# Asymptotic Methods

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# 1 Motivation

Most real problems don't have closed form solutions or if they do, they are unworkably complicated or difficult to interpret. Numerical approximations can be useful but might to expensive computationally and results might be unsuitable. Taylor series approximations can be limited in applicability or have desperately slow convergence.

In asymptotic approximations, the emphasis is usefulness and accurate.

**Example** $\Gamma(z) = \int_0^\infty t^{z-1}\,\mathrm{d}t$ for $Re(z) > 0$ specialise to $z = n$ an integer and take $n$ large

$$\Gamma(n+1) = n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \dots\right).$$

**Example** $x^2y'' + xy' - x^2y = 0$ have solution

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2}.$$

What does it do when $x \to \infty$?

$$I_0(x) \sim (2\pi x)^{-1/2} e^x$$

**Example** $y'' + k^2 e^{-x^2/2} y = 0$, with $y \to y_0$ as $|x| \to \infty$. $k$ is an eigenvalue, $k = k_1, k_2, k_3, \dots$. For large $n$,

$$k_n \sim \frac{\sqrt{\pi}}{2}\left(n + \frac{1}{2}\right).$$

Convergent methods give arbitrarily accurate approximations with more terms/iterations. Asymptotic methods give arbitrarily accurate approximations as some variable approaches a limit.

# 2 Definitions and Notations

## 2.1 Definitions

**Definition 2.1** (Asymptotic equality/equivalence) For real valued functions $f(x), g(x)$, we write $f(x) \sim g(x)$ about $x = x_0$ (or $x \to x_0$) if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

**Remarks**
- $\sim$ is an equivalence relation
- properties: linearity, symmetry, transitivity, respects products ($f_1 \sim g_1, f_2 \sim g_2 \Rightarrow f_1 f_2 \sim g_1 g_2$).
- definition has problems when dealing with oscillatory functions with zeros.

**Definition 2.2** (Little o) We write $f(x) = o(g(x))$ as $x \to x_0$ assuming $g(x) > 0$ close enough to the limit if

$$\lim_{x \to x_0} \left|\frac{f(x)}{g(x)}\right| = 0.$$

Alternatively, we can write $f(x) \ll g(x)$ as $x \to x_0$.

**Definition 2.3** (Big O) We write $f(x) = \mathcal{O}(g(x))$ as $x \to x_0$ assuming $g(x) > 0$ near $x_0$ if $\exists\, c > 0$ such that $|f(x)| \leq c g(x)$ as $x \to x_0$. " $f$ is not radically different in size to $g$ in this limit."

**Remark** In some areas of physics, people use $\mathcal{O}$ and $\sim$ interchangeably.

**Exercise.** Using definitions, show

(i) $f(x) \sim g(x) \Leftrightarrow f - g = o(g)$

(ii) $\sin x \ll x^2$ as $x \to \infty$

(iii) $12x^5 + 5x^3 + x = \mathcal{O}(x^5)$ as $x \to \infty$

## 2.2 Asymptotic Expansions

We have complicated/unusable $f(x)$ and I want to know its behaviour in some limit, in particular

- want to find a "simpler" $g(x)$ so that $f \sim g$
- want to break up $g$ into a hierarchy of approximations such that $g(x) = \sum_{k=0}^{\infty} a_k \phi_k(x)$ where $\phi_k(x)$ are "nice" and adding extra $\phi_k$s improves according by a known size.

**Definition 2.4** (Asymptotic sequence) A sequence of functions $\{\phi_k(x)\}_{k=0}^{\infty}$ is called an asymptotic sequence as $x \to x_0$ if $\phi_{k+1}(x) \ll \phi_k(x)$ as $x \to x_0$ for all $k$.

**Example** $\phi_k = (x-a)^k$, $x$ around $a$. Consider $\left|\dfrac{\phi_{k+1}}{\phi_k}\right| = |x-a| \to 0$ as $x \to a$.

**Example** $\phi_k = e^{-kx}x^k$ as $x \to \pm\infty$. Consider $\left|\dfrac{\phi_{k+1}}{\phi_k}\right| = |e^{-x}x| \to 0$ as $x \to \infty$ but $|e^{-x}x| \not\to 0$ as $x \to -\infty$

**Example** $\phi_k = \cos(kx)$. $\left|\dfrac{\phi_{k+1}}{\phi_k}\right| \not\to 0$ as $x \to \infty$ or $x \to 0$.

**Example** $\{|x|, \log|x|, \log|\log|x||, \dots\}$ as $x \to \infty$

**Definition 2.5** (Asymptotic approximation) Suppose we have a function $f(x)$ which we want to understand and let $\{\phi_k\}$ be an asymptotic sequence as $x \to x_0$. We say that $\sum_{k=0}^{N} a_k \phi_k(x)$ is an *asymptotic expansions* of $f(x)$ as $x \to x_0$ if

$$f(x) = \sum_{k=0}^{N} a_k \phi_k(x) + o(\phi_N(x))$$

**Definition 2.6** (Asymptotic expansion/series) If the definition of asymptotic approximation holds for all $N$, we say that the asymptotic approximation is called an asymptotic expansion/series, and we write $f(x) \sim \sum_{k=0}^{\infty} a_k \phi_k(x)$.

**Remark** We make no claim for convergence. It's just a notation. Furthermore, the limit $N \to \infty$ is never taken. Instead, we have an infinite number of approximations (each of different $N$) and the limit $x \to x_0$ is taken in each separately.

### 2.2.1 Properties of expansions

1. Mathematical operations
   (i) Addition and subtraction: Suppose $\phi_k$ is an asymptotic sequence as $x \to x_0$ and let $f \sim \sum_{k=0}^{\infty} a_k \phi_k$, $g \sim \sum_{k=0}^{\infty} b_k \phi_k$, then $\alpha f + \beta g \sim \sum_{k=0}^{\infty} (\alpha a_k + \beta b_k)\phi_k$. (linearity)
   (ii) Multiplication and division: Let $\phi_k = (x-x_0)^k$, then $fg \sim \sum_{k=0}^{\infty} c_k \phi_k$, where $c_k = \sum_{j=0}^{k} a_j b_{k-j}$. For division, see later.
   (iii) Term by term differentiation doesn't generally work.
   (iv) Term by term integration does work.

2. Different asymptotic representations: Suppose $\{\phi_k\}$ and $\{\psi_k\}$ are different asymptotic sequences as $x \to x_0$ and $f \sim \sum a_k\phi_k$ and $f \sim \sum b_k\psi_k$ as $x \to x_0$, then $a_k$ and $b_k$ are not the same.

> **Example** $f(x) = 1 + \dfrac{1}{x}$ and consider its behaviour as $x \to \infty$.
> - Define $\phi_k = \dfrac{1}{x^k}$. $f(x) \sim \sum a_k\phi_k$, $a_0 = 1, a_1 = 1, a_i = 0\,\forall\, i \geq 2$.
> - Define $\psi_k = \dfrac{1}{(1+x)^k}$. $f(x) = \dfrac{1}{1-(1+x)^{-1}} = 1 + \dfrac{1}{1+x} + \dfrac{1}{(1+x)^2} \sim \sum b_k\psi_k$, $b_k = 1\,\forall\, k$ as $x \to \infty$.

3. Uniqueness of coefficients: Let $\{\phi_k\}$ be an asymptotic sequence as $x \to x_0$ and $f(x) \sim \sum a_k\phi_k$ as $x \to x_0$, then $a_k$ unique.
In fact,

$$a_0 = \lim_{x\to x_0}\left|\frac{f(x)}{\phi_0(x)}\right|$$

$$a_n = \lim_{x\to x_0}\left|\frac{f(x) - \sum_{k=0}^{n-1} a_k\phi_k(x)}{\phi_n(x)}\right|$$

for $n \geq 1$.

> **Proof** Use induction on $N$. Let $N = 0$, then from definition of asymptotic series,
> $$\lim_{x\to x_0}\left|\frac{f(x) - a_0\phi_0(x)}{\phi_0(x)}\right| = 0$$
> $$\Rightarrow \left|\frac{f(x)}{\phi_0(x)} - a_0\right| \to 0.$$
> Let $N = n-1$. Suppose true for $a_k$ for all $k = 1, ..., n-1$.
> $$\lim_{x\to x_0}\left|\frac{f(x) - \sum_{k=0}^{n} a_k\phi_k(x)}{\phi_n(x)}\right| = 0$$
> $$\Rightarrow \left|\frac{f(x) - \sum_{k=0}^{n-1} a_k\phi_k(x)}{\phi_n(x)} - a_n\right| \to 0.$$
> Here, $a_n$ is the expression I want, which involves unique $a_k$, so $a_n$ is also unique. $\square$

4. An asymptotic expansion does not uniquely determine a function: Suppose $f(x) \sim g(x)$ as $x \to x_0$, also $h(x) \ll f(x)$ as $x \to x_0$, then $f(x) + h(x) \sim g(x)$. So if $g$ is an asymptotic expansion, it can describe both $f$ and $f + h$.

> **Example** $f_1(x) = \dfrac{1}{1+x}$ and $f_2(x) = \dfrac{1+e^{-x}}{1+x}$, both are $\sim \sum(-1)^k x^{-(k+1)}$ as $x \to \infty$.

5. Power series: In this course, we will be mostly dealing with $\phi_k = (x - x_0)^k$ as $x \to x_0$ or $\phi_k = \dfrac{1}{x^k}$ as $x \to \infty$. In fact, Taylor series about a point is an asymptotic expansion about that point, and
$$f(x) - \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = o(|x - x_0|^N)$$
as $x \to x_0$.
6. Terminology: Something is said to be *small beyond all orders in some limit* if it can't be represented by any power series or if it is smaller than any power.

**Example** $e^{-x}$ is small beyond all orders in inverse powers of $x$ as $x \to \infty$.

7. Optimal truncation: If $f(x) \sim \sum a_n \psi_n(x)$ as $x \to x_0$ and we want the best approximation to $f(x)$ at $x = x_1$, where $x_1$ is fixed but close to $x_0$, then we truncate the series at the smallest term (or sometimes one before)
8. Care is needed when approximating oscillating functions around zero. Suppose $f$ oscillates around zero and $f(x) \sim g(x)$ as $x \to \infty$, then $\left|\frac{f(x)}{g(x)}\right| \to 1$ as $x \to \infty$. The definitions are deficient when zeros of $f$ and $g$ are not exactly the same.

**Example** $J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right)$ as $x \to \infty$.

## 2.3 Asymptotic expansion in $\mathbb{C}$ and the Stokes phenomenon

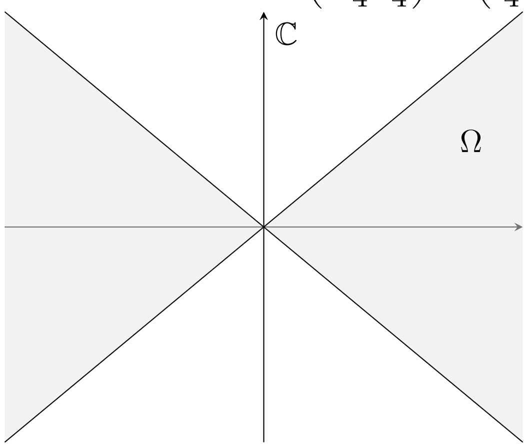
So far, all definitions relied on real variables and real limits, e.g. $x \to \infty$ and $x \to x_0$. When we use complex variables, i.e. $z \in \mathbb{C}$ and $z \to z_0$, typically $z$ must be restricted to a set of $\mathbb{C}$ around $z_0$, for the definitions to work. Even if $f(z)$ is analytic at $z_0$, it may have different asymptotic expansions when $z_0$ is approached in different sectors of $\mathbb{C}$ (This is called the Stokes phenomenon).

**Example** $f(z) = \sinh z^2 = \dfrac{e^{z^2} - e^{-z^2}}{2}$, $z \in \mathbb{C}$. What is $f$'s behaviour as $|z| \to \infty$? Rephrase: which of $e^{z^2}$ or $e^{-z^2}$ is dominant?

Consider $\left|\dfrac{e^{-z^2}}{e^{z^2}}\right| = \left|e^{-2z^2}\right| = e^{-2\operatorname{Re}(z^2)}$. Where does this go as $|z| \to \infty$?

(a) Suppose $z \to \infty$ and $\operatorname{Re}(z^2) > 0$, then $e^{-2\operatorname{Re}(z^2)} \to 0$ as $|z| \to \infty$, so $e^{-z^2} \ll e^{z^2} \Rightarrow f(z) \sim \frac{1}{2} e^{z^2}$.

Note also that $\operatorname{Re}(z^2) > 0$ corresponds to $\arg z \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \bigcup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$.




(b) Suppose $|z| \to \infty$ and $\operatorname{Re}(z^2) < 0$ ($z \in \Omega^c$), $f(z) \sim \frac{1}{2} e^{-z^2}$

So

$$f(z) \sim \begin{cases} \frac{1}{2} e^{z^2} & |z| \to \infty, z \in \Omega \\ \frac{1}{2} e^{-z^2} & |z| \to \infty, z \in \Omega^c \end{cases}$$

# 3 Approximations of Integrals

## 3.1 Taylor expanding the integrand – Stieltjes integrals

Consider

$$I(x) = \int_0^\infty \frac{e^t}{1 + xt} \, \mathrm{d}t$$

for $x \geq 0$.

How does it behave as $x \to 0^+$?

$$\sum_{n=0}^{N}(-xt)^n = \frac{1-(-xt)^{N+1}}{1+xt} = \frac{1}{1+xt} - \underbrace{\frac{(-xt)^{N+1}}{1+xt}}_{r_N}$$

So

$$\begin{aligned}
I &= \int_0^\infty e^{-t}\left(\sum_{n=0}^{N}(-1)^n x^n t^n + r_n\right)\mathrm{d}t \\
&= \sum_{n=0}^{N}(-1)^n x^n \underbrace{\left(\int_0^\infty t^n e^{-t}\,\mathrm{d}t\right)}_{=\Gamma(n+1)=n!} + \int_0^\infty r_N e^{-t}\,\mathrm{d}t \\
&= \sum_{n=0}^{N}(-1)^n n! x^n + R_N \\
&= \sum_{n=0}^{N} c_n \phi_n(x) + R_N,
\end{aligned}$$

where $c_n = (-1)^n n!$, $\phi_n = x^n$.
For this to be an asymptotic approximation as $x \to 0^+$, we need $R_N = o(x^N)$ as $x \to 0^+$.

$$\begin{aligned}
\left|\frac{R_N}{x^n}\right| &= \frac{x^{N+1}}{x^N}\int_0^\infty \frac{e^{-t}t^{N+1}}{1+xt}\,\mathrm{d}t \\
&\le x\int_0^\infty e^{-t}t^{N+1}\,\mathrm{d}t && (1+xt>1) \\
x^2(N+1)! &\to 0 \text{ as } x \to 0^+
\end{aligned}$$

This holds for all $N$. So

$$I(x) \sim \sum_{n=0}^{\infty}(-1)^n n! x^n$$

as $x \to 0^+$.

## 3.2 Integration by parts

Recap:

$$\int_a^b f'(t)g(t)\,\mathrm{d}t = [fg]_a^b - \int_a^b f(t)g'(t)\,\mathrm{d}t$$

Generalisation:

$$I = \int_a^b f(t)g^{(n)(t)}\,\mathrm{d}t = \sum_{k=1}^{n}(-1)^{k-1}\left[f^{(k-1)}(t)g^{(n-k)}(t)\right]_a^b + (-1)^n\int_a^b f^{(n)}(t)g(t)\,\mathrm{d}t$$

The idea is that repeated IBP can generate an asymptotic approximation composed of boundary terms under certain circumstances.

> **Example** $F(x) = \int_x^\infty e^{-t^2/2}\,\mathrm{d}t$. How does this behave as $x \to \infty$?
>
> $$\begin{aligned}
> F(x) &= \int_x^\infty e^{-t^2/2}\,\mathrm{d}t \\
> &= \underbrace{\frac{e^{-x^2/2}}{x}}_{\text{our asymptotic approximation}} + \underbrace{\int_x^\infty \frac{e^{-t^2/2}}{t^2}\,\mathrm{d}t}_{\text{error to be small}}
> \end{aligned}$$

Need to check the error term is $o\left(\frac{e^{-x^2/2}}{x}\right)$.

$$\begin{aligned}
\frac{e^{-x^2/2}}{x} &= \int_x^\infty \frac{te^{-t^2/2}}{t^3}\,\mathrm{d}t \\
&\le \frac{1}{x^3}\int_x^\infty te^{-t^2/2}\,\mathrm{d}t && t \ge x \text{ in the integral} \\
&\le \frac{1}{x^3}e^{-x^2/2} \\
&= o\left(\frac{e^{-x^2/2}}{x}\right) \text{ as } x \to \infty.
\end{aligned}$$

Therefore,

$$F(x) \sim \frac{e^{-x^2/2}}{x} \text{ as } x \to \infty.$$

Repeated IBP gives

$$F(x) \sim \frac{e^{-x^2/2}}{x}\sum_{n=0}^\infty (-1)^n \frac{(2n-1)!!}{x^{2n}} \text{ as } x \to \infty$$

This can be shown by induction. (Exercise)

**Example** $I(x) = \int_a^b f(t)e^{xt}\,\mathrm{d}t$ as $x \to \infty$, where $b > a$ and $f(b) \neq 0$ and $f$ is nice.

Note $e^{bx} \gg e^{ax}$ as $x \to \infty$, so $I(x) \approx f(b)\int_a^b e^{xt}\,\mathrm{d}t \approx f(b)\frac{e^{xb}}{x}$. This is too rough!

Now IBP repeatedly.

$$\begin{aligned}
I(x) &= \sum_{k=1}^N (-1)^{k-1}\left[f^{(k-1)}(t)e^{xt}x^{-k}\right]_a^b + \frac{(-1^N)}{x^N}\int_a^b f^{(N)}(t)e^{xt}\,\mathrm{d}t \\
&= e^{xb}\left(\sum_{k=1}^N (-1)^{k-1}f^{(k-1)}(b)x^{-k} - e^{-x(b-a)}\sum_{k=1}^N (-1)^{k-1}f^{(k-1)}(a)x^{-k} + \frac{(-1)^N}{x^N}\int_a^b f^{(N)}(t)e^{x(x-b)}\,\mathrm{d}t\right) \\
&= e^{xb}\left(J_1 + J_2 + J_3\right)
\end{aligned}$$

$J_1$ could be an asymptotic approximation with $\phi_k = \frac{1}{x^k}$ as $x \to \infty$. We need to show $J_2 = o(x^{-N})$ and $J_3 = o(x^{-N})$.

$$\begin{aligned}
\frac{J_2}{x^{-N}} &= e^{-x(b-a)}\sum_{k=1}^N (-1)^{k-1}f^{(k-1)}(a)x^{N-k} \\
&\to 0 \text{ as } x \to \infty
\end{aligned}$$

$$\begin{aligned}
\left|\frac{J_3}{x^{-N}}\right| &= \left|\int_a^b f^{(N)}(t)e^{x(t-b)}\,\mathrm{d}t\right| \\
&\le \int_a^b |f^{(N)}|e^{x(t-b)}\,\mathrm{d}t \\
&\le \max_{a<t<b}|f^{(N)}(t)|\int_a^b e^{x(t-b)}\,\mathrm{d}t \\
&\le \max_{a<t<b}|f^{(N)}(t)|\frac{1}{x}\left(1 - e^{-x(b-a)}\right) \\
&\to 0 \text{ as } x \to \infty
\end{aligned}$$

This is true for all $N$. Therefore,

$$\int_a^b f(t)e^{xt}\,\mathrm{d}t \sim \sum_{k=0}^\infty (-1)^{k-1}e^{bx}\frac{f^{(k-1)(b)}}{x^k}$$

## 3.3 The Gamma function

Recall that

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t$$

for $\mathrm{Re}(z) > 0$.

Basic properties of $\Gamma$:

(a) $\Gamma$ is analytic for $\mathrm{Re}(z) > 0$.

(b) $\Gamma(z+1) = z\Gamma(z)$ (recursion relation).

(c) $\Gamma(n+1) = n!$ for $b \in \mathbb{N}$ and $\Gamma(1) = 1$.

(d) $\displaystyle\int_0^\infty t^{z-1} e^{-xt} \, \mathrm{d}t = \frac{\Gamma(z)}{x^z}$ for $x > 0$.

## 3.4 Watson's Lemma

Consider

$$F(x) = \int_0^T f(t) e^{-xt} \, \mathrm{d}t$$

as $x \to \infty$ and $0 < T < \infty$.

**Theorem 3.1** (Watson's Lemma) Suppose $f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n}$ as $t \to 0^+$ with $\alpha > -1, \beta > 0$. Also, either

1. $|f(t)| < K e^{bt} \, \forall \, t$ for some $K, b > 0$, or
2. $\displaystyle\int_0^T |f(t)| \, \mathrm{d}t < \infty$,

then

$$F(x) = \int_0^T f(t) e^{-xt} \, \mathrm{d}t \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}}$$

**Proof**

$$F(x) = \underbrace{\int_0^\epsilon f(t) e^{-xt} \, \mathrm{d}t}_{F_1(x)} + \underbrace{\int_\epsilon^T f(t) e^{-xt} \, \mathrm{d}t}_{F_2(x)}$$

for some $\epsilon > 0$ small.

Focus on $F_2$ first.

Case 1:

$$\begin{aligned}
|F_2| &\leq \int_\epsilon^T K e^{bt} e^{-xt} \, \mathrm{d}t \\
&\leq \int_\epsilon^\infty K e^{bt} e^{-xt} \, \mathrm{d}t && \text{assume } x > b \\
&= \frac{K}{x-b} e^{(b-x)\epsilon} = \mathcal{O}\left(\frac{e^{-\epsilon x}}{x}\right)
\end{aligned}$$

Case 2:

$$\begin{aligned}
|F_2| &\leq \int_\epsilon^T |f| e^{-xt} \, \mathrm{d}t \\
&\leq e^{\epsilon x} \underbrace{\int_0^T |f| \, \mathrm{d}t}_{\text{constant}} \\
&= \mathcal{O}(e^{-\epsilon x}) \text{ as } x \to \infty.
\end{aligned}$$

So $F_2$ is small beyond all orders in $x^{-k}$.

We want $F_1 = \sum_{n=0}^{N} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} + o(x^{-\alpha-\beta N-1})$, or $\left| \frac{F_1 - \sum^{N} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}}}{x^{-\alpha-\beta N-1}} \right| \to 0$ as $x \to \infty$.

Recall property (d) of $\Gamma$ function. Then the numerator become

$$\begin{aligned}
&\left|\int_0^\epsilon f e^{-xt}\,\mathrm{d}t - \int_0^\infty t^{\alpha+\beta n} e^{-xt}\,\mathrm{d}t\right| \\
&\qquad = \left|\int_0^\epsilon e^{-xt}\left[f - \sum^{N} a_n t^{\alpha+\beta n}\right]\mathrm{d}t - \int_\epsilon^\infty \sum a_n t^{\alpha+\beta n} e^{-xt}\,\mathrm{d}t\right| \\
&\qquad \le \int_0^\epsilon e^{-xt}\left|f - \sum^{n} a_n t^{\alpha+\beta n}\right|\mathrm{d}t + \left|\int_\epsilon^\infty e^{-xt}\sum a_n t^{\alpha+\beta n}\,\mathrm{d}t\right| \\
&\qquad \le R_1 + R_2
\end{aligned}$$

Consider $R_2$.

$$\begin{aligned}
R_2 &= e^{-\epsilon x}\left|\int_\epsilon^\infty e^{-x(t-\epsilon)}\sum a_n t^{\alpha+\beta n}\,\mathrm{d}\right| \\
&= \frac{e^{-\epsilon x}}{x}\left|\int_0^\infty e^{-u}\sum^{N} a_n\left(\epsilon + \frac{u}{x}\right)^{\alpha+\beta n}\mathrm{d}u\right| \\
&\le \frac{e^{-\epsilon x}}{x}\left|\int_0^\infty e^{-u}\sum^{N} a_n(\epsilon+u)^{\alpha+\beta n}\,\mathrm{d}u\right|
\end{aligned}$$

This can be written in terms of $\Gamma, \epsilon, \alpha, \beta$ and is bounded. So

$$\begin{aligned}
&R_2 = \mathcal{O}(e^{-\epsilon x}/x) \text{ as } x \to \infty \\
\Rightarrow\ & \frac{R_2}{x^{-\alpha-\beta N-1}} \to 0 \text{ as } x \to \infty.
\end{aligned}$$

We know $f - \sum^{N} a_n t^{\alpha+\beta n} = o(t^{\alpha+\beta N}) = \mathcal{O}(t^{\alpha+\beta(N+1)})$ as $x \to \infty$ for all $N$, so

$$\begin{aligned}
R_1 &< K_3 \int_0^\epsilon e^{-xt} t^{\alpha+\beta(N+1)}\,\mathrm{d}t && \text{for some } K_3 > 0 \\
&< K_3 \int_0^\infty e^{-xt} e^{\alpha+\beta(N+1)}\,\mathrm{d}t \\
&= K_3 \frac{\Gamma(\alpha+\beta(N+1)+1)}{x^{\alpha+\beta(N+1)+1}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&R_1 = \mathcal{O}(x^{-\alpha-\beta(N+1)-1}) \\
\Rightarrow & \frac{R_1}{x^{-\alpha-\beta N-1}} \to 0 \text{ as } x \to \infty \\
\Rightarrow & F_1 \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \text{ as } x \to \infty,
\end{aligned}$$

but $F_2 = o(F_1)$, so $F \sim F_1$.

□

**Example** $I(x) = \int_0^\infty e^{-xt}\sin t\,\mathrm{d}t$ as $x \to \infty$. Have $f = \sin(t)$ is exponentially bounded (case 1).

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1},$$

so $\alpha = 1, \beta = 2$. So we can use Watson.

$$\begin{aligned} I(x) &\sim \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2n+2)}{(2n+1)!} \frac{1}{x^{2n+2}} \\ &= \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{x^{2n}} \\ &= \frac{1}{x^2} \frac{1}{1+x^{-2}} \\ &= \frac{1}{1+x^2} \end{aligned}$$

**Example** $xy'' + (a+1-x)y' - ay = 0$, $a > 0$. The solution to this is

$$\mathcal{M} = \frac{e^x}{\Gamma(a)} \int_0^1 e^{-xt}(1-t)^{a-1}\,\mathrm{d}t.$$

We want to know its behaviour as $x \to \infty$.

$$f(t) = (1-t)^{a-1} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(a)}{\Gamma(a-n)n!} t^n,$$

so $\alpha = 0, \beta = 1$. This is case 2. So

$$\begin{aligned} \mathcal{M} &\sim \frac{e^x}{\Gamma(a)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(a)\Gamma(n+1)}{\Gamma(a-n)n!} \frac{1}{x^{n+1}} \\ &\sim e^x \sum_{n=0}^{\infty} (-1)^n \frac{1}{\Gamma(a-n)} \frac{1}{x^{n+1}} \\ &\sim \frac{e^x}{x\Gamma(a)} \end{aligned}$$

## 3.5 Laplace integrals and Laplace's method

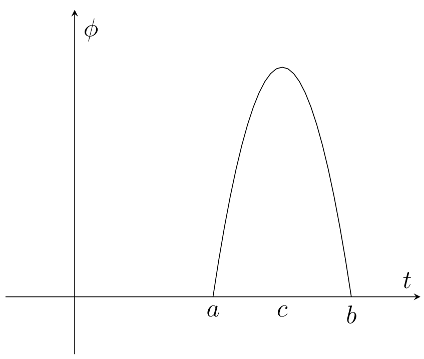
Laplace integrals are of the form

$$I(x) = \int_a^b f(t) e^{x\phi(t)}\,\mathrm{d}t,$$

where $b > a$ and $f$ and $\phi$ are "nice enough" functions, in particular $\displaystyle\int_a^b |f|\,\mathrm{d}t < K < \infty$ for some $K$.

Watson's lemma deals with the special case $\phi(t) = -t$, $a = 0, b = T$. $I(x)$ in this case is determined by the contributions near $t = 0$. In general, we have the *principle of localisation* (Laplace): *The asymptotic expansion of $I(x)$ as $x \to \infty$ is determined by contributions from small neighbourhood of the maximum of $\phi$.*

1. Watson's case: the endpoint dominates (max of $\phi = -t$)
2. Suppose $\phi$ looks like the following (max of $\phi$ at an internal point).




Then as $x$ increase, the integrand is dominated by a small region around $t = c$.

Now look at different cases.

(a) Suppose $\phi(t)$ is monotonic on $[a,b]$. If $\phi'(t) > 0$ on $t \in [a,b]$ and $f(b) \neq 0$, then we expect $I(x)$ to be dominated at the endpoint $t = b$.

$$\begin{aligned} I(x) &= \int_a^b f(t) e^{x\phi(t)} \, \mathrm{d}t \\ &= \int_{\phi(a)}^{\phi(b)} \frac{f(t)}{\phi'(t)} e^{xu} \, \mathrm{d}u && u = \phi(t) \\ &\sim \frac{f(b) e^{x\phi(b)}}{x\phi'(b)} \text{ as } x \to \infty && \text{from section 3.2} \end{aligned}$$

If $\phi'(t) < 0$ and $f(a) \neq 0$, then endpoint $t = a$ dominates $I$, and

$$I(x) \sim \frac{f(a) e^{x\phi(a)}}{x|\phi'(a)|} \text{ as } x \to \infty.$$

(b) Suppose $\phi$ has an internal maximum at $t = c$ such that $\phi'(c) = 0$, $\phi''(c) < 0$, and $f(c) \neq 0$.

$$I(x) = \int_{c-\epsilon}^{c+\epsilon} f(t) e^{x\phi(t)} \, \mathrm{d}t + \int_{\Omega_\epsilon} f(t) e^{x\phi(t)} \, \mathrm{d}t = I_1 + I_2,$$

where $\Omega = [a, c-\epsilon] \cup [c+\epsilon, b]$.

Look at $I_2$ first. Let $\max\limits_{t \in \Omega_\epsilon} \phi = L_\epsilon \Rightarrow L_\epsilon < \phi(c)$.

$$\begin{aligned} |I_2| &\leq e^{xL_\epsilon} \int_{\Omega_\epsilon} |f(t)| \, \mathrm{d}t \\ &\leq e^{xL_\epsilon} \int_a^b |f(t)| \, \mathrm{d}t \\ &< K e^{xL_\epsilon} \end{aligned}$$

Therefore, $I_2 = \mathcal{O}(e^{xL_\epsilon}) \ll e^{x\phi(c)}$ as $x \to \infty$.

Now look at $I_1$. Expand $f$ and $\phi$ around $t = c$.

$$\begin{aligned} \phi(t) &= \phi(c) + \phi'(c)(t-c) + \frac{\phi''(c)}{2}(t-c)^2 + \dots \\ f(t) &= f(c) + \dots \end{aligned}$$

So

$$\begin{aligned} I_1 &= \int_{c-\epsilon}^{c+\epsilon} [f(c) + \dots] \exp\left[x\phi(c) + \frac{1}{2} x\phi''(c)(t-c)^2 + \dots\right] \mathrm{d}t \\ &\approx f(c) e^{x\phi(c)} \int_{c-\epsilon}^{c+\epsilon} \exp\left[\frac{1}{2} x\phi''(c)(t-c)^2\right] \mathrm{d}t \\ &= \frac{f(c) e^{x\phi(c)}}{\sqrt{-x\phi''(c)}} \int_{-[-x\phi''(c)]^{1/2}\epsilon}^{[-x\phi''(c)]^{1/2}\epsilon} e^{-s^2/2} \, \mathrm{d}s && (s = [-x\phi''(c)]^{1/2}(t-c)) \\ &\sim \frac{f(c) e^{x\phi(c)}}{\sqrt{-x\phi''(c)}} \int_{-\infty}^{\infty} e^{-s^2/2} \, \mathrm{d}s && (*) \\ &\sim f(c) e^{x\phi(c)} \sqrt{\frac{2\pi}{-x\phi''(c)}} \end{aligned}$$

when taking $x$ large and $\epsilon$ finite. $(*)$ is justified because the error is of the form $\int_z^\infty e^{-t^2/2} \, \mathrm{d}t \sim \frac{1}{z} e^{-z^2/2}$ as $z \to \infty$, which is very small.

Therefore, $I_1 \gg I_2 \Rightarrow I(x) \sim I_1 \sim f(c) e^{x\phi(c)} \sqrt{\frac{2\pi}{-x\phi''(c)}}$.

**Example** $I(x) = \int_{-\infty}^{\infty} e^{-xt^2} e^{at} \, \mathrm{d}t$ as $x \to \infty$.

Here $\phi(t) = -t^2, f(t) = e^{at}$. $\phi$ has an internal maximum at $t = 0$, $\phi''(0) = 2, f(0) = 1, \phi(0) = 0$.

$$I \sim \sqrt{\frac{\pi}{x}}$$

as $x \to \infty$. (The actual answer is $I(x) = e^{a^2/4x}\sqrt{\frac{\pi}{x}}$.)

**Example** $K_0(x) = \int_0^\infty e^{-t-x^2/4t}\,\mathrm{d}t$ as $x \to \infty$.

Try: $f(t) = e^{-t}, \phi(t) = -\frac{1}{4t}$? But $\phi'(t) > 0$ for all $t \in [0, \infty)$!

So instead, let $\psi = -t - \frac{x^2}{4t}$, $\psi' = -1 + \frac{x^2}{4t^2}$, so $\psi$ has a maximum at $t_{max} = \frac{x}{2}$.

As $x \to \infty$, max $\to \infty$. So let's rescale to a new variable moves with $x$. Now try: $xu = 2t$, then

$$K_0 = \frac{x}{2}\int_0^\infty e^{x(-\frac{u}{2}-\frac{1}{2u})}\,\mathrm{d}u$$

$\varphi'(u) = 0 \Rightarrow u = c = 1$, so

$$K_0(x) \sim \frac{\sqrt{2\pi x}}{2}e^{-x}$$

as $x \to \infty$.

(c) What if $\phi$ achieves its absolute maximum at an internal point $t = c$, and also at an endpoint $t = b$? Then $\phi(b) = \phi(c) = \max_{t\in[a,b]} \phi$ with $\phi'(c) = 0, \phi'(b) \neq 0$, and $f(c) \neq 0$, $f(b) \neq 0$. In this case, the internal point dominates the integral as $x \to \infty$. This is because the end point contribution is proportional to $\frac{e^{x\phi(b)}}{x}$, and internal point proportional to $\frac{e^{x\phi(c)}}{\sqrt{x}}$, and we have

$$\frac{e^{x\phi(b)}}{x} \ll \frac{e^{x\phi(c)}}{\sqrt{x}}.$$

(d) What if $\phi$ have many internal maximum, say $\phi'(c_i) = 0$ and $\phi''(c_i) < 0$ for $i = 1, \dots, n$? In this case, only the absolute maximum matters.

**Example** Suppose $\phi(c_1) < \phi(c_2) < \phi(c_3)$, and the contribution is proportional to $\frac{e^{x\phi(c_i)}}{\sqrt{x}}$, so the local maxima that are not the absolute maxima contributes terms that are exponentially smaller.

**Example** Suppose $\phi(c_1) = \phi(c_2)$, both are absolute maximum, their contributions must be summed up as they all are dominant. So

$$I(x) \sim \sqrt{\frac{2\pi}{-x\phi''(c_1)}}e^{x\phi(c_1)} + \sqrt{\frac{2\pi}{-x\phi''(c_2)}}e^{x\phi(c_2)}.$$

(e) What if an endpoint is a (local and absolute) maximum, i.e. $\phi'(b) = 0, \phi'(b) < 0$ and $f(b) \neq 0$.

$$\begin{aligned} I(x) &\sim \int_{b-\epsilon}^{b} f e^{x\phi}\,\mathrm{d}t \\ &\sim \frac{f(b)e^{x\phi(b)}}{\sqrt{-x\phi''(b)}}\int_{-\infty}^{\infty} e^{-s^2/2}\,\mathrm{d}s \\ &\sim \frac{1}{2}\frac{f(b)e^{x\phi(b)}}{\sqrt{-x\phi''(b)}}\sqrt{2\pi} \end{aligned}$$

**Example** $I(x) = \int_0^{3\pi/2} e^{x\sin^2 t}\,\mathrm{d}t$ for $\phi = \sin^2 t$, so maximum at $\frac{\pi}{2}, \frac{3\pi}{2}$.

So

$$
\begin{aligned}
I(x) &\sim \underbrace{\frac{e^x\sqrt{2\pi}}{\sqrt{2x}}}_{\text{case (b)}} + \underbrace{\frac{e^x\sqrt{2\pi}}{2\sqrt{2x}}}_{case(e)} \\
&\sim \frac{3}{2}\frac{e^x\sqrt{2\pi}}{\sqrt{2x}}.
\end{aligned}
$$

(f) What if $\phi'(c) = \phi''(c) = \cdots = 0$ and $\phi^{(p)}(c) < 0$ with $p$ even positive integer, and $f(c) \neq 0$? Taylor expand and get

$$
\phi = \phi(c) + \frac{\phi^{(c)}(t-c)^p}{p!}.
$$

Therefore,

$$
\begin{aligned}
I &\sim \int_{c-\epsilon}^{c+\epsilon} f(t)e^{x\phi(t)}\,\mathrm{d}t \\
&\sim \int_{c-\epsilon}^{c+\epsilon} f(t)e^{x\phi(c)}e^{x\phi^{(p)}(c)(t-c)^p/p!}\,\mathrm{d}t.
\end{aligned}
$$

Let $s = \left[-x\phi^{(p)}(c)/p!\right]^{1/p}(t-c)$ and take $x \to \infty$,

$$
I \sim \left[\frac{p!}{-x\phi^{(p)}(c)}\right]^{1/p} f(c)e^{x\phi(c)}\int_{-\infty}^{\infty} e^{-sp}\,\mathrm{d}s.
$$

Note that $\int_0^\infty e^{-sp}\,\mathrm{d}s = \int_0^\infty \frac{e^{-u}u^{\frac{1}{p}-1}}{p}\,\mathrm{d}u = \frac{1}{p}\Gamma\left(\frac{1}{p}\right) = \Gamma\left(1+\frac{1}{p}\right)$. Then

$$
I \sim 2f(c)e^{x\phi(c)}\left[\frac{p!}{-x\phi^{(p)}(c)}\right]^{1/p}\Gamma\left(1+\frac{1}{p}\right)
$$

as $x \to \infty$.

(g) How do I get higher order terms? Keep HOT in the Taylor expansions of $f$ and $\phi$ (around $c$). This is ugly in general, but see p.22 - 23 in Chapling.

**Example** Consider $J(x) = \int_0^\pi e^{x\sin t}\,\mathrm{d}t$ as $x \to \infty$. $f = 1, \phi = \sin t$, max at $t = \frac{\pi}{2}$. Case (b) gives leading order term $J \sim \sqrt{\frac{2\pi}{x}}e^x$.

$$
\begin{aligned}
J(x) &\sim \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} e^{x\sin t}\,\mathrm{d}t \\
&= \int_{-\epsilon}^{\epsilon} e^{x\cos u}\,\mathrm{d}u && \left(u = t - \frac{\pi}{2}\right) \\
&\sim \int_{-\epsilon}^{\epsilon} \exp\left[x - \frac{xu^2}{2} + \frac{xu^4}{4!} + \dots\right]\mathrm{d}u \\
&\sim e^x\int_{-\epsilon}^{\epsilon} e^{-xu^2/2}\left(1 + \frac{xu^4}{4!} + \dots\right)\mathrm{d}u \\
&\sim \frac{e^x}{\sqrt{x}}\left[\int_{-\epsilon\sqrt{x}}^{\epsilon\sqrt{x}} e^{-s^2/2} + \frac{s^4e^{-s^2/2}}{4!x} + \dots\,\mathrm{d}s\right] && \left(s = u\sqrt{x}\right) \\
&\sim \frac{e^x}{\sqrt{x}}\left[\int_{-\infty}^{\infty} e^{-s^2/2}\,\mathrm{d}s + \frac{1}{4!x}\int_{-\infty}^{\infty} s^4e^{-s^2/2}\,\mathrm{d}s\right]
\end{aligned}
$$

Noting that

$$
\int_{-\infty}^{\infty} x^{2n}e^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi}(2n-1)!!,
$$

we get

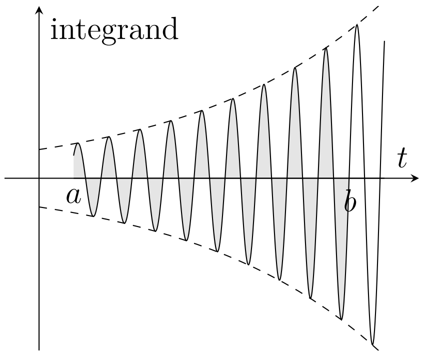
$$
J(x) \sim \sqrt{\frac{2\pi}{x}}e^x\left(1 + \frac{1}{8x} + \dots\right).
$$

## 3.6 Oscillating integrals and the method of stationary phase

Consider "Fourier-type" integrals like

$$F(x) = \int_a^b f(t)e^{ix\phi(t)}\,\mathrm{d}t$$

with $b > a$ and $x$ is large. As $x \to \infty$, the modulus of $e^{ix\phi}$ stays 1 but the integral oscillates more and more and goes crazy. In fact, you get catastrophic cancellation in integral and generally $F(x) \to 0$.




As $x$ gets very big, the 'wavelength' of oscillation becomes much much smaller than the longitude of variation of $f$ and $\phi$, so locally it looks like we are integrating sinuoids, which will integrate to zero.

**Theorem 3.2** (Riemann-Lebesgue lemma) $\displaystyle\int_a^b e^{ixt}f(t)\,\mathrm{d}t \to 0$ as $x \to \infty$ if $\displaystyle\int_a^b |f|\,\mathrm{d}t < \infty$ and $-\infty \leq a < b \leq \infty$.

- How fast does $F \to 0$ as $x \to \infty$?

$$\begin{aligned} F(x) &= \left[f\frac{e^{ixt}}{ix}\right]_a^b - \frac{1}{ix}\int_a^b f(t)e^{ixt}\,\mathrm{d}t \\ &= \frac{1}{ix}\left[f(b)e^{-xb} - f(a)e^{ixa} - \int_a^b f'(t)e^{ixt}\,\mathrm{d}t\right] \end{aligned}$$

  Assuming $f(a) \neq 0$ and $f(b) \neq 0$, and noting that $\displaystyle\int_a^b f'(t)e^{ixt}\,\mathrm{d}t \to 0$ as $x \to \infty$ by RL,

$$\begin{aligned} F(x) &\sim \frac{1}{ix}\left[f(t)e^{ixt}\right]_a^b \\ &= \mathcal{O}\left(\frac{1}{x}\right). \end{aligned}$$

- $f$'s behaviour at the endpoints controls $F$'s leading order asymptotes.
- If $f \to 0$ at endpoints then $F$ is small beyond all orders in $x^{-k}$. In particular, if $F(x) = \displaystyle\int_{-\infty}^{\infty} f(t)e^{ixt}\,\mathrm{d}t$, we usually assume that $f \to 0$ as $t \to \pm\infty$, so

$$F(x) = \mathcal{O}(x^{-N})\,\forall\, N \in \mathbb{Z}^+$$

  if $f$ is infinitely differentiable.

**Lemma 3.3** Consider general $\phi$ but monotonic on $[a, b]$, i.e. $\phi'(t) \neq 0$, then

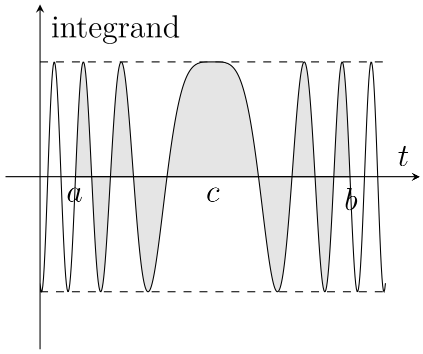
$$F(x) = \int_a^b f(t)e^{ixt}\,\mathrm{d}t \sim \frac{1}{ix}\left[\frac{f(t)}{\phi'(t)}e^{ix\phi(t)}\right]_a^b = \mathcal{O}\left(\frac{1}{x}\right).$$

**Proof** Letting $u = \phi(t)$,

$$\begin{aligned} F(x) &= \int_a^b f(t)e^{ixt}\,\mathrm{d}t \\ &= \int_{\phi(a)}^{\phi(b)} \frac{f(t)}{\phi'(t)} e^{ixu}\,\mathrm{d}u \\ &\sim \frac{1}{ix}\left[\frac{f(t)}{\phi'(t)}e^{ix\phi(t)}\right]_a^b = \mathcal{O}\left(\frac{1}{x}\right). \end{aligned}$$

□

What if $\phi$ has one stationary point $c \in (a,b)$, i.e. $\phi'(c) = 0$? Around the point $c$, the catastrophic cancellation in $F$ is less pronounced.




Localisation principle for oscillating integrals: "the asymptotic expansion of $F$ for large $x$ is controlled by points of stationary phase (and possibly the endpoints)." (Stokes, Kelvin)

$$F(x) = \underbrace{\int_a^{c-\epsilon} f(t)e^{ix\phi(t)}\,\mathrm{d}t}_{=\mathcal{O}(1/x)} + \int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\phi(t)}\,\mathrm{d}t + \underbrace{\int_{c+\epsilon}^{b} f(t)e^{ix\phi(t)}\,\mathrm{d}t}_{=\mathcal{O}(1/x)}.$$

Taylor expand $\phi$.

$$\begin{aligned} \phi(t) &= \phi(c) + \frac{\phi''(c)(t-c)^2}{2} + \dots \\ f(t) &= f(c) + \dots \end{aligned}$$

Then

$$\begin{aligned} \int_{c-\epsilon}^{c+\epsilon} f(t)e^{ix\phi(t)}\,\mathrm{d}t &\approx f(c)e^{ix\phi(c)}\int_{c-\epsilon}^{c+\epsilon}\exp\left[\frac{ix\phi''(c)(t-c)^2}{2}\right]\mathrm{d}t \\ &\approx \frac{f(c)e^{ix\phi(c)}}{|x\phi''(c)/2|^{1/2}}\int_{-|x\phi''(c)/2|^{1/2}}^{|x\phi''(c)/2|^{1/2}} e^{i\sigma s^2}\,\mathrm{d}s \qquad \left(s = \left|\frac{x\phi''(c)}{2}\right|^{1/2}(t-c), \sigma = \operatorname{sgn}(\phi''(c))\right) \\ &\sim \frac{f(c)e^{ix\phi(c)}}{|x\phi''(c)/2|^{1/2}}\underbrace{\int_{-\infty}^{\infty} e^{i\sigma s^2}\,\mathrm{d}s}_{=\sqrt{\pi}e^{i\sigma\pi/4}} \end{aligned}$$

for $\sigma = \pm 1$. Therefore,

$$F(x) \sim f(c)e^{ix\phi(c)}\sqrt{\frac{2\pi}{-ix\phi''(c)}}$$

This is the method of stationary phase.

**Example** $F_0(r) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{ir\sin t}\,\mathrm{d}t$, $f = 1, \phi = \sin t$. $\phi$ has two stationary points at $t = \frac{\pi}{2}, \frac{3\pi}{2}$, a max or min but both contribute to leading order.

At $t = \dfrac{\pi}{2}$, the contribution is $\sim \sqrt{\dfrac{2\pi}{ir}}e^{ir} = \sqrt{\dfrac{2\pi}{r}}e^{i(r-\pi/4)}$.

At $t = \dfrac{3\pi}{2}$, the contribution is $\sim \sqrt{\dfrac{2\pi}{-ir}} e^{ir} = \sqrt{\dfrac{2\pi}{r}} e^{-i(r-\pi/4)}$.

Add them together,

$$J_0(r)\text{``} \sim \text{''} \frac{2}{\sqrt{\pi r}} \cos\left(r - \frac{\pi}{4}\right).$$

Final points:

- *All* stationary points of $\phi$ contribute to leading order in the asymptotic expansion (compared with Laplace's method).
- Endpoints can contribute at next order in the expansion (at $\mathcal{O}(1/x)$).
- Be aware of the problem in the definition of $\sim$ if we find $F$ is oscillating and has many zeros.

## 3.7 Method of steepest descent

Consider integrals of the form $I(x) = \int_C f(z) e^{x\phi(z)}\,\mathrm{d}z$ as $x \to \infty$, where $C$ is a contour in $\mathbb{C}$, and $f$ and $\phi$ are complex-valued functions analytic in a domain containing $C$.

In what follows, $z \in \mathbb{C}$, $z = p + iq$, $p, q \in \mathbb{R}$ and $\phi = u(p,q) + iv(p,q)$, where $u, v$ are real functions. Then

$$I(x) = \int_C \underbrace{e^{xu}}_{\text{amplitude part}} \underbrace{e^{ixv}}_{\text{oscillating part}} \mathrm{d}z$$

**Remark** (Cauchy-Riemann equations) If $\phi$ is analytic, then $u_p = v_q$ and $u_q = -v_p$. Then for $\nabla u = (u_p, u_q)$ and $\nabla v = (v_p, v_q)$,

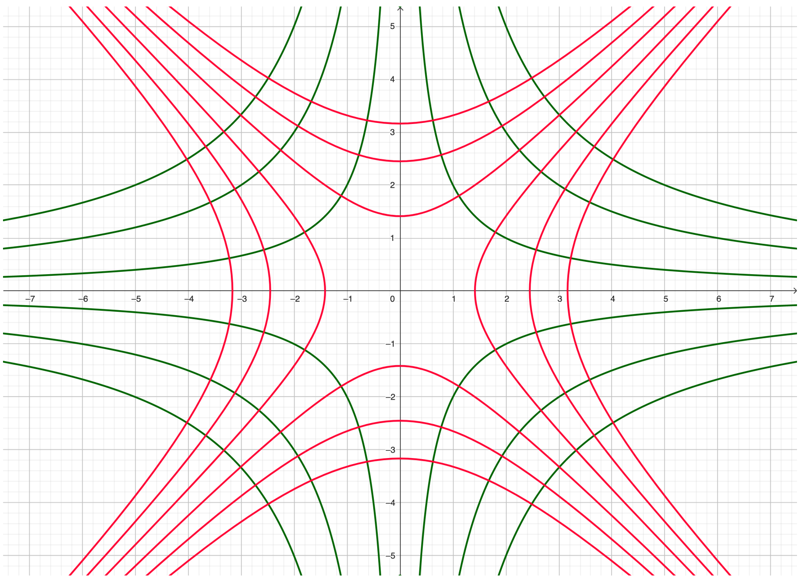
$$\nabla u \cdot \nabla v = 0,$$

i.e. level contours of $u$ and $v$ are orthogonal.

Also, $\nabla v$ is tangent to curves of constant $u$ and $\nabla u$ is tangent to curves of constant $v$.

On constant $v$ curves, $u$ is maximally decreasing or increasing, and vice versa.

**Example** Consider $\phi = z^2$, $u = p^2 - q^2$, $v = 2pq$.




**Remark** (Cauchy's theorem) Consider $\oint_C f e^{\phi x}\,\mathrm{d}z = 0$ if $f$ and $\phi$ are analytic in closed domain bounded by $C$. In particular, suppose $C_1$ and $C_2$ begin and end both at $z_1$ and $z_2$, integrating along both paths gives the same result.

(a) When $\operatorname{Im}\phi = v$ is constant on $C$.

Let $v = v_0 \in \mathbb{R}$ constant on $C$, then $C$ is a steepest descent / ascent curve for $u$. Parameterise $C$: $z \in \mathbb{C}$, $z = z(t) = p(t) + iq(t)$ for $t \in (a, b)$, $t \in \mathbb{R}$ and $a, b$ can be $\pm\infty$.

$$I(x) = e^{ixv_0} \int_a^b e^{xu} \operatorname{Re}\left(f \frac{\mathrm{d}z}{\mathrm{d}t}\right) \mathrm{d}t + e^{ixv_0} \int_a^b e^{xu} \operatorname{Im}\left(f \frac{\mathrm{d}z}{\mathrm{d}t}\right) \mathrm{d}t$$

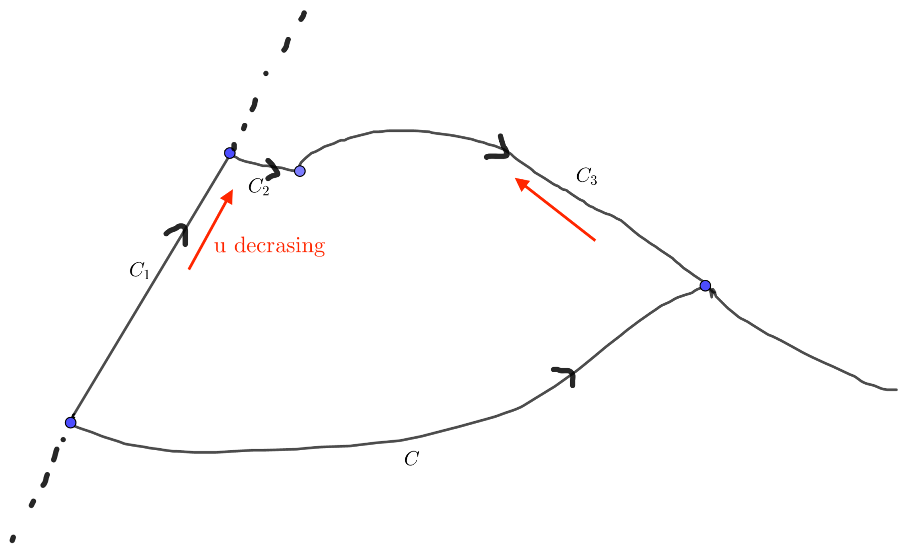
Both integrals are real and can be attacked by Laplace's method or Watson's lemma.

(b) Deforming $C$ to a steepest descent contour of $u$.

(i) If $v$ is not constant on $C$, deform $C$ to a curve on which it is using Cauchy's theorem. If $v$ is identical on $C$'s endpoints, this might work. Suppose $v = v_1$ at $z_1$ and $z_2$, then we can deform it to form a curve $C'$ on which $v = v_1$ and become case (a).

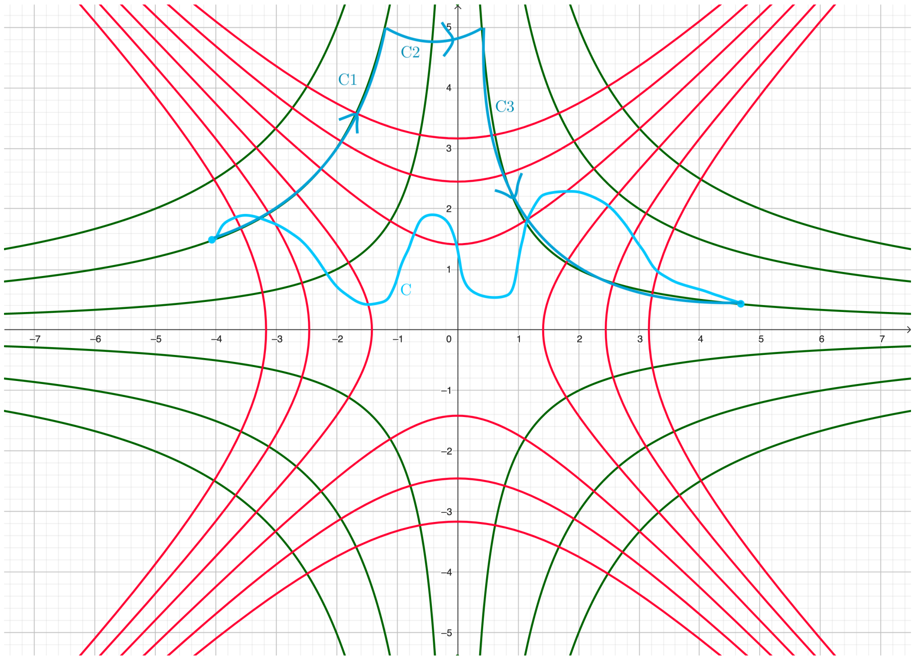
(ii) But this is not always possible and certainly doesn't work for $v$ takes different values at the two endpoints. Then we deform the curve $C$ to a curve that follows several constant $v$ curves joined up by short bridging curves (on which $v$ is not a constant.)

**Example** Suppose $v = v_1$ at $z_1$ and $v = v_2$ at $z_2$. We have $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$. We aim to show the contribution by $C_2$ to is very little.




- $u$ is maximally increasing / decreasing on $v$-constant curves ($C_1$ and $C_3$). We select curves so that $u$ is decreasing as we move away from the endpoints.
- Why? Assuming $\nabla u \neq 0$, Laplace's method tells us that the absolute maximum dominates the asymptotic expansions, and these will correspond to the endpoints.

**Example** Consider $I = \int_C f(z) e^{ixz^2} \, \mathrm{d}z$ as $z \to \infty$. We have $\phi = z^2$.




We have

$$I = \underbrace{\int_{C_1}}_{=\mathcal{O}(e^{xu_1}/x)} + \int_{C_2} + \underbrace{\int_{C_2}}_{=\mathcal{O}(e^{xu_2}/x)} .$$

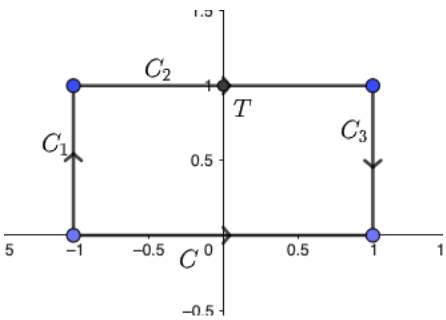
Is $C_2$ small? Take $C_1$ and $C_3$ to infinity and join them up with an infinitely short $C_2$ which contributes nothing.

**Example** $J_0(r) = \dfrac{1}{\pi}\displaystyle\int_0^\pi e^{ir\cos t}\,\mathrm{d}t$ as $r \to \infty$. The method of stationary phase gives us $J_0(r) \sim \sqrt{\dfrac{2}{\pi r}}\cos\left(r - \dfrac{\pi}{4}\right)$.

First let $s = \cos t$, then

$$J_0 = \frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-s^2}}e^{irs}\,\mathrm{d}s$$

Then let $s = z \in \mathbb{C}$ and $\phi = iz \Rightarrow u = -q, v = p$. Constant $v$ curves are vertical lines.




Let $C_1 : z = -1 + it, t \in [0, T]$, $C_2 : z = t + iT, t \in [-1, 1]$ and $C_3 : z = 1 - it, t \in [-T, 0]$. Cauchy's theorem gives $\displaystyle\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$.

$$\int_{C_1} = \frac{i}{\pi}\int_0^T \frac{e^{ir(-1+it)}}{\sqrt{t(t+2i)}}\,\mathrm{d}t = ie^{-ir}\int_0^T \frac{e^{-rt}}{\sqrt{t(t+2i)}}\,\mathrm{d}t$$

$$\int_{C_3} = -\frac{i}{\pi}\int_{-T}^0 \frac{e^{ir(1-it)}}{\sqrt{t(t-2i)}}\,\mathrm{d}t = -ie^{ir}\int_0^T \frac{e^{-rt}}{\sqrt{t(t-2i)}}\,\mathrm{d}t$$

and

$$\int_{C_2} = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{irt-rT}}{\sqrt{1-(t+iT)^2}}\,\mathrm{d}t$$

$$\left|\int_{C_2}\right| \le e^{-rT}\int_{-1}^{1}\left|\frac{1}{\sqrt{1-(t+iT)^2}}\right|\mathrm{d}t.$$

Let $T\to\infty$, then $\left|\dfrac{1}{\sqrt{1-(t+iT)^2}}\right| \to 0$, so

$$\begin{aligned}
J_0(r) &= \frac{1}{\pi} i e^{-ir}\int_0^\infty \frac{e^{-rt}}{\sqrt{t(t+2i)}}\,\mathrm{d}t - \frac{ie^{ir}}{\pi}\int_0^\infty \frac{e^{-rt}}{\sqrt{t(t-2i)}}\,\mathrm{d}t \\
&= \frac{2}{\pi}\operatorname{Re}\left(e^{ir-i\pi/2}\int_0^\infty \frac{e^{-it}}{\sqrt{t(t-2i)}}\,\mathrm{d}t\right) \\
&\sim \frac{2}{\pi}\operatorname{Re}\left(e^{ir-\pi/2}r^{-1/2}\sum_{n=0}^{\infty}\frac{a_n\Gamma\left(n+\frac{1}{2}\right)}{r^n}\right),
\end{aligned}$$

where $\dfrac{1}{\sqrt{t-2i}} = \sum a_n t^n$. So

$$J_0(r) \sim \sqrt{\frac{2}{\pi r}}\left[\cos\left(r-\frac{\pi}{4}\right) + \frac{1}{8r}\sin\left(r-\frac{\pi}{4}\right) - \frac{9}{128r^2}\cos\left(r-\frac{\pi}{4}\right) - \dots\right].$$

(c) Saddle points

- Laplace's method tells us that the absolute maximum of $u$ dominates $I(x)$. This might occur at endpoints or internal points.
- Note if $f=0$ at endpoints, they don't contribute to asymptotic expansion.
- If $C$ goes from $\infty$ to $\infty$ in $\mathbb{C}$, there are no endpoints.

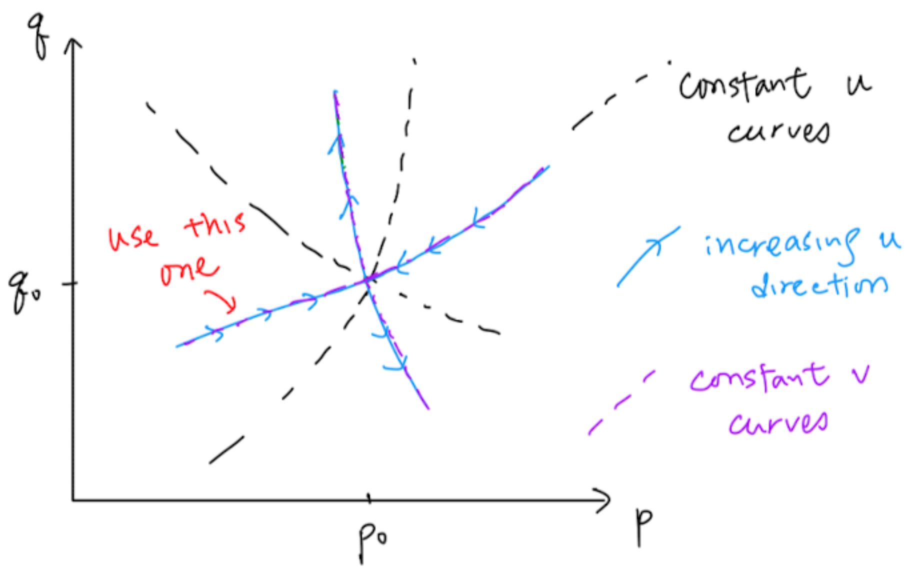
Now, consider the 'internal' maximum of $u$ into procedure. Recall that

(i) If $u$ has a critical point in $(p,q)$ plane, $\nabla u = 0$ there. But also, $v$ have a critical point there as well. It follows from Cauchy Riemann equations.

(ii) Critical points of $u$ and $v$ are saddles in the $(p,q)$ plane.
First, we have $\nabla^2 u = 0$ by CR. Note that for $u$ to have maximum or minimum, then $u_{pp}u_{qq} - u_{pq}^2 > 0$ (consider the Hessian matrix). Then $-u_{pp}^2 - u_{pq}^2 > 0$ and that's a contradiction. So critical points of $u$ are saddles. This also holds for $v$.

(iii) If $z_0 = (p_0, q_0)$ is a critical point of $u$ and $v$, then $\dfrac{\mathrm{d}\phi}{\mathrm{d}z} = 0$ there from CR.

Suppose $(p_0,q_0)$ is a simple saddle of $u$ and $v$, i.e. $\dfrac{\mathrm{d}^2\phi}{\mathrm{d}z^2} \neq 0$,




If we are to deform $C$ to pass through $(p_0, q_0)$, we select the constant $v$-curve on which $u$ is decreasing away from that point on either side.

Procedure:

(a) Find locations of all critical points by $\frac{\mathrm{d}\phi}{\mathrm{d}z} = 0$.

(b) If possible, deform $C$ such that it passes through the saddle points.

(c) The contribution from the saddle point can be calculated from local information at that saddle point $z_0$.

In particular, we can linearise $C'$ (treat it like a straight tangent line through $z_0$) and we parameterise the curve simply

$$z(t) = z_0 + te^{i\theta},$$

where $\theta$ is the angle (inclination) of the tangent line, and then throw into $I(x)$ and use Laplace's method.

Along $v$ constant curves,

$$\phi(z) - \phi(z_0) \in \mathbb{R}$$

and for simple saddle ($\phi'' \neq 0$),

$$\begin{aligned} \phi(z) - \phi(z_0) &= \frac{1}{2}\phi''(z_0)t^2 e^{2i\theta} + \dots \\ &= \frac{1}{2}\left|\phi''(z_0)\right| t^2 e^{2i\theta+\alpha}, \end{aligned}$$

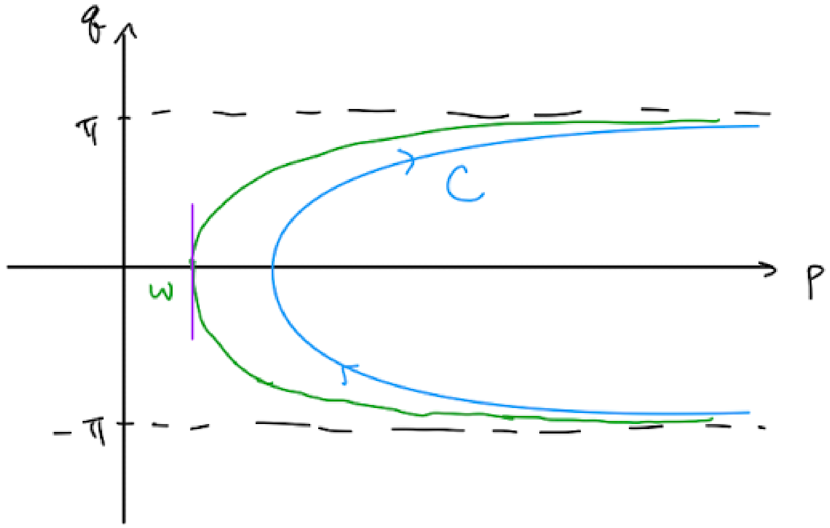
where $\alpha = \arg(\phi''(z_0))$, so

$$\sin(2\theta + \alpha) = 0 \Rightarrow \theta = -\frac{\alpha}{2} + \frac{n\pi}{2},$$

where $n = 0, 1, 2, 3$, and $n = 1, 3$ are the steepest descent rays. We then put $z(t)$ into $I(x)$ and use Laplace's method for $n = 1, 3$.

For non-simple cases, see p.37-38 in Stuart.

**Example** (Debye integral) $I(x) = \int^C e^{x(\operatorname{sech} w \sinh z - z)}\,\mathrm{d}z$, where $w \in \mathbb{R}, w \neq 0$ and $x > 0$ is very large, $z = p + iq$.




Note that $f = 1$, $\phi = \operatorname{sech} w \sinh z - z$. $\phi' = \operatorname{sech} w \cosh z - 1 = 0 \Rightarrow z_0 = w$ saddle point.

Which is the steepest descent curve through $z_0$? When $z = w + te^{i\theta}$. Let $\phi_0 = \phi(z_0) = \tanh w - w$. Then by Taylor,

$$\phi - \phi_0 = \tanh w(\cos 2\theta + i \sin 2\theta)\frac{t^2}{2} + \dots$$

We need $\sin 2\theta = 0\,(\cos 2\theta < 0) \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Now apply Laplace's method to each $\theta$.

For the upper bit ($\theta = \pi/2$),

$$I_1 = \int_0^\epsilon e^{x\phi}\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t,$$

where $\frac{\mathrm{d}z}{\mathrm{d}t} = e^{i\theta} = i$, so

$$\begin{aligned} I_1 &\approx \int_0^\epsilon e^{-\tanh \frac{wt^2x}{2}}\,\mathrm{d}t \\ &\sim ie^{x\phi_0}\sqrt{\frac{2}{x\tanh w}}\int_0^\infty e^{-s^2}\,\mathrm{d}s \\ &\sim \frac{ie^{x\phi_0}}{2}\sqrt{\frac{2\pi}{x\tanh w}}. \end{aligned}$$

For the lower bit ($\theta = 3\pi/2$),

$$I_2 = -\int_0^\epsilon e^{x\phi}\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t.$$

Combining,

$$I \sim ie^{x(\tanh w - w)}\sqrt{\frac{2\pi}{x\tanh w}}.$$

**Remark** In fact, generally, the contribution from a saddle point to $I(x)$ may be expressed as

$$\sim e^{x\phi(z_0)}\sqrt{\frac{-2\pi}{x\phi''(z_0)}}f(z_0).$$

## 3.8 The asymptotes of the Airy function

The Airy equation is

$$y'' - xy = 0.$$

It provides the canonical description of turning points as wave transitions. One solution is

$$A_i(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{1}{3}it^2+xit}\,\mathrm{d}t.$$

The other solution is $B_i(x)$.

We will be interested in the large $x$ asymptotics.

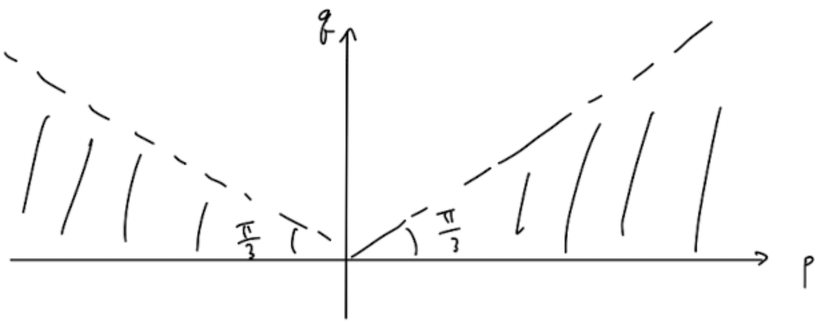
(a) $x > 0$ large

First make a substitution $s = \sqrt{x}t$, then

$$A_i = \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{\infty} e^{ix^{3/2}\left(\frac{1}{3}s^3+s\right)}\,\mathrm{d}s.$$

Stationary phase doesn't work, so try steepest descent with $s = z \in \mathbb{C}$.

Note that the new contour in $\mathbb{C}$ must have $\mathrm{Im}(z^3) > 0$ far from the origin.

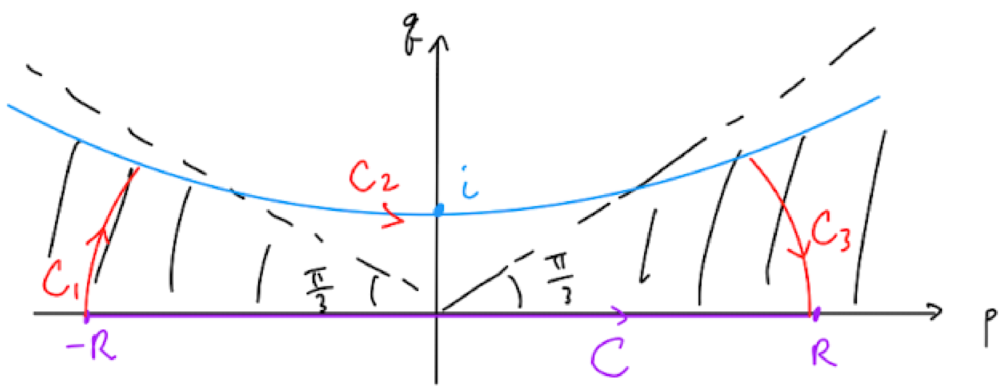



We want to find the saddle points for $\phi = i\left(\frac{1}{3}z^3 + z\right)$. $\phi' = i(z^2+1) = 0 \Rightarrow z = \pm i$.

Now find the constant $v$ curves. $\phi = u + iv$ and $z = p + iq$, then

$$u = \frac{1}{3}q^2 - p^2 q - q$$
$$v = \frac{1}{3}p^3 - pq^2 + p = \text{const}.$$

Note that $v = 0$ on the upper saddle point. Then the constant $v$ curves are $p = 0$ (the steepest ascent curves for $u$) or $q = \pm\sqrt{\frac{1}{3}p^2+1}$ (the steepest descent curves for $u$).



Note that $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$. As $R \to \infty, \int_{C_1}, \int_{C_3} \to 0$ because the integrand $\sim e^{-R^3}$.

Now use formula for saddle point contribution to find $\int_{C_2}$.

$$A_i = \int_C \sim \int_{C_2} \sim \frac{\sqrt{x}}{2\pi} e^{x^{3/2}\phi_0} \sqrt{\frac{-2\pi}{x^{3/2}\phi_0''}}$$
$$\sim \frac{1}{2\pi} x^{-1/4} e^{-\frac{2}{3}x^{3/2}}$$

as $x \to +\infty$.

(b) $x < 0$ and large

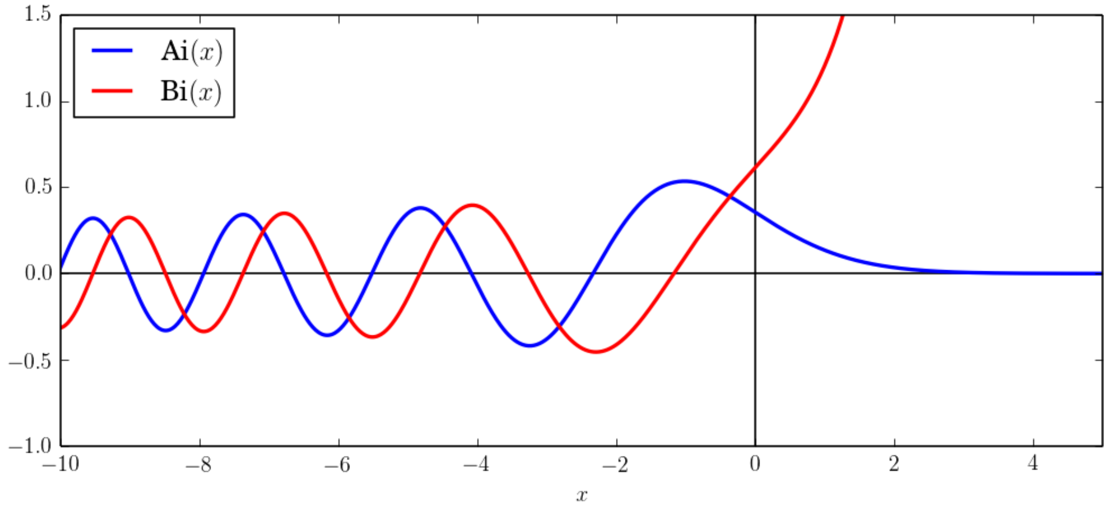
Substitute $s = \sqrt{-x}t$, and

$$A_i(x) = \frac{\sqrt{-x}}{2\pi} \int_{-\infty}^{\infty} \exp\left(i(-x)^{3/2}\left(\frac{1}{3}s^3 - s\right)\right) \mathrm{d}s.$$

Stationary phase will work. $\phi = \frac{1}{3}s^3 - s \Rightarrow s = \pm 1$ are the points of stationary phase. Then

$$A_i(x) \sim \frac{\sqrt{-x}}{2\pi}\left[\sqrt{\frac{2\pi}{-(-x)^{3/2}\phi''(1)}} e^{i(-x)^{3/2}\phi(1)} + \sqrt{\frac{2\pi}{-(-x)^{3/2}\phi''(-1)}} e^{i(-x)^{3/2}\phi(-1)}\right]$$
$$\sim \frac{(-x)^{-1/4}}{\sqrt{\pi}} \cos\left[\frac{2}{3}(-x)^{3/2} - \frac{\pi}{4}\right]$$

as $x \to \infty$.



# 4 Approximating Solutions of ODEs

We look at 2 problems:

1. How to construct 'local' asymptotic solutions near an ODE's irregular singular point
2. How to construct 'global' solutions to ODEs with a small / large parameters

## 4.1 Review of 2nd order ODEs

Consider $y'' + p(x)y' + q(x)y = 0$, where $p, q$ are known functions.

- Often we want to construct 'local' solutions around some point $x_0$.
- If $p$ and $q$ are analytic at $x_0$, then $x_0$ is an ordinary point, and so we are guaranteed 2 series solutions of the form
$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$
- If $p$ and $q$ are not analytic at $x_0$, but $(x - x_0)p$ and $(x - x_0)^2 q$ are, then $x_0$ is a regular singular point and we are guaranteed 1 solution of the form
$$y = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+\sigma}.$$
- Otherwise, $x_0$ is an irregular singular point, and we don't know what $y$ is.
- We can categorise $\infty$ similarly by setting $t = 1/x$ and checking $t = 0$.
Let $Y(t) = y(x)$, $\dfrac{\mathrm{d}Y}{\mathrm{d}x} = -t\dfrac{\mathrm{d}Y}{\mathrm{d}t}$, $\dfrac{\mathrm{d}^2Y}{\mathrm{d}x^2} = t^4\dfrac{\mathrm{d}^2Y}{\mathrm{d}t^2} + 2t^3\dfrac{\mathrm{d}Y}{\mathrm{d}t}$, then we have
$$Y'' + \tilde{p}(t)Y' + \tilde{q}(t)Y = 0,$$
where $\tilde{p} = \dfrac{2}{t} - \dfrac{p(1/t)}{t^2}$ and $\tilde{q} = \dfrac{q(1/t)}{t^4}$, then categorise $t = 0$.

**Example** $y'' + \dfrac{1}{x}y' + (\dfrac{1}{x} - \dfrac{1}{x^2})y = 0$. $x = 0$ is a regular singular point.

**Example** $y'' + \dfrac{7}{x^2}y' + y = 0$. $x = 0$ is an irregular singular point.

**Example** $y'' - xy = 0$. $x \to \infty$ is what? The equation can be written as
$$Y'' + \frac{2}{t}Y' - \frac{1}{t^5}Y = 0,$$
so $t = 0$ is an irregular singular point, and so is $x = \infty$.

ODE takes the form $y'' + p(x)y' + q(x)y = 0 \quad (*)$ can be reduced to the canonical form
$$u'' = Q(x)u.$$
Let $y = u(x)v(x)$, substitute into $(*)$ and get
$$vu'' + (2v' + pv)u' + (v'' + pv' + qv)u = 0.$$
Choose $v$ such that $2v' + pv = 0 \Rightarrow v = \exp\left(-\dfrac{1}{2}\displaystyle\int p\,\mathrm{d}x\right)$, then
$$u'' = \underbrace{-\left(\frac{v'' + pv' + qv}{v}\right)}_{Q} u.$$

## 4.2 Irregular singular points and the Liouville-Green method

Consider $y'' - Q(x)y = 0 \quad (**)$ and $x = x_0$ is an irregular singular point (so $Q$ blows up badly there).

**Example** $y'' = \dfrac{1}{x^4}y$, and $x = 0$ is an irregular singular point, but
$$y = Axe^{1/x} + Bxe^{-1/x}.$$
This can be showed by letting $t = 1/x$.

Noting the poles in the exponential, we try

$$y = e^{S(x)}$$

and we stick with (∗∗). $y' = e^S S'$ and $y'' = e^S S'' + e^S (S')^2$, and (∗∗) becomes

$$S'' + (S')^2 = Q(x).$$

Try $S \sim S_0 + S_1 + S_2 + \dots$, where $S_i$ form an asymptotic sequence as $x \to x_0$. To leading order,

$$S_0'' + (S_0')^2 \sim Q.$$

But assume $S_0'' \ll (S_0')^2$ as $x \to x_0$ (to be checked later),

$$\Rightarrow (S_0')^2 \sim Q \Rightarrow S_0 \sim \pm \int \sqrt{Q}\,\mathrm{d}x.$$

In fact, we get

$$S_0 = \pm \int \sqrt{Q}\,\mathrm{d}x.$$

Consistrnecy check:

$$S_0'' = \pm \frac{1}{2}\frac{Q'}{\sqrt{Q}} \ll Q = (S_0')^2$$

since if we suppose $Q \propto (x - x_0)^b$ where $b > 2$ near $x = x_0$.

To get $S_1$, we have

$$\begin{aligned}(S_0 + S_1)'' + (S_0' + S_1')^2 &\sim Q \\ \Rightarrow S_1'' + (S_1')^2 + 2S_0 S_1 &\sim -S_0'' = \mp \frac{Q'}{2\sqrt{Q}}.\end{aligned}$$

Now assume $S_1 \ll S_0$, $S_1' \ll S_0'$ and $S_1'' \ll 2S_0'S_1'$ (exercise),

$$\Rightarrow 2S_0'S_1' \sim \mp \frac{Q'}{2\sqrt{Q}}$$

$$\Rightarrow S_1' \sim -\frac{1}{4}Q'/Q$$

$$\Rightarrow S_1 = -\frac{1}{4}\ln Q$$

General asymptotic solution to (∗∗) is

$$y \sim AQ^{-1/4}e^{\int \sqrt{Q}\,\mathrm{d}x} + BQ^{-1/4}e^{-\int \sqrt{Q}\,\mathrm{d}x}$$

as $x \to x_0$.

> **Example** $y'' - \dfrac{a^2}{x^3}y = 0$, $a \in \mathbb{R}^+$ as $x \to 0^+$ (irregular singular point).
>
> Assume $y = e^{S(x)}$, $S = S_0 + S_1 + S_2 + \dots$. Assume $S_0'' \ll (S_0')^2$.
>
> - Leading order $(S_0')^2 = \dfrac{a^2}{x^3} \Rightarrow S_0 = \pm 2ax^{-1/2}$
> - Next order $S_1 = \dfrac{3}{4}\ln x$
>
> Checking the assumptions $\dfrac{S_1}{S_0} \propto \sqrt{x}\ln x$, $\dfrac{S_1'}{S_0'} \propto \sqrt{x}$, $\dfrac{S_1''}{S_0'S_1'} \propto \sqrt{x}$, then
>
> $$y \sim Ax^{3/4}e^{-2a/\sqrt{x}} + Bx^{3/4}e^{2a/\sqrt{x}}$$
>
> as $x \to 0^+$.

> **Example** $y'' = xy$ as $x \to \pm\infty$. We have
>
> $$\begin{aligned}y &\sim AQ^{-1/4}e^{\int \sqrt{Q}\,\mathrm{d}x} + BQ^{-1/4}e^{-\int \sqrt{Q}\,\mathrm{d}x} \\ Q &= x\end{aligned}$$
>
> Case 1: $x > 0$ and large
>
> $$y \sim Ax^{1/4}e^{-\frac{2}{3}x^{3/2}} + Bx^{-1/4}e^{\frac{2}{3}x^{3/2}}$$

This agrees with integral formula for $A_i$ but if $A = \dfrac{1}{2\sqrt{\pi}}$ and $B = 0$.

Case 2: $x < 0$ and large.
Then $\sqrt{Q} = i\sqrt{|Q|}$ and
$$y \sim C|x|^{-1/4}e^{\frac{2}{3}i|x|^{3/2}} + D|x|^{-1/4}e^{-\frac{2}{3}i|x|^{3/2}}.$$
Exercise: find $C$ and $D$ so it agrees with $A_i$ asymptotic formula as $x \to -\infty$.

## 4.3 ODEs with a small / large parameter

We consider ODEs of the form
$$y'' + \alpha(\epsilon, x)y' + b(\epsilon, x)y = 0,$$
or equivalently,
$$y'' = Q(\epsilon, x)y,$$
where $0 < \epsilon \ll 1$ and we have BCs or ICs.

Often solutions exhibit variations on 2 different scales – a short/fast scale and also a long/slow scale, sometimes at different regions of the domain.

(a) Regular vs singular singularities

Consider $y'' + 2\epsilon y' + y = 0$ as $\epsilon \to 0$.

- The 'zeroth order' solution, i.e. $\epsilon = 0$, then
$$y = Ae^{ix} + Be^{-ix}$$
- General solution is $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$, where
$$\begin{aligned}\lambda_{1,2} &= -\epsilon \pm i\sqrt{1-\epsilon^2}\\ &= \pm i - \epsilon \mp \frac{1}{2}\epsilon^2 + \dots\end{aligned}$$
The $\epsilon = 0$ solution matches smoothly to the $\epsilon \to 0$ limit. This is a *regular asymptotic* solution.

Now consider $\epsilon y^2 + y = 0$ as $\epsilon \to 0$.

- Zeroth order solution is $y = 0$ (but check BCs!)
- Genreal solution is
$$y = A\cos\left(\frac{x}{\epsilon}\right) + B\sin\left(\frac{x}{\epsilon}\right).$$
This doesn't match smoothly to the $\epsilon = 0$ solution. This is a *singular perturbation* problem. Generally, when $\epsilon$ multiplies the highest derivative in the ODE, you have a singular perturbation problem. Usually, asymptotic approaches are the only way forward.

(b) Asymptotic matching

The principle of asymptotic matching:

- Suppose the domain is broken into two overlapping regions.
- Suppose we have asymptotic approximation on each region / interval.
- Matching requires that the asymptotic approximations have the same functional form on the overlapping region.

## 4.4 The WKBJ method

Consider the BVP
$$\epsilon^2 y'' = q(x)y \tag{†}$$
for $\epsilon$ small, usually on an infinite domain and with BCs. $y \to 0$ as $x \to \pm\infty$. $q$ is a known function without singularities, which can be thought of as a potential and thus parallel to motion of a particle.

(a) Turning points

A point $x = a$ is a turning point if $q(a) = 0$. Usually a solution undergoes a qualitative shift in its behaviour as it goes through $x = a$. 'wavelike' behaviour for $q < 0$, and exponentially growing / decaying behaviour for $q > 0$.

**Example** $y'' = \operatorname{sgn}(x)y$ with $y \to 0$ as $x \to \infty$.

**Example** $y'' = xy$ (Airy's equation).

(b) SKBJ solution with one turning point

- Suppose (†) has 1 turning point at $x = a$, i.e. $q(a) = 0$, then (†) can be written as
$$y'' = \frac{q}{\epsilon^2}y = Qy,$$
where $Q$ is very large as $\epsilon \to 0$. This motivates a solution like
$$y = e^{S(x)/\epsilon}.$$
Substituting into (†),
$$\epsilon S'' + (S')^2 = q. \tag{‡}$$
We assume $S = \sum_{n=0}^{\infty} \epsilon^n S_n$, then $(\dots) + \epsilon(\dots) + \epsilon^2(\dots) + \dots = 0$, so
  - zeroth order gives $(S_0')^2 = q \Rightarrow S = \pm \int \sqrt{q}\,\mathrm{d}x$.
  - At order $\epsilon$, we have
$$\begin{aligned} S_0'' + 2S_0'S_1' &= 0 \\ \Rightarrow S_1' &= -\frac{1}{2}(\ln S_0')' \\ \Rightarrow S_1 &= -\frac{1}{4}\ln q, \end{aligned}$$
so
$$y \sim Aq^{-1/4}e^{\frac{1}{\epsilon}\int \sqrt{q}\,\mathrm{d}x} + Bq^{-1/4}e^{-\frac{1}{\epsilon}\int \sqrt{q}\,\mathrm{d}x} + \dots.$$
This solution is only good far from turning points.
When $q > 0$, then $y$ is exponentially decaying / growing. When $q < 0$, $y$ oscillates with local frequency $\approx q/\epsilon$ (recall if $y \propto e^{i\phi(x)}$, then local frequency $= \dfrac{\mathrm{d}\phi}{\mathrm{d}x}$).
- This approximation is invalid near and at $q = 0$ because asymptotic ordering in (‡) assumed $|q| \gg \epsilon$.

  Question: How close to $x = a$ is our approximation valid?

  Assume $|x - a| \ll 1$, then $q \approx q'(x)(x - a)$. Plugging this into $|q| \gg \epsilon$, we have $|x - a| \gg \epsilon$. At $q < 0$, we have
$$\begin{aligned} y_1 &= A|q|^{-1/4}e^{\frac{i}{\epsilon}\int \sqrt{|q|}\,\mathrm{d}x} + B|q|^{-1/4}e^{\frac{i}{\epsilon}\int \sqrt{|q|}\,\mathrm{d}x} \\ &= C|q|^{-1/4}\cos\left(\frac{1}{\epsilon}\int_x^a \sqrt{|q|}\,\mathrm{d}x' - \gamma\right), \end{aligned}$$
where $\gamma, C$ are constants. For $q > 0$, we have
$$y_2 = Dq^{-1/4}e^{-\frac{1}{2}\int_a^x \sqrt{q}\,\mathrm{d}x'}$$
- Solutions near the turning point

  Assume $|x - a| \ll 1$. (†) can be approximated by
$$\epsilon^2 y'' = \mu(x - a)y,$$

where $\mu = q'(a)$. Let $z = (\mu/\epsilon^2)^{1/3}(x-a)$, then this becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = zy.$$

This is the Airy equation! So the solution around the turning point is

$$y_3 = EAi(z).$$

We throw away $B_i$ since it blows up at $\infty$. We need to match / glue $y_1$ to $y_3$ to $y_2$ to get a consistent global solution to (†).

- Matching

  For the overlapping of zone 1 to zone 3,

  $$\epsilon \ll a - x \ll 1.$$

  Here, $q \approx \mu(x-a)$. Taylor expanding,

  $$y_1 \approx C(\mu(a-x))^{-1/4} \cos\left(\frac{\sqrt{\mu}}{\epsilon}\frac{2}{3}(a-x)^{3/2} - \gamma\right).$$

  For large negative $z$,

  $$\begin{aligned} y_3 &\approx E\frac{1}{\sqrt{\pi}}(-z)^{-1/4}\cos\left(\frac{2}{3}(-z)^{3/2} - \frac{\pi}{4}\right) \quad \text{(true for } |z| \gg 1) \\ &\approx \frac{E}{\sqrt{\pi}\left((\mu/\epsilon^2)^{1/3}(a-x)\right)^{1/4}}\cos\left(\frac{2}{3}\frac{\sqrt{\mu}}{\epsilon}(a-x)^{3/2} - \frac{\pi}{4}\right). \end{aligned}$$

  Since $-z \ll 1$, $a - x \gg \epsilon^{2/3} \gg \epsilon$. This approximation for $y_3$ lies in the overlapping region because $\epsilon^{2/3} \ll 1$ as $\epsilon \to 0$.
  So matching requires $\gamma = \pi/4$ and $E = \sqrt{\pi}(\mu\epsilon)^{-1/2}C$.

  For the overlapping of zone 3 and zone 2, we have $\epsilon \ll x - a \ll 1$, and

  $$\begin{aligned} y_2 &\approx D\left(\mu(x-a)\right)^{-1/4}\exp\left(-\frac{\sqrt{\mu}}{2}\frac{2}{3}(x-a)^{3/2}\right) \\ y_3 &\approx \frac{E}{2\sqrt{\pi}z^{1/4}}e^{-\frac{2}{3}z^{3/2}} \quad (z \gg 1) \\ y_3 &\approx \frac{E}{2\sqrt{\pi}\left((\mu/\epsilon^2)^{1/3}(x-a)\right)^{1/4}}\exp\left(-\frac{2}{3}\frac{\sqrt{\mu}}{\epsilon}(x-a)^{3/2}\right) \quad (x - a \gg \epsilon^{2/3}) \end{aligned}$$

  Therefore, we get $E = 2\sqrt{\pi}(\mu\epsilon)^{-1/6}D$.

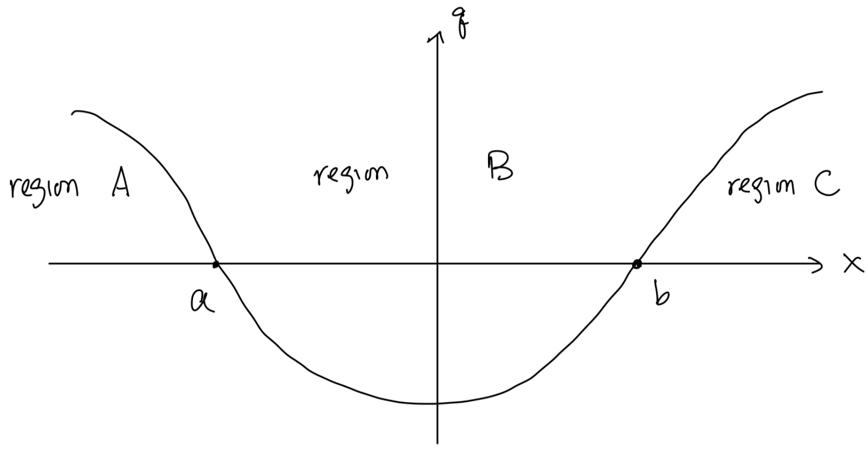
  In sum, we have the "connection formulae"

  $$y \approx \begin{cases} Dq^{-1/4}\exp\left(-\frac{1}{\epsilon}\int_a^x \sqrt{\mu}\,\mathrm{d}x'\right) & x - a \gg \epsilon \\ 2\sqrt{\pi}(\mu\epsilon)^{-1/6}D\mathrm{Ai}\left(\epsilon^{-2/3}\mu^{1/3}(x-a)\right) & |x-a| \ll 1 \\ 2D|q|^{-1/4}\cos\left(\frac{1}{\epsilon}\int_x^a \sqrt{|q|}\,\mathrm{d}x' - \frac{\pi}{4}\right) & a - x \gg \epsilon \end{cases},$$

  and $D$ remains undetermined because we have not imposed a BC at $x \to -\infty$.
- If $q'(a) < 0$, transform $x \mapsto -x$ and apply same process and then transform back.

(c) WKBJ method with two turning points

Suppose $a < b$ with $q(a) = q(b) = 0$ and $q'(a) < 0$, $q'(b) > 0$, and $q$ does not diverge in $[a, b]$.




What is $y$ in this situation when we insist that $y \to 0$ as $x \to \pm\infty$?
In region A,

$$y_A \sim Aq^{-1/4} \exp\left(-\frac{1}{\epsilon}\int_x^a \sqrt{q}\,\mathrm{d}x'\right)$$

valid for $a - x \gg \epsilon$. In region C,

$$y_C \sim Cq^{-1/4} \exp\left(-\frac{1}{\epsilon}\int_b^x \sqrt{q}\,\mathrm{d}x'\right)$$

valid for $x - b \gg \epsilon$. In region B,

$$y_B \sim B|q|^{-1/4} \cos\left(\frac{1}{\epsilon}\int \sqrt{q}\,\mathrm{d}x - \gamma\right)$$

valid for $x - a \gg \epsilon$ and $b - x \gg \epsilon$.

Matching $y_A$ to $y_B$ and $y_B$ to $y_C$ gives different expressions, so they have to agree.

Matching $y_A$ to $y_B$:

$$\begin{aligned} y_B = y_{AB} &\sim 2A|q|^{-1/4} \cos\left(\frac{1}{\epsilon}\int_a^x \sqrt{|q|}\,\mathrm{d}x' - \frac{\pi}{4}\right) \\ &= 2A|q|^{-1/4}\cos\theta \end{aligned}$$

Matching $y_B$ to $y_C$:

$$\begin{aligned} y_B = y_{BC} &\sim 2C|q|^{-1/4} \cos\left(\frac{1}{\epsilon}\int_x^b \sqrt{|q|}\,\mathrm{d}x' - \frac{\pi}{4}\right) \\ &= 2C|q|^{-1/4}\cos(\phi - \theta), \end{aligned}$$

where $\phi = \dfrac{1}{\epsilon}\displaystyle\int_a^b \sqrt{|q|}\,\mathrm{d}x - \dfrac{\pi}{2}$.

In region B that $y_{AB} = y_{BC}$, so $A = \pm C$, then

$$\cos\theta \pm \cos(\phi - \theta) = 0.$$

Both cases ($\pm$) are removed by $\cos^2(\phi - \theta) - \cos^2\theta = 0 \Rightarrow \sin(2\theta - \phi)\sin\phi = 0$.
If $\sin\phi = 0$, this holds for all $x$, then $\phi = n\pi$, where $n = 0, 1, 2, 3\ldots$, and

$$\frac{1}{\epsilon}\int_a^b \sqrt{|q|}\,\mathrm{d}x = \left(n + \frac{1}{2}\right)\pi.$$

This 'quantisation condition' must be met if there exist bound solutions to (†) with the specified BCs: If $q = q(x, \lambda)$, this condition specifies what values $\lambda$ can take.

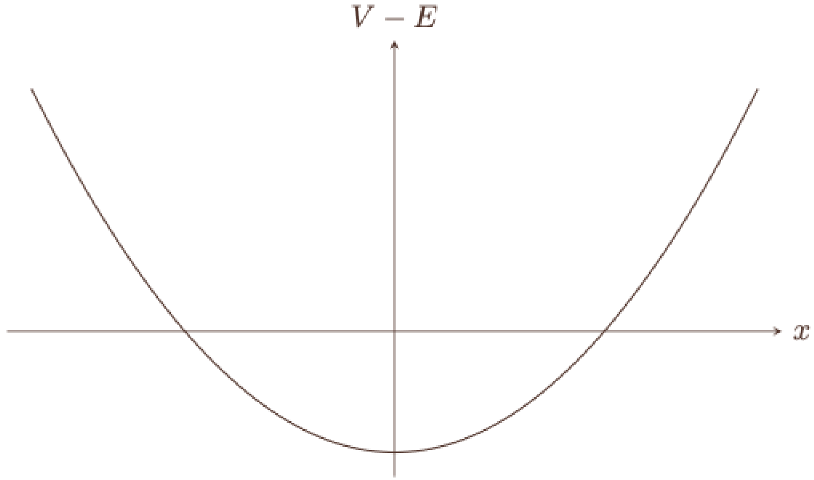
We have $A = (-1)^n C$. If $A = C$, then we have the 'even solution'. If $A = -C$, we have the 'odd solution'.

### 4.4.1 1D bound states in QM

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2}+V(x)\Psi=\Psi$$

is the time-independent Schrodinger equaiton. It describes the wave function $\Psi$ of a particle of mass $m$ and energy $E$ in a 1D potential $V(x)$, and $\hbar$ is the Planck's constant. We want bound states $\Psi \to 0$ as $|x| \to \infty$.

We identify $\epsilon = \hbar$ and $q = 2m(V - E)$. Consider $V(x)$ such that for some $E$, $V - E$ is well shaped.




For a given $\epsilon > 0$, there are 2 turning points $a$ and $b$, these are determined by $V(x) = E$ so $a = a(E)$ and $b = b(E)$.

A bound state requires

$$\int_{a(E)}^{b(E)} \sqrt{2m(E-V(x))}\,\mathrm{d}x = \left(n+\frac{1}{2}\right)n\hbar \qquad (*)$$

for $n = 0, 1, 2, \ldots$. For each $n$, we then solve $E = E_n$. We have $E = E_0, E_1, E_2, \ldots$ with $E_{n+1} > E_n$. $(*)$ is the Bohr-Sommerfeld quantisation condition.

In classical mechanics,

- a bound particle oscillates strictly between $a$ and $b$.
- $E = p^2/2m + V$, with $p$ the momentum, so $p^2 = 2m(E - V)$. The quantisation condition can then be written as

  $$\oint p\,\mathrm{d}x = (2n+1)\pi\hbar$$

  and integrating over 1 period of oscillation.
- The local wavenumber is

  $$\begin{aligned} k &= \sqrt{|q|}/\epsilon \\ &= \sqrt{2m(V-E)}/\hbar \\ &= p/\hbar \end{aligned}$$

  and the wavelength is $\dfrac{2\pi}{k} = \dfrac{2\pi\hbar}{p}$. This is the de Brogle wavelength.

### 4.4.2 The QM harmonic oscillator

$V = \frac{1}{2}m\omega^2x^2$, where $\omega$ is a constant. Newton's second law gives

$$\ddot{x} + \omega^2 x = 0.$$

We alao have

$$E = \frac{p^2}{2m} + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

For fixed $E$, a particle follows an elliptical orbit in phase space $(x, p)$. The area enclosed by an elliptical orbit is

$$\iint_{\text{ellipse}} \mathrm{d}x\mathrm{d}p = \oint p\,\mathrm{d}x = (2n+1)\pi\hbar = \frac{2\pi E}{\omega}.$$

The semi-axis are $\sqrt{2mE}$ and $\sqrt{\frac{2E}{m\omega^2}}$, so

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

for $n = 0, 1, 2, \ldots$.

# 5 More on Stokes phenomena

## 5.1 Stokes and anti-stokes lines

Recall $f(z) \sim g(z)$ for $z \in \mathbb{C}$ and $z \to z_0$ is generally valid in a limited sector centred on $z_0$, or if $z_0 = \infty$ in a sector of $\mathbb{C}$.

**Example** $\cosh z = \frac{1}{2}\left(e^z + e^{-z}\right)$ and $|z| \to \infty$.

- $e^{-z}$ is exponentially small for $-\pi/2 < \arg z < \pi/2$, $\cosh z \sim \frac{e^z}{2}$.
- $e^z$ is exponentially small for $\pi/2 < \arg z < 3\pi/2$, $\cosh z \sim \frac{e^{-z}}{2}$.

**Example** $y'' = zy$, then

$$y \sim Az^{-1/4}e^{-\frac{2}{3}z^{3/2}} + Bz^{-1/4}e^{\frac{2}{3}e^{3/2}} \tag{$*$}$$

if $z \in \mathbb{R}$ and $z \to \infty$. But if $z \in \mathbb{C}$, this is not immediately obvious which term is dominant.

On crossing a sector boundary, an exponentially small term becomes exponentially large. More formally, $f = g + (f - g)$. If $f \sim g$ in some sector, then $f - g \ll f$ and $g$ is *dominant* and $f - g$ is *recessive*. At a sector boundary, $f - g$ becomes same order as $g$ and on the other side $f - g$ becomes dominant.

Consider a LG approximation to ODEs of form

$$f(z) \sim Ah(z)e^{S_1(z)} + Bh(z)e^{S_2(z)} \tag{$\ddagger$}$$

as $|z| \to \infty$. Suppose in some sector of $\mathbb{C}$, $\mathrm{Re}(S_2) \ll \mathrm{Re}(S_1)$, then $e^{S_2}$ term is recessive and the $e^{S_1}$ term is dominant.

We define a *Stokes line* to be where

$$\mathrm{Re}(S_1) = \mathrm{Re}(S_2),$$

thus

$$\left|\frac{e^{S_1}}{e^{S_2}}\right| = e^{\mathrm{Re}(S_1 - S_2)} = 1.$$

Along this line, the ratio of the two terms is purely oscillatory and of the same order. The dominant and recessive terms trade places as we cross a Stokes line.

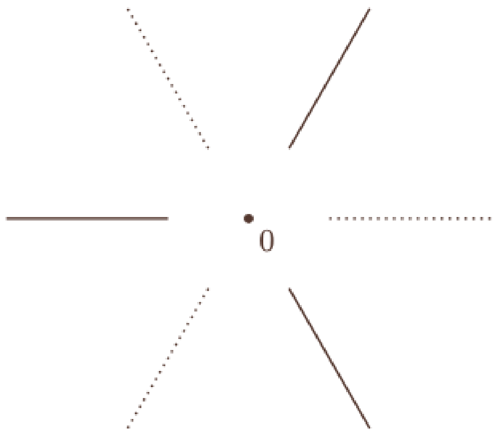
Another problem: $S_1$ and $S_2$ are not usually entire functions even if $f(z)$ is. (recall $S_1, 2 = \pm\int\sqrt{Q}\,\mathrm{d}x$)

Therefore, the approximation undergoes jumps when we cross branch cuts, and ($\ddagger$) cannot be true globally for fixed $A$, $B$. $A$, $B$ need to take different values at different places. It is best to change values of recessive coefficient where the real part of $S_1$ and $S_2$ differ the most. Often $\mathrm{Re}(S_1 - S_2)$ is maximally different near $\mathrm{Im}(S_1 - S_2) = 0$.

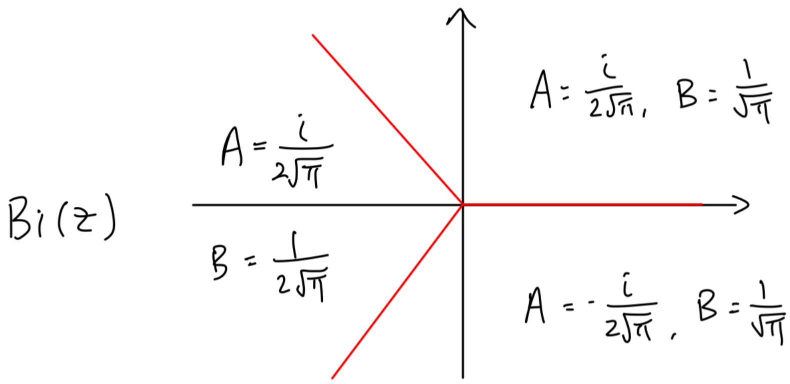
An *anti-Stokes line* is defined to be where $\mathrm{Im}(S_1) = \mathrm{Im}(S_2)$.

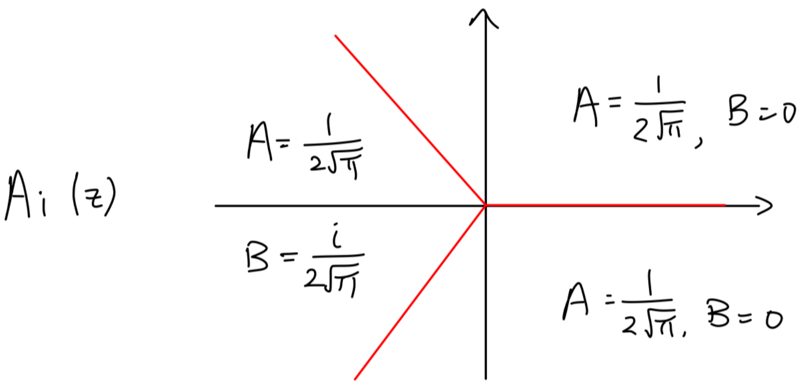
Warning: many others swap the terminology of Stokes and anti-Stokes lines.

**Example** $y'' = zy$, LG approximations is given by $(*)$.
Stokes lines are where $\mathrm{Re}(z^{3/2}) = 0 \Rightarrow \arg z = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$, and the anti-Stokes lines are where $\mathrm{Im}(z^{3/2}) = 0 \Rightarrow \arg z = 0, \frac{2}{3}\pi, \frac{4}{3}\pi$.




Two solutions of the Airy equation are $\mathrm{Ai}(z), \mathrm{Bi}(z)$.




## 5.2 *Super-asymptotics*

An optimally truncated series is one in which you retain the smallest term, discarding all terms of higher degree. This is called a superasymptotic approximation and the error is typically of $o(e^{-a/\epsilon})$, where $a$ is a constant and $\epsilon$ is our small parameter.

**Example** (Stieltjes integral)

$$I(\epsilon) = \int_0^\infty \frac{e^{-t}}{1+\epsilon t}\,\mathrm{d}t$$

where $0 < \epsilon \ll 1$. Recall

$$I(x) = \sum_{n=0}^{N} \underbrace{(-1)^n n! \epsilon^n}_{a_n} + E_N,$$

where

$$\begin{aligned} E_n &= \int_0^\infty \frac{e^{-t}(-\epsilon t)^{N+1}}{1+\epsilon t}\,\mathrm{d}t \\ &\le \epsilon^{N+1} \int_0^\infty t^{N+1} e^{-t}\,\mathrm{d}t \\ &\le \epsilon^{N+1}(N+1)!. \end{aligned}$$

For what $n$ is $\left|\dfrac{a_{n+1}}{a_n}\right| > 1$?

$$\Rightarrow (n+1)\epsilon > 1 \Rightarrow n > \frac{1}{\epsilon} - 1$$

So we set $N = \left\lfloor \dfrac{1}{\epsilon} - 1 \right\rfloor$ ($\sim \dfrac{1}{\epsilon}$ as $\epsilon \to 0$), then

$$\begin{aligned} E_N &\le \left(\frac{1}{\epsilon}\right)^{-1/\epsilon} \left(\left\lfloor \frac{1}{\epsilon} \right\rfloor\right)! \\ &= \text{‰}(\sqrt{\epsilon} e^{-1/\epsilon}). \end{aligned}$$

In this example, truncation gives exponentially small error, not algebraically (in $\epsilon$) small error.

## 5.3 *Hyperasymptotics*

A hyper asymptotic approximation achieves higher accuracy than a super-asymptotic one by adding terms from a 2nd (or a 3rd and 4th) series generated from the error terms $E_N$ but with different scaling assumptions.

**Example** (Stieltjes integral) When $N + 1 = \left\lfloor \dfrac{1}{\epsilon} \right\rfloor$, the integral in $E_N$ has a maximum at $t \approx \dfrac{1}{\epsilon}$, so

$$\frac{1}{1+\epsilon t} = \frac{1}{2\left(1 + \frac{1}{2}(\epsilon t - 1)\right)} = \frac{1}{2} \sum \left(-\frac{1}{2}\right)^k (\epsilon t - 1)^k$$

and

$$I(\epsilon) = \sum_{n=0}^{N} a_n + \sum_{n=0}^{m} b_n$$