{d}{d\not{p}}\Sigma(\not{p})\Big|_{p^2=M^2} &= 0 && \text{Fixes the electron wavefunction normalization} \\
\Pi^{\mu\nu}(k) &= 0 && \text{Fixes the photon wavefunction normalization} \\
-ie\Gamma^\mu(p = p') &= -ie\gamma^\mu && \text{Fixes the electron charge}
\end{aligned} \tag{D.1}$$

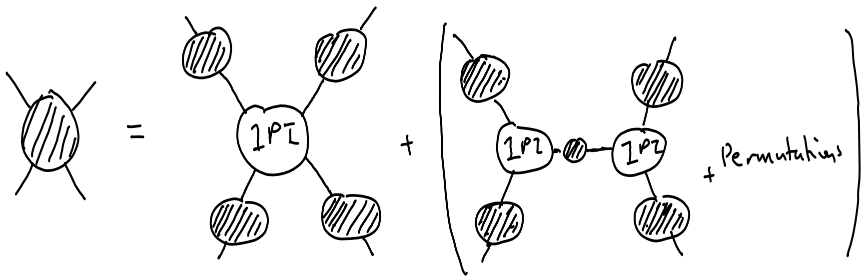
Note that the gauge symmetry ensures the photon remains massless, despite quantum effects. In order to impose these conditions on the counter-terms, we need to regularize and evaluate these diagrams to the desired order.

D.1 Callan-Symanzik Revisited

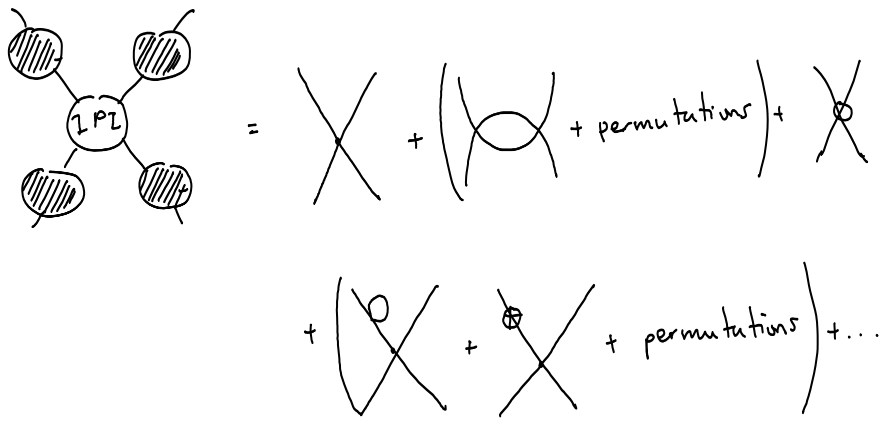
We saw that, in order to compute the beta-function in ϕ^4 theory, we saw that many of the details of the four-point correlation function did not play a significant role. The details of the beta-function seemed to arise simply out of the counter-terms alone. In general, the renormalized connected correlation function at one loop, takes the form

$$\begin{aligned} G_n = &\ (\text{Tree-level diagram}) + (\text{1PI loop diagrams}) + (\text{Vertex Counterterm}) \\ &+ (\text{External leg corrections}) \end{aligned} \tag{D.2}$$

This may be made exact using the decomposition of a connected correlation function into 1PI components



For ϕ^4 theory, the three-point correlation function vanishes and this expression may be written as the diagrammatic expansion, whose leading terms are



At one-loop, there is no wavefunction renormalization in ϕ^4 theory and so the diagrams in the second line did not play a role in the calculation of the beta-function in ϕ^4 theory and the Callan-Symanzik equation is

$$\left(M\frac{\partial}{\partial M} + \beta_\lambda \frac{\partial}{\partial \lambda}\right) G_n(x_1, ..., x_n) = 0.$$

More generally, the wavefunction renormalization plays an important role and there is a contribution to wavefunction renormalization from each external leg

$$\left(M\frac{\partial}{\partial M} + \beta_\lambda \frac{\partial}{\partial \lambda} + \sum_{i=1}^{n} \gamma_i\right) G_n(x_1, ..., x_n) = 0,$$

where $i = 1, 2, .., n$ labels the external vertices. Recalling that

$$\gamma = \frac{1}{2} M \frac{\partial}{\partial M} \delta_Z,$$

we can write the Callan-Symanzik equation as

$$\left(M \frac{\partial}{\partial M} + \beta_\lambda \frac{\partial}{\partial \lambda} + \frac{1}{2} \sum_{i=1}^{n} M \frac{\partial}{\partial M} \delta_{Z_i} \right) G_n(x_1, ..., x_n) = 0.$$

If the coupling of the theory is taken to be g and let the coupling be fixed by a renormalization condition on the n-point function. Then we may write

$$G_n = \left[-ig - (\text{1PI 1-loop diagram}) - i\delta g - ig \sum_i (\text{1PI loop diagram} - \delta_{Z_i}) \right] \prod_{i=1}^{n} \frac{1}{p_i^2} + ...$$

The only M-dependence comes form the counter-terms and so

$$\begin{aligned} \left(M \frac{\partial}{\partial M} + \frac{1}{2} \sum_{i=1}^{n} M \frac{\partial}{\partial M} \delta_{Z_i} \right) G_n(x_1, ..., x_n) &= M \frac{\partial}{\partial M} \left(1 + \frac{1}{2} \sum_{i=1}^{n} \delta_{Z_i} \right) G_n(x_1, ..., x_n) \\ &= -iM \frac{\partial}{\partial M} \left(\delta g - \frac{1}{2} g \sum \delta_{Z_i} \right) \qquad \text{(D.3)} \end{aligned}$$

We also have that

$$\frac{\partial}{\partial \lambda} G_n(x_1, ..., x_n) = -i + \mathcal{O}(g),$$

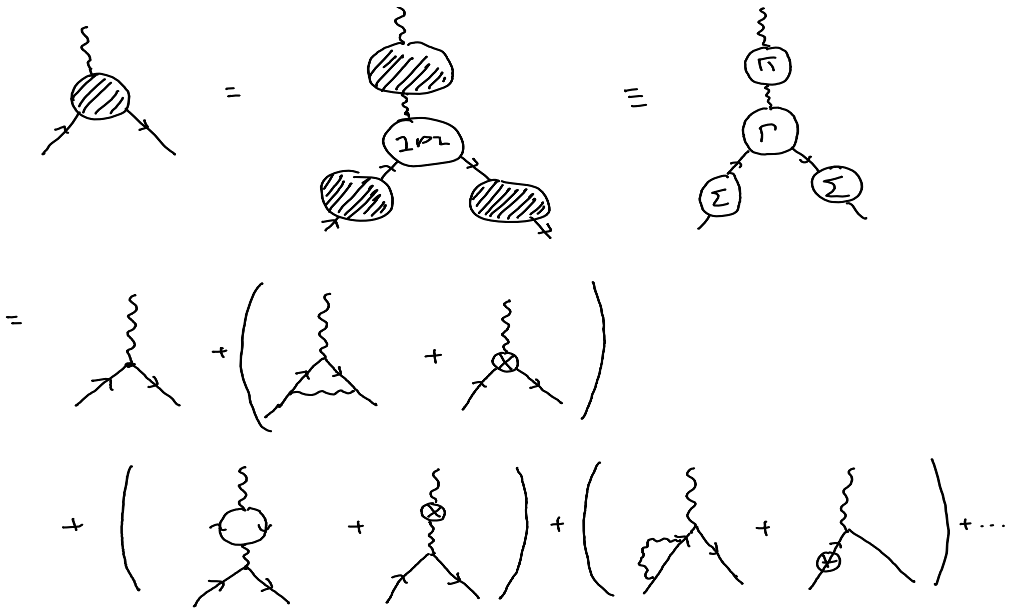
The Callan-Symanzik equation then gives

$$M \frac{\partial}{\partial M} \left(\delta_g - g \sum_i \delta_{Z_i} \right) + \beta_g + g \sum_i \frac{1}{2} M \frac{\partial}{\partial M} \delta_{Z_i} = 0$$

and so to leading order in g

$$\boxed{\beta(g) = M \frac{\partial}{\partial M} \left(-\delta g + \frac{1}{2} g \sum_{i=1}^{n} \delta_{Z_i} \right).}$$

The analogous diagrams in QED apply to the three-point function between two fermions and a photon



Considering the three-point vertex (n=3), with two fermion and one photon lines, the beta function for the electron charge ($g = e$) is given by

$$\begin{aligned}\beta(e) &= M\frac{\partial}{\partial M}\Big(-e\delta_1 + \frac{1}{2}e(2\times\delta_2 + \delta_3)\Big)\\ &= M\frac{\partial}{\partial M}\Big(-e\delta_1 + e\delta_2 + \frac{1}{2}e\delta_3\Big)\end{aligned} \tag{D.4}$$

Our goal will be to calculate this beta-function and to understand the running (i.e. scale-dependence) of the coupling. To this end, in the next section we shall determine the one-loop counter-terms in QED.

D.1.1 δ_2 Counter-term

Recalling that

$$-i\Sigma_2(p) = -i\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-d/2)}{[(1-x)m^2 - x(1-x)p^2]^{2-d/2}}[(4-\epsilon)m - (2-\epsilon)x\not{p}]$$

The first renormalization condition in (D.1) requires

$$\not{p}\delta_2 - \delta m = \Sigma_2(M)$$

at $p^2 = M^2$. To pull out the δ_2 electron wavefunction normalization term we evaluate this for massless fermions (i.e. set $m = 0$), so that

$$\begin{aligned}\not{p}\delta_2 &= -\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-d/2)}{[x(1-x)M^2]^{2-d/2}}[(2-\epsilon)x\not{p}]\\ &= -\not{p}\frac{e^2}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}}(2-\epsilon)\int_0^1 dx\,\frac{x}{[x(1-x)]^{2-d/2}}\end{aligned} \tag{D.5}$$

and so, taking $d = 4 - \epsilon$

$$\boxed{\delta_2 = -\frac{e^2}{16\pi^2}\int_0^1 dx\, x(2-\epsilon)\Gamma(\epsilon/2)\left(\frac{4\pi}{(1-x)m^2 - x(1-x)M^2}\right)^{\frac{\epsilon}{2}}}$$

We would like to isolate the M-dependence in this expression. For small ϵ, we have

$$\delta_2 = \frac{e^2}{8\pi^2}\int_0^1 dx\, x \ln[(1-x)m^2 - x(1-x)M^2] + ...,$$

where the ellipsis denote terms that do not depend on M and/or vanish in the $\epsilon \to 0$ limit. And so

$$\gamma_2 := \frac{1}{2}M\frac{\partial}{\partial M}\delta_2 = \frac{e^2}{16\pi^2}.$$

With very little further work, we could find the δ_m counter-term but we will not need it for the calculation of β_e so it is left as an excercise.

D.1.2 δ_3 Counter-term

If we set $p^2 = M^2$ in the photon self-energy, then

$$\Pi(M^2) = -\frac{8e^2}{(4\pi)^{d/2}}\int_0^1 dx\, x(1-x)\frac{\Gamma(2-\frac{d}{2})}{[m^2 - x(1-x)M^2]^{2-\frac{d}{2}}}$$

Using the renormalization condition $\Pi(M^2) = 0$, we see that $\delta_3 = \Pi(M^2)$, i.e.

$$\delta_3 = -\frac{e^2\Gamma(\epsilon/2)}{2\pi^2}\int_0^1 dx\, x(1-x)\left(\frac{4\pi}{m^2 - x(1-x)M^2}\right)^{\epsilon/2}$$

Expanding in small ϵ to extract the M dependence in this limit, as above, we find

$$\delta_3 = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln[m^2 - x(1-x)M^2] + ...$$

We then calculate

$$\gamma_3 := M\frac{\partial}{\partial M}\delta_3 = -\frac{e^2}{\pi^2}\int_0^1 dx\, x(1-x)\frac{x(1-x)M^2}{m^2 - x(1-x)M^2} + ...$$

If we choose M such that $M >> m$, which we are free to do, then this simplifies to

$$\gamma_3 = \frac{e^2}{6\pi^2},$$

to leading order.

D.2 Calculating the Beta Function

Using the Callan-Symanzik equation above, which gives the beta-function of QED in terms of the counter-terms as

$$\beta_e = M\frac{\partial}{\partial M}\Big(e\delta_1 - e\delta_2 - \frac{1}{2}e\delta_3\Big)$$

We saw that $\delta_1 = \delta_2$ and so, we only need the logarithmic derivative of δ_3

$$\beta_e = \frac{1}{2}M\frac{\partial}{\partial M}e\delta_3$$

which gives, at one-loop

$$\boxed{\beta_e = \frac{e^3}{12\pi^2}}$$

E More on the Beta-Function

E.1 β-Functions

The Lagrangian $\mathcal{L}_0$ may be written as

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \qquad \Big\} \text{ Classical theory} \\
&\quad + \frac{1}{2}\delta_{Z_\phi}(\partial\phi)^2 + \frac{1}{2}\delta_{m^2}\phi^2 + \frac{\delta_\lambda}{4!}\phi^4 \qquad \Big\} \text{ Counter-terms} \\
&= \frac{1}{2}(1+\delta_{Z_\phi})(\partial\phi)^2 + \frac{1}{2}(m^2+\delta_{m^2})\phi^2 + \frac{1}{4!}(\lambda+\delta_\lambda)\phi^4
\end{aligned} \tag{E.1}$$

Schematically, the quantity $\lambda + \delta\lambda$ that appeared in our original Lagrangian $\mathcal{L}_0$ is independent of a choice of renormalization scale μ. It must therefore be that λ depends on μ in such a way that

$$\mu\frac{d\lambda_0}{d\mu} = \mu\frac{d}{d\mu}(\lambda+\delta\lambda) = 0.$$

More precisely, the terms in the quantum effective action $\widetilde{\Gamma}_n(p)$ should not depend on the choice of μ. In the next section we will explore the consequences of this fact.

E.2 The Beta Function from Dimensional Regularization

Recalling (**??**), we can write the 4 point 1PI vertex as

$$\widetilde{\Gamma}_4(s,t,u) = \lambda + \frac{\lambda^2}{32\pi^2}\int_0^1 dx\, \ln(\Delta_s\Delta_t\Delta_u) - \frac{3\lambda^2}{32\pi^2}\int_0^1 dx\, \ln(\Delta_{\mu^2}) + ...$$

where we have chosen to fix the counter-term such that

$$\widetilde{\Gamma}_4(\mu^2,\mu^2,\mu^2) = \lambda.$$

The mass and wavefunction counterterms do not depend on the energy scale we measure them at at one-loop so we focus on the coupling counter-term, which does exhibit interesting physics. We can think of this as a renormalized coupling

$$\lambda_0 = \lambda + \delta_\lambda,$$

that appears in our action that defines the theory. We see that the value the effective coupling takes naively depends on the energy scale we observe it at and the renormalization scale μ. There are interesting physical reasons why we expect the couplings to run with energy as we shall discuss in the next section; however, we do not want our definition of the theory to depend on an arbitrary choice of μ. Thus, we impose the condition that the effective coupling does not depend on μ. We shall see that this condition (independence from μ) determines how the coupling runs with energy. We shall require that

$$\mu\frac{d}{d\mu}\widetilde{\Gamma}_4(s,t,u) = 0$$

As such

$$
\begin{aligned}
0 = \mu\frac{d}{d\mu}\widetilde{\Gamma}_4(s,t,u) &= \mu\frac{d\lambda}{d\mu} + \frac{2\lambda}{32\pi^2}\int_0^1 dx \ln\left(\frac{\Delta_s\Delta_t\Delta_u}{\Delta_{\mu^2}^3}\right) - \frac{3\lambda^2}{32\pi^2}\int_0^1 dx\, \frac{\mu}{\Delta_{\mu^2}}\left(\frac{d}{d\mu}\Delta_{\mu^2}\right) + ... \\
&= \beta_\lambda + \beta_\lambda\frac{\lambda}{16\pi^2}\int_0^1 dx \ln\left(\frac{\Delta_s\Delta_t\Delta_u}{\Delta_{\mu^2}^3}\right) + \frac{3\lambda^2}{32\pi^2}\int_0^1 dx\, \frac{2\mu^2 x(1-x)}{\Delta_{\mu^2}} + ... \\
&= \beta_\lambda\left(1 + \frac{\lambda}{16\pi^2}\int_0^1 dx \ln\left(\frac{\Delta_s\Delta_t\Delta_u}{\Delta_{\mu^2}^3}\right)\right) + \frac{3\lambda^2}{32\pi^2}\int_0^1 dx\, \frac{2\mu^2 x(1-x)}{\Delta_{\mu^2}} + ...
\end{aligned}
\tag{E.2}
$$

where we have used the definition of the β function

$$
\beta_\lambda := \mu\frac{\partial\lambda}{\partial\mu}
$$

so that we have

$$
\begin{aligned}
\beta_\lambda &= -\frac{3\lambda^2}{16\pi^2}\int_0^1 dx\, \frac{\mu^2 x(1-x)}{\Delta_{\mu^2}}\left(1 + \frac{\lambda}{16\pi^2}\int_0^1 dx \ln\left(\frac{\Delta_s\Delta_t\Delta_u}{\Delta_{\mu^2}^3}\right)\right)^{-1} + ... \\
&= -\frac{3\lambda^2}{16\pi^2}\int_0^1 dx\, \frac{\mu^2 x(1-x)}{\Delta_{\mu^2}} + ...
\end{aligned}
\tag{E.3}
$$

where we have expanded out the right hand side to order λ^2. Assuming that $\mu >> m$, we can write $\Delta_{\mu^2} = m^2 - x(1-x)\mu^2 \approx -x(1-x)\mu^2$ and so we have our final expression for the 1-loop Beta function

$$
\boxed{\beta_\lambda = \frac{3\lambda^2}{16\pi^2} + ...}
$$

This is a differential equation that can be solved to relate the coupling λ at different values of μ

$$
\boxed{\lambda(\mu') = \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2}\lambda(\mu)\ln(\mu'/\mu)}.}
$$

It is one of the remarkable successes of quantum field theory that running couplings such as these have predicted and then accurately observed in realistic quantum field theories such as QED.

E.3 The Beta-function from Momentum Cut-off Regularization

We end this section by showing how the beta-function is also recovered in momentum cut-off regularization. The 4-point 1PI amplitude is given by

$$
i\mathcal{M} = -i\lambda + (-i\lambda)^2[iV(s) + iV(t) + iV(u)] - i\delta_\lambda + ...
$$

where

$$V(p^2) = \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}\frac{1}{(k+p)^2-m^2}.$$

We have calculated this before in dimensional regularization, we shall now do so again with a momentum cut-off regularization. Using the same manipulations as in dimensional regularization, we can bring this integral to the form

$$V(p^2) = -\frac{\pi^2}{2(2\pi)^4}\int_0^1 dx \int_0^{\Lambda^2} d(\ell_E^2)\frac{\ell_E^2}{(\ell_E^2+\Delta)^2}$$

where we have used the formula for the area of a sphere in $\mathbb{R}^4$ and have introduced a momentum cut-off Λ. $\Delta = m^2 - x(1-x)p^2$ as usual. The integral can be done simply by the substitution $y = \ell_E^2 + \Delta$, to give

$$V(p^2) = -\frac{1}{32\pi^2}\int_0^1 dx \left[\ln\left(1+\frac{\Lambda^2}{\Delta}\right) - \frac{\Lambda^2}{\Lambda^2+\Delta}\right].$$

As previously, it is useful to write this as

$$V(p^2) = -\frac{1}{32\pi^2}\int_0^1 dx \left[\ln\left(\frac{\Lambda^2}{\Delta}\right) + \ln\left(1+\frac{\Delta}{\Lambda^2}\right) - \frac{\Lambda^2}{\Lambda^2+\Delta}\right].$$

To decide how to fix the counter-term, consider

$$\lim_{\Lambda\to\infty} V(p^2) = -\frac{1}{32\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{\Delta}\right) - 1\right]$$

so we choose, at the renormalization scale $s = t = u = \mu^2$

$$\delta_\lambda = \frac{3\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{\Delta_{\mu^2}}\right) - 1\right], \qquad \Delta_{\mu^2} = m^2 - x(1-x)\mu^2.$$

The amplitude is then

$$\begin{aligned} i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \Bigg[\ln\left(\frac{\Delta_s\Delta_t\Delta_u}{\Delta_{\mu^2}^3}\right) - \ln\left[\left(\frac{\Delta_s}{\Lambda^2}+1\right)\left(\frac{\Delta_t}{\Lambda^2}+1\right)\left(\frac{\Delta_u}{\Lambda^2}+1\right)\right] \\ +3\ln\left(\frac{\Delta_{\mu^2}}{\Lambda^2}+1\right) - \left(\frac{\Delta_s}{\Lambda^2+\Delta_s}+\frac{\Delta_t}{\Lambda^2+\Delta_t}+\frac{\Delta_u}{\Lambda^2+\Delta_u}-\frac{3\Delta_{\mu^2}}{\Lambda^2+\Delta_{\mu^2}}\right)\Bigg] + ... \end{aligned} \quad \text{(E.4)}$$

We see that this is finite as the cut-off is removed and gives λ when $\Lambda \to \infty$ and $p^2 = \mu^2$. Sending Λ to infinity gives

$$i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \ln\left(\frac{\Delta_s\Delta_t\Delta_u}{\Delta_{\mu^2}^3}\right) + ..., \quad \text{(E.5)}$$

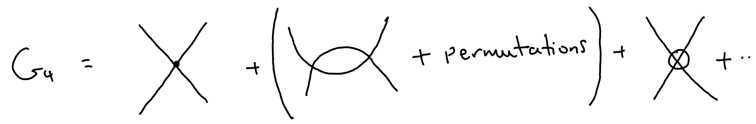
which is exactly the result we found in dimensional regularization. Repeating the same steps as before we reproduce the beta-function

$$\beta_\lambda = \frac{3\lambda^2}{16\pi^2} + ...$$

as found previously.

E.4 The beta-function from Callan-Symanzik

As previously, we choose to define the coupling λ in terms of the four-point correlation function. The external legs play no role so we can consider the amputated connected correlation function



$$= -i\lambda + (-i\lambda)^2 \Big[iV(s) + iV(t) + iV(u)\Big] - i\delta_\lambda + ...$$

where $V(s)$ is given by

$$V(s) = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2}\frac{i}{(k+p)^2},$$

where the total momentum flowing into the diagram is $p^2 = s$. The renormalization condition requires $G_4 = -i\lambda$ at $s = t = u = \mu^2$. Therefore;

$$i\delta_\lambda = (-i\lambda)^2\, 3iV(\mu^2).$$

We have evaluated this integral and others like it many time before and we shall just give the answer, calculated using dimensional regularization. We find

$$\delta_\lambda = \frac{3\lambda^2}{16\pi^2}\left(\frac{1}{\epsilon} - \frac{1}{2}\ln(\mu^2) + ...\right)$$

It is this counter term that gives G_4 its dependence on μ. As found previously, there is no wavefunction renormalization for ϕ^4 theory at one loop, so the Callan-Symanzik equation becomes

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda}\right) G_4(x_1, ..., x_4) = 0.$$

Using the above result

$$\mu\frac{\partial G_4}{\partial\mu} = \frac{3i\lambda^2}{16\pi^2} + ...$$

and

$$\frac{\partial G_4}{\partial\lambda} = -i - 2i\lambda\mathcal{F}(s,t,u;\mu^2) + ...,$$

where we have written

$$\mathcal{F}(s,t,u;M^2) := iV(s) + iV(t) + iV(u) - 3iV(-M^2).$$

We therefore have

$$\frac{3\lambda^2}{16\pi^2} = \beta(\lambda)\Big(1 + 2\lambda\mathcal{F}(s,t,u;M^2)\Big) + ...$$

and so

$$\beta_\lambda = \frac{3\lambda^2}{16\pi^2}\Big(1 - 2\lambda\mathcal{F}(s,t,u;\mu^2) + ...\Big) + ...$$

so, to order λ^2, we find

$$\boxed{\beta_\lambda = \frac{3\lambda^2}{16\pi^2}.}$$

Now that we have the beta-function, we can calculate how the coupling depends on the renormalization scale in ϕ^4 theory. The equation

$$\beta_\lambda = \mu \frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}$$

is separable. Integrating between renormalization scales μ_1 and μ_2 gives

$$\boxed{\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 - \frac{3}{16\pi^2}\lambda(\mu_1)\ln(\mu_2/\mu_1)}}$$

and so we see that the positivity of the beta function means that the coupling logarithmically increases as we go to higher energies. Similarly, if we measure λ at an energy scale M, the effective coupling we experience (and use in perturbation theory) at a momentum p is

$$\lambda(p) = \frac{\lambda(M)}{1 - \frac{3\lambda(M)}{16\pi^2}\ln(p/M)}.$$

The running of the coupling may seem a little strange at first. Later we will try to gain some intuition of the physics behind it. Before we do that, let us look at how these results apply to a more physically relevant theory.

E.5 Asymptotic Freedom from Callan-Symanzik

An alternative, but clearly equivalent way to calculate the beta-function is by using the Callan-Symanzik equation.

$$\beta(g) = M \frac{\partial}{\partial M}\left(-\delta g + \frac{1}{2}\sum_{i=1}^{n}\delta Z_i\right),$$

where, for a given n-point function δg is the coupling constant counter-term and δZ_i is the wavefunction counter-term for the external legs of the diagram. So, for the tri-valent vertex with two fermions and one gluon[77] we have (as with QED)

$$\beta(g) = gM \frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right),$$

We then have that

$$\begin{aligned} -\delta_1 + \delta_2 + \frac{1}{2}\delta_3 &= \frac{g^2}{(4\pi)^2}\frac{\Gamma(2-d/2)}{M^\epsilon}\left(C_2(r) + C_2(\text{Adj}) - C_2(r) + \frac{5}{6}C_2(\text{Adj}) - \frac{2}{3}n_f C(r)\right) \\ &= \frac{g^2}{(4\pi)^2}\frac{\Gamma(2-d/2)}{M^\epsilon}\left(\frac{11}{6}C_2(\text{Adj}) - \frac{2}{3}n_f C(r)\right). \qquad \text{(E.6)} \end{aligned}$$

[77] So that $\sum_i \delta Z_i = g\delta_2 + g\delta_2 + g\delta_3$.

We can then use the Laurent expansion of $\Gamma(2 - d/2)$ near $d = 4$ to write

$$
\begin{aligned}
-\delta_1 + \delta_2 + \frac{1}{2}\delta_3 &= \frac{1}{2}\frac{g^2}{(4\pi)^2}\left(\frac{11}{3}C_2(\text{Adj}) - \frac{4}{3}n_f C(r)\right)\left(\frac{2}{\epsilon} + \gamma + ...\right)(1 - \epsilon\ln(M) + ...) \\
&= \frac{1}{2}\frac{g^2}{(4\pi)^2}\left(\frac{11}{3}C_2(\text{Adj}) - \frac{4}{3}n_f C(r)\right)\left(\frac{2}{\epsilon} + \gamma - 2\ln(M) + ...\right). \qquad \text{(E.7)}
\end{aligned}
$$

And so

$$\beta(g) = -\frac{g^2}{(4\pi)^2}\left(\frac{11}{3}C_2(\text{Adj}) - \frac{4}{3}n_f C(r)\right)$$

as found above.

F Casimirs and Indexes

A Casimir is an element of the centre of a Lie algebra; an object, constructed from the generators, that commutes with all other generators of the group. Given the Lie algebra

$$[T_a, T_b] = -f_{ab}{}^c T_c,$$

a simple way to construct an invariant is by taking a product of generators ad then taking a trace over free indices. We will only need Casimirs that are quadratic in the generators and, since $(T_a)^i{}_j$ has two sets of indices, it is unsurprising that there are two obvious invariants to construct. The first example is the quadratic Casimir

$$T^2 := \sum_{a=1}^{\dim(G)} T^a T^a,$$

for which

$$[T_a, T^2] = [T_a, T_b]T_b + T_b[T_a, T_b] = i f_{ab}{}^c\{T_c, T_b\} = 0$$

due to the antisymmetry (symmetry) of $f_{ab}{}^c$ ($\{T_c, T_b\}$). Since T^2 commutes with all operators it must be proportional to the identity

$$(T^2)^i{}_j = C_2(r)\delta^i_j,$$

where $i, j = 1, 2, ...N$ for and $N \times N$ matrix representation and $C_2(r)$ is a constant, where r denotes the representation of the T_a. For the adjoint representation we identify the indices a and i and $(T_a)^i{}_j \to (T_a)^b{}_c = f_{ac}{}^b$ and so

$$f_{acd} f_{bcd} = C_2(\text{Adj})\delta_{ab}. \qquad \text{(F.1)}$$

A second invariant is given by

$$\text{Tr}(T_a T_b) = (T_a)^i{}_j (T_b)^j{}_i.$$

For $G = SU(N)$, the generators are traceless Hermitian matrices, and we can choose a basis such that $\text{Tr}(T_a T_b) \propto \delta_{ab}$ and so we can write

$$\text{Tr}(T_a T_b) = C(r)\delta_{ab}$$

A useful result is that

$$\mathrm{Tr}([T_a, T_b]T_c) = if_{ab}{}^{d}\mathrm{Tr}(T_d T_c) = if_{abc}C(r).$$

A relationship between the two constants $C_2(r)$ and $C(r)$ may be found as follows. We have that

$$(T_a)^i{}_j(T_b)^j{}_i = C(r)\delta_{ab},$$

summing over the generator indices,

$$\delta^{ab}(T_a)^i{}_j(T_b)^j{}_i = \mathrm{Tr}(T^2) = C(r)\mathrm{Tr}(\delta_{ab}) = C(r)\mathrm{dim}(G).$$

But,

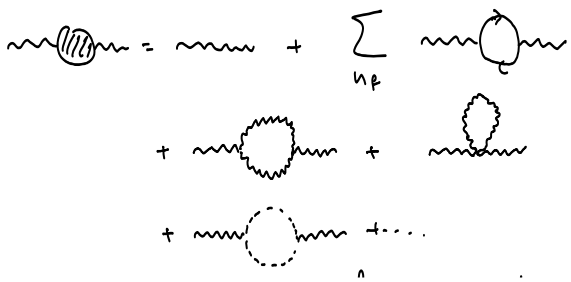
$$\mathrm{Tr}(T^2) = C_2(r)\mathrm{Tr}(\delta^i_j) = C_2(r)\mathrm{dim}(r),$$

where $\mathrm{dim}(r)$ is the dimension of the representation.

G QCD Loop Calculations

G.1 Gluon Self-Energy

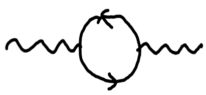
In terms of Feynman diagrams, the gluon two-point function may be written as



where the sum is over the fermion species (n_f) that couple to the Yang-Mills field. We consider these terms one-by-one.

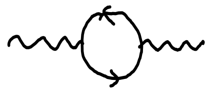
G.1.1 Fermion Loop

This is a similar calculation to the analogous diagram in QED, but with the insertion of generators in the fundamental representation at each vertex. For a single fermion species propagating in the loop



$$\begin{aligned} &:= i\Pi_2^{\mu\nu}(q) \\ &= -(ig)^2\mathrm{Tr}(T_a T_b)\int \frac{d^dk}{(2\pi)^d}\mathrm{tr}\left[\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma^\nu \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\right] \end{aligned} \quad \text{(G.1)}$$

where the factor of -1 at the beginning comes from the fact there is a fermion loop, the Tr is a trace over group indices and tr is a trace of gamma matrix (spinor) indices. We note that $\mathrm{Tr}(T_a T_b) = C(r)\delta_{ab}$, where $C(r)$ denotes the index of the representation r. In the calculations that follow, we shall neglect fermion masses as such terms do not affect the UV divergences we are studying and so we need to evaluate



$$\approx -g^2 C(r)\delta_{ab} \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma^\mu(\not{k})\gamma^\nu(\not{k}+\not{q})\right]}{k^2(k+q)^2}$$

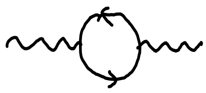
where we choose to work in $d = 4-\epsilon$ dimensions. We can evaluate the trace identities in $4-\epsilon$ dimensions

$$\mathrm{tr}\left[\gamma^\mu(\not{k})\gamma^\nu(\not{k}+\not{q})\right] = (4-\epsilon)\Big[k^\mu(k^\nu+q^\nu)^\nu_k(k^\mu+q^\mu) - g^{\mu\nu}k\cdot(k+q)\Big].$$

The order ϵ term in this trace will not contribute when we send ϵ to zero at the end of the calculation, so we will drop it from here. WE deal with the denominator using Feynman parameterisation

$$\frac{1}{k^2(k+q)^2} = \int_0^1 dx\, \frac{1}{[(1-x)k^2 + x(k+q)^2]^2} = \int_0^1 dx\, \frac{1}{[\ell^2 + x(1-x)q^2]^2},$$

where $\ell_\mu = k_\mu + xq_\mu$, so that

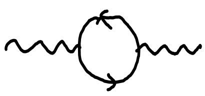


$$= -4g^2\delta_{ab} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{2\ell^\mu\ell^\nu - 2x(1-x)q^\mu q^\nu - g^{\mu\nu}[\ell^2 - x(1-x)q^2]}{(\ell^2-\Delta)^2}$$

where $\Delta := -x(1-x)q^2$. Since $g^{\mu\nu}$ is the only isotropic two-tensor it must be that, for any function $f(\ell^2)$,

$$\int d^d\ell\, \ell^\mu\ell^\nu f(\ell^2) = \frac{1}{d} g^{\mu\nu} \int d^d\ell\, \ell^2 f(\ell^2),$$

and so

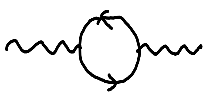


$$= -4g^2\delta_{ab} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(2/d)g^{\mu\nu}\ell^2 - 2x(1-x)q^\mu q^\nu - g^{\mu\nu}[\ell^2 - x(1-x)q^2]}{(\ell^2-\Delta)^2}$$

To evaluate the integral, we go to Euclidean space (note that $\ell^2 = -\ell_E^2$) and define

$$G^{\mu\nu} := -2x(1-x)q^\mu q^\nu + g^{\mu\nu}x(1-x)q^2],$$

so that



$$= -4g^2\delta_{ab} \int_0^1 dx \int \frac{d^d\ell_E}{(2\pi)^d} \left[-\left(\frac{2}{d}-1\right)\frac{g^{\mu\nu}\ell_E^2}{(\ell_E^2+\Delta)^2} + \frac{G^{\mu\nu}}{(\ell_E^2+\Delta)^2}\right]$$

We can then use the standard integrals (we saw how to prove such integrals in ϕ^4 theory)

Some useful integrals

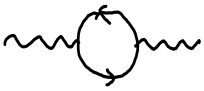
$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2} \tag{G.2}$$

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n-d/2-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2-1} \tag{G.3}$$

Using these integral and the fact that

$$\Gamma(1-d/2) = \frac{\Gamma(2-d/2)}{1-d/2},$$

which follows from the general statement $\Gamma(z+1) = z\Gamma(z)$, we find



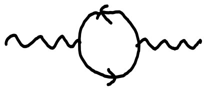
$$= -\frac{4ig^2}{(4\pi)^{d/2}}\delta_{ab}C(r)\Gamma(2-d/2)\int_0^1 dx\,\frac{G^{\mu\nu}-\Delta g^{\mu\nu}}{\Delta^{2-d/2}}$$

It is not hard to show that

$$G^{\mu\nu}-\Delta g^{\mu\nu} = 2x(1-x)(g^{\mu\nu}q^2 - q^\mu q^\nu),$$

and so, assuming that there are n_f species of fermion, which (except for masses, which we neglect) are all the same,

$$\sum_{\text{fermions}}$$



$$= \Big(g^{\mu\nu}q^2 - q^\mu q^\nu\Big)\delta_{ab}C(r)i\Pi(q^2),$$

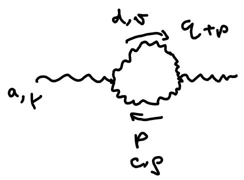
where the sum is taken over the fermions in the loop and

$$\Pi(q^2) = -\frac{8g^2}{(4\pi)^{d/2}}n_f\Gamma(2-d/2)\int_0^1 dx\,\frac{2x(1-x)}{[x(x-1)q^2]^{2-d/2}}$$

It is encouraging to see the factor of $g^{\mu\nu}q^2-q^\mu q^\nu$ in the expression for the gluon propagator. Our understanding of this factor is that it arises from the gauge invariance of the theory and we would not expect this to be broken by perturbative effects. In QED, a contribution f this kind would be all that we could expect at one loop. In Yang-Mills, the gluon self-interactions allow for other contributions.

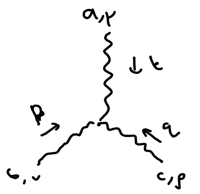
G.1.2 Cubic Interaction Gluon Loop

The one-loop contribution coming from the gluon cubic interactions is given by the single diagram



$$= \frac{1}{2}g^2 f^{acd}f_{cd}{}^{b}\int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2}\frac{-i}{(p+q)^2}N^{\mu\nu}$$

where we have used propagators for gluons in the Feynman gauge and the vertex



$$= gf^{abc}\Big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\Big].$$

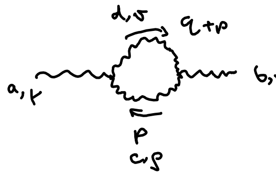
The numerator is

$$N^{\mu\nu} = \Big[g^{\mu\rho}(q-p)^\sigma + g^{\rho\sigma}(2p+q)^\mu + g^{\sigma\mu}(-q-2q)^\rho\Big] \times \Big[\delta^\nu_\rho(p-q)_\sigma + g_{\rho\sigma}(-2p-q)^\nu + \delta^\nu_\sigma(p+2q)_\rho\Big]$$

We use the Feynman propagator to write

$$\frac{1}{p^2(p+q)^2} = \int_0^1 dx\, \frac{1}{[(1-x)p^2 + x(p+q)^2]^2} := \int_0^1 dx\, \frac{1}{\ell^2 - \Delta^2},$$

with $\ell_\mu = p_\mu + xq_\mu$ and $\Delta = -x(1-x)q^2$ as before. Then,

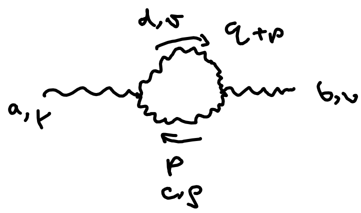


$$= -\frac{g^2}{2} C_2(\mathrm{Adj})\delta^{ab} \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{N^{\mu\nu}}{\ell^2 - \Delta^2}$$

where we have introduced the quadratic Casismir $f^{acd}f_{cd}{}^b = C_2(\mathrm{Adj})\delta^{ab}$. Replacing $\ell^\mu\ell^\nu \to \frac{1}{d}\ell^2 g^{\mu\nu}$ in the integral and dropping terms linear in ℓ^μ, which vanish (by symmetry) in the integral, we find we can write the numerator is equivalent in the integral to

$$N^{\mu\nu} \sim -g^{\mu\nu}\ell^2\, 6\left(1 - \frac{1}{d}\right) - g^{\mu\nu}q^2\Big[(2-x)^2 + (1+x)^2\Big] + q^\mu q^\nu\Big[(2-d)(1-2x)^2 + 2(1+x)(2-x)\Big]$$

We go to Euclidean space to evaluate the integral and using the standard integrals given above, we find the somewhat horrific expression



$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(\mathrm{Adj})\delta_{ab} \int_0^1 dx\, \frac{F^{\mu\nu}(q,x)}{[x(x-1)q^2]^{2-d/2}}$$

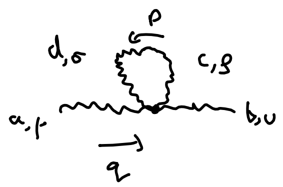
where

$$F^{\mu\nu}(q,x) = \Gamma(1-d/2)g^{\mu\nu}q^2\left[\frac{3}{2}(d-1)x(1-x)\right]$$
$$+\Gamma(2-d/2)g^{\mu\nu}q^2\left[\frac{1}{2}(2-x)^2+\frac{1}{2}(1+x)^2\right]$$
$$-\Gamma(2-d/2)q^\mu q^\nu\left[\left(1-\frac{d}{2}\right)(1-2x)^2+(1-x)(2-x)\right] \qquad \text{(G.4)}$$

This looks awful and there is not the expected prefactor present. We shall see that, combined with the remaining two diagrams, the final expression simplifies dramatically.

G.1.3 Quadratic Interaction Gluon Loop

The one-loop contribution coming from the gluon quadratic interactions is given by the single diagram



$$= \frac{1}{2}(-ig^2)H^{ab\mu\nu\rho\sigma}\int\frac{d^4p}{(2\pi)^4}\frac{-ig_{\rho\sigma}}{p^2}$$

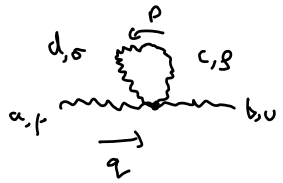
The symmetry factor of 1/2 is included and

$$\begin{aligned}H^{ab\mu\nu\rho\sigma} &= \delta^{cd}\Big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\Big]\\ &= \delta^{cd}\Big[f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\Big]\\ &= 2C_2(\text{Adj})\delta^{ab}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}) \qquad \text{(G.5)}\end{aligned}$$

where in the second line we have used the fact that $\delta^{cd}f^{cde}=0$ in the first term in the brackets and $f^{acd}f^{bcd}=C_2(\text{Adj})\delta^{ab}$ has been used in the last line. Contracting with $g_{\rho\sigma}$ gives

$$g_{\rho\sigma}H^{ab\mu\nu\rho\sigma} = 2C_2(\text{Adj})\delta^{ab}g_{\rho\sigma}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}) = 2C_2(\text{Adj})\delta^{ab}g^{\mu\nu}(d-1),$$

so we have

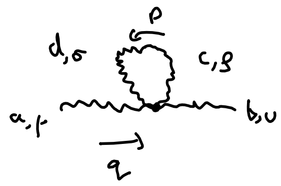


$$= -g^2C_2(\text{Adj})\delta^{ab}g^{\mu\nu}(d-1)\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}$$

In $d=4$ this integral gives zero but we will not discard it. Instead, we shall use it to write zero in a very helpful way. Let us insert

$$1 = \frac{(p+q)^2}{(p+q)^2},$$

and use Feynman parameterization to write the integral in terms of the variable $\ell_\mu = p_\mu + xq_\mu$ as in the other expressions. After some straightforward algebra, we find



$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(\mathrm{Adj})\delta^{ab} \int_0^1 dx\, \frac{J^{\mu\nu}(x,q)}{[x(x-1)q^2]^{2-d/2}},$$

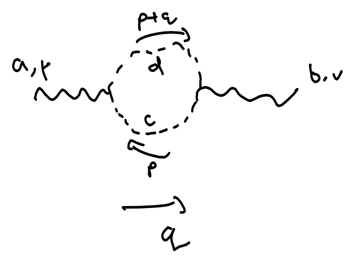
where

$$\begin{aligned} J^{\mu\nu}(x,q) \;=\; & -\Gamma(1-d/2)g^{\mu\nu}q^2\left[\frac{1}{2}d(d-1)x(1-x)\right] \\ & -\Gamma(2-d/2)g^{\mu\nu}q^2\Big[(d-1)(1-x)^2\Big] \end{aligned} \tag{G.6}$$

This looks unwieldy, but we can now see the possibility of cancellations between this and the term constructed from cubic vertices.

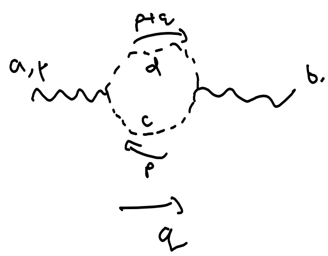
G.1.4 Ghost Loop Contribution

There is one final contribution. This comes from the ghost fields running in the loop.



$$= (-1)g^2 f^{dac} f^{cbd} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2} \frac{i}{(p+q)^2} (p+q)^\mu p^\nu$$

where we note the factor of -1 due to the fact that the ghosts are fermionic so we have a fermion loop. Proceeding as before, the integral can be evaluated to give



$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(\mathrm{Adj})\delta^{ab} \int_0^1 dx\, \frac{i}{p^2} \frac{K^{\mu\nu}(x,q)}{[x(x-1)q^2]^{2-d/2}},$$

where

$$\begin{aligned} K^{\mu\nu}(x,q) \;=\; & \Gamma(1-d/2)G^{\mu\nu}q^2\left[\frac{1}{2}x(1-x)\right] \\ & +\Gamma(2-d/2)q^\mu q^\nu\Big[x(1-x)\Big] \end{aligned} \tag{G.7}$$

which has a similar structure to the previous two terms.

G.1.5 Putting It All Together

Let us consider the sum of the three diagrams

$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(\mathrm{Adj})\delta^{ab} \int_0^1 dx \, \frac{\mathcal{X}^{\mu\nu}(x,q)}{\Delta^{2-d/2}},$$

where

$$\mathcal{X}^{\mu\nu}(x,q) = F^{\mu\nu}(x,q) + J^{\mu\nu}(x,q) + K^{\mu\nu}(x,q)$$

is given by the previous expressions for each amplitude added together. The explicit expression for $\mathcal{X}^{\mu\nu}(x,q)$ looks intractable; however, if we collect coefficients of the Gamma functions, we start to see some simplifications emerge.

There are several tricks we can use to simplify $\mathcal{X}^{\mu\nu}(x,q)$

- The Gamma function identity $\Gamma(1-d/2) = \frac{\Gamma(2-d/2)}{1-d/2}$ can be used to write each term in the expression in terms of $\Gamma(2-d/2)$.

- The integral over x is invariant under the exchange $x \leftrightarrow 1-x$. This means that, under the integral, we can make simplifications such as

$$2 - x \sim 1 + x$$

or

$$x = \frac{1}{2}x + \frac{1}{2}x \sim \frac{1}{2}x + \frac{1}{2}(1-x) = \frac{1}{2}.$$

and so terms such as

$$\frac{1}{2}(d-2)(-2x+1) \sim 0.$$

After much tedious algebraic manipulation, one finds that

$$\mathcal{X}^{\mu\nu}(x,q) = \Big(q^2 g^{\mu\nu} - q^\mu q^\nu\Big)\Gamma(2-d/2)\left[\left(1-\frac{d}{2}\right)(1-2x)^2 + 2\right],$$

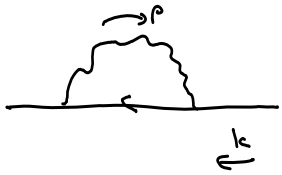
and we see the expected projective pre-factor $\mathcal{P}^{\mu\nu} := q^2 g^{\mu\nu} - q^\mu q^\nu$ emerges, giving the final result

$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(\mathrm{Adj})\delta^{ab}\Big(q^2 g^{\mu\nu} - q^\mu q^\nu\Big) \int_0^1 dx \, \frac{\Gamma(2-d/2)}{[x(x-1)q^2]^{2-d/2}} \left[\left(1-\frac{d}{2}\right)(1-2x)^2 + 2\right]$$

It is worth emphasising that the ghost term was crucial in giving the correct projective term to the sum of diagrams.

G.2 Fermion Self-Energy

The one-loop correction to the fermion propagator is much simpler to calculate as there in only one diagram to evaluate



$$= \int \frac{d^4p}{(2\pi)^4}(ig)^2\gamma^\mu T_a \frac{i(\not{p}+\not{k})}{(p+k)^2}\gamma_\mu T_a \frac{-i}{p^2}$$

By now the way to evaluate such diagrams should be straightforward. The final result is

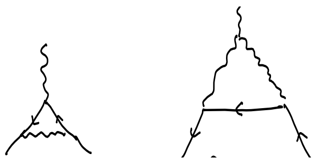
$$\frac{ig^2}{(4\pi)^{d/2}}C_2(r)\not{k}\int_0^1 dx\,\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}(1-x)(d-2)$$

where

$$\Delta := -x(1-x)k^2.$$

G.3 One-Loop Vertex Corrections

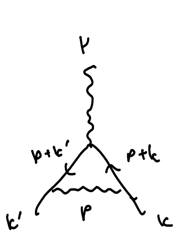
At one loop, there are two diagrams that contribute to the 2 fermion, 1 gluon vertex.



We shall treat these diagrams in turn. Since the evaluation of the diagrams does not involve any fundamentally new ideas, we shall be very brief.

Fermion-Fermion-Gluon Vertex Loop

The first diagram we shall discuss has a divergence coming from a loop involving two fermion lines and a gluon. The diagram is

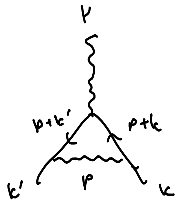


$$= g^3 T_b T_a T_b \int \frac{d^4p}{(2\pi)^4}\frac{\gamma^\nu(\not{p}+\not{k}')\gamma^\mu(\not{p}+\not{k})\gamma_\nu}{(p+k')^2(p+k)^2p^2}$$

Note that we can write

$$\begin{aligned} T_bT_aT_b &= T_bT_bT_a + T_b[T_a,T_b] = C_2(r)T_a + if_{ab}{}^cT_bT_c \\ &= C_2(r)T_a - \frac{1}{2}f_{ab}{}^c f_{bc}{}^d T_d = \Big(C_2(r) - \frac{1}{2}C_2(\mathrm{Adj})\Big)T_a. \end{aligned} \tag{G.8}$$

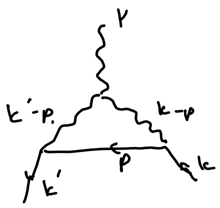
This diagram diverges as



$$= \frac{ig^3}{8\pi^2\epsilon}\left(C(r) - \frac{1}{2}C_2(\mathrm{Adj})\right)T_a\gamma^\mu + \ldots$$

G.3.1 Fermion-Gluon-Gluon Vertex Loop

We also have the diagram



$$= \int \frac{d^4p}{(2\pi)^4}(ig\gamma_\nu T_b)\frac{i\not{p}}{p^2}(ig\gamma_\rho T_c)\frac{-i}{(k'-p)^2}\frac{-i}{(k-p)^2}(igf^{abc})N^{\mu\nu\rho}$$

where

$$N^{\mu\nu\rho} = g^{\mu\nu}(2k'-k-p)^\rho + g^{\nu\rho}(-k'-k+2p)^\mu + g^{\rho\mu}(2k-k'-p)^\nu$$

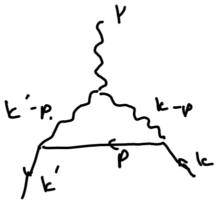
It will be useful to write

$$f^{abc}T_bT_c = \frac{i}{2}f^{abc}f_{bc}{}^dT_d = \frac{i}{2}C_2(\mathrm{Adj})T^a,$$

so that we have the integral

$$\frac{ig^3}{2}C_2(\mathrm{Adj})T^aN^{\mu\nu\rho}\int\frac{d^dp}{(2\pi)^d}\frac{\gamma_\nu\not{p}\gamma_\rho}{p^2(k'-p)^2(k-p)^2}.$$

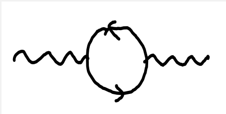
This diverges as



$$= \frac{3ig^3}{32\pi^2\epsilon}C_2(\mathrm{Adj})T_a\gamma^\mu + \ldots$$

Summary of divergences

Gluon Polarization

$$\sum_{\text{fermions}}$$  $$= \Big(g^{\mu\nu}q^2 - q^\mu q^\nu\Big)\delta_{ab}C(r)i\frac{-g^2}{6\pi^2\epsilon} + ...,$$

where $C(r)$ is given by

$$tr(T_aT_b) = (T_a)_i{}^j(T_b)_j{}^i = C(r)\delta_{ab}.$$

In this, the fundamental representation, $C(r) = \frac{1}{2}$. The other terms in the gluon polarization are

 $$= \frac{ig^2}{8\pi^2}C_2(\text{Adj})\delta^{ab}\Big(q^2g^{\mu\nu} - q^\mu q^\nu\Big)\frac{5}{3\epsilon} + ...$$

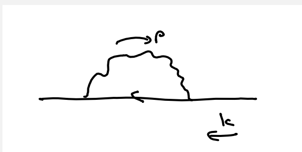
where $C_2(r)$, the quadratic Casimir, is given by

$$(T_a)_i{}^k(T_a)_k{}^j = C_2(r)\delta_i{}^j.$$

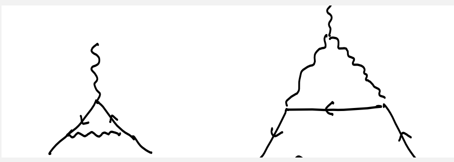
For the Adjoint representation, where $(T_a)_b{}^c = f_{ab}{}^c$, of $SU(N)$, we have $C_2(\text{Adj}) = N$.
Putting them together gives

$$i\Big(q^2g^{\mu\nu} - q^\mu q^\nu\Big)\delta^{ab}\frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(\text{Adj}) - \frac{4}{3}n_fC(r)\right) + ...$$

Fermion self-energy

 $$= \frac{ig^2}{8\pi^2\epsilon}C_2(r)\not{k} + ...$$

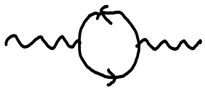
Vertex
The two one-loop diagrams may be written as

 $$= \frac{ig^3}{8\pi^2\epsilon}\Big(C_2(r) + C_2(\text{Adj})\Big)T_a\gamma^\mu + ...$$

G.4 Counter-terms in the MS Scheme

We can use the asymptotic expansion to extract the divergent parts of the diagrams

G.5 Gluon Polarization

$\sum_{\text{fermions}}$  $= \left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\delta_{ab}C(r)i\Pi(q^2),$

where the sum is taken over the fermions in the loop,

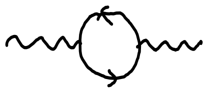
$$\Pi(q^2) = -\frac{8g^2}{(4\pi)^{d/2}}n_f\Gamma(2-d/2)\int_0^1 dx\,\frac{x(1-x)}{[x(x-1)q^2]^{2-d/2}}$$

and $C(r)$ is given by

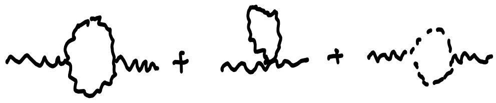
$$tr(T_aT_b) = (T_a)_i{}^j(T_b)_j{}^i = C(r)\delta_{ab}.$$

In this, the fundamental representation, $C(r) = \frac{1}{2}$.

Isolating the divergence, the expression can be written as

$\sum_{\text{fermions}}$  $= \left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\delta_{ab}C(r)i\frac{-g^2}{6\pi^2\epsilon} + ...,$

The other terms in the gluon polarization are



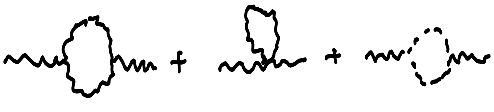
$$= \frac{ig^2}{(4\pi)^{d/2}}C_2(\mathrm{Adj})\delta^{ab}\left(q^2g^{\mu\nu} - q^\mu q^\nu\right)\int_0^1 dx\,\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\left[\left(1-\frac{d}{2}\right)(1-2x)^2+2\right]$$

where $C_2(r)$, the quadratic Casimir, is given by

$$(T_a)_i{}^k(T_a)_k{}^j = C_2(r)\delta_i{}^j.$$

For the Adjoint representation, where $(T_a)_b{}^c = f_{ab}{}^c$, of $SU(N)$, we have $C_2(\mathrm{Adj}) = N$. This is just the contractions of structure constants

which simplifies to

 $= \frac{ig^2}{8\pi^2}C_2(\mathrm{Adj})\delta^{ab}\left(q^2g^{\mu\nu} - q^\mu q^\nu\right)\frac{5}{3\epsilon} + ...$

Putting these parts together, along with the counter-terms gives

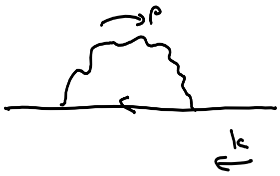
$$i\left(q^2g^{\mu\nu} - q^\mu q^\nu\right)\delta^{ab}\frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(\mathrm{Adj}) - \frac{4}{3}n_fC(r)\right) - i\left(q^2g^{\mu\nu} - q^\mu q^\nu\right)\delta^{ab}\delta_3 + ...$$

and so the MS scheme gives

$$\boxed{\delta_3 = \frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(\mathrm{Adj}) - \frac{4}{3}n_f C(r)\right)}$$

G.5.1 Fermion self-energy

For massless fermions (the mass counter-term plays no role in determining the beta function)



$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(r) \not{k} \int_0^1 dx \, \frac{\Gamma(2-d/2)}{[x(1-x)k^2]^{2-d/2}}(1-x)(d-2)$$

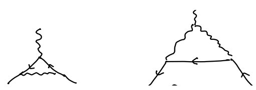
$$= \frac{ig^2}{8\pi^2\epsilon} C_2(r) \not{k} + ... \tag{G.9}$$

and, including the counter-term $i\not{k}\delta_2$ gives in the MS scheme

$$\boxed{\delta_2 = -\frac{g^2 C_2(r)}{8\pi^2\epsilon}.}$$

G.5.2 Gluon-Fermion Vertex

The two one-loop diagrams may be written as



$$= \frac{ig^3}{8\pi^2\epsilon}\Big(C_2(r) + C_2(\mathrm{Adj})\Big) T_a \gamma^\mu + ...$$

The divergence can be canceled by the counter-term $igT_a\gamma^\mu\delta_1$, where

$$\boxed{\delta_1 = -\frac{g^2}{8\pi^2\epsilon}\Big(C_2(r) + C_2(\mathrm{Adj})\Big)}$$

MS scheme counter-terms

Including the counter-terms, the renormalized one-loop contributions to Z_1, Z_2 and Z_3 are:

Gluon polarization

$$i\Big(q^2 g^{\mu\nu} - q^\mu q^\nu\Big)\delta^{ab}\frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(\text{Adj}) - \frac{4}{3}n_f C(r)\right) - i\Big(q^2 g^{\mu\nu} - q^\mu q^\nu\Big)\delta^{ab}\delta_3 + ... = \text{Finite}$$

Fermion self-energy

$$\frac{ig^2}{8\pi^2\epsilon}C_2(r)\not{k} + i\not{k}\delta_2 ... = \text{Finite}$$

Fermion-gluon vertex

$$\frac{ig^3}{8\pi^2\epsilon}\Big(C_2(r) + C_2(\text{Adj})\Big)T_a\gamma^\mu + ig\gamma^\mu T_a\delta_1 ... = \text{Finite}$$